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Справочник по высшей математике, часть II содержит следующие разделы: «Неопределенный, определенный и несобственные интегралы», «Дифференциальные уравнения», «Комплексные числа и ТФКП», «Интеграл по фигуре и Теория поля», «Ряды».

В конце справочника приведены русско-английский и англо-русский словари, включающие термины всех рассмотренных тем.

Данный справочник предназначен для студентов, изучающих математику на английском языке, а также магистрантам и аспирантам.

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CONTENTS

I. INDEFINITE INTEGRAL

Definitions. Properties

	Names, definitions, theorems	Formulas
1	An **antiderivative** (a primitive) of a given function *f(x)* in a given interval is any function *F(x)* whose derivative is equal to the given function for any point of this interval.	$F'(x) = f(x)$
2	Let $F_1(x) \neq F_2(x)$ are antiderivatives of a function *f(x)* in any interval then	$F_2(x) = F_1(x) + C$
3	An **indefinite integral** of a function $f(x)$ is a set of all antiderivatives of this function and is denoted by the symbol $\int f(x)dx$. The function $f(x)$ is called the **integrand**, the expression $f(x)\,dx$ is the **element of integration**, and the variable *x* is the **variable of integration.**	$\int f(x)dx$ It is read: the indefinite integral of a function $f(x)$ with respect to *x*.
Properties of Indefinite Integral		
4	The derivative of an indefinite integral equals the integrand	$\left(\int f(x)dx\right)' = f(x)$
5	The differential of an indefinite integral equals the element of integration	$d\int f(x)dx = f(x)dx$
6	The integral of a differential of a function *u* is *u* plus an arbitrary constant C	$\int du = u + C$
7	A constant may be moved across the integral sign	$\int Cf(x)dx = C\int f(x)dx$ (C $\neq$ 0).
8	The integral of a sum of a finite number of functions is equal to the sum of the integrals of these functions	$\int \sum_{k=1}^{n} f_k(x)dx = \sum_{k=1}^{n} \int f_k(x)dx$

The Table of Integrals

I	**Rules of integration**	**1.** $\int 0 \cdot dx = C$
		2. $\int (u \pm v)dx = \int udx \pm \int vdx$
		3. $\int Af(x)\,dx = A\int f(x)\,dx$, A is a constant
		4. $\left.\begin{matrix}\int u \cdot vdx \\ \int \frac{u}{v}dx\end{matrix}\right\}$ here are no formulas in the general case
II	**Integrals of power functions**	**1.** $\int u^{\alpha}\,du = \frac{u^{\alpha+1}}{\alpha+1} + C, (\alpha \neq -1)$
		2. $\int u^{-1}du = \int \frac{du}{u} = \ln\lvert u\rvert + C$
		3. $\int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C$
		4. $\int \frac{du}{u^k} = -\frac{1}{(k-1)u^{k-1}} + C, \quad k \neq 1$
III	**Integrals of exponential functions**	**1.** $\int a^u du = \frac{a^u}{\ln a} + C$
		2. $\int e^u du = e^u + C$

IY	**Integrals of trigonometric functions**	**1.** $\int \sin u du = -\cos u + C$
		2. $\int \cos u du = \sin u + C$
		3. $\int \frac{du}{\cos^2 u} = \tan u + C$
		4. $\int \frac{du}{\sin^2 u} = -\cot u + C$
		5. $\int \tan u du = -\ln\lvert\cos u\rvert + C$
		6. $\int \cot u du = \ln\lvert\sin u\rvert + C$
		7. $\int \frac{du}{\sin u} = \ln\left\lvert\tan\frac{u}{2}\right\rvert + C$
V	**1. "Formula of antysine"**	$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$
	2. "Formula of antytangent"	$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$
	3. "Long logarithm"	$\int \frac{du}{\sqrt{u^2 \pm a^2}} = \ln\left\lvert u + \sqrt{u^2 \pm a^2}\right\rvert + C$
	4. "High logarithm"	$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left\lvert\frac{u-a}{u+a}\right\rvert + C$
VI	**Integration of hyperbolic functions**	**1.** $\int \sinh u du = \cosh u + C$
		2. $\int \cosh u du = \sinh u + C$
		3. $\int \frac{du}{\cosh^2 u} = \tanh u + C$
		4. $\int \frac{du}{\sinh^2 u} = -\coth u + C$
VII	If $\int f(x)dx = F(x) + C$ then	$\int f(kx+b)dx = \frac{1}{k}F(kx+b) + C$

1.3. Trigonometric Formulas

I	**Fundamental Trigonometric Identities**	**1.** $\sin^2 x + \cos^2 x = 1$ **2.** $\tan x = \dfrac{\sin x}{\cos x}$ **3.** $\cot x = \dfrac{\cos x}{\sin x}$ **4.** $1 + \tan^2 x = \dfrac{1}{\cos^2 x}$ **5.** $1 + \cot^2 x = \dfrac{1}{\sin^2 x}$
II	**Sum and Difference Identities**	**1.** $\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$ **2.** $\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$ **3.** $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}$
III	**Co-function Identities**	**1.** $\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$ **2.** $\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$ **3.** $\tan\left(\dfrac{\pi}{2} - x\right) = \cot x$ **4.** $\cot\left(\dfrac{\pi}{2} - x\right) = \tan x$
IY	**Lowering of the Order**	**1.** $\sin x \cos x = \dfrac{\sin 2x}{2}$ **2.** $\cos^2 x = \dfrac{1 + \cos 2x}{2}$ **3.** $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
Y	**Product-to-Sum Identities**	**1.** $\cos\alpha \cos\beta = \dfrac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta))$ **2.** $\sin\alpha \sin\beta = -\dfrac{1}{2}(\cos(\alpha + \beta) - \cos(\alpha - \beta))$ **3.** $\cos\alpha \sin\beta = \dfrac{1}{2}(\sin(\alpha + \beta) - \sin(\alpha - \beta))$ **4.** $\sin\alpha \cos\beta = \dfrac{1}{2}(\sin(\alpha + \beta) + \sin(\alpha - \beta))$

1.4. Methods of Integration

	Type of integral	Recommendations
A	$\int \frac{dx}{ax^2+bx+c}$ $\int \frac{dx}{\sqrt{ax^2+bx+c}}$	1) Complete the square of quadratic expression 2) Use formulas (VII) and (V). **Completing the square:** $ax^2+bx+c=a\left(x^2+\frac{bx}{a}\right)+c=a\left(\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}\right)+c=$ $=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}$
B	$\int \frac{(Mx+N)dx}{ax^2+bx+c}$ $\int \frac{(Mx+N)dx}{\sqrt{ax^2+bx+c}}$	1) Write in the numerator the derivative of quadratic form. Equate the coefficients. 2) Write the integral as the sum of two integrals. a) Calculate the first of them using the formulas $\int \frac{u'dx}{u}=\ln\lvert u\rvert+C$ or $\int \frac{u'dx}{\sqrt{u}}=2\sqrt{u}+C$ b) Calculate the second integral using the method (A).
C	$\int f(x)g(x)\,dx$	Simplify $f(x)\cdot g(x)$ such that to use the formulas (VII) and (I) – (Y). Rewrite the given integral in the form $\int f(x)\,d(G)$, using the formula $g(x)dx=d\left(\int g(x)dx\right)=d(G(x))$. Then a) if $f(x)=f_1(G(x))$ and $\int f_1(G)\,d(G)$ is a table integral with respect to the new variable of integration *G,* integrate it. b) If (a) does not take place, use the method integration by parts $\int udv=uv-\int v\,du$, where $\begin{bmatrix} u=f(x)\Rightarrow & du=f'(x)dx \\ dv=dG\Rightarrow & v=G \end{bmatrix}$ or use the methods which are denoted below.

<table>
<tr>
<td>D</td>
<td>Integration of a rational fraction

$\int \frac{P_n(x)}{Q_m(x)}\,dx$</td>
<td>1) Verify if this fraction is a proper or an improper fraction. If it is an improper fraction, use the long division to receive a whole part and a proper fraction.

2) Factorize the denominator of the proper fraction using the factors of the kind
$x-\alpha$; $(x-\alpha)^k$; x^2+px+q, or $(x^2+px+q)^l$, where $D=p^2-4q<0$.

3) Verify if the proper fraction is reducible or not. If it is reducible, short it.

4) Write the irreducible proper fraction as the sum of partial fractions

$\frac{R(x)}{(x-\alpha)^k(x^2+px+q)^l}=\frac{A_k}{(x-\alpha)^k}+\frac{A_{k-1}}{(x-\alpha)^{k-1}}+\ldots+\frac{A_1}{x-\alpha}+\frac{M_1x+N_1}{(x^2+px+q)^l}+\frac{M_2x+N_2}{(x^2+px+q)^2}+\ldots++\frac{M_lx+N_l}{(x^2+px+q)}$

5) Find coefficients of partial fractions using “finger’s rule” or method of indefinite coefficients</td>
</tr>
<tr>
<td>E</td>
<td colspan="2">$\int R(\cos x, \sin x)\,dx$</td>
</tr>
<tr>
<td>E_1</td>
<td>$\int \sin ax \cdot \cos bx\,dx$,

$\int \cos ax \cdot \cos bx\,dx$,

$\int \sin ax \cdot \sin bx\,dx$,

$\int \sin^{2m} x \cdot \cos^{2n} x\,dx$</td>
<td>Simplify the integrand using “Product-to-sum Identities” (page 9)</td>
</tr>
</table>

E_2	$a)\ \boldsymbol{R(-\sin x,\cos x)} = = -\boldsymbol{R(\sin x,\cos x)}$	Substitution : $\cos x = t \Rightarrow \sin x dx = -dt$, $\sin^2 x = 1 - t^2$
	$\boldsymbol{b)\ R(\sin x,-\cos x) = = -R(\sin x,\cos x)}$	Substitution : $\cos x = t \Rightarrow \sin x dx = -dt$ $\sin^2 x = 1 - t^2$
E_3	$\boldsymbol{R(-\sin x,-\cos x) = = R(\sin x,\cos x)}$	Substitution: $a)\ \tan x = \mathrm{t} \Rightarrow dx = \frac{dt}{1+t^2}$, $\sin x = \frac{t}{\sqrt{1+t^2}}$, $\cos x = \frac{1}{\sqrt{1+t^2}}$ or $b)\ \cot x = t \Rightarrow dx = -\frac{dt}{1+t^2}$, $\sin x = \frac{1}{\sqrt{1+t^2}}$, $\cos x = \frac{t}{\sqrt{1+t^2}}$
E_4	**Universal Substitution**	$\tan\frac{x}{2} = t \Rightarrow x = 2\arctan t \Rightarrow dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

F	Integration of Irrational Functions	
F_1	$\int R\left(x, \sqrt[k_1]{ax+b},..., \sqrt[k_n]{ax+b}\right) dx$	Substitution: $ax+b=t^N$, where $N = LCM(k_1,...k_n)$ (the least common myltiple) and $x=\dfrac{t^N-b}{a} \Rightarrow dx = N\,t^{N-1}dt$, $\sqrt[k_i]{ax+b} = t^{N/k_i}$
F_2	$\int \dfrac{dx}{x^k\sqrt{ax^2+bx+c}}$ $k = 1,2$	Substitution: $\dfrac{1}{x}=t \Rightarrow dx = -\dfrac{dt}{t^2}$ $\sqrt{ax^2+bx+c} = \dfrac{\sqrt{ct+bt+a}}{t}$
F_3	$\int \dfrac{dx}{x^k\sqrt{ax^2+bx+c}}$ $k = 1,2$	$\dfrac{1}{x}=t \Rightarrow dx = -\dfrac{dt}{t^2}$, $\sqrt{ax^2+bx+c} = \dfrac{\sqrt{ct+bt+a}}{t}$
F_4	$I = \int \dfrac{P_n(x)\,dx}{\sqrt{ax^2+bx+c}}$	$I = Q_{n-1}(x)\cdot\sqrt{ax^2+bx+c} + \lambda\int\dfrac{dx}{\sqrt{ax^2+bx+c}}$ (1) To find coefficients of $Q_{n-1}(x)$ and λ use the next algorithm: 1) differentiate the both parts of the equality (1), using the formula $\left(\int f(x)dx\right)' = f(x)$; 2) reduce these fractions to the common denominator and use the method of the indefinite coefficients
F_5	$\int R\left(x, \sqrt{x^2-a^2}\right) dx$	Substitution: $x=\dfrac{a}{\cos t} \Rightarrow dx = \dfrac{a\sin t\,dt}{\cos^2 t}$, $\sqrt{x^2-a^2} = \dfrac{a\sin t}{\cos t}$

