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Textbook on Sections

INTEGRAL CALCULUS. DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS

For Students Studying a Course of Higher Mathematics in English

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В пособии в краткой форме представлены основные сведения по темам: «Интегральное исчисление», «Дифференциальные уравнения и их приложения» для студентов академии, изучающих высшую математику на английском языке. Основные теоремы и формулы приведены с доказательством, а также даны решения типовых примеров и задания для самостоятельного решения.

Компьютерная верстка ***Корнейчук Е. С.***

PREFACE

This course is designed as a textbook of methodics for engineering students on special sections of mathematical analysis, such as integral calculus and ordinary differential equations.

The examples are presented demonstrate applications of mathematical analysis to various problems of mechanics and physics.

The study of these examples is very important since the main interest of an engineer lies in solving concrete applied problems.

In doing the exercises by themselves, students find that they are required to devote considerable time to calculation.

CONTENTS

PART I

INDEFINITE INTEGRAL

§ 1. Antiderivative. Indefinite Integral

We have solved the problem of *finding the derivative of a given function*. Now we proceed to the inverse problem: given ***the derivative of a function, to find this function***. The solution of this problem is of great importance for mathematical analysis and its applications.

Definition. An **antiderivative** (a primitive) of a given function *f(x)* in a given interval is any function *F(x)* whose derivative is equal to the given function for any point of this interval:

$$F'(x) = f(x) \tag{1.1.1}$$

For example

$f(x) = \sin x \Rightarrow F(x) = -\cos x + C$, **as** $(-\cos x + C)' = \sin x.$

Definition. The operation of finding antiderivatives is called **integration.**

Theorem. Let $F_1(x) \neq F_2(x)$ are antiderivatives of a function *f(x)* in any interval then $F_2(x) = F_1(x) + C$.

To prove it denote $F_2(x) - F_1(x)$ as $\varphi(x)$. So we have

$$\varphi(x) = F_2(x) - F_1(x) \Rightarrow \varphi'(x) = (F_2(x) - F_1(x))' = F_2'(x) - F_1'(x) =$$
$$f(x) - f(x) = 0 \Rightarrow \varphi(x) = C.$$

Thus $F_2(x) = F_1(x) + C.$ The theorem is proved.

Definition. A set of all antiderivatives of a function $f(x)$ is called an **indefinite integral** of this function and is denoted by the symbol $\int f(x)dx.$

It is read: **indefinite integral of a function** $f(x)$ **with respect to *x*.**

The function *f(x)* is called the *integrand*, the expression *f(x) dx* is the *element of integration,* and the variable *x* is the *variable of integration.*

§ 2. The Basic Table of Integrals

(1) $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C, (\alpha \neq -1)$

(2) $\int x^{-1} dx = \int \frac{dx}{x} = \ln|x| + C$

(3) $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C$

(4) $\int a^x dx = \frac{a^x}{\ln a} + C.$

(5) $\int e^x dx = e^x + C.$

(6) $\int \sin x dx = -\cos x + C.$

(7) $\int \cos x dx = \sin x + C.$

(8) $\int \frac{dx}{\sin^2 x} = -\cot x + C.$

(9) $\int \frac{dx}{\cos^2 x} = \tan x + C.$

(10) $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C.$

(11) $\int \frac{dx}{1+x^2} = \arctan x + C.$

(12) $\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln\left|x + \sqrt{x^2 \pm 1}\right| + C.$

(13) $\int \frac{dx}{x^2 - 1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$

Each of these integration formulas is easily checked by differentiation.

(*) Note that if $\int f(x)dx = F(x) + C$ **then** $\int f(kx+b)dx = \frac{1}{k}F(kx+b) + C.$

§ 3. Properties of Indefinite Integral

Using the definition of antiderivative and properties of derivatives we can prove the next theorems.

Theorem 1.3.1. **The derivative of an indefinite integral equals the integrand:**

$$\left(\int f(x)dx\right)' = f(x).$$

Example. $\left(\int \frac{x^3+x}{2^x} dx\right)' = \frac{x^3+x}{2^x}.$

Theorem 1.3.2. **The differential of an indefinite integral equals the element of integration:**

$$d\int f(x)dx = f(x)dx.$$

Theorem 1.3.3. **The integral of a differential of a function** *u* **is** *u* **plus an arbitrary constant** *C:*

$$\int du = u + C.$$

Example. $\int d(\cos x) = \cos x + C.$

Theorem 1.3.4. **A constant may be moved across the integral sign:**

$$\int Cf(x)dx = C\int f(x)dx \quad (C \neq 0).$$

Example. $\int 5e^x dx = 5\int e^x dx = 5e^x + C.$

Theorem 1.3.5. **The integral of a sum of a finite number of functions is equal to the sum of the integrals of these functions:**

$$\int \sum_{k=1}^{n} f_k(x)dx = \sum_{k=1}^{n} \int f_k(x)dx.$$

Example.

$$\int\left(x + \frac{1}{\sqrt{x}} + \sin x\right)dx = \int xdx + \int\frac{dx}{\sqrt{x}} + \int \sin xdx =$$

$$= \frac{x^2}{2} + 2\sqrt{x} - \cos x + C.$$

§4. Integration by Parts

The general method, called *integration by parts,* depends upon the formula for the differential of a product: $d(uv) = udv + vdu$ or $udv = d(uv) - vdu$. When this is integrated, we have

$$\int udv = uv - \int vdu \qquad (1.4.1)$$

Formula (1.4.1) expresses one integral, $\int udv$, in terms of a second integral $\int vdu$.

If, by proper choice of u and dv, the second integral is simpler than the first, we may be able to evaluate it quite simply and thus arrive at an answer.

Case 1. If an integrand is a product of a polynomial by one of the trigonometric or exponential functions *we denote the polynomial* as ***u*** and the other part of element of integration as ***dv.***

Case: 2. If an integrand is a product of a polynomial by one of an inverse functions, namely $\arcsin x,\ \arccos x,\ \arctan x,\ arc\cot x,\ \log_a x,\ \ln x$ *we denote the inverse function as* ***u.***

Example 1.4.1.

$$\int \ln x dx = \left[\begin{array}{l} u = \ln x \Rightarrow du = \dfrac{dx}{x} \\ dv = dx \Rightarrow v = x \end{array}\right] = x\ln x - \int \frac{xdx}{x} = x\ln x - x + C.$$

Example 1.4.2.

$$\int (1-3x)\sin(2x+1)dx = \left[\begin{array}{l} u = 1-3x \Rightarrow du = -3dx \\ dv = \sin(2x+1)dx \Rightarrow v = -\dfrac{1}{2}\cos(2x+1) \end{array}\right] =$$

$$= -\frac{(1-3x)\cos(2x+1)}{2} - \frac{3}{2}\int \cos(2x+1)dx = -\frac{(1-3x)\cos(2x+1)}{2} -$$

$$-\frac{3}{4}\sin(2x+1) + C.$$

Sometimes to obtain the desired result it is necessary to use integration by parts several times. If this is Case 1 it is possible to use the Tabular Integration:

Example 1.4.3.

$$\int (x^2+3x-1)\cos x dx =$$

$$\left|\begin{array}{cc} (2x+3) & (\sin x) \\ 2 & (-\cos x) \\ 0 & (-\sin x) \end{array}\right|$$

$$=(x^2+3x-1)\sin x - (2x+3)(-\cos x) + 2(-\sin x) + C =$$

$$= (x^2+3x-3)\sin x + (2x+3)\cos x + C.$$

As you see here are derivatives of the polynomial in the first column and the antiderivatives of the trigonometric function in the second. The first product is taken with its own sign, the second is with opposite sign and so on**.**

§ 5. Method of substitution (Integration by change of variable)

When computing integrals we have resorted to the theorem on the invariance of integration formulas. If we succeed in writing the element of integration in the form

$$f(\varphi(x))\varphi'(x)dx = f(u)du$$

where $u = \varphi(x)$ and if the integral

$$\int f(u)du = F(u) + C$$

of the expression on the right-hand side is known, the original integral is equal to

$$\int f(\varphi(x))\varphi'(x)dx = F(\varphi(x)) + C.$$

Remark: When you evaluate integrals by substitution do not forget to return to the original variable!

Example 1.5.1.

$$\int x^2\sqrt[3]{4-3x^3}\,dx = \begin{bmatrix} 4-3x^3 = u \Rightarrow -9x^2dx = du \Rightarrow \\ \Rightarrow x^2dx = -\dfrac{du}{9} \end{bmatrix} =$$

$$= -\frac{1}{9}\int u^{1/3}du = -\frac{1}{12}u^{4/3} + C =$$

$$= -\frac{1}{12}\left(4-3x^3\right)^{4/3} + C = -\frac{1}{12}\sqrt[3]{\left(4-3x^3\right)^4} + C.$$

Up to now we have used the method of substitution by replacing the variable of integration x by another variable u using the formula $u = \varphi(x)$. But it is also possible to make a substitution not by expressing u in terms of x but, by taking x as a function of u. That is by putting

$$x = \psi(u) \Rightarrow dx = \psi'(u)\,du$$

(it is supposed that $\psi(u)$ and $\psi'(u)$ are continuous). Then

$$f(x)dx = f(\psi(u))\psi'(u)du$$

and

$$\int f(x)dx = \int f(\psi(u))\psi'(u)du \qquad (1.5.1)$$

If the integral on the right-hand side of (1.5.1) is found and expressed as *F(u)+C*, the given integral can be found by returning to the variable *x*. To do it we are need to express *u* in terms of *x* from the equation $x = \psi(u)$.

Example 1.5.2.

$$\int \sqrt{1-x^2}\,dx = \left[\begin{array}{l} x = \sin u \Rightarrow \ dx = \cos u du \\ \sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \cos u \\ u = \arcsin x \end{array}\right] =$$

$$= \int \cos^2 u du = \frac{1}{2}\int (1+\cos 2u)du =$$

$$= \frac{1}{2}\left(u + \frac{\sin 2u}{2}\right) + C = \frac{1}{2}\left(\arcsin x + \frac{\sin 2\arcsin x}{2}\right) + C =$$

$$= \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + C.$$

§ 6. Integrals Involving $ax^2 + bx + c$

The general quadratic $f(x) = ax^2 + bx + c,\ a \neq 0$ can be reduced to the form $a(u^2 + B)$ by completing the square, as follows:

$$ax^2 + bx + c = a\left(x^2 + 2\frac{b}{2a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a},$$

and substituting $\quad u = x + \frac{b}{2a}, \quad B = \frac{4ac-b^2}{4a}, \quad$ which gives us

$$f(x) = a(u^2 + B).$$

Example 1.6.1.

$$\int \frac{(x+1)dx}{\sqrt{2x^2 - 6x + 4}} =$$

$$\left[\begin{array}{l} 2x^2-6x+4=2\left(x^2-3x+2\right)=2\left((x-3/2)^2-1/4\right) \\ x-3/2=u \Rightarrow \ x=u+3/2 \Rightarrow \ dx=du \\ x+1=u+5/2 \end{array}\right]$$

$$=\frac{1}{\sqrt{2}}\int\frac{u+5/2}{\sqrt{u^2-1/4}}du=\frac{1}{2\sqrt{2}}\int\frac{\left(u^2-1/4\right)'du}{\sqrt{u^2-1/4}}+\frac{5}{2\sqrt{2}}\int\frac{du}{\sqrt{u^2-1/4}}=$$

$$=\frac{1}{\sqrt{2}}\sqrt{u^2-1/4}+\frac{5}{2\sqrt{2}}\ln\left|u+\sqrt{u^2-1/4}\right|+C=$$

$$=\frac{1}{\sqrt{2}}\sqrt{x^2-3x+2}+\frac{5}{2\sqrt{2}}\ln\left|x-\frac{3}{2}+\sqrt{x^2-3x+2}\right|+C.$$

Remark. It is possible to calculate integrals of a type $\int\frac{Mx+N}{ax^2+bx+c}dx$, $\int\frac{Mx+N}{\sqrt{ax^2+bx+c}}dx$ without substitution. To do it use the next algorithm :

1. Write the derivative of quadratic form in the numerator.
2. Equate the coefficients.
3. Write the integral as the sum of two integrals, calculating the first of them by formulas (2) or (3) (see § 2).
4. To calculate the second integral complete the square and use one of the formulas (10)-(13) in § 2.

Example 1.6.2.

$$\int\frac{5x-1}{3x^2+12x+7}=$$

$$=\int\frac{(6x+12)\cdot\frac{5}{6}-\frac{12\cdot 5}{6}-1}{3x^2+12+7}dx=\frac{5}{6}\int\frac{\left(3x^2+12x+7\right)'}{3x^2+12x+7}dx-11\int\frac{dx}{3x^2+12x+7}=$$

$$\left[3x^2+12x+7=3\left(x^2+4x+\frac{7}{3}\right)=3\left((x+2)^2-4+\frac{7}{3}\right)=3\left((x+2)^2-\frac{5}{3}\right)\right]$$

$$=\frac{5}{6}\ln\left|3x^2+12x+7\right|-\frac{11}{3}\int\frac{dx}{(x+2)^2-\left(\sqrt{5/3}\right)^2}=$$

$$=\frac{5}{6}\ln(3x^2+12x+7)-\frac{11}{3\cdot 2\sqrt{5/3}}\ln\left|\frac{x+2-\sqrt{5/3}}{x+2+\sqrt{5/3}}\right|+C.$$

§ 7. Integration of Rational Fractions

Definition. A ratio of two polynomials is called a ***rational function*** or ***rational fraction.***

Definition. If a degree of a numerator is less than a degree of a denominator this fraction is called a ***proper fraction***. If this is not so a fraction is called an ***improper fraction***.

For example, the fractions $\frac{3x^2+5x-7}{8x^2+9x+15}, \frac{x^3+1}{x-14}$ are improper fractions, and the fraction $\frac{1-x^3}{x^4+12x-121}$ is the proper fraction.

Every improper fraction is the sum of polynomial and a proper fraction.

The fraction $\frac{2x^3+2x^2+5x-6}{x^2-3x+5}$ is improper. The division of the numerator by the denominator gives us:

$$\begin{array}{l|l} 2x^3+2x^2+5x-6 & x^2-3x+5 \\ \cline{2-2} -\left(2x^3-6x^2+10x\right) & 2x+8 \\ \quad 8x^2-5x-6 & \\ \quad -\left(8x^2-24x+40\right) & \\ \qquad 19x-46 & \end{array}$$

Thus we have

$$\frac{2x^3+2x^2+5x-6}{x^2-3x+5} = 2x+8+\frac{19x-46}{x^2-3x+5}.$$

Definition. The fractions of a kind $\frac{A}{(x-\alpha)^k}, \frac{Mx+N}{\left(x^2+px+q\right)^l}$ where $p^2-4q<0$, are called the ***partial fractions.***

It is known that any polynomial with real coefficients can be expressed as a product of real linear and quadratic factors. Namely:

$$Q_n(x) = (x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}\ldots\left(x^2+p_1x+q_1\right)^{l_1}\ldots,$$

where $k_1+k_2+\ldots+2l_1+\ldots=n$.

