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FUNCTIONS OF SEVERAL VARIABLES

Textbook

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Методическое пособие содержит основные определения, формулы и теоремы дифференциального исчисления функции многих переменных и предназначено для студентов академии, изучающих математику на английском языке.

Основные теоремы и формулы приведены с доказательством, а также даны решения типовых задач, задания для самостоятельной работы. Кроме того прилагаются комплексной контрольной работы.

CONTENTS

1. Definitions

Until now we have studied functions of a single variable. Many phenomena in the physical world can be described by such functions, but most quantities actually depend on more than one variable. For example, the volume of a rectangular box depends on its length, width, and height; the temperature at a point of a metal plate depends on the coordinates of the point (and possibly on time as well). Any quantity that depends on several other quantities can be thought as determining a function of several variables.

A function of several variables consists of two parts: a **domain** $(D(u))$, which is a set of points in the plane or in space, and a **rule**, which assigns to each member of the domain one and only one real number. This rule can be written in the form $u = f(P)$, where $P \in D(u)$.

If $D(u)$ is a part of the plane XOY then any point of this region has two coordinates: $P = P(x, y)$ and we have a **function of two independent variables**. As a rule it is written in the form $z = f(x, y)$.

If $D(u)$ is a part of the space XYZ then any point of this region has three coordinates: $P = P(x, y, z)$ and we have a **function of three independent variables**. Let us write it in the form $u = f(x, y, z)$.

If $D(u)$ is a part of an n-dimensional space then any point of this region has n coordinates: $P = P(x_1, x_2, ..., x_n)$ and we have a **function of *n* independent variables**. Let us write it in the form $u = f(x_1, x_2, ..., x_n)$.

Independent variables are equivalent and they are called the arguments.

To find a domain of a function of two variables we can use the same rules as for a function of one variable.

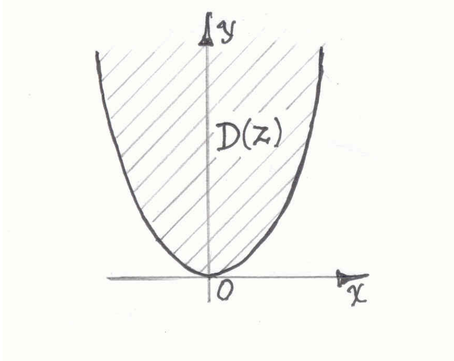
Example 1.1. Find the domain of the given function. $z = f(x, y)$ *if*

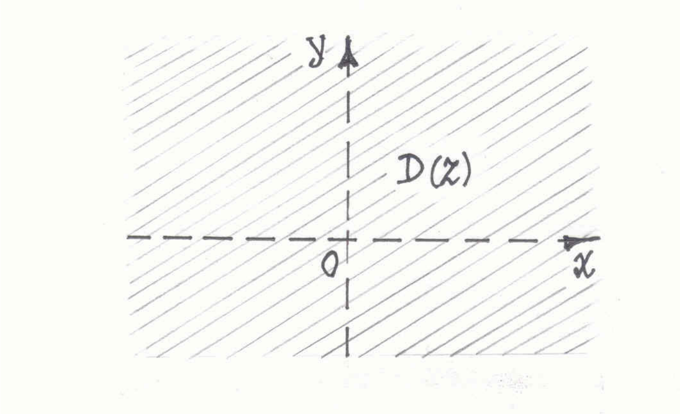
a) $f(x, y) = \sqrt{y - x^2}$ b) $f(x, y) = \dfrac{1}{xy}$

	Function	Domain
a)	$z = \sqrt{y - x^2}$	$D(z): y \geq x^2$
b)	$z = \dfrac{1}{xy}$	$D(z): xy \neq 0$

Fig. 1.1

a) The domain of the function $f(x, y) = \sqrt{y - x^2}$ are all points of xy-plane which are on the parabola $y = x^2$ and in the interior region of this parabola.





b) The domain of the function $z = \dfrac{1}{xy}$ are all points of xy-plane except the
points of x-axis and y-axis

2. Ways of Representation Functions of Two Variables

A function of two variables, like a function of one variable, can be specified by a **table**, by a formula (**analytically**) or by its **graph**.