F_6	$\int R\left(x, \sqrt{a^2 + x^2}\right) dx$	Substitution: $x = a\tan t \Rightarrow dx = \frac{a\,dt}{\cos^2 t}$, $\sqrt{x^2 + a^2} = \frac{a}{\cos t}$

II. Definite Integral and its Applications

2.1. Definition and Calculation of a Definite Integral

1	**Definition of a definite integral**	The limit of the integral sums $\sum_{k=1}^{n} f(\xi_k)\Delta x_k$ as $\lambda \to 0$ $\left(\lambda = \max_k \Delta x_k\right)$ is called the definite integral of the function *f(x)* with respect to *x* over the interval [*a,b*] and it is denoted $$\lim_{\lambda \to 0}\sum_{k=1}^{n} f(\xi_k)\Delta x_k = \int_a^b f(x)dx$$ This is read as the integral of *f(x)dx* from *a* to *b*, *a* is the lower limit, *b* is the upper limit. • If a function *f(x)* is continuous on an interval [*a,b*], then its definite integral over [*a,b*] exists.
2	**Newton – Leibniz formula**	$\int_a^b f(x)dx = F(x)\Big\vert_a^b = F(b) - F(a)$, where $F(x)$ is an antiderivative of $f(x)$ on an interval [*a,b*]. The symbol $\Big\vert_a^b$ indicates that the value of the function corresponding to the lower index must be subtracted from the one corresponding to the upper index.
		a) $\int_a^a f(x)dx = 0$

3	**Properties of definite integrals**	**b)** $\int_b^a f(x)dx = -\int_a^b f(x)dx$ **c)** $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ **d)** $\int_a^b \left(f(x) \pm g(x)\right)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$ **e)** If $f(x) \geq g(x)$ on [*a, b*] then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$ **f)** $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ **g)** $\int_{-a}^a f(x)dx = \begin{cases} 0 \text{ if } f(-x) = -f(x) \\ 2\int_0^a f(x)dx, \text{if } f(-x) = f(x) \end{cases}$
4	**Integration by parts**	$\int_a^b udv = uv\Big\|_a^b - \int_a^b vdu$
5	**Integration by substitution**	Let *f(x)* be continuous function on a closed interval [*a, b*]. Assume that $x = \varphi(x)$ satisfies the conditions a) $\varphi(x)$ and $\varphi'(x)$ are continuous on a closed interval $[\alpha, \beta]$; b) $a \leq \varphi(t) \leq b$ when $\alpha \leq t \leq \beta$; c) $\varphi(\alpha) = a, \varphi(\beta) = b$. Then we have $\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$
6.	**Geometrical meaning of definite integral**	The definite integral $\int_a^b f(x)\,dx$ equals the area *S* of a region bounded above by the graph of the function $y = f(x) > 0$, on the sides by vertical lines through $x = a$ and $x = b$, and below by the *x*-axis $S = \int_a^b f(x)\,dx$

2. 2. Geometrical Applications of Definite Integral

1	**Area of a plane region**	$S=\begin{cases}\int\limits_a^b\left(y_2(x)-y_1(x)\right)dx,\ \text{if } D:\begin{Bmatrix} y=y_2(x),\, y=y_2(x),\\ a\le x\le b\end{Bmatrix}, \\ \textbf{the Cartesian system of coordinates;} \\ \int\limits_\alpha^\beta y(t)x'(t)dt,\ \text{if } D:\{x=x(t),\, y=y(t),\, \alpha\le t\le\beta\}, \\ \textbf{the parametric form;} \\ \int\limits_\alpha^\beta\left(r_2^2(\varphi)-r_1^2(\varphi)\right)d\varphi,\ \text{if } D:\begin{Bmatrix} r=r_1(\varphi),\, r=r_2(\varphi),\\ \varphi=\alpha,\, \varphi=\beta\end{Bmatrix}, \\ \textbf{the polar system of coordinates}\end{cases}$
2	**Length of a plane curve**	$L=\begin{cases}\int\limits_a^b\sqrt{1+(y')^2}\,dx,\ \text{if } \ell:\{y=y(x),\, a\le x\le b\}, \\ \textbf{the Cartesian system of coordinates;} \\ \int\limits_\alpha^\beta\sqrt{(x')^2+(y')^2}\,dt,\ \text{if } \ell:\{x=x(t),\, y=y(t),\, \alpha\le t\le\beta\}, \\ \textbf{the parametric form;} \\ \int\limits_\alpha^\beta\sqrt{r^2+(r')^2}\,d\varphi,\ \text{if } \ell:\{r=r(\varphi),\, \alpha\le\varphi\le\beta\}, \\ \textbf{the polar system of coordinates}\end{cases}$
3	**Volume of a solid**	Let there be given a body bounded by a closed surface and let the area $S(x)$ $(a\le x\le b)$ of its any cross-section by a plane perpendicular to the *x*-axis be known then the volume of this body is $V=\int\limits_a^b S(x)dx.$

4	**Volume of a solid of revolution**	Let a solid be obtained by revolving a curvilinear trapezoid bounded by a curve $y = y(x)$ with the base $[a, b]$ about **a)** the x-axis **b)** the y-axis then the volume of these bodies are calculated by the formulas : **a)** $V_X = \pi \int_a^b y^2(x)dx$ **b)** $V_Y = 2\pi \int_a^b xydx$
5	**Area of a surface of revolution**	Let the surface be obtained by revolving a curve $y = y(x)$ with the base $[a, b]$ about the x-axis, then $S_X = 2\pi \int_a^b y\sqrt{1 + (y')^2}\,dx$

III. IMPROPER INTEGRALS

1	**Improper integral of the first type**	a) $\int_a^{+\infty} f(x)dx = \lim\limits_{B \to +\infty} \int_a^B f(x)dx =$ $= \lim\limits_{B \to +\infty} (F(B) - F(a)) = F(+\infty) - F(a)$ b) $\int_{-\infty}^b f(x)dx = \lim\limits_{A \to -\infty} \int_A^b f(x)dx =$ $= \lim\limits_{A \to -\infty} (F(b) - F(A)) = F(b) - F(-\infty)$ c) $\int_{-\infty}^{+\infty} f(x)dx = F(+\infty) - F(-\infty)$, where $F(+\infty)$, $F(-\infty)$ are, respectively, the limits (if they exist) as $x \to +\infty$ and $x \to -\infty$.

2.	**Improper integral of the second type**	a) $\int_a^b f(x)dx = \begin{bmatrix} f(x) \text{ does not exist} \\ \text{in the point } b \end{bmatrix} = \lim_{\varepsilon \to 0} \int_a^{b-\varepsilon} f(x)dx$, $\varepsilon > 0$. b) $\int_a^b f(x)dx = \begin{bmatrix} f(x) \text{ does not exist} \\ \text{in the point } a \end{bmatrix} = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{b} f(x)dx$, $\varepsilon > 0$. c) $\int_a^b f(x)dx = \begin{bmatrix} f(x) \text{ does not exist} \\ \text{in the point } c,\ a < c < b \end{bmatrix} =$ $= \lim_{\varepsilon \to 0} \left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^{b} f(x)dx \right)$, $\varepsilon > 0$.

Note: If at least one of these limits does not exist or is infinite then the corresponding improper integral is called divergent.

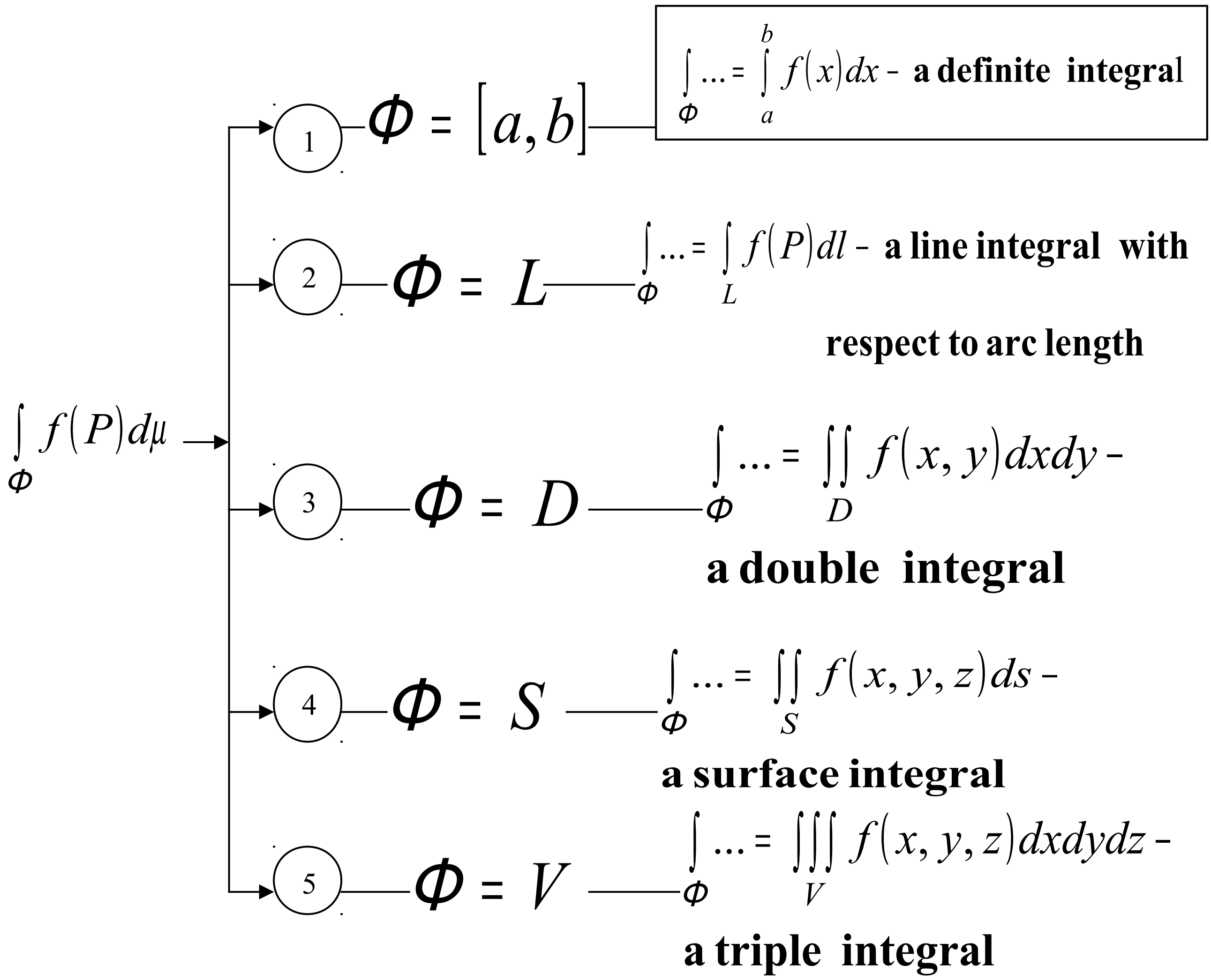
IV. INTEGRAL OVER a FIGURE (Multiple Integrals)