Suppose that there is a fraction

$\frac{P_m(x)}{(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_k)}$, $m<k, \alpha_1 \neq \alpha_2 \neq \ldots \neq \alpha_k$, then

$$\frac{P_m(x)}{(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_k)} = \frac{A_1}{x-\alpha_1}+\frac{A_2}{x-\alpha_2}+\ldots+\frac{A_k}{x-\alpha_k} \qquad (1.7.1)$$

To find the coefficients $A_1, A_2, \dots A_k$ we can use so called *"finger's rule"*:

$$A_1 = \frac{P_m(\alpha_1)}{(\alpha_1 - \alpha_2)\dots(\alpha_1 - \alpha_k)}, \dots, A_k = \frac{P(\alpha_k)}{(\alpha_k - \alpha_1)\dots(\alpha_k - \alpha_{k-1})}.$$

$\dfrac{P_m(x)}{(x-\alpha)^k}, m < k$, then

$$\frac{P_m(x)}{(x-\alpha)^k} = \frac{A_k}{(x-\alpha)^k} + \frac{A_{k-1}}{(x-\alpha)^{k-1}} + \dots + \frac{A_1}{x-\alpha} \qquad (1.7.2)$$

Here only coefficient A_k can be calculating using the "finger's rule". To find the rest coefficients we use the method of *indefinite coefficients.* To do this reduce the fractions of the right hand side of (1.7.2) to the common denominator. Then equate the coefficients of corresponding powers of *x*, and solve the resulting equations for the undetermined coefficients.

$$\frac{P_m(x)}{(x^2 + px + q)^k}, m < 2k, p^2 - 4q < 0,$$

then

$$\frac{P_m(x)}{(x^2 + px + q)^k} = \frac{M_k x + N_k}{(x^2 + px + q)^k} + \frac{M_{k-1} x + N_{k-1}}{(x^2 + px + q)^{k-1}} + \dots + \frac{M_1 x + N_1}{x^2 + px + q} \qquad (1.7.3)$$

Here all of coefficients are calculated by the method of indefinite coefficients.

Example 1.7.1. Express the fraction $\dfrac{-2x+4}{(x^2+1)(x-1)^2}$ as a sum of partial fractions**.**

The given fraction is the proper irreducible fraction. Using the decompositions (1.7.2) and (1.7.3) we have

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Mx+N}{x^2+1} + \frac{1}{(x-1)^2} + \frac{A}{x-1} \Rightarrow$$

$$-2x + 4 = (Mx+N)(x-1)^2 + x^2 + 1 + A(x^2+1)(x-1) =$$

$$= (M+A)x^3 + (-2M + N - A + 1)x^2 + (M - 2N + A)x + (N - A + 1).$$

To order for this to be an identity in *x*, it is necessary and sufficient that the coefficient of each power of *x* be the same on the left side of the equation as it is on the right side. Equating these coefficients leads to the following equations:

$$\begin{array}{c|ll} x^3 & 0 = M + A & (1) \\ x & -2 = M - 2N + A & (2) \\ x^0 & 4 = N - A + 1 & (3) \end{array}$$

$(2)-(1) \Rightarrow\ -2N=-2 \Rightarrow\ N=1,$
$(3) \Rightarrow\ A=N-3 \Rightarrow\ A=-2,$
$(1) \Rightarrow\ M=-A \Rightarrow\ M=2.$

Hence

$$\frac{-2x+4}{\left(x^2+1\right)(x-1)^2}=\frac{2x+1}{x^2+1}+\frac{1}{(x-1)^2}-\frac{2}{x-1}.$$

Example 1.7.2. Evaluate

$$\int \frac{x^4+x^3-2}{x^4+x^3-x-1}dx.$$

Solution. The integrand is a rational fraction, but it is not a proper fraction. Hence we divide first:

$$\frac{x^4+x^3-2}{x^4+x^3-x-1}=\frac{\left(x^4+x^3-x-1\right)+(x-1)}{x^4+x^3-x-1}=1+\frac{x-1}{x^4+x^3-x-1}.$$

The denominator factors as follows:

$$x^4+x^3-x-1=x^3(x+1)-(x+1)=(x+1)\left(x^3-1\right)=$$
$$=(x+1)(x-1)\left(x^2+x+1\right).$$

Then

$$\frac{x-1}{x^4+x^3-x-1}=\frac{x-1}{(x+1)(x-1)\left(x^2+x+1\right)}=\frac{1}{(x+1)\left(x^2+x+1\right)}=$$
$$=\frac{1}{x+1}+\frac{Mx+N}{x^2+x+1}\Rightarrow$$
$$\Rightarrow\ x^2+x+1+(x+1)(Mx+N)=x-1\Rightarrow$$
$$\Rightarrow\ (1+M)x^2+(1+M+N)x+(1+N)=x-1.$$

Equating the coefficients of corresponding powers of x, we get

$$\begin{array}{l|l} x^2 & 1+M=0 \\ x^0 & 1+N=1 \end{array}$$

Solving this system of equation with respect to M and N we receive

$$M=-1, N=0.$$

Hence,

$$\int \frac{x^4+x^3-2}{x^4+x^3-x-1}dx=\int\left(1+\frac{1}{x+1}+\frac{x}{x^2+x+1}\right)dx=\text{[see § 6]}=$$

$$= x + \ln|x+1| + \frac{1}{2}\int \frac{\left(x^2 + x + 1\right)'}{x^2 + x + 1}dx - \frac{1}{2}\int \frac{dx}{(x+0.5)^2 + \left(\sqrt{3}/2\right)^2} =$$

$$= x + \ln|x+1| + \frac{1}{2}\ln\left(x^2 + x + 1\right) - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{x+0.5}{\sqrt{3}/2} + C =$$

$$= x + \ln|x+1| + \ln\sqrt{x^2 + x + 1} - \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C.$$

§ 8. Integration of Function Rational with respect to Trigonometric Functions

There are two ways to evaluate an integral of a kind

$$\int R(\sin x, \cos x)dx \qquad (1.8.1)$$

where the integrand is a function rational with respect to sin x and cos x.

1. Transformation an integrand using trigonometric formulas.
2. Trying to use the substitution.

Example 1.8.1.

$$\int \sin^2 x \cdot \cos^2 x dx = \left[\begin{array}{l} \sin^2 x \cdot \cos^2 x = (\sin x \cos x)^2 = \left(\frac{\sin 2x}{2}\right)^2 = \frac{\sin^2 2x}{4} = \\ = \frac{1 - \cos 4x}{8} \end{array} \right] =$$

$$= \frac{1}{8}\int (1 - \cos 4x)dx = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C = \frac{x}{8} - \frac{\sin 4x}{32} + C.$$

$\int R(\sin x, \cos x)d$ onsider **some special trigonometric substitutions.**

$R(-\sin x, \cos x) =$

$= -R(\sin x, \cos x)$

$\left[\begin{array}{l} substitution: \\ \cos x = t \Rightarrow \\ \sin x dx = -dt, \\ \sin^2 x = 1 - t^2 \end{array} \right]$

$\sin x = t / \sqrt{1+t^2}$,

$\cos x = 1 / \sqrt{1+t^2}$

Example 1.8.2. ***Evaluate the integral***

$$\int \frac{\cos^3 x}{4 + \sin^2 x} dx$$

The integrand is the odd function with respect to cos *x*, so that we try the substitution sin *x* = *t*:

$$\int \frac{\cos^3 x}{4+\sin^2 x} = \left[\begin{array}{l} \sin x = t \Rightarrow \ \cos x dx = dt, \cos^2 x = 1 - t^2, \\ \dfrac{\cos^3 x dx}{4+\sin^2 x} = \dfrac{\cos^2 x(\cos x dx)}{4+\sin^2 t} = \dfrac{(1-t^2)dt}{4+t^2} \end{array} \right] =$$

$$= \int \frac{-t^2+1}{t^2+4} dt = -\int \frac{t^2-1}{t^2+4} dt = -\int \frac{(t^2+4)-5}{t^2+4} dt = -\int \left(1 - \frac{5}{t^2+2^2}\right) dt =$$

$$= -t + \frac{5}{2} \arctan \frac{t}{2} + C,$$

where *t* = sin *x*.

Example 1.8. 3. Evaluate the integral

$$\int \frac{dx}{4 + 3\cos^2 x + 5\sin^2 x}$$

The integrand is the even function with respect to sin *x* and cos *x*. So try the substitution tan *x* = *t*:

$$\int \frac{dx}{4+3\cos^2 x+5\sin^2 x} =$$

$$\left[\begin{array}{l} \tan x = t \Rightarrow \ dx = \dfrac{dt}{1+t^2}, \cos^2 x = \dfrac{1}{1+t^2}, \sin^2 x = \dfrac{t^2}{1+t^2} \Rightarrow \\ \Rightarrow \ 4+3\cos^2 x+5\sin^2 x = 4+\dfrac{3}{1+t^2}+\dfrac{5t^2}{1+t^2} = \dfrac{9t^2+7}{1+t^2} \Rightarrow \\ \Rightarrow \ \dfrac{dx}{4+3\cos^2 x+5\sin^2 x} = \dfrac{dt(1+t^2)}{(1+t^2)(9t^2+7)} = \dfrac{dt}{(3t)^2+\left(\sqrt{7}\right)^2} \end{array}\right]$$

$$= \int \frac{dt}{\left(\sqrt{7}\right)^2+(3t)^2} = \frac{1}{3\sqrt{7}}\arctan\frac{3t}{\sqrt{7}}+C = \frac{1}{3\sqrt{7}}\arctan\frac{3\tan x}{\sqrt{7}}+C.$$

It has been discovered that the substitution

$$t = \tan\frac{x}{2} \qquad (1.8.2)$$

enables us to reduce the problem of integrating any rational function of sin x and cos x to a problem involving a rational function of t. This in turn can be integrated by the method of partial fractions. In fact:

$$\int R(\sin x, \cos x)dx = \left[\begin{array}{l} t = \tan\dfrac{x}{2} \Rightarrow \ x = 2\arctan t \Rightarrow \ dx = \dfrac{2}{1+t^2}dt \\ \sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{2t}{1+t^2}, \cos x = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{1-t^2}{1+t^2} \end{array}\right] =$$

$$= \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2}{1+t^2}dt = \int R_1(t)dt.$$

Thus the substitution (1.8.2) is a very powerful tool. This method is cumbersome, however, and is used only when the simpler methods outlined previously have failed. The substitution (1.8.2) is called the *Euler's substitution* (*universal substitution*). This method is very convenient for computing integrals of the form

$$\int \frac{dx}{a\cos x+b\sin x+c}.$$

Example 1.8.4.

$$\int \frac{dx}{\cos x + \sin x - 1} =$$

$$\left[\begin{array}{l} x = \tan\frac{x}{2} \Rightarrow \; dx = \frac{2dt}{1+x^2}, \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2} \\ \frac{dx}{\cos x + \sin x - 1} = \frac{2dt}{\left(1+t^2\right)\left(\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} - 1\right)} = \frac{2dt}{1 - t^2 + 2t - 1 - t^2} = \\ = \frac{2}{2t - t^2}dt = \left(\frac{1}{t} + \frac{1}{1-t}\right)dt \end{array}\right]$$

$$= \int \left(\frac{1}{t} - \frac{1}{t-1}\right)dt = \ln|t| - \ln|t-1| + C = \ln\left|\frac{t}{t-1}\right| + C = \ln\left|\frac{\tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right| + C.$$

§ 9. Integration of Some Irrational functions

1. To evaluate integrals of the form

$$\int R\left(x.\sqrt[k_1]{ax+b}, \sqrt[k_2]{ax+b}, \ldots, \sqrt[k_n]{ax+b}\right)dx \qquad (1.9.1)$$

we can try the substitution

$$ax + b = t^n,$$

where n is the least common multiple of the indices of radicals.

Example 1.9.1.

$$\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}} = \left[\begin{array}{l} x = t^6 \Rightarrow \; dx = 6t^5 dt \\ \sqrt{x} = t^3, \sqrt[3]{x} = t^2, here \\ t = \sqrt[6]{x} \end{array}\right] = 6\int \frac{t^5}{t^3 + t^2}dt = 6\int \frac{t^3}{t+1}dt =$$

$$= 6\int \frac{\left(t^3+1\right) - 1}{t+1}dt = 6\int \frac{(t+1)\left(t^2 - t + 1\right) - 1}{t+1}dt = 6\int \left(t^2 - t + 1 - \frac{1}{t+1}\right)dt =$$

$$= 6\left(\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right) + C = 2t^3 - 3t^2 + 6t - 6\ln|t+1| + C =$$

$$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\left|\sqrt[6]{x} + 1\right| + C.$$

2. **Integrals involving** $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$ may be simplified by trigonometric substitutions:

a) $\sqrt{a^2 - x^2} = [x = a\sin t] = a\cos t$ (1.9.2)

b) $\sqrt{a^2 + x^2} = [x = a\tan t] = \dfrac{a}{\cos t}$ (1.9.3)

c) $\sqrt{x^2 - a^2} = \left[x = \dfrac{a}{\cos t}\right] = a\tan t$ (1.9.4)

Example 1.9. 2. Evaluate the integral

$$\int \frac{\sqrt{(x^2+9)^3}}{x^6}dx$$

Here we have the case (1.9.2) where *a* = *3*. Hence

$$\int \frac{\sqrt{(x^2+9)^3}}{x^6}dx = \begin{bmatrix} x = 3\tan t \Rightarrow \ dx = \dfrac{3dt}{\cos^2 t} \\ \sqrt{(x^2+9)^3} = \dfrac{3}{\cos t} \\ t = \arctan\dfrac{x}{3} \end{bmatrix} =$$

$$= \frac{1}{9}\int \frac{\cos t dt}{\sin^6 t} = \frac{1}{9}\left(-\frac{1}{5\sin^5 t}\right) + C = -\frac{1}{45\sin^5 t} + C,$$

where $t = \arctan\dfrac{x}{3}$.

Even when it is not clear at the start that a substitution will work, it is advisable to try one that seems reasonable. Sometimes a chain of substitutions will reduce the given integral to the table integrals.

Miscellaneous Problems

Evaluate the following integrals:

1. $\displaystyle\int \frac{dx}{(5x-7)^{10}}$

2. $\displaystyle\int \frac{\cos 7x}{(1-\sin 7x)^5}dx$

3. $\displaystyle\int \frac{\sqrt[5]{3\tan 2x - 5}}{\cos^2 2x}dx$

4. $\displaystyle\int \frac{dx}{21x - 0.5}$

5. $\int \frac{e^{2x}dx}{5-3e^{2x}}$

6. $\int \frac{dx}{\sqrt{\frac{1}{2}\ln x+1}\cdot x}$

7. $\int \frac{x^3 3^{-x^3}+3}{x^3}dx$

8. $\int \frac{2e^{3x}-7}{e^{2x}}dx$

9. $\int \frac{dx}{e^{3-4x}}$

10. $\int e^{2x}\cos\left(1+3e^{2x}\right)dx$

11. $\int \frac{dx}{\cot(5-4x)}$

12. $\int \cos^4 xdx$

13. $\int \frac{dx}{x^2+6x}$

14. $\int \frac{dx}{\sqrt{x}\sin^2\left(5\sqrt{x}-3\right)}$

15. $\int \frac{dx}{(1-3x)^2+3}$

16. $\int \frac{dx}{\sqrt{1-2x-x^2}}$

17. $\int \frac{x^4dx}{\sqrt{x^{10}-25}}$

18. $\int \frac{dx}{(5x+1)^2-4}$

19. $\int \frac{dx}{x\sqrt{\ln^2 5x-1}}$

20. $\int \frac{2^{5x}dx}{\sqrt{1-2^{10x}}}$

21. $\int \frac{\sin 2xdx}{5+\cos^2 2x}$

22. $\int \frac{(3x-1)dx}{\sqrt{3-2x-x^2}}$

23. $\int \frac{(5x-3)dx}{\sqrt{x^2+6x+20}}$

24. $\int \frac{(2x+1)dx}{x^2+2x+2}$

25. $\int \frac{(3x+4)dx}{x^2-2x-15}$

26. $\int \frac{(x+1)dx}{\cos^2 x}$

27. $\int \log_3 xdx$

28. $\int \left(3x^3-x^2\right)\sin 3xdx$

29. $\int (1-5x)2^x dx$

30. $\int \frac{14x^3+10x^2+4}{7x+5}dx$

31. $\int \frac{x+1}{x^3-4x^2+3x}dx$

32. $\int \frac{4x^2+4x+3}{x^3+x^2+2x+2}dx$

33. $\int \frac{3x^2-28x+85}{(x+5)\left(x^2-10x+25\right)}dx$

34. $\int \frac{dx}{\sin x+4\cos x+4}$

35. $\int \frac{dx}{1-\sin 2x+3\cos^2 x}$

36. $\int \sin^2 3x\cdot\sin^2 5xdx$

37. $\int \frac{dx}{(2+x)\sqrt{1+x}}$

38. $\int \frac{\sin^3 x}{\sqrt[3]{\cos x}}dx$

39. $\int \frac{\sqrt{x}}{\sqrt[4]{x}+1}dx$

40. $\int \tan^5 xdx$

41. $\int \sin^3 2xdx$

42. $\int \frac{\cos 6xdx}{\sin^2 3x\cos^2 3x}$

43. $\int \frac{dx}{\sqrt{\left(1+x^2\right)^7}}$

44. $\int \sqrt{25-x^2}dx$

45. $\int \frac{x^2dx}{\sqrt{(x^2-25)^5}}$

46. $\int \frac{xdx}{\sqrt{1-x^2}}$

47. $\int \sin\frac{x}{2}\cos 2xdx.$

PART 2

DEFINITE INTEGRAL

§ 1. Definite Integral. Existence Theorem

Let there be a bounded function *f(x)* on a closed interval [*a,b*]. We partition the interval into *n* subintervals by choosing $n-1$ points, $x_1, x_2, \ldots, x_{n-1}$, between *a* and *b* subject only to the condition that

$$a < x_1 < x_2 < \ldots < x_{n-1} < b.$$

To make the notation consistent, we denote *a* by x_0 and *b* by x_n.

The set P = $\left[x_0, x_1, \ldots, x_n\right]$ is called a partition of [*a, b*].

The typical subinterval $\left[x_{k-1}, x_k\right]$ is called the *k*th subinterval of P. Its length is $\Delta x_k = x_k - x_{k-1}$. Next we take in each subinterval an arbitrary point , denoting these points by $\xi_1, \xi_2, \ldots, \xi_n$. Now we form the sum

$$S_n = \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k \qquad (2.1.1)$$

This sum which depends on P and the choice of the numbers ξ_k is called an **integral sum**. We denote $\max_k \Delta x_k$ by λ and call it a diameter of partition.

Definition. The limit of the integral sums (2.1.1) as $\lambda \to 0$ is called the definite integral of the function *f(x)* with respect to x over the interval [*a,b*] and is denoted as

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} f(\xi_k)\,\Delta x_k = \int_a^b f(x)dx \qquad (2.1.2)$$

This is read as the integral of *f(x)dx* from *a* to *b*.
f(x) is called an **integrand** and *f(x)dx* is called an **element of integration**,

a is the **lower limit**, *b* is the **upper limit**.

The symbol $\int$ is an integral sign. Leibniz chose it because it resembled S in the German word for summation.

It is possible to prove

Theorem (the existence of definite integral).

If a function *f(x)* is continuous on an interval [*a,b*], then its definite integral over [*a,b*] exists.

§ 2. The Newton – Leibniz theorem

Let *f(x)* be a continuous function in the closed interval [*a,b*] and *F*(*x*) is an antiderivative of *f(x)* that is $F'(x) = f(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a) \qquad (2.2.1)$$

Obviously

$$F(b) - F(a) = \sum_{k=1}^{n} \left(F(x_k) - F(x_{k-1})\right) = \sum_{k=1}^{n} F'(\xi_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} F'(\xi_k)\Delta x_k =$$

$$= \sum_{k=1}^{n} f(\xi_k)\Delta x_k .$$

If $\lambda \to 0$ we have

$$\lim_{\lambda \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta x_k = \lim_{\lambda \to 0}(F(b) - F(a)) \Rightarrow \int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$$

The symbol $\Big|_a^b$ with the two indices *a* and *b* is the so-called sign of double substitution. It indicates that the value of the function corresponding to the lower index must be subtracted from the one corresponding to the upper index.

Example 2.2.1. A leaky 5-lb. bucket is lifted from the ground into the air by pulling in 20 ft at a constant speed. The rope weighs 0.08lb./ft. The bucket starts with 2 gal of water (16 lb.) and leaks at a constant rate. It finishes draining just as reaches the top. How much work was spent

a) lifting the water alone;
b) lifting the water and bucket together;
c) lifting the water, bucket, and rope?

When a body moves a distance *d* along a straight line as the result of being acted on by a force that has a constant magnitude *F* in the direction of the motion, the work *W* done by the force in moving the body is *F* by *d*

$$W = Fd \qquad (2.2.2)$$

The work done by a continuous force $F(x)$ directed along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)dx \qquad (2.2.3)$$

a) <u>The water alone</u>. The force required to lift the water's weight, which varies steadily from 16 to 0 lb over the 20-ft lift. When the bucket is x ft off the ground, the water weighs $F(x) = 16\left(\frac{20 - x}{20}\right) = 16 - \frac{4x}{5} lb.$ The work done is

$$W = \int_a^b F(x)dx = \int_o^{20}\left(16 - \frac{4x}{5}\right)dx = 320 - 160 = 160 ft \cdot lb.$$

b) <u>The water and bucket together.</u> According to Eq.(2.2.2), it takes 5×20 = 100 ft.lb to lift a 5-lb weight 20 ft. Therefore 160 + 100 = 260 ft.lb of work were spent lifting the water and bucket together.

c) <u>The water, bucket, and rope.</u> Now the total weight at level x is

$$F(x) = \left(16 - \frac{4x}{5}\right) + 5 + (0.08)(20 - x), \text{ where}$$

$16 - \frac{4x}{5}$ - is variable weight of water,

5 - is the constant weight of bucket,

$0.08(20 - x)$ - is a weight of rope paid out at elevation x.

The work lifting the rope is

$$\text{Work on rope} = \int_0^{20} 0.08(20 - x)dx = 32 - 16 = 16 ft \cdot lb.$$

The total work for the water, bucket, and rope combined is 160+100+16=276 $ft \cdot lb.$

Example 2.2.2.. A spring has a natural length of 1m. A force of 24N stretches the spring to a length of 1.8 m.

a) Find the spring constant *k;*

b) How much work does it take to stretch the spring 2m beyond its natural length?

c) How far will a 45N force stretch the spring?

Hook's law says that the amount of force F it takes to stretch or compress a spring x length units from its natural (unstressed) length is proportional to x. In symbols, $F = kx$.

The number k, measured in force units per unit length, is a constant characteristic of the spring, called the **spring constant**. Hook's law gives good results as long as the force doesn't distort the metal in the spring. We shall assume that the forces in this section are too small to do that.

a) **The spring constant.** We find the spring constant from the equation $F = kx$. A force of $24N$ stretches the spring $0.8m$, so

$$24 = k \cdot 0.8 \Rightarrow k = \frac{24}{0.8} = 30N/m.$$

b) **The work to stretch the spring 2*m***. We imagine the unstressed spring hanging along the x – axis with its free end at $x = 0$. Then the force required to stretch the spring $x\ m$ beyond its natural length is the force required to pull the free end of the spring x units from the origin. Hook's law with $k = 30$ tells us that this force is

$$F(x) = 30x.$$

The work required to apply this force from $x = 0m$ to $x = 2m$ is

$$W = \int_0^2 30x\,dx = 15x^2 \Big|_0^2 = 60N \cdot m.$$

a) **How far will a 45-N force stretch the spring?** We substitute $F = 45$ in the equation $F = 30x$ to find

$$45 = 30x, \text{ or } x = 1.5m.$$

A 45- N force will stretch the spring $1.5m$.

§ 3. Rules of Algebra for Definite Integrals

1. $\int_a^a f(x)dx = 0$ (a definition)

2. Order of integration: $\int_b^a f(x)dx = -\int_a^b f(x)dx$ (also a definition)

3. Constant multiples: $\int_a^b kf(x)dx = k\int_a^b f(x)dx$ (for any number k)

4. Sums and difference: $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5. Domination: $f(x) \geq g(x)$ on $[a,b] \Rightarrow \int_a^b f(x)dx \geq \int_a^b g(x)dx$

6. Additivity: If $f(x)$ is integrable on the intervals joining a, b, and c, then

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$$

7. **The mean value theorem for definite integrals.** If $f(x)$ is continuous on the closed interval $[a, b]$, then at some point ξ in the interval $[a, b]$

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)dx \qquad (2.3.1)$$

The number on the right-hand side of Eq.(2.3.1) is called the average value or mean value of $f(x)$ on $[a, b]$. Notice that the average value of $f(x)$ on $[a, b]$ is the integral of f divided by the length of the interval.

8. **The integral with variable upper limit:**

If *f(t)* is an integrable function, its integral from any fixed number *a* to another number *x* defines a function *Φ* whose value at *x* is

$$\Phi(x) = \int_a^x f(t)dt.$$

The derivative of the integral with respect to its upper limit is equal to the integrand, that is $\Phi'(x) = f(x)$.

9. $\int_{-a}^{a} f(x)dx = 0$ if $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(-x) = f(x)$$

All these properties one can easily prove with the aid of the formula Newton – Leibniz

$$\int_a^b f(x)dx = F(x)\Big|_a^b \text{ , where } F'(x) = f(x).$$

For an example let us prove the property 3:

$$\int_a^b kf(x)dx = kF(b) - kF(a) = k(F(b) - F(a)),$$

that is

$$k\int_a^b f(x)dx = k(F(b) - F(a)),$$

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx.$$

Thus this property is proved.

§ 4 Methods of Evaluating Definite Integrals

1. Integration by parts.

We have $$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu \qquad (2.4.1)$$

Proof: The relation

$$\int_a^b udv = \left(\int udv\right)\Big|_a^b = \left(uv\Big|_a^b - \int_a^b vdu\right)$$

directly implies the formula we set out to prove.

2. Change of Variable in the Definite Integral (integration by substitution)

Theorem. Let *f(x)* be continuous function on a closed interval [*a, b*]. Assume that $x = \varphi(t)$ satisfies the conditions

1) $\varphi(t)$ and $\varphi'(t)$ are continuous on a closed interval $[\alpha, \beta]$;
2) $a \le \varphi(t) \le b$ when $\alpha \le t \le \beta$;
3) $\varphi(\alpha) = a$, $\varphi(\beta) = b$.

Then we have

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt \qquad (2.4.2)$$

Proof. Let *F(x)* be an antiderivative of a function *f(x)*, that is $F'(x) = f(x)$. Then using the Newton – Leibniz formula we get

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a),$$

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = \int_a^b f(\varphi(t))d(\varphi(t)) = F(\varphi(t))\Big|_\alpha^\beta = F(\varphi(\beta)) - F(\varphi(\alpha)) =$$
$$= F(b) - F(a).$$

Comparing the equalities, we arrive at the formula (2.4.2).

Examplt 2.4.1. Consider the integral $\int_0^{1/2} x^2\sqrt{1-x^2}\,dx.$

We put $x = \sin t$ and find the new limits of integration t_1 and t_2 from the equations $0 = \sin t$ and $\frac{1}{2} = \sin t$. So t_1 can be taken equal to *0*, and t_2 equal to $\frac{\pi}{6}$. As *t* varies from 0 to $\frac{\pi}{6}$ the variable $x = \sin t$ runs throughout the given interval of integration $\left[0, \frac{1}{2}\right]$. Thus

$$\int_0^{\frac{1}{2}} x^2\sqrt{1-x^2}\,dx = \left[\begin{array}{l} x = \sin t \Rightarrow \ dx = \cos t dt \\ t_1 = \arcsin 0 = 0; t_2 = \arcsin\frac{1}{2} = \frac{\pi}{6} \\ \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \cos t \end{array}\right] = \int_0^{\pi/6} \sin^2 t\cos^2 t dt =$$

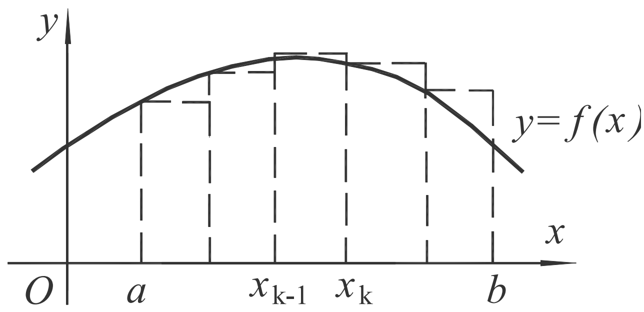
$$= \left[\begin{array}{l} \sin^2 t \cos^2 t = \frac{1}{4}\sin^2 2t = \\ = \frac{1}{8}(1 - \cos 4t) \end{array} \right] = \frac{1}{8}\int_0^{\pi/6}(1-\cos 4t)dt = \frac{1}{8}\left(t - \frac{\sin 4t}{4}\right)\Bigg|_0^{\pi/6} =$$

$$= \frac{1}{8}\left(\frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3}\right) = \frac{1}{16}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}.\right)$$

§ 5. Geometrical Meaning of Definite Integral

In geometry we learned how to find areas of certain polygons: rectangles, triangles, parallelograms, trapezoids. Indeed, the area of any polygon can be found by cutting it into triangles.

The area of a circle is easily computed from the formula $S = \pi R^2$. But the idea behind this simple formula isn't so simple. In fact, it is the subtle concept of a *limit,* the area of the circle being defined as the limit of areas of inscribed (or circumscribed) regular polygons as the number of sides increases without bound. A similar idea is involved in the definition we now introduce for other plane areas.



Let $y = f(x)$ define a continuous function of x on the closed interval [*a, b*]

For simplicity, we shall also suppose that *f(x)* is positive for any $x \in [a,b]$.

We consider the problem of calculating the area bounded above by the graph of the function *y = f(x)*, on the sides by vertical lines through $x = a$ and $x = b$, and below by the *x*-axis.