4.1. Definitions

Types of figures		
№	Name	Definition, formula
1	**One-dimensional figure**	• a segment of a straight line: $\Phi = [a,b]$ • a part of a plane curve : $\Phi = L$ • a part of a curve in a surface of a space : $\Phi = L$
2	**Two-dimensional figure**	• a domain in a plane: $\Phi = D$ • a surface in a space: $\Phi = S$

3	**Three-dimensional figure**	a spatial body: $\Phi = V$
Measure of a figure $(\mu(\Phi))$		
4	**Measure of one-dimensional figures**	Length
5	**Measure of two-dimensional figures**	Area
6	**Measure of three-dimensional figures**	Volume
7	**Diameter of a figure**	Diameter of a figure is the greatest distance between two points of this figure.
8	**Diameter of a partition of a figure** (λ)	$\lambda = \max(\lambda_1, \lambda_2, ..., \lambda_n)$, where λ_k is the diameter of the k-th subfigure of the given figure
9	**Integral over a figure**	$\int_{\Phi} f(P)d\mu = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(P_k)\Delta\mu_k$, where $\lambda = \max_k \Delta\mu_k$
10	**Properties of integral over a figure**	• $\int_{\Phi} d\mu = \mu(\Phi)$ • $\int_{\Phi} kf(P)d\mu = k\int_{\Phi} f(P)d\mu$, k is a constant • $\int_{\Phi} (f(P) \pm g(P))d\mu = \int_{\Phi} f(P)d\mu \pm \int_{\Phi} g(P)d\mu$ • If $f(P) \le g(P)$, for any $P \in \Phi$, then $\int_{\Phi} f(P)d\mu \le \int_{\Phi} g(P)d\mu$ • If $\Phi = \Phi_1 \cup \Phi_2$ and $\Phi_1 \cap \Phi_2 = \varnothing$, then $\int_{\Phi} f(P)d\mu = \int_{\Phi_1} f(P)d\mu + \int_{\Phi_2} f(P)d\mu$

4.2. Special Types of Integral over a Figure



Note,

a) a line integral with respect to arc length sometimes termed a **line integral of the first type** or cuvelinear integral.

b) a surface integral $\iint_S f(x,y,z)ds$ sometimes termed a **surface integral of the first type**.

4.3. Iterated (Repeated) Integrals

1	**Iterated (repeated) integrals are integrals of a kind**	a) $\int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy$ b) $\int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx$
2	**Calculation of iterated integrals**	a) To calculate $\int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy$, it is necessary: • calculate the inner integral: $\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy = F(x)$, taking *x* as constant; • calculate the outer integral: $\int_a^b F(x)dx$
		b) To calculate $\int_c^d dy \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx$, it is necessary: • calculate the inner integral: $\left.\int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\right\|_{y=const} = F(y)$; • calculate the outer integral: $\int_c^d F(y)dy$

4.4. Calculation of an Integral over a Figure

1	**Calculation of definite integral**	$\int_a^b f(x)dx = F(x)\Big\|_a^b = F(b) - F(a)$. where $F(x)$ is an antiderivative of $f(x)$.

2.	**Calculation of line integral with respect to the arc length** $\int\limits_L f(P)\,dl$	Let a plane curve L be given in the Cartesian system of coordinates $L: y = y(x),\quad a \le x \le b$, then $\int\limits_L f(x,y)\,dl = \int\limits_a^b f(x, y(x)) \cdot \sqrt{1 + (y'(x))^2}\,dx$
		b) Let a plane curve L be given in the parametric form $L: \begin{cases} x = x(t) \\ y = y(t) \end{cases},\quad \alpha \le t \le \beta$, then $\int\limits_L f(x,y)\,dl = \int\limits_\alpha^\beta f(x(t), y(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$
		c) Let a plane curve L be given in the polar coordinates $L: r = r(\varphi), \alpha \le \varphi \le \beta$, then $\int\limits_L f(x,y)\,dl = \begin{bmatrix} x = r\cos\varphi,\ y = r\sin\varphi \\ dl = \sqrt{r^2 + (r')^2}\,d\varphi \end{bmatrix} =$ $= \int\limits_\alpha^\beta f(r\cos\varphi, r\sin\varphi) \cdot \sqrt{r^2 + (r')^2}\,d\varphi$
		d) Let a space curve *L* be given in the parametric form, $L: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}\quad \alpha \le t \le \beta$, then $\int\limits_L f(x,y,z)\,dl =$ $= \int\limits_\alpha^\beta f(x(t), y(t), z(t)) \cdot \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt$

3	**Calculation of double integral** $\iint\limits_D f(x,y)dxdy$	**a)** $\iint\limits_D f(x,y)dxdy = \int\limits_a^b dx \int\limits_{y_1(x)}^{y_2(x)} f(x,y)\,dy$, where $[a,b] = \mathrm{Pr}_{Ox}\, D$, $y = y_1(x)$ is the lower boundary of the region D, $y = y_2(x)$ is the upper boundary of the region D.
		b) $\iint\limits_D f(x,y)dxdy = \int\limits_c^d dy \int\limits_{x_1(y)}^{x_2(y)} f(x,y)\,dx$, where $[c,d] = \mathrm{Pr}_{Oy}\, D$, $x = x_1(y)$ is the left boundary of the region D, $x = x_2(y)$ is the right boundary of the region D.
		c) A double integral in polar coordinates: $x = r\cos\varphi,\ y = r\sin\varphi,\ dxdy = rdrd\varphi$ $\iint\limits_D f(x,y)dxdy = \int\limits_\alpha^\beta d\varphi \int\limits_{r_1(\varphi)}^{r_2(\varphi)} f(r\cos\varphi, r\sin\varphi)\,rdr,$ if $D: \begin{cases} \alpha \le \varphi \le \beta, \\ r_1(\varphi) \le r \le r_2(\varphi) \end{cases}$
4.	**Calculation of surface integral of the first type:** $\iint\limits_S f(x,y,z)ds$	**a)** Let $S: z = z(x,y),\quad (x,y) \in D_{xy} = \mathrm{Pr}_{xy}\, S$, then $\iint\limits_S f(x,y,z)ds =$ $\iint\limits_{D_{xy}} f(x,y,z(x,y))\sqrt{1 + (z'_x)^2 + (z'_y)^2}\,dxdy$
		b) Let $S: y = y(x,z),\quad (x,z) \in D_{xz} = \mathrm{Pr}_{xz}\, S$, then $\iint\limits_S f(x,y,z)ds =$ $\iint\limits_{D_{xy}} f(x,y(x,z),z)\sqrt{1 + (y'_x)^2 + (y'_z)^2}\,dxdz$
		c) Let $S: x = x(y,z),\quad (y,z) \in D_{yz} = \mathrm{Pr}_{yz}\, S$, then $\iint\limits_S f(x,y,z)ds =$ $\iint\limits_{D_{yz}} f(x(y,z),y,z)\sqrt{1 + (x'_y)^2 + (x'_z)^2}\,dydz$

5	**Calculation of triple integral** $\iiint\limits_V f(x,y,z)dxdydz$	a) $\iiint\limits_V f(x,y,z)dxdydz = \iint\limits_{D_{xy}} dxdy \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz$, where D_{xy} is the projection of V on the *xy*- plane;
		b) $\iiint\limits_V f(x,y,z)dxdydz = \iint\limits_{D_{xz}} dxdz \int\limits_{y_1(x,z)}^{y_2(x,z)} f(x,y,z)\,dy$, where D_{xz} is the projection of V on the *xz*- plane;
		c) $\iiint\limits_V f(x,y,z)dxdydz = \iint\limits_{D_{yz}} dydz \int\limits_{x_1(y,z)}^{x_2(y,z)} f(x,y,z)\,dx$, where D_{yz} is the projection of V on the *yz*- plane.
6	**Triple integral in cylindrical coordinates** $\begin{cases} x = r\cos\varphi, \\ y = r\sin\varphi, \\ z = z \end{cases}$	If $x = r\cos\varphi$, $y = r\sin\varphi$, $z = z$, then $dv = r\,dr\,d\varphi\,dz$ and $\iiint\limits_V f(x,y,z)dv = \iiint\limits_V f(r\cos\varphi, \sin\varphi, z)rdrd\varphi\,dz$
7	**Triple integral in spherical coordinates** $\begin{cases} x = r\sin\theta\cos\varphi, \\ y = r\sin\theta\sin\varphi, \\ z = r\cos\theta \end{cases}$	If $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$, then $dv = r^2\sin\theta\cdot dr\cdot d\varphi\cdot d\theta$, and $\iiint\limits_V f(x,y,z)dv =$ $= \iiint\limits_V f(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)r^2\sin\theta\,drd\varphi\,d\theta$

4.5. Geometrical Interpretation of Integral over a Figure

№	Formula	Explanation
1	$\int\limits_{\Phi} d\mu = \mu(\Phi)$ is the measure of a figure Φ	Particular cases: **1.** $\Phi = [a,b]$, then $\int\limits_{\Phi} d\mu = \int\limits_{a}^{b} dx = b - a$ is the length of $[a,b]$
		2. $\Phi = L$, then $\int\limits_{\Phi} d\mu = \int\limits_{L} dl$ is the length of the curve L
		3. $\Phi = S_D$, then $\int\limits_{\Phi} d\mu = \iint\limits_{D} dxdy = S_D$ is the area of the region D
		4. $\Phi = S$, then $\int\limits_{\Phi} d\mu = \iint\limits_{S} ds = S$ is the area of the surface of S
		5. $\Phi = V$, then $\int\limits_{\Phi} d\mu = \iiint\limits_{V} dxdydz = \mathrm{V}$ is the volume of the solid V

4.6. Physical Applications of an Integral over a Figure

№	Formula	Explanation
1	**Mass of a figure Φ**	$m(\Phi) = \int\limits_{\Phi} \rho(P)d\mu$, where $\rho(P)$ is the density of the given figure.
2	**Definite Integral**	
2.1	**The distance traveled by a body between $t=a$ and t moving with velocity** $v = v(t)$	$s = \int\limits_{a}^{b} v(t)\,dt$
2.2	**Work of a Variable over Power** $F(x)$	$A = \int\limits_{a}^{b} F(x)\,dx$

2.3	**Static moments of a plane curve** $y = f(x), a \le x \le b$ **with a density** $\gamma(x, y)$	about x axis	$M_x = \int_a^b \gamma(x, y) y\, dl = \int_a^b \gamma(x, y) y \sqrt{1 + (y')^2}\, dl$
		about y axis	$M_y = \int_a^b \gamma(x, y) x\, dl = \int_a^b \gamma(x, y) x \sqrt{1 + (y')^2}\, dl$
2.3. (a)	**Static moments of a plane trapezoid bounded by** $y = f(x), x = a,$ $x = b, y = 0$	about x axis	$M_x = \frac{1}{2}\int_a^b y^2\, dx$
		about y axis	$M_y = \int_a^b x \cdot y\, dx$
2.4	**Static moments of a plane region bounded by** $y = f(x), y = g(x),$ $f(x) \ge g(x)\ a \le x \le b$	about x axis	$M_x = \frac{1}{2}\int_a^b \left(f^2(x) - g^2(x)\right) dx$
		about y axis	$M_y = \int_a^b x\left(f(x) - g(x)\right) dx$
2.5	**Center of gravity of a curve part (the center of mass)**	$x_C = \frac{M_y}{\mu}; \quad y_C = \frac{M_x}{\mu}$, where μ is the length of a figure	
3	**Double Integral**		
3.1	**Static moments of a plane region *D* with a density at a point:** $\gamma(x, y)$ **(the first moments)**	with respect to the x-axis $M_x = \iint_D y \cdot \gamma(x, y)\, dxdy$	
		with respect to the y-axis $M_y = \iint_D x \cdot \gamma(x, y)\, dxdy$	
3.2	**Center of gravity of a region *D***	$x_C = \frac{M_y}{\mu}; \quad y_C = \frac{M_x}{\mu}$, where μ is the area of a figure	
3.3	**Moments of inertia (the second moments)**	about the x-axis	$I_x = \iint_D y^2 \cdot \gamma(x, y)\, dxdy$
		about the y-axis	$I_y = \iint_D x^2 \cdot \gamma(x, y)\, dxdy$
3.4	**The polar moment of inertia about the origin**	$I_0 = \iint_D r^2 \cdot \gamma(x, y)\, dxdy$	

4	**Triple Integral**	
4.1	**Static moments of a solid *V***	with respect to the *xy*-plane $M_{xy} = \iiint_V z\gamma(x,y,z)dv$
		with respect to the *xz*-plane $M_{xz} = \iiint_V y\gamma(x,y,z)dv$
		with respect to the *yz*-plane $M_{yz} = \iiint_V x\gamma(x,y,z)dv$
4.2	**Center of gravity of of a solid *V***	$x_C = \frac{M_{yz}}{\mu}$; $y_C = \frac{M_{xz}}{\mu}$, $z_C = \frac{M_{xy}}{\mu}$ where μ is the volume of a figure *V*
4.3	**Moments of inertia**	with respect to the *xy*-plane $I_{xy} = \iiint_V z^2\gamma(x,y,z)dv$
		with respect to the *xz*-plane $I_{xz} = \iiint_V y^2\gamma(x,y,z)dv$
		with respect to the *yz*-plane $I_{yz} = \iiint_V x^2\gamma(x,y,z)dv$
		about the *x*-axis $I_x = \iiint_V (y^2 + z^2)\gamma(x,y,z)dv$
		about the *y*-axis $I_y = \iiint_V (x^2 + z^2)\gamma(x,y,z)dv$
		about the *z*-axis $I_z = \iiint_V (x^2 + y^2)\gamma(x,y,z)dv$

V. FIELD THEORY

5.1. Line Integral of the Second Type (Line Integral with respect to coordinates)

<table>
<tr><td colspan="3">a) Let functions $P(x,y)$ and $Q(x,y)$ be continuous functions defined over a smooth plane curve L</td></tr>
<tr><td>a_1</td><td>Line integral of the second type (a line integral with respect to the coordinates)</td><td>$I=\int_L P(x,y)\,dx+Q(x,y)dy$</td></tr>
<tr><td rowspan="2">a_2</td><td rowspan="2">Calculation of a line integral over a plane curve</td><td>• Let a plane curve L be given in the Cartesian system of coordinates: $L: y=f(x), a\le x\le b$, then
$I=\int_a^b (P(x,f(x))+Q(x,f(x))f'(x))dx$</td></tr>
<tr><td>• Let a plane curve L be given in the parametric form , $L:\begin{cases} x=x(t) \\ y=y(t)\end{cases}, \alpha\le t\le\beta$,
then
$I=\int_\alpha^\beta (P(x(t),y(t))\,x'(t)+Q(x(t),y(t))y'(t))\,dt$</td></tr>
<tr><td>b)</td><td colspan="2">Let functions $P(x,y,z)$, $Q(x,y,z)$ and $Q(x,y,z)$ be continuous functions defined over a smooth space curve L</td></tr>
<tr><td>b_1</td><td>Line integral of the second type (a line integral with respect to the coordinates)</td><td>$\int_L P(x,y,z)\,dx+Q(x,y,z)dy+R(x,y,z)\,dz$</td></tr>
<tr><td>b_2</td><td colspan="2">Calculation of a line integral: let a space curve L be given in the parametric form $L:\begin{cases} x=x(t) \\ y=y(t) \\ z=z(t)\end{cases} \quad \alpha\le t\le\beta$, then
$\int_L P(x,y,z)\,dx+Q(x,y,z)dy+R(x,y,z)\,dz=$
$=\int_\alpha^\beta (P(x(t),y(t),z(t))\,x'(t)+Q(x(t),y(t),z(t))\,y'(t)+R(x(t),y(t),z(t))\,z'(t))\,dt$</td></tr>
</table>

5.2. SURFACE INTEGRAL OF THE SECOND TYPE
(Integrals over an Oriented Surfaces)

1	**Oriented Surface**	A surface S is called the oriented surface if there the unique normal $\overline{n}$ at each point of the given surface is defined
2	**Integral over Oriented Surface *S* or integral of the second type**	$I = \iint\limits_S P(x,y,z)\,dydz + Q(x,y,z)dxdz + R(x,y,z)\,dxdy$
3	**Calculation of a surface integral of the second type**	$I = \pm \iint\limits_{D_{yz}} Pdydz \pm \iint\limits_{D_{xz}} Qdxdz \pm \iint\limits_{D_{xy}} Rdxdy$, where $D_{yz} = \Pr\limits_{y0z} S$, $D_{xz} = \Pr\limits_{x0z} S$, $D_{xy} = \Pr\limits_{x0y} S$
		$I = \iint\limits_S \left(\overline{F}, \overline{n}^0\right) dS$, where, $\iint\limits_S \left(\overline{F}, \overline{n}^0\right) dS$ is the surface of the first type, $\overline{F}(M) = P(x,y,z)\,\overline{i} + Q(x,y,z)\,\overline{j} + R(x,y,z)\,\overline{k}$, $S: f(x,y,z) = 0$ and $\overline{n}^0 = \dfrac{\overline{grad\ f}}{\lvert\overline{grad\ f}\rvert}$ is the ort of a normal to S

5.3. Scalar Fields

№	Name	Formula
1	**Scalar field**	If a scalar point function $u = u(M)$ is defined in the domain V, it is called a **scalar field** in V
3	**Level line of a plane scalar field**	A **level line (curve**) is a set of points of a plane in which scalar function $u(x,y)$ takes the same value: $u(x,y) = C$.

4	**Level surface of a scalar field**	**A level surface** is a set of points of a space in which scalar function $u(x,y,z)$ takes the same value, i.e. $u(x,y,z)=c$, where $c = const$.
5	**Gradient of a scalar field**	**Gradient** of a scalar field is a vector, whose coordinates are equal to the partial derivatives of the function $u(x,y,z)$ at the point M: $\overline{grad}\, u\,(M) = \frac{\partial u(M)}{\partial x}\bar{i} + \frac{\partial u(M)}{\partial y}\bar{j} + \frac{\partial u(M)}{\partial z}\bar{k}, \quad (\overline{grad}\, u = \overline{\nabla} u)$
6	**Directional derivative of a scalar field**	**Directional derivative** of a scalar field $u(x,y,z)$ at the point M in the direction of the vector $\bar{a}$ is a scalar product of gradient of this function by the ort of the given vector: $\frac{\partial u(M)}{\partial \bar{a}} = \left(\overline{grad}\, u(M), \bar{a}^0\right)$, where $\bar{a}^0 = \frac{\bar{a}}{\lvert \bar{a} \rvert}$ $\frac{\partial u(M)}{\partial \bar{a}} = \frac{\partial u}{\partial x}\cdot\cos\alpha + \frac{\partial u}{\partial y}\cdot\cos\beta + \frac{\partial u}{\partial z}\cdot\cos\gamma$, where $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are directional cosines of the vector $\bar{a}$.
7	**Physical meaning of a directional derivative**	**1.** Directional derivative of a scalar field $u(x,y,z)$ at the point M in the direction of the vector $\bar{a}$ is a rate of change of a function $u(x,y,z)$ at a point M in this direction. **2.** If $\frac{\partial u}{\partial \bar{a}} > 0$, then the function u increases in the direction $\bar{a}$, if $\frac{\partial u}{\partial \bar{a}} < 0$, then the function u decreases in this direction. **3.** A value $\left\lvert\frac{\partial u}{\partial \bar{a}}\right\rvert$ is the **instantaneous velocity** of the function u in a direction $\bar{a}$ at a point M.

8	**Physical meaning of a gradient**	**1.** Gradient of a function u denotes the direction of the maximal increasing of this function. **2.** The maximal rate of change of a function u at a point M is equal to module of the gradient of this function at the giving point: $\left\|\overline{grad}\, u(M)\right\| = \sqrt{\left(\frac{\partial u(M)}{\partial x}\right)^2 + \left(\frac{\partial u(M)}{\partial y}\right)^2 + \left(\frac{\partial u(M)}{\partial z}\right)^2}$

5.4. Vector Fields

1	**Vector field**	If, to each point M in some region V, a vector $\overline{F} = \overline{F}(M)$ is assigned, the collection of all such vectors is called a **vector field** $\overline{F}(M) = P(x,y,z)\,\overline{i} + Q(x,y,z)\,\overline{j} + R(x,y,z)\,\overline{k}$
2	**A vector lines of a vector field**	A **vector line** of a vector field $\overline{F}(M) = P(x,y,z)\,\overline{i} + Q(x,y,z)\,\overline{j} + R(x,y,z)\,\overline{k}$ is a curve at whose every point M the direction of its tangent coincides with the direction of the vector $\overline{F}(M)$. Equations of vector lines are the solutions of the following differential equations: $\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$
3	**Flux of a vector field**	**Flux** of a vector field $\overline{F}(M)$ across a surface S is the surface integral of the scalar product of the vector field by the unit normal vector to the surface, taken over this surface , defined by the equation $f(x,y,z) = 0$ $K = \iint\limits_S \overline{F} \cdot \overline{n}^0\, ds$, where $\overline{n}^0 = \frac{\overline{grad}\, f}{\left\|\overline{grad}\, f\right\|}$

4	**Properties of a flux**	**1.** A flux is physical meaning of surface integral of the second type. **2.** A flux K of a vector $\overline{F}(M)$ is a scalar value. **3.** A fluid flux is the amount of fluid flowing through a surface in a unit of time **4.** If a surface S is closed and bounds some volume V, then $K = \oint\!\!\oint_S \overline{F} \cdot \overline{n}^0 ds$ • If $K > 0$, then there are **sources** in V • If $K < 0$, then there are **sinks** in V • If $K = 0$, then the amount of fluid flowing in V is equal to the amount of fluid flowing out V in a unit of time. Such field is called a **solenoidal** field. • Flux is the physical meaning of a surface integral of the second type.
5	**Divergence of a vector field** $\overline{F}$	**Divergence** of a vector field $\overline{F}$ at a point M is the sum of partial derivatives of $\overline{F}$ at the given point: $div\overline{F}(M) = \frac{\partial P(M)}{\partial x} + \frac{\partial Q(M)}{\partial y} + \frac{\partial R(M)}{\partial z}$
6	**Properties of a divergence**	• If $div\overline{F}(M_0) > 0$, then a point M_0 is called a source. • If $div\overline{F}(M_0) < 0$, then a point M_0 is called a sink. • If $div\overline{F} = 0$, then the vector field is a solenoidal field. • Divergence characterizes the capacity density of the source of the vector field.
7	**Circulation of a vector field**	**Circulation** of a vector field $\overline{F}(M)$ over the closed contour L is the line integral of the scalar product of the vector $\overline{F}(M)$ by the vector $\overline{dr}$ tangent to the contour L taken round that contour. $\text{C} = \oint_{\text{L}} \overline{F} \cdot \overline{dr} \Rightarrow \text{C} = \oint_{\text{L}} Pdx + Qdy + Rdz$
8	**A potential field**	If the circulation of a vector field equals zero, then this field is a **potential** field.
		If a curve L is in a force field $\overline{F}$, then the circulation is work