This area we'll call **area under a curve** and denote it by S. To find it we partition the interval [*a,b*] into *n* subintervals by choosing $n+1$ points, $x_0, x_1, x_2, \ldots, x_{n-1}, x_n$, such that

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

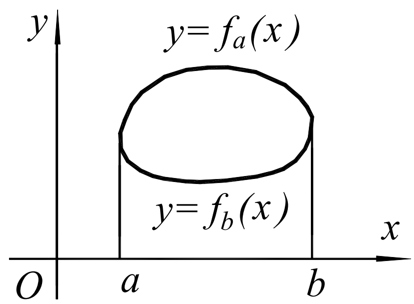
Then we divide the area into *n* thin strips by lines perpendicular to the *x*-axis through these points. Each strip is approximated by a rectangle. Let $S_1, S_2, \ldots, S_n$ be the areas of these rectangles. Thus we have $S_1 = f(x_0)\Delta x_1$, $S_2 = f(x_1)\Delta x_2, \ldots, S_n = f(x_{n-1})\Delta x_n$.

$$S \approx \sum_{k=1}^{n} f(x_{k-1})\,\Delta x_k \qquad (2.5.1)$$

Comparing (2.5.1) with (2.1.1) and using the formula (2.1.2) we have

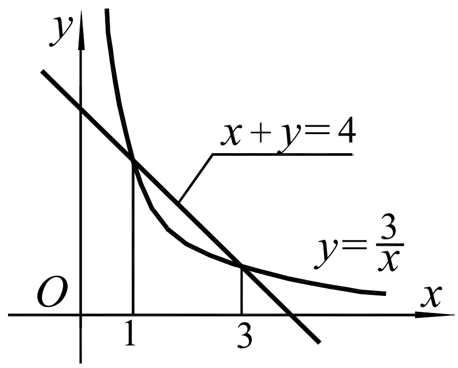
$$S = \int_a^b f(x)dx \qquad (2.5.2)$$

Using the properties of a definite integral and the formula (2.5.2) we are able to prove that the area of a region bounded above by the curve $y = f_{above}(x)$ or $y = f_a(x)$, and below by a curve $y = f_{below}(x)$ or $y = f_b(x)$ is calculated by the formula



$$S = \int_a^b \left(f_a(x) - f_b(x)\right)dx \qquad (2.5.3)$$

Example 2.5.1. Find the area bounded by the curve $y = \frac{3}{x}$ and the line $x + y - 4 = 0$.



$$S = \int_a^b \left(f_a(x) - f_b(x)\right)dx$$

where $f_a(x) = 4 - x, f_b(x)\frac{3}{x}$, Let us find the limits of integration. To do it solve the equation

$$f_a(x) = f_b(x) \Rightarrow \frac{3}{x} = 4 - x \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow x_1 = 1, x_2 = 3.$$

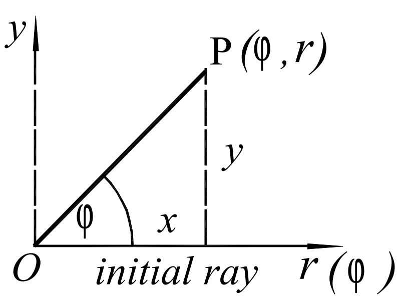
Hence

$$S = \int_1^3 \left(4 - x - \frac{3}{x}\right)dx = \left(4x - \frac{x^2}{2} - 3\ln x\right)\Bigg|_1^3 =$$

$$= 12 - \frac{9}{2} - 3\ln 3 - 4 + \frac{1}{2} - 3\ln 1 = 4 - 3\ln 3.$$

§ 6. Plane Areas in Polar Coordinates

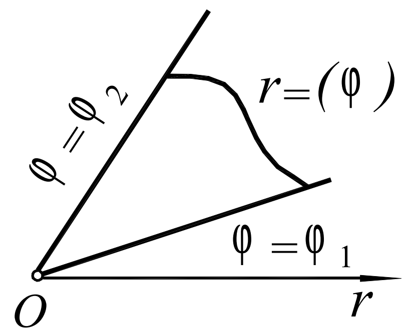
We know that a point can be located in a plane by giving its abscissa and ordinate relative to a given coordinate system $P(x.y)$.



Another useful way to locate a point in a plane is by *polar coordinates.* First, we fix an *origin O* and an *initial ray* **r** from *O*. The point *P* has polar coordinates $r \geq o, \varphi$ where *r* is equal to distance from *O* to *P*, and φ is directed angle from initial ray to *OP*.

It is easily to prove, that

$$\left.\begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}\right\}. \qquad (2.6.1)$$



where (x, y) are Cartesian coordinates and (r, φ) are polar coordinates of one and the same point. It is possible to prove that the area of a plane region *S* bounded by the rays $\varphi = \varphi_1$, $\varphi = \varphi_2$, and the curve $r = r(\varphi)$ can be founded by the formula

$$S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)\,d\varphi \qquad (2.6.2)$$

Example 2.6.1. Find the area that is inside the circle *r* = *a* and outside the cardioid $r = a(1 + \cos\varphi)$.

Solution. Using the formula (2.6.1) and the fact that the area is outside the cardioid and inside the circle we have

$$S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2}\left(r_2^2(\varphi) - r_1^2(\varphi)\right)d\varphi \text{ , where } r_1(\varphi) = a(1 - \cos\varphi),\ r_2(\varphi) = a.$$

We can find φ_1 and φ_2 from the condition $r_1(\varphi) = r_2(\varphi)$:

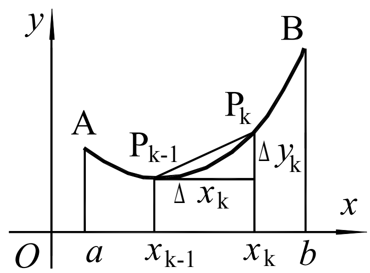
$$a = a(1 - \cos\varphi) \Rightarrow 1 - \cos\varphi = 1 \Rightarrow \cos\varphi = 0 \Rightarrow \varphi_1 = -\frac{\pi}{2}, \varphi_2 = \frac{\pi}{2}$$

where the curves intersect. Hence

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left(a^2 - a^2(1 - \cos\varphi)^2\right)d\varphi = a^2\int_0^{\frac{\pi}{2}}\left(1 - (1 - \cos\varphi)^2\right)d\varphi =$$

$$= a^2\int_0^{\frac{\pi}{2}}\left(2\cos\varphi - \frac{1}{2} + \frac{\cos 2\varphi}{2}\right)d\varphi = a^2\left(2\sin\varphi - \frac{1}{2}\varphi + \frac{\sin 2\varphi}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = a^2\left(2 - \frac{\pi}{4}\right).$$

§ 7. Length of a Plane curve



Divide the arc *AB* into *n* pieces and join the successive points of division by straight lines. A representative line, such as $P_{k-1}P_k$, will have length

$P_{k-1}P_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

The length of the curve $AB = L$ is approximately

$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

When the number of division points is increased indefinitely while the lengths of the individual segments tend to zero, we obtain

$$L = \lim_{\lambda \to 0} \sum_{k=1}^{n} \sqrt{(\Delta x_k{}^2) + (\Delta y_k)^2}\ , \qquad (2.7.1)$$

where $\lambda = \max(\Delta x_1, \Delta x_2, \ldots, \Delta x_n)$, if this limit exists. Suppose that the function $y = f(x)$ is continuous and possesses a continuous derivative at each point of the curve from $A(a, f(a))$ to $B(b, f(b))$. Then there is some point c_k between P_{k-1} and P_k on the curve where the tangent to the curve is parallel to the chord $P_{k-1}P_k$. That is, $f'(x_k) = \dfrac{\Delta y_k}{\Delta x_k}$ or $\Delta y_k = f'(x_k)\Delta x_k$. Hence (2.7.1) may also be written in the form

$$L = \lim_{\lambda \to 0} \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (f'(x_k)\Delta x_k)^2} = \lim_{\lambda \to 0} \sum_{k=1}^{n} \sqrt{1 + (f'(x_k))^2}\,\Delta x_k$$

or

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx \qquad (2.7.2)$$

Example 2.7. 1. Find the length of the curve $y = x^{3/2}$ from $(0,0)$ to (4,8).

Solution.

$$L = \int_a^b \sqrt{1 + (y')^2}\,dx = \left[a = 0, b = 4, y' = \frac{3}{2}x^{1/2} \Rightarrow (y')^2 = \frac{9}{4}x\right] =$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4}x}\,dx = \frac{4}{9}\left(\left(1 + \frac{9}{4}x\right)^{3/2} \cdot \frac{2}{3}\right)\Bigg|_0^4 = \frac{8}{27}\left(10\sqrt{10} - 1\right).$$

There is a particularly useful formula for calculating the length of a curve that is given parametrically:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad \alpha \le t \le \beta$$

Let $x'(t)$ and $y'(t)$ be continuous functions on the closed interval $[\alpha, \beta]$ and $x'(t) > 0$, $x(\alpha) = a, x(\beta) = b$. As we know

$$f'(x) = \frac{dy}{dx} = \frac{y'(t)dt}{x'(t)dt} = \frac{y'(t)}{x'(t)}$$

Such that using the formula (2.7.2) we get

$$L = \int_{\alpha}^{\beta} \sqrt{1 + \left(\frac{y'(t)}{x'(t)}\right)^2}\, x'(t)dt = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt .$$

$$L = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt . \qquad (2.7.3)$$

If a curve

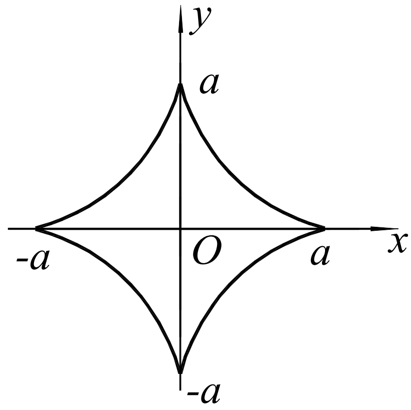
$$\begin{cases} x = x(t) \\ y = y(t) \qquad \alpha \le t \le \beta \\ z = z(t) \end{cases}$$

is in the space its length we can find by the formula

$$L = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt . \qquad (2.7.4)$$

Example 2.7. 2. Calculate the total length of the astroid (hypocycloid)

$$\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases} \qquad 0 \le t \le 2\pi \; .$$



Solution.

$$L = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt =$$

$$= 4\int_{0}^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt =$$

$$= \begin{bmatrix} x = a\cos^3 t \Rightarrow \; x' = -3a\cos^2 t \sin t \Rightarrow \; (x')^2 = 9a^2\cos^4 t \sin^2 t \\ y = a\sin^3 t \Rightarrow \; y' = 3a\sin^2 t\cos t \Rightarrow \; (y')^2 = 9a^2 \sin^4 \cos^2 t \\ (x')^2 + (y')^2 = 9a^2\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t) = (3a\sin t\cos t)^2 = \\ = \left(\frac{3}{2}a\sin 2t\right)^2 \end{bmatrix} =$$

$$= \frac{4\cdot 3}{2}a\int_{0}^{\pi/2}\sin 2t dt = -\frac{6}{2}a(\cos 2t)\Big|_{o}^{\pi/2} = -3a(\cos\pi - \cos 0) = 6a.$$

Example 2.7.3. Let a curve is given in the Polar coordinates:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \alpha \le \varphi \le \beta \ .$$

Prove that

$$L = \int_{\alpha}^{\beta} \sqrt{\left(r'(\varphi)\right)^2 + r^2(\varphi)}\,d\varphi \qquad (2.7.5)$$

Solution. As we know

$$L = \int_{\alpha}^{\beta} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt =$$

$$\left[\begin{array}{l} t = \varphi \Rightarrow dt = d\varphi \\ x = r\cos\varphi \Rightarrow x' = r'\cos\varphi - r\sin\varphi \Rightarrow \left(x'\right)^2 = \\ = (r')^2\cos^2\varphi - 2rr'\sin\varphi\cos\varphi + r^2\sin^2\varphi \\ y = r\sin\varphi \Rightarrow y' = r'\sin\varphi + r\cos\varphi \Rightarrow \left(y'\right)^2 = \\ = (r')^2\sin^2\varphi + 2rr'\sin\varphi\cos\varphi + r^2\cos^2\varphi \\ (x')^2 + (y')^2 = (r')^2\left(\cos^2\varphi + \sin^2\varphi\right) + r^2\left(\sin^2\varphi + \cos^2\varphi\right) = (r')^2 + r^2 \end{array}\right]$$

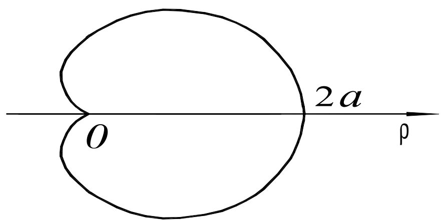
$$= \int_{\alpha}^{\beta} \sqrt{\left(r'(\varphi)\right) + r^2(\varphi)}\,d\varphi \ .$$

The formula (2.7.5) is proved.

Example 2.7.4.

Find the total length of a cardioid:

$r = a\left(1 + \cos\varphi\right), \quad 0 \le \varphi \le 2\pi$.



Solution.

$$L=\int_{\alpha}^{\beta}\sqrt{(r'(\varphi))^2+r^2(\varphi)}\,d\varphi=$$

$$\left[\begin{aligned}&r'=-a\sin\varphi\ \Rightarrow\ (r')^2=a^2\sin^2\varphi\\&r^2=a^2(1+2\cos\varphi+\cos^2\varphi)\\&(r')^2+r^2=a^2(\sin^2\varphi+\cos^2\varphi+1+2\cos\varphi)=\\&=2a^2(1+\cos^2\varphi)=\left(2a\cos\frac{\varphi}{2}\right)^2\end{aligned}\right]$$

$$=2a\int_0^{2\pi}\cos\frac{\varphi}{2}\,d\varphi=4a\int_0^{\pi}\cos\frac{\varphi}{2}\,d\varphi=8a\sin\frac{\varphi}{2}\Big|_0^{\pi}=8a.$$

The answer: $L=8a$.

§ 8. Miscellaneous Problems

I. In problems 1 to 18 compute the integrals.

1. $\int_0^1\sqrt{1+x}\,dx$
2. $\int_{-2}^{-1}\frac{dx}{(11+5x)^3}$
3. $\int_1^e\frac{1+\lg x}{x}dx$
4. $\int_1^2\frac{e^{1/x}}{x^2}dx$
5. $\int_2^3\frac{dx}{2x^2+3x-2}$
6. $\int_{-\pi/2}^{\pi/2}\frac{dx}{1+\cos x}$
7. $\int_0^1 xe^{-x}dx$
8. $\int_0^{\pi}x^3\sin x\,dx$
9. $\int_0^{e-1}\ln(x+1)dx$
10. $\int_0^{\pi/4}\cos^4\varphi\,d\varphi$
11. $\int_4^9\frac{\sqrt{x}}{\sqrt{x}-1}dx$
12. $\int_{\sqrt{2}/2}^{1}\frac{\sqrt{1-x^2}}{x^6}dx$
13. $\int_1^{\sqrt{3}}\frac{\sqrt{1+x^2}}{x^2}dx$
14. $\int_1^2\frac{\sqrt{x^2-1}}{x}dx$
15. $\int_0^{-\ln 2}\sqrt{1-e^{2x}}\,dx$
16. $\int_1^2\frac{dx}{x+x^3}$
17. $\int_0^{1/2}\frac{x^3dx}{x^2-3x+2}$
18. $\int_0^{\pi/2}\frac{dx}{2\cos x+3}$

19. Compute the area of the figure bounded by the curves

a) $y = \frac{x^2}{2}$ and $y = \frac{1}{1+x^2}$; b) $y = 2 - x^2$ and $y^3 = x^2$.