9	**Physical meaning of a circulation**	of this field when a point is displacement along the curve L $A = \int_L P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ If a vector field is potential, then the work does not depend on the form of curve. It depends on the initial and terminal points of this curve.
10	**Rotation (curl) of a vector field** $\overline{F}$	$\overline{rot\overline{F}} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
11	**Physical meaning of a rotation**	• Direction of rotor is the direction, around which circulation has the maximum value comparatively with the circulation around any direction that does not concur with the normal of a plane region bounded by closed contour. • If $\overline{rot\overline{F}} = 0$, then a vector field is potential (irrotational)
12	**Hamiltonian** $\overline{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$	$\overline{\nabla} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$
13	**Properties of Hamiltonian**	**1.** $\overline{\nabla} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) \Rightarrow \overline{\nabla} u = \overline{grad\, u}$ **2.** $\overline{\nabla}\,\overline{F} = div\overline{F}$ **3.** $\overline{\nabla} \times \overline{F} = \overline{rot}\,\overline{F}$ **4.** $div\,\overline{grad u} = \Delta\, u$
14	**Laplace operator** $\Delta = \overline{\nabla} \cdot \overline{\nabla}$	$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$
15	**Green's formula**	Green's formula gives us the connection of the line integral of the second type with the double integral: $\oint_L P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right)dxdy$. L is the boundary of the domain D and the integration along L goes in the positive direction.
		Gauss-Ostrogradsky formula gives us the connection between a

16	**Gauss-Ostrogradsky formula**	surface integral and a triple integral $$\oiint_S \overline{F}\cdot \overline{n}^0 ds = \iiint_V div\overline{F}dxdydz$$ The flux of a vector field through a closed surface S in the direction of the outer normal is equal to the triple integral of the divergence of the field over the spatial domain V bounded by this surface.
17	**Stokes' formula**	$$\oint_L \overline{F}d\overline{r} = \iint_S \overline{rot}\,\overline{F}\cdot \overline{n}^0 ds$$ The flux of the rotation of a vector field $\overline{F}$ through the surface S is equal to the circulation of the vector field over L, which is the boundary of the given surface. *S.*
18	**Helmholtz theorem**	$$\overline{F}(M) = \underbrace{\overline{F_1}(M)}_{\text{potential}} + \underbrace{\overline{F_2}(M)}_{\text{solinoid}}$$ Any vector field $\overline{F}(M)$ can be represented as a sum of two vector fields, one of which is potential, another is solinoid.
19	**Harmonic field**	Vector field $\overline{F}$ is called a harmonic field if it is a potential and a solinoid at the same time, i.e. $\begin{cases}\overline{rot}\,\overline{F} = 0,\\ div\overline{F} = 0\end{cases}$

VI. DIFFERENTIAL EQUATIONS (DE)

6.1. DE of the First Order: $F(x,y,z) = 0$

The name of DE	Conditions	Recommendation for the solution
1. DE of the first order	$y' = f(x,y)$ **(1)** $M(x,y)\,dx = N(x,y)\,dy$ **(2)**	See the types of DE of the first order

2. DE with separable variables	$f(x,y) = X(x)\cdot Y(y)$ or $M(x,y) = X_1(x)\cdot Y_1(y)$ $N(x,y) = X_2(x)\cdot Y_2(y)$	1) reduce the DE to the form **(2),** knowing that $y' = \dfrac{dy}{dx}$; 2) separate the variables 3) integrate the both parts of DE
3. DE with homogeneous coefficients	$f(tx,ty) = f(x,y)$ or $M(tx,ty) = t^k M(x,y)$ $N(tx,ty) = t^k N(x,y)$	1) reduce the DE to the form (**1**); 2) do a substitution: $y = ux \Rightarrow y' = u'x + u$; 3) solve DE with respect to the new unknown function *u* and return to *y*, using the relation $u = \dfrac{y}{x}$
4.Linear DE of the first order	$y' + P(x)y = Q(x)$	1) do a substitution: $y = uv \Rightarrow y' = u'v + uv'$ 2) solve the system: $\begin{cases} v' + P(x)v = 0 \\ u'v = Q(x) \end{cases}$ 3) write out the answer: $y = u(x)\cdot v(x)$
4. Bernoulli's DE	$y' + P(x)y = Q(x)y^n, n \neq 0,1$	1) do a substitution: $y = uv \Rightarrow y' = u'v + uv'$ 2) solve the system: $\begin{cases} v' + P(x)v = 0 \\ u'v = Q(x)u^n v^n \end{cases}$ 3) write out the answer: $y = u(x)\cdot v(x)$

6.2. The Second Order DE $F(x, y, y', y'') = 0$

Kinds of DE	The form of DE	Recommendation for the solution
1. The left–hand side of the equation does not contain *y*	$F(x, y', y'') = 0$	1) Put $y' = z(x) \Rightarrow y'' = z'(x)$, where $z(x)$ is the new unknown function. 2) Solve the first order DE $F(x, z, z') = 0$ with respect the new unknown function $z(x)$. 3)Solve the first order DE $y' = z(x)$.
2. The left–hand side of the equation does not contain *x*	$F(y, y', y'') = 0$	1) Put $y' = z(y)$ where $z(y)$ is the new unknown function depen-ding on variable *y*, then $y'' = z'(y)\, y' \Rightarrow y'' = z'(y)\, z$ 2) Solve the first order DE $F(y, z, z'z) = 0$ with respect the new unknown function $z(y)$ Solve the first order DE $y'(x) = z(y)$
3. The left–hand side of the equation does not contain y and y'	$y'' = f(x)$	Integrate the given equation twice: 1) $y' = \int f(x)dx + C_1$ 2) $y = \int \left(\int f(x)dx \right) dx + C_1 x + C_2$

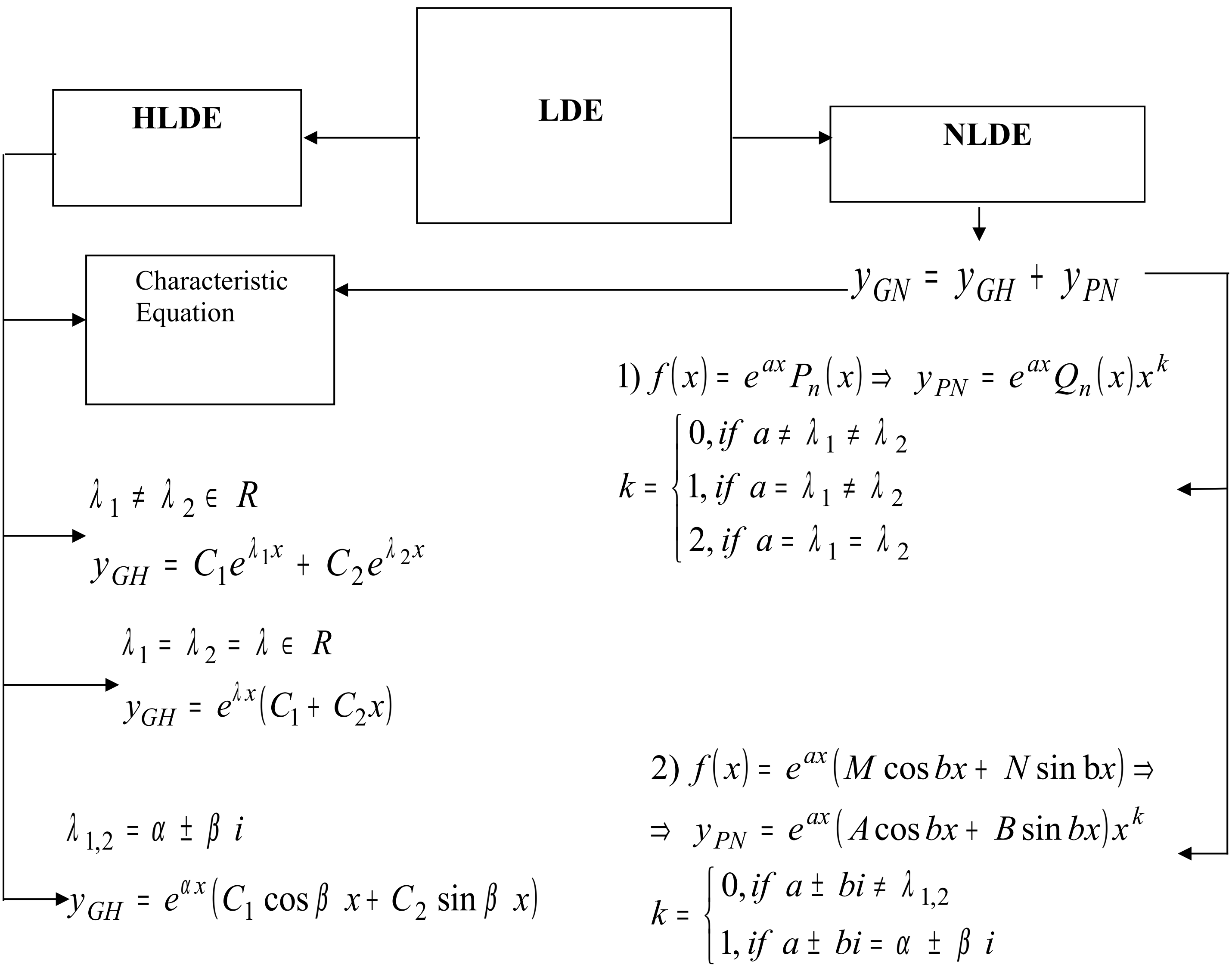
Note: it is possible to solve a DE of a kind $y^{(n)} = f(x)$ by integrating this equation *n* times.

6.3. Particular Solution of DE, Cauchy's Problem

1. $\begin{cases} F(x, y, y') = 0 \\ y\big\vert_{x = x_0} = y_0 \end{cases}$	*1)* Find a general solution (general integral) of the given DE. *2)* Using the initial condition $y\big\vert_{x = x_0} = y_0$ find the value of arbitrary constant *C*. *3)* Write out the particular solution (integral) of the given DE.

2 $\begin{cases} F(x,y,y',y'')=0 \\ y\big\vert_{x=x_0}=y_0 \\ y'\big\vert_{x=x_0}=y'_0 \end{cases}$	1) Find the general solution (general integral) of the given DE. 2) Using the initial conditions $\begin{cases} y\big\vert_{x=x_0}=y_0 \\ y'\big\vert_{x=x_0}=y'_0 \end{cases}$ find the values of the arbitrary constants C_1 and C_2. 3) Write out the particular solution (integral) of the given DE.

6.4. Linear Differential Equations of the Second Order with Constant Coefficients



LDE – linear differential equation
HLDE – homogeneous linear differential equation
NLDE – nonhomogeneous linear differential equation
y_{GH} – general solution of homogeneous linear differential equation

y_{GN} – general solution of nonhomogeneous linear differential equation
y_{PN} – particular solution of nonhomogeneous linear differential equation

6.5. The Method of Variation of Arbitrary Constants

Let a nonhomogeneous equation be given:

$$y'' + py' + q = f(x), \qquad (1)$$

where $f(x)$ is an arbitrary continuous function.

Let the homogeneous equation

$$y'' + py' + q = 0$$

corresponding to equation (1) have the general solution

$$y = C_1 y_1 + C_2 y_2$$

where C_1 and C_2 are arbitrary constants. It is possible to prove that

$$y = C_1(x) y_1 + C_2(x) y_2$$

is a particular solution of the equation (1) if the functions $C_1(x)$ and $C_2(x)$ are the solution of the system

$$\begin{cases} C_1 y_1 + C_2 y_2 = 0 \\ C_1' y_1' + C_2' y_2' = f(x) \end{cases}.$$

6.6. System of Differential Equations

• A system of differential equations is a collection of equations each of which may involve the independent variable, the unknown functions and their derivatives.

It is always assumed that the number of the equations is equal to the number of the unknown functions. For example

$$\begin{cases} x' = a_1 x + b_1 y \\ y' = a_2 x + b_2 \end{cases}, \qquad (1)$$

where $x = x(t), \quad y = y(t)$.

Such form of the system is called the normal form.

• The equation $\begin{vmatrix} a_1 - \lambda & b_1 \\ a_2 & b_2 - \lambda \end{vmatrix} = 0 \qquad (2)$

is called the characteristic equation of the system.

• To find the general solution of the system (1), we can use the next way:

1) Find the roots of the characteristic equation (2).
2) Using the corresponding formula for the solution of HLE find one of the unknown functions of the system.

3) Substitute the found function to any equation of this system and find the second function.

VII. SERIES

7.1. Limits

1	$\lim\limits_{\substack{x\to a\\ x\in D(f)}} f(x) = f(a)$	If a function $f(x)$ is continuous at a point $x = a$ then the limit of this function when $x \to a$ equals the value of the given function at a point $x = a$
2	• $\frac{1}{[0]} = \infty$ • $\frac{1}{[\infty]} = 0$	The connections between infinitesimal and infinite functions
3	$\sin u \sim tgu \sim \arcsin u \sim arctg\, u \sim u$ if $u \to 0$	The trigonometric functions $\sin u$, tgu, $\arcsin u$ and $arctg\, u$ are equivalent their argument if it tends to zero
4	$\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$ $\lim\limits_{x\to\infty}\left(1+\frac{a}{bx+c}\right)^{kx+m} = e^{ak/b}$	The second special limit
5	**Remember!**	$\lim\limits_{n\to\infty}\sqrt[n]{n^{\alpha}} = 1,\ \alpha \in R$
6	$\lim\limits_{\substack{n\to\infty\\ a>1}}\frac{a^{n}}{n^{\alpha}} = \infty$	Any exponential function increases quicker then any power function
7	$\lim\limits_{x\to\infty}\frac{P_n(x)}{Q_m(x)} = \begin{cases}\infty, & n > m\\ 0, & n < m\\ \frac{a_n}{b_m}, & n = m\end{cases}$	Comparison of polynomials when argument tends to infinite
8	$\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \left[\frac{0}{0},\frac{\infty}{\infty}\right] = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$	L\`Hopital's rule

7.2. Some Formulas Containing Factorials

1	$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$	**2**	$0! = 1$
3	$(2n)! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (2n-1) \cdot 2n$	**4**	$(2n+1)! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2n \cdot (2n+1)$
5	$(2n)!! = 2 \cdot 4 \cdot \ldots \cdot (2n-2) \cdot 2n$	**6**	$(2n+1)!! = 1 \cdot 3 \cdot \ldots \cdot (2n-1) \cdot (2n+1)$

7.3. Standard series

1	General harmonic series	$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \begin{cases} \text{converges if } \alpha > 1, \\ \text{diverges if } \alpha \leq 1 \end{cases}$
2	Geometrical series	$\sum_{n=1}^{\infty} q^{n} \begin{cases} \text{converges if } 0 < q < 1, \\ \text{diverges if } q \geq 1 \end{cases}$

7.4. REMEMBER

1	**Necessary test of convergence**	If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$
2	**Sufficient test of divergence**	If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges

7.5. Number Positive Series. Sufficient Tests of Convergence

1	**Comparison Tests**	**1.** Let $0 \leq a_n \leq b_n$, then $\begin{cases} \text{if } \sum_{n=1}^{\infty} b_n \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges,} \\ \text{if } \sum_{n=1}^{\infty} a_n \text{ diverges, then } \sum_{n=1}^{\infty} b_n \text{ diverges} \end{cases}$
		2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$, where $\ell < \infty$, $\ell \neq 0$, then $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ both are convergent or both are divergen.

2	**D'Alembert Test (Ratio Test)**	If $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}=D$, then $\sum\limits_{n=1}^{\infty} a_n \begin{cases}\text{converges for } D<1,\\ \text{diverges for } D>1,\\ \quad ? \qquad \text{for D} = 1.\end{cases}$
3	**Root Caushy Test (Root Test)**	If $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=C$, then $\sum\limits_{n=1}^{\infty} a_n \begin{cases}\text{converges for } C<1,\\ \text{diverges for } C>1,\\ \quad ? \qquad \text{for } C = 1.\end{cases}$
4	**Integral Caushy Test**	Let $a_n = f(n)$, then $\sum\limits_{n=1}^{\infty} a_n$ and improper integral $\int\limits_1^{\infty} f(x)\,dx$ are both convergent or are both divergent.

7.6. Some Recommendations for Using of Sufficient Tests

1	If a_n is a power or an exponential function with respect to *n* use the comparison test.
2	If a_n is a product of factors the number of which depends on *n*, use d'Alambert test. Particular case is when a_n contains *n*!
3	If a_n contains *n* in the base and in the exponent at the same time, but is not the case (**2**) use the root Caushy test.
4	If **1 – 3** do not give answer use the Integral Caushy test.

7.7. Alternating Number Series

1	Definition of alternating number series	A number series of a kind $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$, $a_n \ge 0$ is called an **alternation number series**
2	**Absolutely convergence of an alternating series**	If $\sum_{n=1}^{\infty} \lvert b_n \rvert = \sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ absolutely converges
3	**Leibniz Test**	Let $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is not absolutely convergent, but 1) $a_1 < a_2 < \ldots < a_n < \ldots$ 2) 2) $\lim_{n\to\infty} a_n = 0$, then this series is conditionally convergent and its sum $\left\lvert \sum_{n=1}^{\infty} (-1)^{n-1} a_n \right\rvert < a_1$

7.8. Power Series

1	**Definition**	A series of a kind $\sum_{n=0}^{n=\infty} C_n (x - x_0)^n$ is called a **power series**
2	**Taylor series for a function** $f(x)$	$f(x) = \sum_{n=0}^{n=\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$
3	**Maclaurin series for a function** $f(x)$	$f(x) = \sum_{n=0}^{n=\infty} \frac{f^{(n)}(0)}{n!} x^n$

7.9. Standard Maclaurin Series

1	• $e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!} + \ldots = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \quad t \in (-\infty, +\infty)$

2	• $\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}, \quad t \in (-\infty, +\infty)$
3	• $\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots + (-1)^n \frac{t^{2n}}{(2n)!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!}, \quad t \in (-\infty, +\infty)$
4	• $(1+t)^{\alpha} = 1 + \sum_{n=1}^{\infty} \frac{\alpha \cdot (\alpha - 1) \cdot \ldots \cdot (\alpha - n + 1)}{n!} t^n, \quad t \in (-1, +1)$
5	• $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n, \quad t \in (-1, +1)$
6	• $\ln(1+t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} t^n, \quad t \in (-1, +1)$
7	• $\arcsin t = t + \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2^n \cdot n!(2n+1)} t^{2n+1}, \quad t \in (-1, +1)$
8	• $\arctan t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} t^{2n+1}, \quad t \in (-1, +1)$
9	• $\sinh t = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} t^{2n+1}, \quad t \in (-\infty, +\infty)$
10	• $\cosh t = \sum_{n=0}^{\infty} \frac{1}{(2n)!} t^{2n}, \quad t \in (-\infty, +\infty)$

7.10. Fourier Series for Functions into

1	$y=f(x),\ x\in(-\pi,\pi)\Rightarrow\ f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{n=\infty}(a_n\cos nx+b_n\sin nx)$ where $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx,\ n=0,1,2,\ldots$ $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx,\ n=1,2,\ldots$ $S(\pm\pi)=\frac{1}{2}(f(\pi-0)+f(\pi+0))$ If x_0 is a discontinuous point then $S(x)=\frac{1}{2}(f(x_0-0)+f(x_0+0))$
2	• $y=f(x),\ x\in(-\ell,\ell)\Rightarrow$ $f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{n=\infty}\left(a_n\cos\frac{n\pi x}{\ell}+b_n\sin\frac{n\pi x}{\ell}\right),$ where $a_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\cos\frac{n\pi x}{\ell}dx,\ n=0,1,2,\ldots$ $b_n=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)\sin\frac{n\pi x}{\ell}dx,\ n=1,2,\ldots$
3	**Complex Form of Fourier Series**
3_1	$y=f(x),\ x\in(-\pi,\pi)\Rightarrow$ $f(x)\sim\sum_{n=-\infty}^{\infty}c_ne^{inx}$, $c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx,\ \ n=0,\pm1,\pm2,\ldots$

3_2	$y = f(x),\ x \in (-\ell, \ell) \Rightarrow f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{\ell}}$, where $c_n = \dfrac{1}{2\ell}\int_{-\ell}^{\ell} f(x) e^{-\frac{in\pi x}{\ell}} dx,\quad n = 0, \pm 1, \pm 2, \ldots$
4	**Fourier Integral**
4_1	**Real Form of Fourier Integral** $f(x) = \int_0^{\infty} (A(\omega)\cos\omega x + B(\omega)\sin\omega x) d\omega$, where $A(\omega) = \dfrac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\omega t\, dt$, $B(\omega) = \dfrac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\omega t\, dt$
4_2	**Complex Form of Fourier Integral** $f(x) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$, where $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$

VIII. COMPLEX NUMBERS AND FUNCTIONS

8.1. COMPLEX NUMBERS

№	Definitions, theorems	Formulas

1	**A complex number is an ordered pair of real numbers**	$z=(x,y)$, where x is a real part, y is an imaginary part
2	**Arithmetic operations**	$(x_1,y_1)=(x_2,y_2)\Leftrightarrow\begin{cases}x_1=x_2\\y_1=y_2\end{cases}$, $(x_1,y_1)\pm(x_2,y_2)=(x_1\pm x_2,y_1\pm y_2)$ $(x_1,y_1)\cdot(x_2,y_2)=(x_1x_2-y_1y_2,x_1y_2+y_1x_2)$
3	**Imaginary unit**	$i=(0,1)$
4	**Powers of imaginary unit** ***i***	$i^n=\begin{cases}1, & \text{if } n=4k\\ i, & \text{if } n=4k+1\\ -1, & \text{if } n=4k+2\\ -i, & \text{if } n=4k+3\end{cases}$
5	**Standard or algebraic form of a complex number**	$z=x+yi$
6	**The square root of a complex number**	$\sqrt{z}=\sqrt{x+iy}=\pm\left(\sqrt{\frac{\lvert z\rvert+x}{2}}+i\,\text{sign}y\sqrt{\frac{\lvert z\rvert-x}{2}}\right)$
7	**Modulus, or absolute value of a complex number** ***z***	$\lvert z\rvert=\rho=\sqrt{x^2+y^2}$
8	**The conjugate to** $z=x+yi$	$\bar{z}=x-yi$
9	**Division of complex numbers**	$z=\frac{z_1}{z_2}=\frac{z_1\overline{z_2}}{z_2\overline{z_2}},\ z_2\neq 0$
10	**Principal argument of** ***z***	$\varphi=\arg z=$ $=\begin{cases}\arctan\frac{y}{x}, & \text{if } z\in \text{ I or IY quadrant},\\ \pi+\arctan\frac{y}{x}, & \text{if } z\in \text{ II or III quadrant}.\end{cases}$
11	**The trigonometric form of a**	

	complex number	$z = \rho\left(\cos\varphi + i\sin\varphi\right)$, where $\rho = \lvert z \rvert$
12	**Euler formula**	$e^{iz} = \cos z + i\sin z$
13	**The exponential form of a complex number**	$z = \rho\, e^{i\varphi}$ $(z = \rho\, \exp(i\varphi))$
14	**The product of complex numbers**	• $z_1 z_2 = \rho_1 \rho_2 \left(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right)$ • $z_1 z_2 = \rho_1 \rho_2 e^{i(\varphi_1 + \varphi_2)}$
15	**The ratio of complex numbers**	• $\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}\left(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right)$ • $\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} e^{i(\varphi_1 - \varphi_2)}$
16	**The integral power of a complex number**	• $z^n = \rho^n\left(\cos n\varphi + i\sin n\varphi\right)$ • $z^n = \rho^n e^{in\varphi}$
17	**The integral root of a complex number**	• $\sqrt[n]{z} = w_k = \sqrt[n]{\rho}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right)$, $k = 0,1,2,\ldots,n-1.$ • $\sqrt[n]{z} = w_k = \sqrt[n]{\rho}\,\exp\left(i\frac{\varphi + 2k\pi}{n}\right)$, $k = 0,1,2,\ldots,n-1.$

8.2. COMPLEX FUNCTIONS

№	Names, definitions	Formulas
1	**Complex plane**	The set of all complex numbers is called the complex plane.