20. Find the area of the figure enclosed by the astroid
$x = 2\cos^3 t,\ \ y = 2\sin^3 t.$

21. Compute the area of the figure bounded by the first and the second turns of the spiral of Archimedes $r = a\varphi$ and the segment of the polar axis.

22. Find the length of the curve $y = \ln\sin x$ from $x = \frac{\pi}{3}$ to $x = \frac{2\pi}{3}$.

23. Find the arc length of the evolvent of the circle
$x = 2(\cos t + t\sin t), y = 2(\sin t - t\cos t)$, from $t_1 = 0$ to $t_2 = \pi$.

24. Find the length of the cardioid $r = 4(1 - \sin t)$.

PART 3
DIFFERANTIAL EQUATIONS

§ 1. Definitions

A ***differential equation*** is an equation that contains one or more derivatives of a differentiable function, that is

$$F\left(x.y, y', y'', \dots, y^{(n)}\right) = 0 \qquad (3.1.1)$$

The ***order*** of a differential equation is the order of the equation's highest order derivative.

We call a function $y = \varphi(x)$ a ***solution*** of a differential equation if y and its derivatives satisfy the equation.

Example. Show that for any values of the arbitrary constants C_1 and C_2 the function $y = C_1 \cos x + C_2 \sin x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} + y = 0.$$

Solution.

We differentiate the function twice to find $\frac{d^2y}{dx^2}$:

$$\frac{dy}{dx} = -C_1 \sin x + C_2 \cos x \Rightarrow \frac{d^2y}{dx^2} = -C_1 \cos x - C_2 \sin x.$$

Then we substitute the expression for y and $\frac{d^2y}{dx^2}$ into the differential equation to see whether the left-hand side reduces to zero. It does because

$$\frac{d^2y}{dx^2} + y = \left(-C_1 \cos x - C_2 \sin x\right) + \left(C_1 \cos x + C_2 \sin x\right) = 0.$$

So this function is a solution of the differential equation.

It can be shown that the formula

$$y = C_1 \cos x + C_2 \sin x$$

gives all possible solutions of the equation

$$\frac{d^2y}{dx^2} + y = 0.$$

A formula that gives all the solutions of a differential equation is called the **general solution** of the given equation.

To **solve** a differential equation means to find its general solution.

Notice that the considered equation has order two and its general solution has two arbitrary constants. The general solution of the *n*th order differential equation can be expected to contain *n* arbitrary constants.

§2. FIRST ORDER DIFFERANTIAL EQUATIONS

I. Separable Equations

We shall say that a differential equation of the first order

$$\frac{dy}{dx} = f(x, y)$$

has **variable separable** if the function *f(x, y)* can be written in the form

$$f(x, y) = f_1(x) \cdot f_2(y) \qquad (3.2.1)$$

On multiplying both parts of the equation (3.2.1) by $\frac{dx}{f_2(y)}$ we get

$$\frac{dy}{f_2(y)} = f_1(x)dx. \qquad (3.2.2)$$

Integrating both parts of the equation (3.2.2) we obtain

$$\int \frac{dy}{f_2(y)} = \int f_1(x)dx + C, \qquad (3.2.3)$$

where *C* is an arbitrary constant.

The expression (3.2.3) represents the **general integral** of the equation (3.2.1).

Example 3.2.1. Radioactive substances are those elements that naturally break down into other elements, releasing energy as they do. The rate at which such a substance decays is proportional to the mass of the material present. Let *m* be the amount present and the initial mass of the radioactive substance be m_0. We shall determine the relationship between the amount *m* of the remaining substance and time *t*.

Solution.

According to the above law, we can write the relation

$$\frac{dm}{dt} = -km, \qquad (3.2.4)$$

where $k > 0$ is a proportionality coefficient. It is taken with the minus sign since the amount of the substance *m* decreases as *t* grows, which indicates that the derivative is nonpositive. Separating the variables in the equation thus obtained we write

$$\frac{dm}{m} = -kdt.$$

On integrating we obtain

$$\ln m = -kt + \ln C$$

whence

$$m = Ce^{-kt}$$

The quantity m_0 does not enter into the differential equation; it only appears in the initial condition which has the form

$$m\Big|_{t=t_0} = m_0 .$$

This condition implies that $C = m_0$. Consequently, the particular solution satisfying the condition of the problem is

$$m = m_0 e^{-kt} .$$

The value of the constant *k* can be determined experimentally by measuring the amount of the remaining substance at a time moment *t.*

II. Homogeneous First Order Equations

A first order differential equation is **homogeneous** if it can be put into the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right). \qquad (3.2.5)$$

We can change this equation into a separable equation with the substitutions

$$y = ux \Rightarrow \frac{dy}{dx} = u + x\frac{du}{dx}.$$

Then we have

$$u + x\frac{du}{dx} = f(u),$$

that is

$$x\frac{du}{dx} = f(u) - u.$$

It follows that

$$\frac{du}{f(u) - u} = \frac{dx}{x}$$

and after the integration we get

$$\int \frac{du}{f(u) - u} = \ln|x| + \ln C$$

or

$$\int \frac{du}{f(u) - u} = \ln|Cx|.$$

Example 3.2. 2. Let us solve the homogeneous equation

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2 - 2xy}.$$

Solution. The substitution $y = ux$ leads to the equation

$$u + x\frac{du}{dx} = \frac{u - u^2}{1 - 2u}$$

or, equivalently,

$$\frac{du}{dx} = \frac{1}{x} \cdot \frac{u^2}{1 - 2u}.$$

Separating the variables we receive

$$\frac{1 - 2u}{u^2} du = \frac{dx}{x} \Rightarrow \left(\frac{1}{u^2} - \frac{2}{u}\right) du = \frac{dx}{x}.$$

On integrating we have

$$-\frac{1}{u} - 2\ln|u| = \ln|x| - \ln|C| \Rightarrow \ln\left(e^{1/u}u^2\right) = \ln\left|\frac{C}{x}\right|,$$

and consequently,

$$u^2 e^{1/u} = \frac{C}{x}.$$

On returning to the variable y we arrive at the general integral of the given differential equation:

$$\frac{y^2}{x} e^{x/y} = C.$$

III. Linear First Order Equations

A differential equation that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (3.2.6)$$

is called a **linear first order equation**.

Let us represent the unknown function in the form

$$y = u(x) \cdot v(x), \qquad (3.2.7)$$

and find its derivative $\frac{dy}{dx} = y'$:

$$y' = u'v + uv'. \qquad (3.2.8)$$

On substituting (3.2.7) and (3.2.8) into the equation (3.2.6) we get

$$u'v + u\left(v' + P(x)v\right) = Q(x).$$

Now we take v as a particular solution of the equation

$$v' + P(x)v = 0.$$

On separating variables we obtain

$$\frac{dv}{v} = -P(x)dx$$

whence

$$\ln|v| = -\int P(x)dx,$$

that is

$$v = e^{-\int P(x)dx}. \qquad (3.2.9)$$

To determine the function *u(x)* we have the equation

$$v u' = Q(x).$$

On solving this equation we arrive at the following formula

$$u = \int Q(x)\, e^{\int P(x)dx} dx + C. \qquad (3.2.10)$$

We summarize the results (3.2.9) and (3.2.10) and get

$$y = uv \Rightarrow \; y = e^{-\int P(x)dx}\left(\int Q(x)\, e^{\int P(x)dx} dx + C\right). \qquad (3.2.11)$$

This formula expresses the general solution of the linear equation (3.2.6).

Example 3.2.3. Consider an electric circuit containing a resistance *R*, an inductance *L* and an *electric* – current source with electromotive force *E.*

Solution. As is known from physics, if *I* is the electric current flow then

$$E = RI + L\frac{dI}{dt}.$$

This is a linear differential equation with respect to the unknown function $I = I(t)$ which can be written in the form

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L}.$$

We shall solve this equation with the initial condition

$$I(t)\,|_{t=0} = 0.$$

Thus, we are concerned with the problem on switching on an electric current source. Making use of the general formula (3.2.11) we have

$$I(t) = e^{-\int \frac{R}{L}dt}\left(\int \frac{E}{L} e^{\int \frac{R}{L}dt} dt + C\right).$$

On integrating we obtain

$$I(t) = \frac{E}{R}\left(1 + Ce^{-\frac{R}{L}t}\right).$$

Imposing the initial condition that $I(0) = 0$ determines the value of *C* to be -1 so

$$I(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

We see from this that the current $I(t)$ is always less than $\frac{E}{R}$ but that it approaches $\frac{E}{R}$ as a **steady – state** value:

$$\lim_{t\to\infty}\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)=\frac{E}{R}(1-0)=\frac{E}{R}.$$

The current $I=\frac{E}{R}$ is the current that will flow in the circuit if either $L=0$ (no inductance) or $\frac{dI}{dt}=0$ (steady current, *I* is constant).

§3. DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

Some Particular Types of Equations of the Second Order (Reduction of Order)

I. The right – hand side of the equation does not contain *y* and y'

$$y''=f(x) \qquad (3.3.1)$$

Since $y''=(y')'$ we have

$$y'=\int f(x)dx+C_1 .$$

On integrating once again we obtain

$$y=\int\left(\int f(x)dx\right)dx+C_1x+C_2$$

Example 3.3.1. Solve the differential equation $y''=\sin 2x-e^{x/5}$.

Solution. Here we have $f(x)=\sin 2x-e^{x/5}$, hence

$$y'=\int\left(\sin 2x-e^{x/5}\right)dx+C_1 \Rightarrow y'=-\frac{\cos 2x}{2}-5e^{x/5}+C_1 .$$

On integrating once again we obtain the general solution of the given equation:

$$y=-\frac{\sin 2x}{4}-25e^{x/5}+C_1x+C_2 .$$

П. The right – hand side of the equation does not contain *y*

$$y''=f(x.y') \qquad (3.3.2)$$

Let us put $z=y'$, then $y''=z'$, and the equation (3.3.2) becomes a first – order equation with respect to *z*

$$z'=f(x,z).$$

If the solution $\varphi(x,C_1)$ of this equation is found the sought – for solution of the original equation is obtained by the integration of the equality $y'=z$, that is

$$y=\int\varphi(x,C_1)dx+C_2 .$$

Example 3.3.2. Find the general solution of the differential equation

$$xy'' - y' = x^2 e^x.$$

Solution. Reduce this equation to the form (3.3.2):

$$y'' = \frac{y'}{x} + xe^x.$$

Using the substitution

$$y' = z \Rightarrow y'' = z'$$

we get the linear equation (see (3.2.6)):

$$z' = \frac{z}{x} + xe^x \Rightarrow \begin{bmatrix} z = uv \Rightarrow \\ z' = u'v + uv' \end{bmatrix} \Rightarrow$$

$$u'v + uv' = \frac{uv}{x} + e^x \qquad (*)$$

1). Let $uv' = \frac{uv}{x}$ then $\frac{dv}{dx} = \frac{v}{x} \Rightarrow \frac{dv}{v} = \frac{dx}{x} \Rightarrow v = x.$

2). Substitute $v = x$ into the equation (*):

$$u'x = xe^x \Rightarrow du = e^x dx \Rightarrow u = e^x + C_1.$$

As $z = uv$ we get $z = xe^x + C_x x.$ But $z = y'$ thus we have

$$y' = xe^x + C_1 x.$$

Integrating this equation we get the general solution of the given differential equation:

$$y = xe^x - e^x + \frac{C_1}{2}x^2 + C_2.$$

Ш. The right – hand side of the equation does not contain *x*

$$y'' = f(y, y'). \qquad (3.3.3)$$

The substitutions to use are

$$p = \frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\cdot\frac{dy}{dx} = \frac{dp}{du}\cdot p.$$

Then the equation (3.3.3) takes the form

$$p\frac{dp}{dy} = f(y, p).$$

If the solution $p = \varphi(y, C_1)$ of this equation is determined, the solution of the equation (3.3.3) is found from an equation with variable separable

$$p = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \varphi(y, C_1), \text{ that is } \frac{dy}{\varphi(y, C_1)} = dx$$

and

$$\int \frac{dy}{\varphi(y, C_1)} = x + C_2$$

Example 3.3.3. Solve DE of the second order $y'' \cot y = (y')^2$.

Solution. This equation does not contain x so we use substitution $y' = \frac{dy}{dx} = p(y)$:

$$y'' \cot y = (y')^2 \Rightarrow \begin{bmatrix} y' = p \Rightarrow \\ y'' = \frac{dp}{dy} p \end{bmatrix} \Rightarrow \frac{dp}{dy} p \cot y = p^2 .$$

Separating the variables and integrating we get

$$p = \frac{C_1}{\cos y} .$$

Returning to the function y gives us

$$\frac{dy}{dx} = \frac{C_1}{\cos y} \Rightarrow \cos y\, dy = C_1 dx .$$

Integrating the last equation we get the general integral of the given equation:

$$\sin y = C_1 x + C_2$$

or the general solution

$$y = \arcsin(C_1 x + C_2 .)$$

§4. Some Problems of Particle Dynamics

Let a material point be in a rectilinear motion under the action of a force directed along the trajectory. By Newton's second law we get the differential equation for the law of motion

$$m \frac{d^2 S}{dt^2} = F\left(t, S, \frac{dS}{dt} \right), \tag{3.4.1}$$

where S is the path length, $V = \frac{dS}{dt}$ is the velocity, $W = \frac{d^2 S}{dt^2}$ is the acceleration.

1. Uniformly Acceleration Motion

Let the force F be constant. Denote the ratio $\frac{F}{m}$ as a, then. $\frac{d^2 S}{dt^2} = a$

Integrating we have $\frac{dS}{dt} = at + C_1$ and $S = \frac{at^2}{2} + C_1 t + C_2$. It is obvious that $C_1 = \left. \frac{dS}{dt} \right|_{t=0} = v_0$ and $C_2 = S \left|_{t=0} = S_0$. Thus we have derived the well-known formula for the distance traveled in a uniformly accelerated motion

$$S = \frac{at^2}{2} + V_0 t + S_0 .$$

2. Experiments show that every body moving in a medium undergoes the resistance of medium. When the velocity of motion is high the force of resistance becomes proportional to the square of the velocity

$$F_{res.} = -\gamma V^2 \; (\gamma > 0).$$

Let us consider a body falling on the earth and acted upon by gravitation and air drag. In this case the differential equation of motion (3.4.1) takes the form

$$m\frac{d^2S}{dt^2} = mg - \gamma\left(\frac{dS}{dt}\right)^2,$$

where *mg* is the force of gravity.