2	**A neighborhood of a point** z_0	A neighborhood of a point z_0 is the set $U(z_0,\delta)=\{z:\lvert z-z_0\rvert<\delta\},\ \delta>0$ If $z_0=\infty$, then $U(\infty,\delta)=\{z:\lvert z\rvert>\delta\}$.
3	**The limit of a complex function**	$\lim\limits_{z\to z_0} f(z)=w_0 \Leftrightarrow$ $\Leftrightarrow\ \forall\varepsilon>0\ \exists\delta>0:\forall z:\ 0<\lvert z-z_\circ\rvert<\delta\Rightarrow$ $\lvert f(z)-\omega_\circ\rvert<\varepsilon$.
4	**Continuity of a complex function**	A complex function $w=f(z)$ is called continuous at a point z_0 if $\lim\limits_{z\to z_0} f(z)=f(z_0)$
		A complex function $w=f(z)$ is called continuous in a region *D* if this function is continuous at any point of *D*
5	**Definition of a complex function**	$w(z)=u(x,y)+iv(x,y)$, where $z=x+iy$, $\operatorname{Re} w=u(x,y)$ and $\operatorname{Im} w=v(x,y)$ are real functions
6	**Main Basic Functions with Complex Variables**	
6_1	**Linear function**	$w=az+b$, where *a* and *b* are complex numbers, $z=x+iy$
6_2	**Principal value of logarithmic function**	$\ln z=\ln\lvert z\rvert+i\arg z\quad (\arg z\in(-\pi,\pi))$
	Logarithmic function	$w=Ln\, z=\ln\lvert z\rvert+iArg\, z=\ln\lvert z\rvert+i(\arg z+2k\pi)=$ $=\ln z+i\arg z\quad (k\in \boldsymbol{Z})$
6_3	**Exponential function**	$w=e^z=e^xe^{iy}=e^x(\cos y+i\sin y)$
6_4	**Power function**	$w=z^a=e^{a\,Ln\, z}$

6_5	**Trigonometric functions**	$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ $\cos s\, z = \frac{e^{iz} + e^{-iz}}{2}$ $\tan z = \frac{\sin z}{\cos z}$ $\cot z = \frac{\cos z}{\sin z}$
6_6	**Hyperbolic functions**	$\sinh z = shz = \frac{e^{z} - e^{-z}}{2}$ $\cosh z = chz = \frac{e^{z} + e^{-z}}{2}$ $\tanh z = th\, z = \frac{\sinh z}{\cosh z}$ $\coth z = cth\, z = \frac{\cosh z}{\sinh z}$
7	**Trigonometric functions** $w = \sin z$ **and** $w = \cos z$ **are not bounded in the complex plane.** **Exponential function is** $2\pi\, i -$ **periodic function.**	
8	**Properties of hyperbolic functions**	• $w = \sinh z$ and $w = \cosh z$ are $2\pi\, i$-periodical functions; $w = \tanh z$ and $w = \coth z$ are $\pi\, i$-periodical functions
		• $\cosh^2 z - \sinh^2 z = 1$
		• $\sinh(-z) = -\sinh z$; • $\cosh(-z) = \cosh z$
		• $\sinh 2z = 2\sinh z \cdot \cosh z$ • $\cosh 2z = \cosh^2 z + \sinh^2 z$
		• $\sinh z + \cosh z = e^z$
9	**Connection between hyperbolic and trigonometric functions**	• $\sinh i\, z = i \sin z$ • $\sin z = -i \sinh i\, z$ • $\cosh i\, z = \cos z$ $\sin i\, z = i \sinh z$ • $\cos i\, z = \cosh z$
10	**Separation of Real and Imaginary Parts of Complex Functions**	
10_1	**Trigonometric functions**	$\sin z = \sin x \cosh y + i \cos x \sinh y$

		$\cos z = \cos x \cosh y + i \sin x \sinh y$
10_2	**Hyperbolic functions**	$\sinh z = \sinh x \cos y + i \cosh x \sin y$
		$\cosh z = \cosh x \cos y + i \sinh x \sin y$
10_3	**Principal logarithmic function**	$\ln z = \ln\lvert z \rvert + i \arg z =$ $= \frac{1}{2}\ln\left(x^2 + y^2\right) + i \arctan \frac{y}{x}$
11	**Expression of *x* and *y* through *z***	Let $z = x + iy$, then $x = \frac{z + \overline{z}}{2}; \quad y = \frac{z - \overline{z}}{2i} \quad \left(\overline{z} = x - iy\right)$
12	**Definition of a derivative of a complex function** $w = f(z)$	$f'(z) = \lim\limits_{\Delta z \to 0} \frac{\Delta \omega}{\Delta z} = \lim\limits_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$
13	**Cauchy-Riemann Conditions**	Let $f(z) = u(x, y) + iv(x, y)$, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
		Let $f(z)$ is given in the polar coordinates, then $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \varphi}; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \varphi}$
14	**Derivative of a complex function**	Let $f(z) = u(x, y) + i\,v(x, y)$ then $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$
		Let $f(z)$ is given in the polar coordinates: $f(z) = u(r, \varphi) + i\,v(r, \varphi)$, then $f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$
15	**Analyticity of a complex function at a point**	One-valued complex function $w = f(z)$ is called analytic at a point z_0 if it is differentiable at any neighborhood of this point

16	**Analyticity of a complex function in a region**	One-valued complex function $w = f(z)$ is called analytic in a region D if it is differentiable at all points of this region
17	**Differential of a complex function**	$dw = f'(z)\,dz$
18	**Geometrical meaning of a derivative of a complex function**	$\lvert f'(z_0)\rvert$ determines the coefficient of similarity at a point z_0: if $\lvert f'(z_0)\rvert > 1$, then $\lvert f'(z_0)\rvert$ is the coefficient of tensions; if $\lvert f'(z_0)\rvert < 1$, then $\lvert f'(z_0)\rvert$ is the coefficient of compression
19	**Integral of a complex function along the curve connecting the points z_1 and z_2**	Let $f(z)$ is not analytic then $\int\limits_{\breve{L}=(z_1 z_2)} f(z)dz = \int\limits_{\breve{L}\in R^2} (u\,dx - v\,dy) + i\int\limits_{\breve{L}\in R^2} (v\,dx + u\,dy)$
		Let $f(z)$ is analytic in the domain D and $\breve{L} \subset D$, then $\int\limits_{\breve{L}=(z_1 z_2)} f(z)dz = \int\limits_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$, where $F'(z) = f(z)$
20	**Integral of a complex function along the closed curve *L***	Cauchy theorem : If $f(z)$ is analytic in the one-connected domain D ,then $\oint\limits_L f(z) = 0$ for any closed contour L in D
		If $f(z)$ is analytic in the multiple-connected domain

		D, bounded by Γ (exterior contour) and $\gamma_1, \gamma_2, \ldots, \gamma_n$ (interior contours), then $$\oint_{\Gamma} f(z) = \sum_{k=1}^{n} \oint_{\gamma_k} f(z)dz$$
21	**Cauchy Integral formula and its corollaries**	If $f(z)$ is analytic in the one-connected domain D and L is a boundary of D, then $f(z_0) = \frac{1}{2\pi i}\oint_L \frac{f(z)dz}{z - z_0}$, where $z_0 \in D \Rightarrow$ $$\Rightarrow \oint_L \frac{f(z)dz}{z - z_0} = 2\pi i \cdot f(z_0).$$ The multiple $\frac{1}{z - z_0}$ is called **Cauchy kernel**
		$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_L \frac{f(z)dz}{(z - z_0)^{n+1}},\ n = 1,2,\ldots\ z_0 \notin L \Rightarrow$$ $$\Rightarrow \oint_L \frac{f(z)dz}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)$$
22	**Taylor Series** If $f(z)$ is analytic in the circle $\lvert z - z_0 \rvert \le R$, then $$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$ where z is a point such that $\lvert z - z_0 \rvert = r$ ($r < R$). $$c_n = \frac{f^{(n)}(z_0)}{n!}.$$	
23	**Zeros of Analytic Function** **Zero** of analytic function $f(z)$ is the point z_0 such that $f(z_0) = 0$. In this case $c_0 = f(z_0) = 0$ - coefficient of the Taylor expansion.	
	Zero of Order *m*	

24	Let $f(z) = (z - z_0)^m \cdot \varphi(z),\ \varphi(z_0) \neq 0$. If $c_0 = c_1 = c_2 = ... = c_{m-1} = 0,\ c_m \neq 0$, then the first non-zero term in the Taylor expansion is $c_m(z - z_0)^m$. In this case $f(z)$ is said to have a **zero of order *m*** at $z = z_0$.
25	**Laurent Series** Let C_1 and C_2 be two circles of center z_0 with radii R_1 and R_2 ($R_1 < R_2$). A function $f(z)$ analytic in an annulus $R_1 \leq \lvert z - z_0 \rvert \leq R_2$ may be represented by the expression $$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n,$$ z being any point of the annulus. $c_n = \dfrac{1}{2\pi i} \oint_C \dfrac{f(t)dt}{(t - z_0)^{n+1}}$, where C is any closed contour, lying within the annulus between C_1 and C_2.
26	**Singularity** If a function $f(z)$ is not analytic at a point z_0, then z_0 is called a **singularity** (or a singular point).
27	**Isolated Singularity *m*** Suppose $f(z)$ is analytic in the region *D*, defined by $\lvert z - z_0 \rvert < R$ (i.e. in a neighborhood of z_0), and not at the point z_0. Then the point z_0 is called an **isolated singular** point of $f(z)$. We can draw two concentric circles of center z_0, both lying within *D*. In the annulus between these circles, $f(z)$ may be represented by a Laurent expansion: $$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n = \underbrace{\sum_{n=0}^{\infty} c_n (z - z_0)^n}_{\text{regular part}} + \underbrace{\sum_{n=1}^{\infty} c_{-n} (z - z_0)^{-n}}_{\text{principal part}}$$
28	**Residue** The coefficient of the term $(z - z_0)^{-1}$ is called the residue of a function at the point z_0: $\mathrm{Res}(f(z), z_0) = c_{-1}$.

29	**Removable Singularity** Singularity z_0 is called a **removable** point if $\lim_{z\to z_0} f(z)$ exists and is finite. In this case Laurent expansion does not contain a principal part. Therefore $c_{-1} = \mathrm{Res}(f(z), z_0) = 0$.
30	**Essential Singularity** If $\lim_{z\to z_0} f(z)$ does not exist then z_0 is called an essential singularity. In this case the principal part of Laurent Series contains infinite terms.
31	**Pole of Order *m*** A singular point z_0 is called the pole of order m of a function $f(z)$ if z_0 is a zero of order m of the function $\frac{1}{f(z)}$ (or $\lim_{z\to z_0} f(z)\cdot(z-z_0)^m = \alpha$ exists and it is finite: $\alpha \neq 0$, $\alpha \neq \infty$) In this case the principal part of Laurent Series contains m terms and $c_{-1} = \mathrm{Res}(f(z), z_0) = \frac{1}{(m-1)!}\lim_{z\to z_0}(f(z)\cdot(z-z_0)^m)^{(m-1)}$ If the pole of $f(z)$ is the simple pole $(m=1)$ $c_{-1} = \mathrm{Res}(f(z), z_0) = \lim_{z\to z_0} f(z)\cdot(z-z_0)$
32	**Cauchy Residue Theorem** Let $f(z)$ be analytic inside and on a simple closed contour C, except for a finite number of isolated singular points $z_1, z_2, \ldots, z_n$ located inside C. Then $\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z); z_k)$

IX. Appendix

9.1. English – Russian Vocabulary

A

Absolutely convergence – абсолютная сходимость
alternating number series – знакопеременный числовой ряд
analyticity – аналитичность
antiderivative – первообразная

B

Bernoulli's DE – дифференциальное уравнение Бернулли

C

Cauchy's Problem – задача Коши
Cauchy-Riemann conditions – условия Коши-Римана
center of gravity of a figure – центр тяжести фигуры
characteristic equation – характеристическое уравнение
circulation of a vector field – циркуляция векторного поля
comparison test – признак сравнения
complete the square – выделить полный квадрат
complex function – функция комплексной переменной
complex number – комплексное число
conjugate complex number – сопряженное комплексное число
converge – сходиться
convergence – сходимость
(to be) convergent – сходящийся
curl – ядро

D

D'Alembert test – признак Даламбера
definite integral – определенный интеграл
differential equation – дифференциальное уравнение
directional derivative of a scalar field – производная скалярного поля по направлению
discontinuous point – точка разрыва
diverge – расходиться
divergence – расходимость

divergence of a vector field – циркуляция векторного поля
double integral – двойной интеграл
double integral in polar coordinates – двойной интеграл в полярных координатах

E
essential singularity – существенная особенность
expanding function – доопределение функции
exponential form of a complex number – показательная форма комплексного числа
exterior contour – внешний контур

F
factorial – факториал
flux of a vector field – поток векторного поля
Fourier series – ряд Фурье

G
Gauss-Ostrogradsky formula – формула Гаусса-Остроградского
general solution – общее решение
y_{GH} - **general solution of homogeneous linear differential equation** – общее решение однородного линейного дифференциального уравнения
y_{GN} - **general solution of nonhomogeneous linear differential equation** – общее решение неоднородного линейного дифференциального уравнения
geometrical series – геометрический ряд
gradient of a scalar field – градиент скалярного поля
Green's formula – формула Грина

H
Hamiltonian – оператор Хамельтона
harmonic series – гармонический ряд
Helmholtz' theorem – теорема Гельмгольца
HLDE – **homogeneous linear differential equatio**n – однородное линейное дифференциальное уравнение

I
imaginary unit – мнимая единица
improper integral – несобственный интеграл
indefinite integral – неопределенный интеграл
initial conditions – начальные условия
inner integral – внутренний интеграл
instantaneous velocity – мгновенная скорость
integral Caushy's test – интегральный признак Коши
integral over a figure – интеграл по фигуре

integral power of a complex number $\left(z^n, n \in N\right)$ – целая степень комплексного числа
integral root of a complex number – корень целой степени из комплексного числа
integral sum – интегральная сумма
integral over an oriented surface – интеграл по ориентированной поверхности
integrand – подынтегральная функция
integration – интегрирование
integration by parts – интегрирование по частям
integration by substitution – интегрирование с помощью замены переменных
irrational function – иррациональная функция
isolated singularity – изолированная особенность
iterated integral – повторный интеграл

K
kernel – ядро

L
Laplace operator – оператор Лапласа
Laurent series – ряд Лорана
Leibniz test – признак Лейбница
level line – линия уровня
level surface – поверхность уровня
line integral with respect to the arc length – криволинейный интеграл по длине дуги
line integral of the first type – криволинейный интеграл первого рода
line integral with respect to coordinates – криволинейный интеграл по координатам
linear DE of the first order – линейное дифференциальное уравнение первого порядка
LDE – linear differential equation – линейное дифференциальное уравнение
lowering of the order – понижение степени

M
Maclaurin series – ряд Маклорена
measure of a figure – мера фигуры
method of indefinite coefficients – метод неопределенных коэффициентов
method of variation of arbitrary constants – метод вариации постоянных
modulus – модуль
multiple integrals – кратные интегралы
multy-connected domain – многосвязная область

N
neighborhood - окрестность

Newton – Leibniz formula – формула Ньютона-Лейбница
NLDE – nonhomogeneous linear differential equation – неоднородное линейное дифференциальное уравнение
number positive series – числовой знакоположительный ряд

O
one-dimensional figure – одномерная фигура
ordered pair – упорядоченная пара
outer integral – внешний интеграл
P
partial fraction – простейшая дробь
particular solution – частное решение
y_{PN} - **particular solution of nonhomogeneous linear differential equation** частное решение однородного линейного дифференциального уравнения
partition – разбиение
pole of order m **–** полюс m- го порядка
potential field – потенциальное поле
power series – степенной ряд
principal argument – главный аргумент

R
removable singular point – устранимая особая точка
residue – вычет
root Caushy's test – радикальный признак Коши
rotation (curl) of a vector field – ротор векторного поля

S
scalar field – скалярное поле
second order DE – дифференциальное уравнение второго порядка
separable variables – разделяющиеся переменные
series – ряд
singularity (singular point) – особенность, особая точка
sink – сток
solenoid field – соленоидальное поле
solid of revolution – тело вращения
source - источник
static moment – статистический момент
Stokes' formula – формула Стокса
sufficient test – достаточный признак
surface integral of the first type – поверхностный интеграл первого рода
surface of revolution – поверхность вращения
system of differential equations – система дифференциальных уравнений

T
Taylor series – ряд Тейлора

three-dimensional figure – трехмерная фигура
trigonometric form of a complex number – тригонометрическая форма комплексного числа
trigonometric identities – тригонометрические тождества
triple integral – тройной интеграл
triple integral in cylindrical coordinates – тройной интеграл в цилиндрических координатах
triple integral in spherical coordinates– тройной интеграл в сферических координатах
two-dimensional figure – двумерная фигура

U
universal substitution – универсальная подстановка

V
variable – переменная
vector field – векторное поле
vector lines of a vector fie**ld** – векторные линии векторного поля

Z
zero of order m – ноль порядка m
zeros of analytic function – нули аналитической функции

9.2. Russian – English Vocabulary

А
абсолютная сходимость – absolutely convergence
аналитичность – analyticity

В
векторное поле – vector field
векторные линии векторного поля – vector lines of a vector field
внешний интеграл – outer integral
внешний контур – exterior contour
внутренний интеграл – inner integral
выделить полный квадрат – complete the square
вычет – residue

Г
гармонический ряд – harmonic series
геометрический ряд – geometrical series
главный аргумент – principal argument
градиент скалярного поля – gradient of a scalar field

Д

двойной интеграл – double integral
двойной интеграл в полярных координатах – double integral in polar coordinates
двумерная фигура – two-dimensional figure
дифференциальное уравнение – differential equation
дифференциальное уравнение Бернулли – Bernoulli's DE
дифференциальное уравнение второго порядка – second order DE
доопределение функции – expanding function
достаточный признак – sufficient test

З

задача Коши – Cauchy's Problem
знакопеременный числовой ряд – alternating number series

И

изолированная особенность – isolated singularity
интеграл по ориентированной поверхности – integral over oriented surface
интеграл по фигуре – integral over figure
интегральная сумма – integral sum
интегральный признак Коши – integral Caushy's test
интегрирование – integration
интегрирование по частям – integration by parts
интегрирование с помощью замены переменных – integration by substitution
иррациональная функция – irrational function
источник – source

К

комплексное число – complex number
корень целой степени из комплексного числа – integral root of a complex number
кратные интегралы – multiple integrals
криволинейный интеграл первого рода – line integral of the first type
криволинейный интеграл по длине дуги – line integral with respect to the arc length
криволинейный интеграл по координатам – line integral with respect to coordinates

Л

линейное дифференциальное уравнение LDE – linear differential equation –
линейное дифференциальное уравнение первого порядка – linear DE of the first order
линия уровня – level line

М

мгновенная скорость – instantaneous velocity
мера фигуры – measure of a figure
метод вариации постоянных – method of variation of arbitrary constants
метод неопределенных коэффициентов – method of indefinite coefficients
мнимая единица – imaginary unit
многосвязная область – multy-connected domain
модуль – modulus

Н

начальные условия – initial conditions
неопределенный интеграл – indefinite integral
несобственный интеграл – improper integral
ноль порядка m **–** zero of order m
нули аналитической функции – zeros of analytic function

О

общее решение – general solution
одномерная фигура – one-dimensional figure
окрестность – neighborhood
оператор Гамельтона – Hamiltonian
оператор Лапласа – Laplace operator
определенный интеграл – definite integral
особенность, особая точка – singularity (singular point)

П

первообразная – antiderivative
переменная – variable
поверхностный интеграл первого рода – surface integral of the first type
поверхность вращения – surface of revolution
поверхность уровня – level surface
повторный интеграл – iterated integral
подынтегральное выражение, функция – integrand
показательная форма комплексного числа – exponential form of a complex
полюс ***m*****- го порядка** – pole of order m
понижение степени – lowering of the order
потенциальное поле – potential field
поток векторного поля – flux of a vector field
признак Даламбера – D'Alembert's test (ration test)
признак Лейбница – Leibniz' test
признак сравнения – comparison test
производная скалярного поля по направлению – directional derivative of a scalar field

Р

радикальный признак Коши – root Caushy’s test (root test)
разделяющиеся переменные – separable variables
расходимость – divergence
расходиться – diverge
ротор векторного поля – rotation (curl) of a vector field
ряд – series
ряд Лорана – Laurent Series –
ряд Маклорена – Maclaurin series
ряд Тейлора – Taylor series
ряд Фурье – Fourie Series

С

система дифференциальных уравнений – system of differential equations
скалярное поле – scalar field
соленоидальное поле – solenoid field
сопряженное комплексное число – conjugate complex number
статистический момент – static moments
степенной ряд – power series
сток – sink
существенная особенность – essential singularity
сходимость – convergence
сходиться – converge
сходящийся – to be convergent

Т

тело вращения – solid of revolution
теорема Гельмгольца – Helmholtz’ theorem
точка разрыва – discontinuous point
трехмерная фигура – three-dimensional figure
тригонометрическая форма комплексного числа – trigonometric form of a complex
тригонометрические тождества – trigonometric identities
тройной интеграл – triple integral
тройной интеграл в сферических координатах – triple integral in spherical coordinates
тройной интеграл в цилиндрических координатах – triple integral in cylindrical coordinates

У

универсальная подстановка – universal substitution
упорядоченная пара – ordered pair
условия Коши-Римана – Cauchy-Riemann Conditions
устранимая особая точка – removable singular point

Ф

факториал – factorial
формула Гаусса-Остроградского – Gauss-Ostrogradsky formula
формула Грина – Green's formula
формула Ньютона-Лейбница – Newton – Leibniz formula
формула Стокса – Stokes' formula
функция комплексной переменной – complex function

Х

характеристическое уравнение – characteristic equation

Ц

целая степень комплексного числа – integral power of a complex number $\left(z^n, n \in N\right)$
центр тяжести фигуры – center of gravity of a figure
циркуляция векторного поля – circulation of a vector field
циркуляция векторного поля – divergence of a vector field

Ч

числовой знакоположительный ряд – number positive series

Я

ядро – curl, kernel

For Notes