On making the substitutions $\frac{dS}{dt} = V$, $\frac{d^2S}{dt^2} = \frac{dV}{dt}$ we arrive at the first-order equation

$$m\frac{dV}{dt} = \frac{\gamma}{m}\left(\frac{mg}{\gamma} - V^2\right).$$

Putting $\frac{\gamma}{m} = a$, $\frac{mg}{\gamma} + b^2$ and separating the variables we obtain

$$\frac{dV}{b^2 - V^2} = a dt.$$

On integrating we receive

$$\frac{1}{2b}\ln\frac{b+V}{b-V} = at + C.$$

Since $V\Big|_{t=0} = 0$, we have $C = 0$. Then $\frac{b+V}{b-V} = e^{2abt}$ and

$$V = b\frac{e^{2abt} - 1}{e^{abt} + 1} = b\tanh(abt).$$

This formula shows that the velocity is always less than *b* and tends to this value as $t \to \infty$. Hence, the velocity does not increase indefinitely and tend to a definite limit referred to as the terminal velocity of fall:

$$V_{term} = b = \sqrt{\frac{mg}{\gamma}}.$$

§5. LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

I. Definitions and General Properties

Definition. A linear differential equation of the second order is an equation of the form

$$y'' + a_1(x)y' + a_2(x)y = f(x) \qquad (3.5.1)$$

When the function $f(x)$ is identically equal to zero, the equation (3.5.1) is called a **homogeneous linear equation**; otherwise it is called a **nonhomogeneous equation.**

If the functions $a_1(x)$, $a_2(x)$ and $f(x)$ are continuous in an interval (a,b) the equation (3.5.1) possesses a unique solution satisfying arbitrary initial conditions

$$y\Big|_{x=x_0} = y_0,\ y'\Big|_{x=x_0} = y_0' \qquad (3.5.2)$$

II. Homogeneous Linear Equations

We shall start with a homogeneous linear equation

$$y'' + a_1(x)y' + a_2(x)y = 0 \qquad (3.5.3)$$

Definition. A system of two particular solutions of the equation (3.5.3) $y_1(x)$ and $y_2(x)$ is said to be a **fundamental system** of solutions of this equation in an interval (a,b) if the determinant

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

is not equal to zero at any point of the interval (a,b).

The determinant $W(x)$ is called the Wronski determinant or Wronskian.

Theorem. The general solution of the equation (3.5.3) has the form

$$y = C_1y_1(x) + C_2y_2(x) \qquad (3.5.4)$$

where C_1 and C_2 are arbitrary constants; $y_1(x)$ and $y_2(x)$ form a fundamental system of solutions of this equation.

Proof. Differentiate the function (3.5.4) twice:

$$y' = C_1y_1' + C_2y' \Rightarrow \quad y'' = C_1y_1'' + C_2y_2''.$$

On substituting y, y' and y'' into the left-hand side of the equation (3.5.3) we get

$$C_1y_1'' + C_2y_2'' + a_1(C_1y_1' + C_2y_2') + a_2(C_1y_1 + C_2y_2) =$$
$$= C_1(y_1'' + a_1y_1' + a_2y_1) + C_2(y_2'' + a_1y_2' + a_2y_2).$$

The expressions in the parentheses are the results of the substitution of the functions y_1 and y_2 in the equation (3.5.3). Since these functions are the solutions of the equation (3.5.3) both expressions are identically equal to zero and hence the function $y = C_1y_1 + C_2y_2$ is the solution of the equation (3.5.3) for any C_1 and C_2.

Now let us prove that for any initial conditions (3.5.2) there exist the constants C_1^0 and C_2^0 such that the solution $y = C_1^0y_1 + C_2^0y_2$ satisfies these initial conditions. To prove this we substitute the function (3.5.4) into the initial conditions (3.5.2). This results in the following system of linear algebraic equations with respect to C_1 and C_2

$$\begin{cases} C_1y_1(x_0) + C_2y_2(x_0) = y_0 \\ C_1y_1'(x_0) + C_2y_2'(x_0) = y_0' \end{cases}.$$

The determinant of this system is

$$W(x) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} \neq 0$$

since $y_1(x)$ and $y_2(x)$ form a fundamental system of solutions of the equation (3.5.3). So it is possible to find unique C_1^0 and C_2^0 that the solution $y = C_1^0 y_1 + C_2^0 y_2$ satisfies the initial conditions (3.5.2).

§6. Nonhomogeneous Linear Equations

Consider a nonhomogeneous linear equation

$$y'' + a_1(x)y' + a_2(x)y = f(x) \qquad (3.6.1)$$

We shall say that the homogeneous equation

$$y'' + a_1(x)y' + a_2(x)y = 0$$

obtained from the equation (3.6.1) **corresponds** to this equation.

Theorem. The general solution of the nonhomogeneous equation (3.6.1) is a sum of the general solution of the corresponding homogeneous equation and a particular solution of the given equation.

Prove this theorem by your own.

I. Solution of Linear Homogeneous Equation with Constant Coefficients

We solve the linear differential equation with constant coefficients

$$y'' + a_1 y' + a_2 y = 0 \qquad (3.6.2)$$

Let us try to find a solution of (3.6.2) in the form

$$y = e^{\lambda x},$$

where λ is a real or complex number. We have

$$y' = \lambda e^{\lambda x}, \quad y'' = \lambda^2 e^{\lambda x}$$

and, consequently, there must be the identity

$$e^{\lambda x}\left(\lambda^2 + a_1\lambda + a_2\right) = 0.$$

Since $e^{\lambda x} \neq 0$, it follows that

$$\lambda^2 + a_1\lambda + a_2 = 0. \qquad (3.6.3)$$

The equation (3.6.3) is called the **characteristic equation** associated with (3.6.2).

There are three cases in connection with the roots λ_1 and λ_2 of the characteristic equation (3.6.3).

1. The roots λ_1 and λ_2 are real and distinct: $\lambda_1 \neq \lambda_2$. In this case either root can be taken as the exponent λ in the function $e^{\lambda x}$ and thus we obtain two solutions of the equation (3.6.2):

$$y_1 = e^{\lambda_1 x} \text{ and } y_2 = e^{\lambda_2 x}.$$

These solutions form a fundamental system since the Wronski determinant is not equal to zero. In fact

$$W(x) = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix} = e^{(\lambda_1+\lambda_2)x}(\lambda_2 - \lambda_1) \neq 0$$

Therefore the general solution in this case is given by the formula

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}. \qquad (3.6.4)$$

2. The roots λ_1 and λ_2 are real and coincide: $\lambda_1 = \lambda_2 = \lambda$. In this case the above procedure only yields one solution $y = e^{\lambda x}$. It is easy to show that the function $y_2 = xe^{\lambda x}$ can be taken as the second solution of the equation. Do it by your own. Here it is also readily checked that the Wronskian does not vanish for any value of λ:

$$W(x) = \begin{vmatrix} e^{\lambda x} & xe^{\lambda x} \\ \lambda e^{\lambda x} & e^{\lambda x} + \lambda xe^{\lambda x} \end{vmatrix} = e^{2\lambda x} \neq 0.$$

Hence the general solution of the equation (3.6.2) is

$$y = (C_1 + C_2 x)\, e^{\lambda x} \qquad (3.6.5)$$

Example 3.6.1. Solve the equation

$$y'' + 4y' + 4y = 0.$$

Solution. The characteristic equation

$$\lambda^2 + 4\lambda + 4 = 0$$

has one two-fold root $\lambda_1 = \lambda_2 = -2$ and, consequently, the general solution has the form

$$y = (C_1 + C_2 x)e^{-2x}.$$

Example 3.6.2. Solve the equation

$$y'' + 5y' + 6y = 0.$$

Solution. The characteristic equation

$$\lambda^2 + 5\lambda + 6 = 0$$

has two real and distinct roots $\lambda_1 = -3$ and $\lambda_2 = -2$. Therefore the general solution of the given equation is

$$y = C_1 e^{-3x} + C_2 e^{-2x}.$$

3. The roots of the characteristic equation are conjugate complex numbers: $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$ $(\beta \neq 0)$. In this case the equation (3.6.4) applies once again to give

$$y = \tilde{C}_1 e^{(\alpha + i\beta)x} + \tilde{C}_2 e^{(\alpha - i\beta)x} = e^{\alpha x}\left(\tilde{C}_1 e^{i\beta x} + \tilde{C}_2 e^{-i\beta x}\right) \qquad (3.6.6)$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are complex constants.

By Euler's formula

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \text{ and } e^{-i\beta x} = \cos\beta x - i\sin\beta x.$$

Hence we may replace the equation (3.6.6) by

$$y = e^{\alpha x}\left((\tilde{C}_1 + \tilde{C}_2)\cos\beta x + i(\tilde{C}_1 - \tilde{C}_2)\sin\beta x\right). \qquad (3.6.7)$$

Finally, we introduce new arbitrary constants $C_1 = \widetilde{C}_1 + \widetilde{C}_2$ and $C_2 = i\left(\widetilde{C}_1 - \widetilde{C}_2\right)$, to give the solution of the equation (3.6.7) in a shorter form

$$y = e^{\alpha x}\left(C_1 \cos\beta x + C_2 \sin\beta x\right). \qquad (3.6.8)$$

The constants C_1 and C_2 are real because $\widetilde{C}_1$ and $\widetilde{C}_2$ are complex conjugate.

Example 3.6.3. (Simple Harmonic Motion). Suppose we have a spring of natural length *L* and spring constant *k*, with its upper end fastened to a rigid support. We hang a mass *m* from the spring. The weight of the mass stretches the spring to a length *L*+ *S* when allowed to come to rest in a new equilibrium position. By Hooke's law, the tension in the spring is *kS*. The force of gravity pulling down on the mass is *mg*. Equilibrium requires

$$kS = mg. \qquad (1)$$

How will the mass behave if we pull it down an additional amount x_0 beyond the equilibrium position and release it? To find out , let *x*, positive direction downward, denote the displacement of the mass from equilibrium *t* seconds after the motion has started. Then the forces acting on the mass are

+ *mg* (weight due to gravity),

-*k(S + x)* (spring tension).

By Newton's second law, the sum of these forces is $m\frac{d^2x}{dt^2}$, so

$$m\frac{d^2x}{dt^2} = mg - kS - kx = 0 \qquad (2)$$

Since $mg = kS$ from the equation (1), the equation (2) simplifies to

$$m\frac{d^2x}{dt^2} + kx = 0. \qquad (3)$$

In additional to satisfying this differential equation, the position of the mass satisfies the initial conditions

$$x = x_0 \text{ and } \frac{dx}{dt} = 0, \text{ when } t = 0. \qquad (4)$$

If we divide both sides of the equation (3) by *m* and write ω for $\sqrt{k/m}$, the equation becomes

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad (5)$$

The roots of the characteristic equation $\lambda^2 + \omega^2 = 0$ are $\lambda = \pm i\omega$, so the general solution of the equation (5) is

$$x = C_1 \cos t\omega + C_2 \sin t\omega \qquad (6)$$

applying the initial conditions in the equation (4) determines the constants to be

$$C_1 = x_0 \text{ and } C_2 = 0.$$

The mass's displacement from equilibrium *t* seconds into the motion is

$$x = x_0 \cos t\omega \qquad (7)$$

This equation represents a **simple harmonic motion** of amplitude x_0 and period $T = \frac{2\pi}{\omega}$. We normally combine the two terms in the general solution in the equation (6) into a single term, using the trigonometric identity

$$C\cos(t\omega - \varphi) = C\cos\varphi \cos t\omega + C\sin\varphi \sin t\omega .$$

To apply the identity, we take

$$C_1 = C\cos\varphi , C_2 = C\sin\varphi , \qquad (8)$$

where

$$C = \sqrt{C_1^2 + C_2^2} , \tan\varphi = \frac{C_2}{C_1} . \qquad (9)$$

With these substitutions, the equation (6) becomes

$$x = C\cos(t\omega - \varphi) . \qquad (10)$$

We treat C and φ as two new arbitrary constants.

The equation (10) represents a simple harmonic motion of amplitude C and period $T = \frac{2\pi}{\omega}$. The angle φ is the **phase angle** of the motion.

§ 7. Nongomogeneous Linear Differential Equations with Constant Coefficients

We consider a linear equation of the form

$$y'' + a_1 y' + a_2 y = f(x) \qquad (3.7.1)$$

The general solution of this equation is the sum of the general solution of the corresponding homogeneous equation and a particular solution of the equation (3.7.1). It is already known how to find the general solution of the homogeneous equation. Now it remains to determine a particular solution of the given equation. We shall start with some special cases.

1. Let the right-hand side of the equation (3.7.1) be of the form

$$f(x) = P_n(x)e^{kx}, \qquad (3.7.2)$$

where $P_n(x)$ is a polynomial of the *n*th degree. Then the equation has a particular solution

$$y_p = x^m Q_n(x)\, e^{kx},$$

where $Q_n(x)$ is a polynomial of the same degree as $P_n(x)$; if the number k is not a root of the characteristic equation, then $m = 0$, and if it is a root then m is equal to the multiplicity of that root. We find the coefficients of the polynomial $Q_n(x)$ with the aid of the method of undetermined coefficients.

Example 3.7.1. Find a particular solution of the equation

$$y'' - 6y' + 9y = e^{3x} .$$

Solution.

The characteristic equation

$$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0$$

has $\lambda = 3$ as a double root. The appropriate choice for y_p in this case is Ax^2e^{3x}. When we substitute

$$y_p = Ax^2e^{3x}$$

and its derivatives in the given differential equation, we get

$$\left(9Ax^2e^{3x} + 12Axe^{3x} + 2Ae^{3x}\right) - 6\left(3Ax^2e^{3x} + 2Axe^{3x}\right) + 9Ax^2e^{3x} =$$

$$= e^{3x} \Rightarrow 2Ae^{3x} = e^{3x} \Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}.$$

Our solution is $y_p = \frac{1}{2}x^2e^{3x}$.

2. Let the right-hand side of the equation (3.7.1) be of the form

$$f(x) = M\cos bx + N\sin bx,$$

then the equation (3.7.1) has a particular solution

$$y_p = \left(A\cos bx + B\sin bx\right)x^m,$$

where A and B are unknown coefficients; $m = 0$ if the numbers $\pm ib$ are not characteristic roots and $m = 1$ if the numbers $\pm ib$ satisfy the characteristic equation.

Example 3.7.2. Find a particular solution of the differential equation

$$y'' + 4y' + 13y = 5\sin 2x.$$

Solution. The characteristic equation $\gamma^2 + 4\lambda + 13 = 0$ has the roots $\lambda_{1,2} = -2 \pm 3i$. Since the numbers $\pm 2i$ are not roots of the characteristic equation we look for a particular solution of the form

$$y_p = A\cos 2x + B\sin 2x.$$

Differentiating y_p twice we obtain

$$y' = -2A\sin 2x + 2B\cos 2x,$$

$$y'' = -4A\cos 2x - 4B\sin 2x.$$

The substitution in equation yields

$$-4A\cos 2x - 4B\sin 2x - 8A\sin 2x + 8B\cos 2x +$$
$$+ 13A\cos 2x + 13B\sin 2x = 5\sin 2x$$

.

Equating the coefficients in $\sin 2x$ and $\cos 2x$ on both sides of the equality we get

$$\begin{array}{l|l} \sin 2x & -8A + 9B = 5 \\ \cos 2x & 9A + 8B = 0 \end{array}$$

whence $A = -\dfrac{8}{29}$ and $B = \dfrac{9}{29}$. Hence the particular solution is

$$y_p = -\frac{8}{29}\cos 2x + \frac{9}{29}\sin 2x.$$

Example 3.7.3. A particle slides freely in a tube, which rotates in a vertical plane about its midpoint with constant angular velocity ω. If x is the distance of the particle from the midpoint of the tube at time t, and if the tube is horizontal with $t = 0$ the motion of the particle along the tube is given by

$$\frac{d^2x}{dt^2} - \omega^2 x = -g\sin t\omega .$$

Solve this equation if $x = x_0$, $\dfrac{dx}{dt} = V_0$, when $t = 0$. The characteristic equation is

$\lambda^2 + \omega^2 = 0 \Rightarrow \lambda_1 = \omega, \lambda_2 = -\omega$ and the general solution of the corresponding homogeneous equation is

$$x_c = C_1 e^{t\omega} + C_2 e^{-t\omega}.$$

Furthermore, the particular solution appears as

$$x_p = A\sin t\omega + B\cos t\omega ,$$

where C_1, C_2, A and B are constants to be determined. Differentiating twice the last expression we obtain

$$\frac{dx_p}{dt} = A\omega\cos t\omega - B\omega\sin t\omega ,$$

$$\frac{d^2x_p}{dt^2} = -\omega^2\left(A\sin t\omega + B\cos t\omega\right).$$

Substitution in the main differential equation gives

$$-2A\omega^2\sin t\omega - 2B\omega^2\cos t\omega = -g\sin t\omega ,$$

which implies that:

$$-2A\omega^2 = -g, \text{ or } A = \frac{g}{2\omega^2}, \text{ and } B = 0.$$

Thus,

$$x = C_1 e^{t\omega} + C_2 e^{-t\omega} + \frac{g}{2\omega^2}\sin t\omega .$$

To find the values C_1 and C_2, we use the initial conditions for $t = 0, \dfrac{dx}{dt} = V_0$ and $x = x_0$.

And since

$$\frac{dx}{dt} = C_1\omega\ e^{t\omega} - C_2\omega\ e^{-t\omega} + \frac{g}{2\omega}\cos t\omega,$$

$$\frac{dx}{dt}\Big|_{t=0} = C_1\omega - C_2\omega + \frac{g}{2\omega} = V_0,$$

and

$$x\big|_{t=0} = C_1 + C_2 = x_0.$$

These are simultaneous equations with C_1 and C_2 unknown. Substitution of $C_2 = x_0 - C_1$ in the expression for V_0 yields:

$$C_1 = \frac{V_0 - \frac{g}{2\omega} + x_0\omega}{2\omega}, \text{ and } C_2 = \frac{x_0\omega - V_0 + \frac{g}{2\omega}}{2\omega}.$$

At last we can write the general solution of the equation in such a way

$$x = \frac{1}{2}x_0\left(e^{t\omega} + e^{-t\omega}\right) + \left(\frac{V_0}{2\omega} - \frac{g}{4\omega^2}\right)\left(e^{t\omega} - e^{-t\omega}\right) + \frac{g}{2\omega^2}\sin t\omega\ ,$$

or

$$x = x_0\cosh t\omega + \left(\frac{V_0}{\omega} - \frac{g}{2\omega^2}\right)\sinh t\omega + \frac{g}{2\omega^2}\sin t\omega\ .$$

For the motion to be the simple harmonic type, we must have the relationship:

$$\left|\omega^2 x\right| >> \left|g\sin t\omega\right|, \text{ or } \left|\frac{\omega^2 t}{g}\right| >> 1.$$

Example 3.5.4. Find a form of the general solution of a differential equation if the roots of characteristic equation are $\lambda_{1,2} = -3 \pm \sqrt{5}i$ and the right hand-side is equal to $2e^{-3x}\sin\sqrt{5}x$ and explain the answer.

The answer.

$$y_g = e^{-3x}(C_1\cos\sqrt{5}x + C_2\sin\sqrt{5}x) + xe^{-3x}(M\cos\sqrt{5}x + N\sin\sqrt{5}x).$$

Miscellaneous Problems

Solve the following differential equations.

1). $\sqrt{21-8x-4x^2}\,dy = dx$

2). $\left(x - y\cos\frac{y}{x}\right)dx + x\cos\frac{y}{x}\,dy = 0$

3). $y' = \frac{x+2y}{2x-y}$

4). $y' + \frac{y}{x+1} = -\frac{1}{2}(x+1)^3 y^3$

5). $e^y\left(1+x^2\right)y' - 2x\left(1+e^y\right) = 0$

6). $y' - \frac{y}{x} = -\frac{12}{x^2}$

7). $y^2 - 1 + (2xy + 3y)y' = 0$

8). $2(y' + xy) = (x-1)e^x y^2$

9). $y' - \frac{y}{\sin x} = \sin\frac{x}{2}$

Find the particular solution of the following equations

10). $x\ln y \cdot y' = x^3 y, \quad y\big|_{x=1} = e$

11). $(xy' - y)\arctan\frac{y}{x} = x, \quad y\big|_{x=1} = 2$

Solve the following second order equations

12). $(1+\sin x)y'' = y'\cos x$

13). $xy'' - y' = x^2 e^x$

14). $2y''y + (y')^2 + (y')^3 = 0$

15). $y''\tan x = 2y'$

Solve the following homogeneous linear equations

16). $2y'' + y = 0$

17). $y'' + 2y' = 0$

18). $y'' + 6y' + 9y = 0$

19). $\begin{cases} 4y'' + 8y' + 5y = 0 \\ y(0) = y'(0) = 1 \end{cases}$

20). $\begin{cases} y'' - 4y' - 5y = 0 \\ y(0) = y'(0) = 2 \end{cases}$

Solve the following nonhomogeneous linear equations

21). $y'' - y = \left(-2x^2 + x + 4\right) + 2e^x$

22). $y'' - 2y' + 10y = (26x - 5)\,e^{3x}$

23). $y'' - 5y' + 6y = 8\sin 2x + 72\cos 2x$

24). $y'' + y' = 9x^2 + 92x + 9$

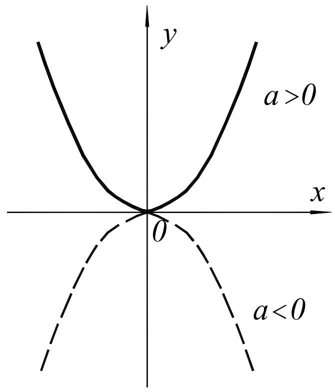
APPENDIXES.

Graphs of Some Functions.

1. Power Functions

Parabolas

a). $y = ax^{2n}, n = 1,2,\ldots$



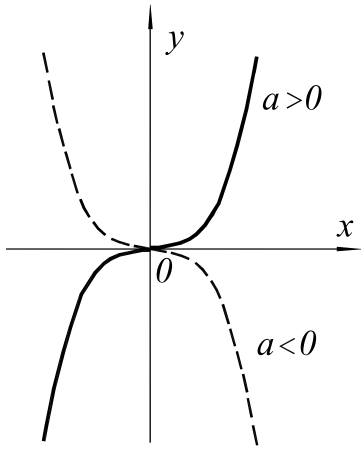
Domain of definition:

$D(y) = (-\infty, +\infty)$

Range of values

$E(y) = [0, +\infty)$

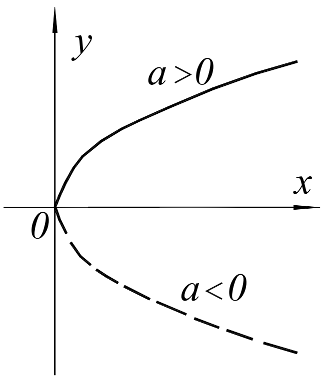
b). $y = ax^{2n+1}, n = 1,2,\ldots$



$D(y) = (-\infty, +\infty)$

$E(y) = (-\infty, +\infty)$

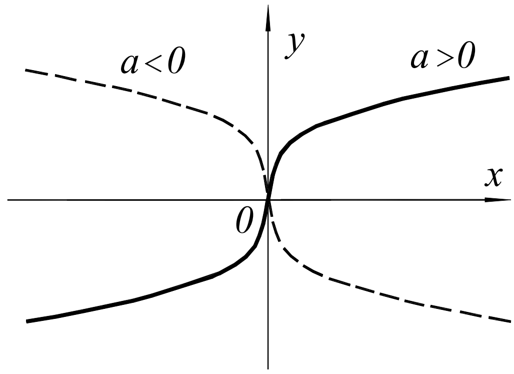
c). $y = a\sqrt[2n]{x}, n = 1,2,\cdots$



$D(y) = [0, +\infty)$

$$E(y) = \begin{cases} [0, +\infty), a > 0 \\ (-\infty, 0], a < 0 \end{cases}$$

d). $y = a\sqrt[2n+1]{x}, n = 1,2,\cdots$

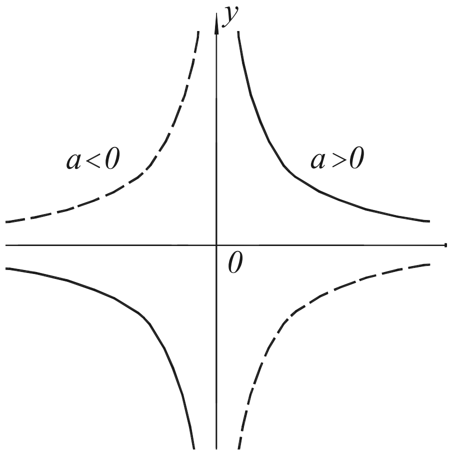


$D(y) = (-\infty, +\infty)$

$E(y) = (-\infty, +\infty)$

Hyperbolas

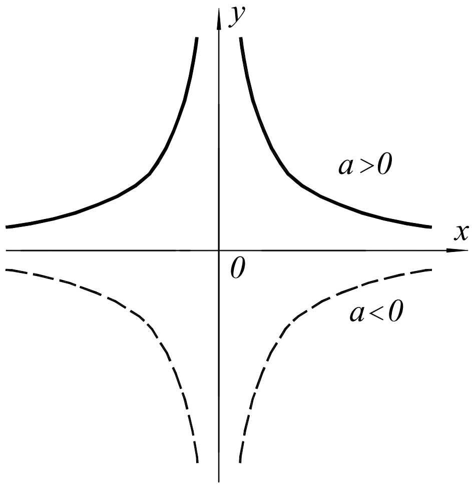
e). $y = \dfrac{a}{x^{2n-1}}, n = 1,2,\ldots$



$D(y) = (-\infty, 0) \cup (0, +\infty)$

$E(y) = (-\infty, 0) \cup (0, +\infty)$

f). $y = \dfrac{a}{x^{2n}}, n = 1,2,\ldots$

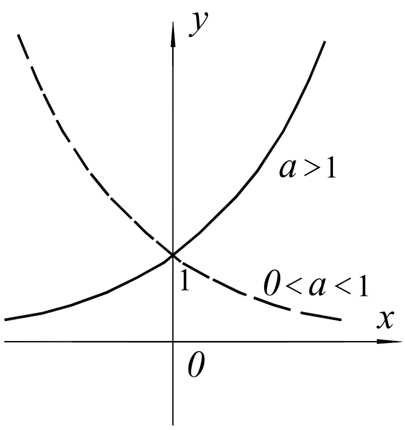


$D(y) = (-\infty, 0) \cup (0, +\infty)$

$$E(y) = \begin{cases} (0, +\infty), a > 0 \\ (-\infty, 0), a < 0 \end{cases}$$

2. Exponential Function

$y = a^x, a > 0, a \neq 1$

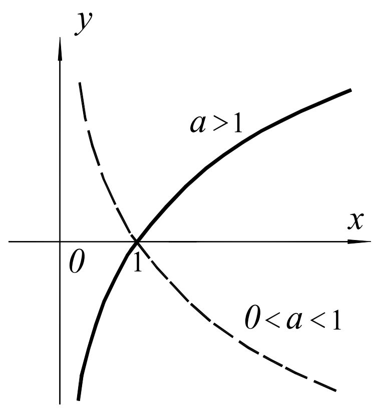


$D(y) = (-\infty, +\infty)$

$E(y) = (0, +\infty)$

3. Logarithm Function

$y = \log_a x, a > 0, a \neq 1$

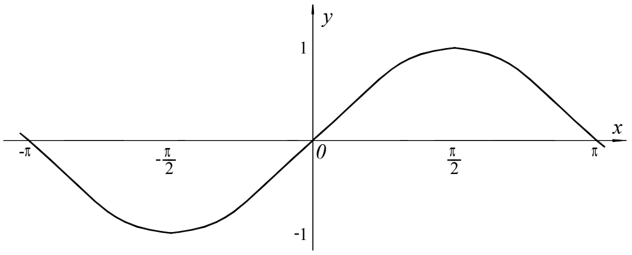


$D(y) = (0, +\infty)$

$E(y) = (-\infty, +\infty)$

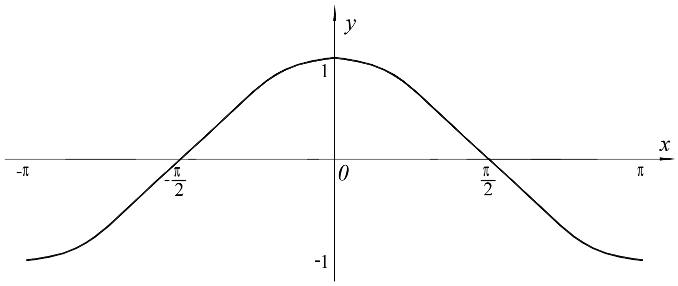
4. Trigonometric Functions

a). Sinusoid (sine curve, harmonic curve) $y=\sin x$



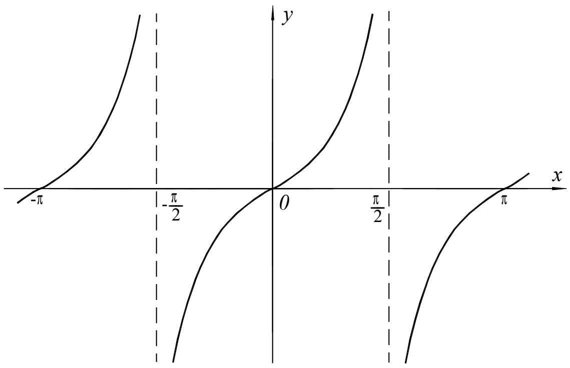
$$D(y)=(-\infty,+\infty),$$
$$E(y)=[-1,+1]$$

b). Cosine curve $y=\cos x$



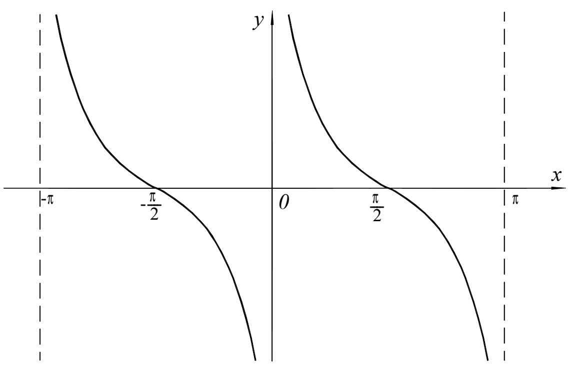
$$D(y)=(-\infty,+\infty),\quad E(y)=[-1,+1]$$

c). Tangent curve $y=\tan x$



$$D(y)=\left(-\frac{\pi}{2}+k\pi,\frac{\pi}{2}+k\pi\right),k=0,\pm1,\pm2,\dots$$
$$E(y)=(-\infty,+\infty)$$

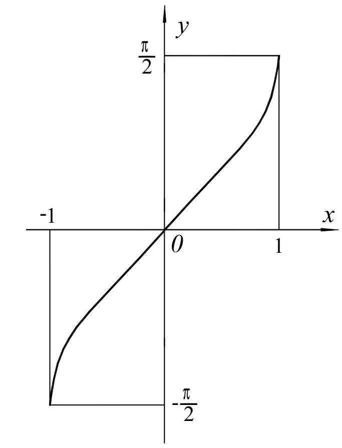
d). Cotangent curve $y=\cot x$



$$D(y)=(k\pi,(k+1)\pi),k=0,\pm1,\pm2,\dots$$
$$E(y)=(-\infty,+\infty)$$

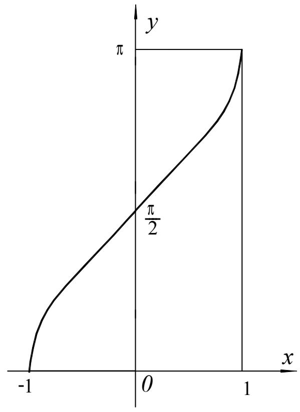
5. Inverse Trigonometric Functions

a). $y = \sin^{-1} x$



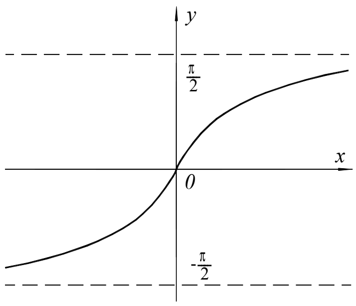
$D(y) = [-1;1], \quad E(y) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

b). $y = \cos^{-1} x$



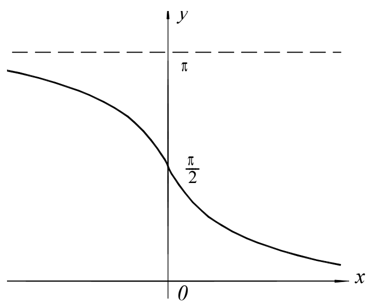
$D(y) = [-1;1], \; E(y) = [0;\pi]$

c). $y = \tan^{-1} x$



$D(y) = (-\infty, +\infty), \; E(y) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

d). $y = \cot^{-1} x$



$D(y) = (-\infty, +\infty), E(y) = (0, \pi)$

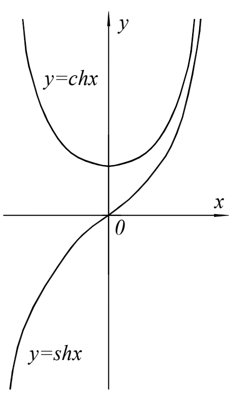
6. Hyperbolic Functions

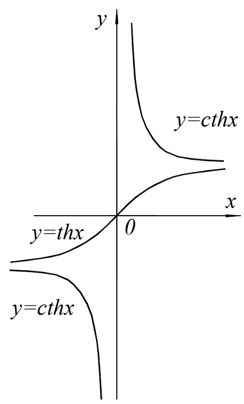
a). $y = \sinh x \; (shx)$

b). $y = \cosh x \; (chx)$

c). $y = \tanh x \; (thx)$

d). $y = \coth x \; (cthx)$





a). $\sinh x = \dfrac{e^x - e^{-x}}{2}$

$D(\sinh x) = (-\infty, +\infty)$

c). $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$D(\tanh x) = (-\infty, +\infty)$

$E(\sinh x) = (-\infty, +\infty)$

$E(\tanh x) = (-1, +1)$

b). $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

$D(\cosh x) = (-\infty, +\infty)$

$E(\cosh x) = [1, +\infty)$

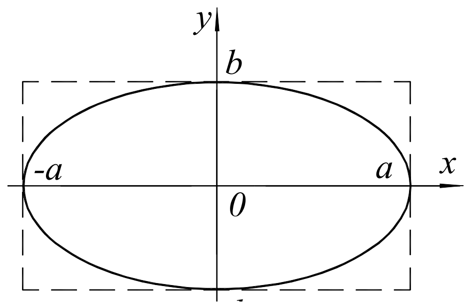
d). $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$

$D(\coth x) = (-\infty, 0) \cup (0, +\infty)$

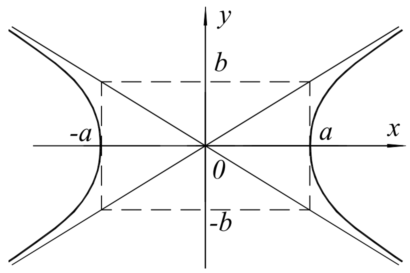
$E(\coth x) = (-\infty, -1) \cup (1. +\infty)$

7. Curves of the Second Order

a). Ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



b).Hyperbola: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

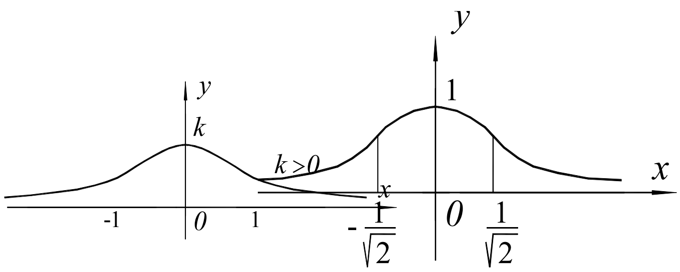


8. Witch of Agnesi:

$y = \dfrac{k}{1 + x^2}$

9). Curve of Gauss:

$y = e^{-x^2}$

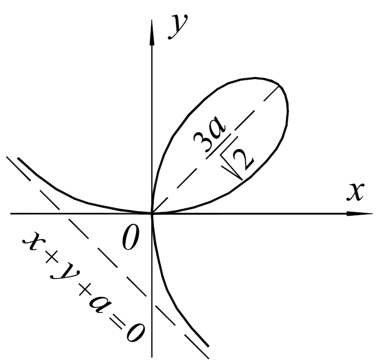


10. Loops

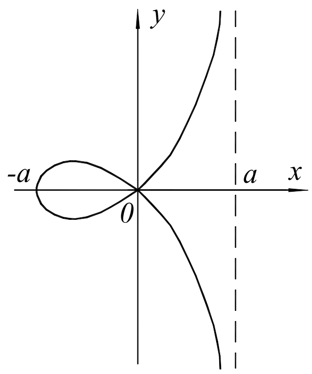
a). Folium of Descartes

$x^3 + y^3 - 3axy = 0$, or

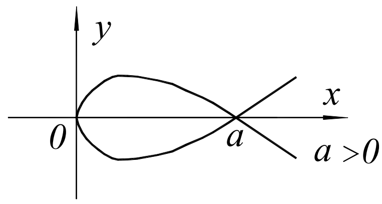
$$\begin{cases} x = \dfrac{3at}{1 + t^3} \\ y = \dfrac{3at^2}{1 + t^2} \end{cases}$$



b) $y^2 = x^2 \cdot \dfrac{a + x}{a - x}$



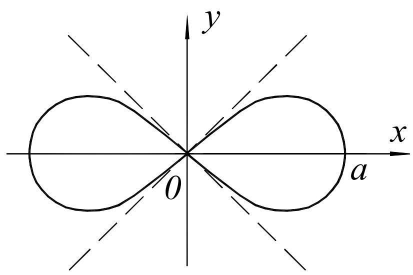
c). $a^2 y^2 = x\left(a - x^2\right), a > 0$



11. Lemniscate of Bernoulli

$\left(x^2 + y^2\right)^2 = a^2\left(x^2 - y^2\right)$

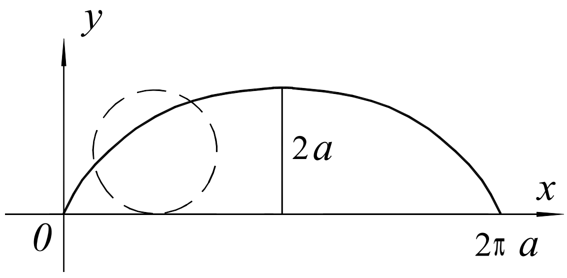
or $r^2 = a^2 \cos 2\varphi$



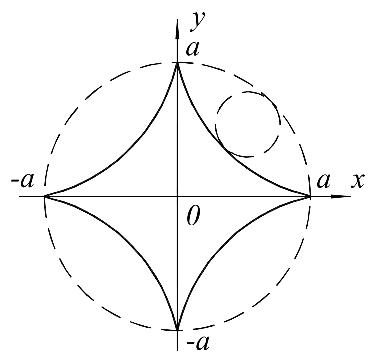
12. Parametric Equations of Curves

a). Cycloid:

$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, a > 0$

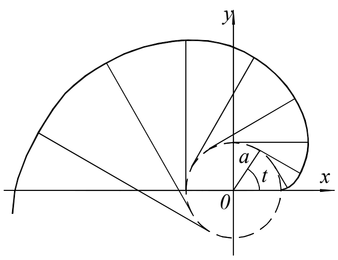


b). Astroid: $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}, a > 0$

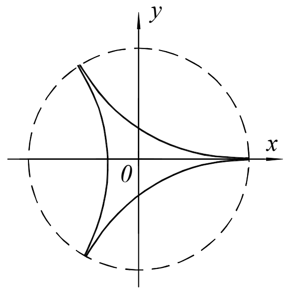


c). Evolvent of Circle:

$\begin{cases} x = a(\cos t + t\sin t) \\ y = a(\sin t - t\cos t) \end{cases}, a > 0$



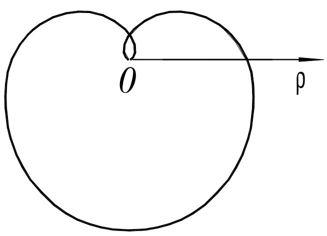
d). $\begin{cases} x = R\cos\frac{t}{3}\cdot\left(2+\cos\frac{t}{3}\right) \\ y = R\sin\frac{t}{3}\cdot\left(2-\sin\frac{t}{3}\right) \end{cases}$



12. Curves in the Polar System of Coordinates

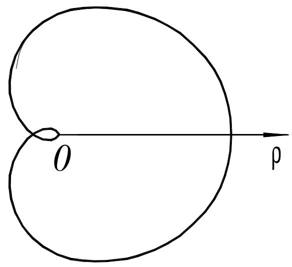
a). $\rho = a\sin^3\frac{\varphi}{3}$,

$a > 0, \varphi \in [0, 3\pi]$



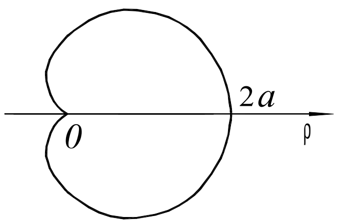
b). $\rho = a\cos^3\varphi$,

$a > 0, \varphi \in \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$

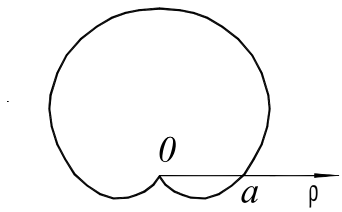


Cardioids

c) $\rho = a(1+\cos\varphi), a > 0$

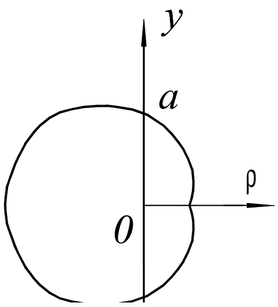


d) $\rho = a(1+\sin\varphi), a > 0$

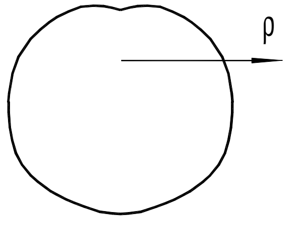


Limacons

a). $\rho = 1-\cos\varphi$



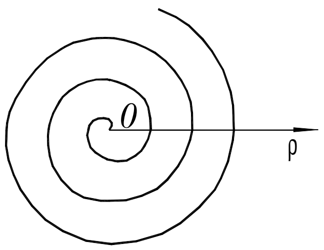
b). $\rho = 1-\sin\varphi$

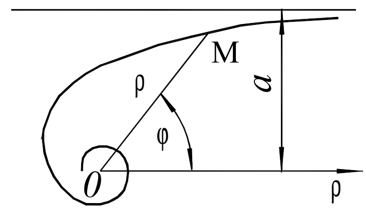


Spirals

a). $\rho = a\varphi, a > 0$

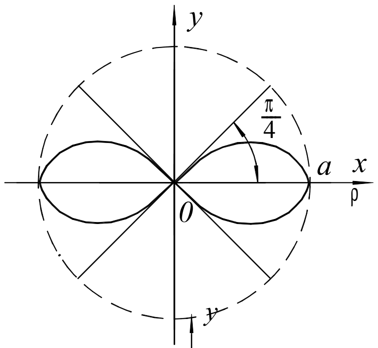
b). $\rho = \dfrac{a}{\varphi}, a > 0$

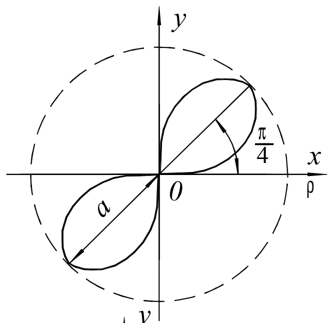


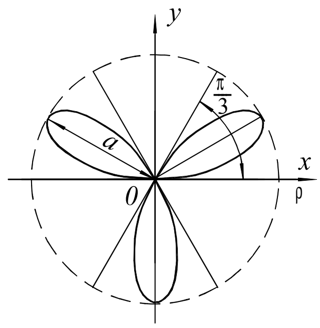


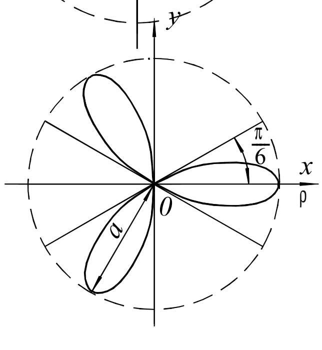
Roses

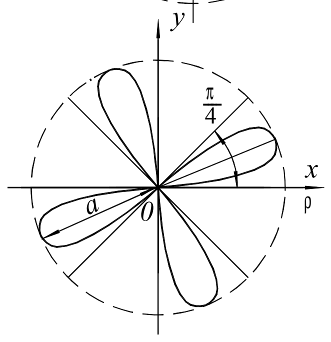
a). $\rho = a\sin 2\varphi, a > 0$ b). $\rho = a\cos 2\varphi, a > 0$ c). $\rho = a\sin 3\varphi, a > 0$

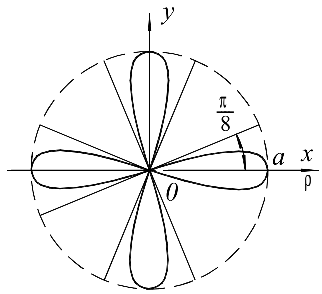












d).

$\rho = a\cos 3\varphi, a > 0$ e). $\rho = a\sin 4\varphi, a > 0$ f). $\rho = a\cos 4\varphi, a > 0$