A. A tabular representation of a function indicates the values of the function for a number of pairs of values of the independent variables. For example, the table 2.1 shows dependence of area S_{ij} on two independent variables
b_i – a width and a_j – a length.

	a_1	a_2	…	a_n
b_1	S_{11}	S_{12}	…	S_{1n}
b_2	S_{21}	S_{22}	…	S_{2n}
…	…	…	…	…
b_m	S_{m1}	S_{m2}	…	S_{mn}

Table 2.1

B. A function of two variables can be specified by formula. In the analytical specification of a function we use a formula determining the values of the function depending on the values of the independent variables. For example, $z = \dfrac{xy}{x^2 + y^2}$.

C. The function of two variables can be represented by its graph too.

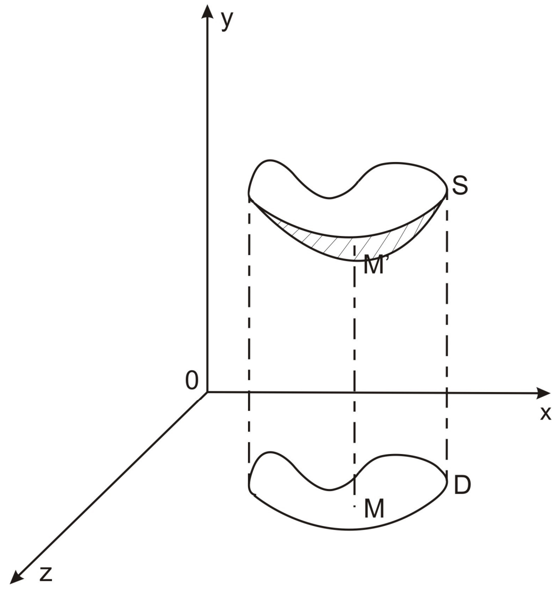


Fig.2.1

The **graph** of a function $z = f(x, y)$ is the collection of points $(x, y, f(x, y))$ and is represented by a surface $\boldsymbol{S}$. Each point of this surface M' has coordinates (x, y, z). The domain of the function $z = f(x, y)$ is the region $\boldsymbol{D}$ of xy – plane. When point $M(x, y)$ runs through region $\boldsymbol{D}$, than point $M'(x, y, z)$ runs through the surface $\boldsymbol{S}$, whose equation is $z = f(x, y)$.

Example 2.1. The graph of a function $Ax + By + Cz + D = 0$ is the plane passing through the points $\left(-\dfrac{D}{A}, 0, 0\right)$, $\left(0, -\dfrac{D}{B}, 0\right)$, $\left(0, 0, -\dfrac{D}{A}\right)$.
In *Fig.2.2* is drawn the portion of this plane in the first octant.

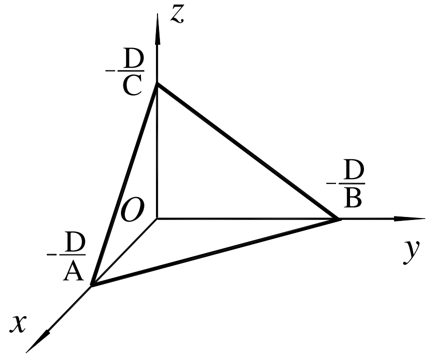


Fig.2.2

3. A Level Curves

Let f be a function of two independent variables x and y and denote the dependent variable by z. The equation $z = f(x, y)$ may be interpreted as defining a surface S in xyz-space. If we cut the surface with planes $z = h_1, z = h_2$, we get **contour lines** on the surface. And if we project them onto the xy-plane, we obtain a **level curves** $f(x, y) = h_1$, $f(x, y) = h_2$ in the domain of $z = f(x, y)$ (*Fig. 3.1*).

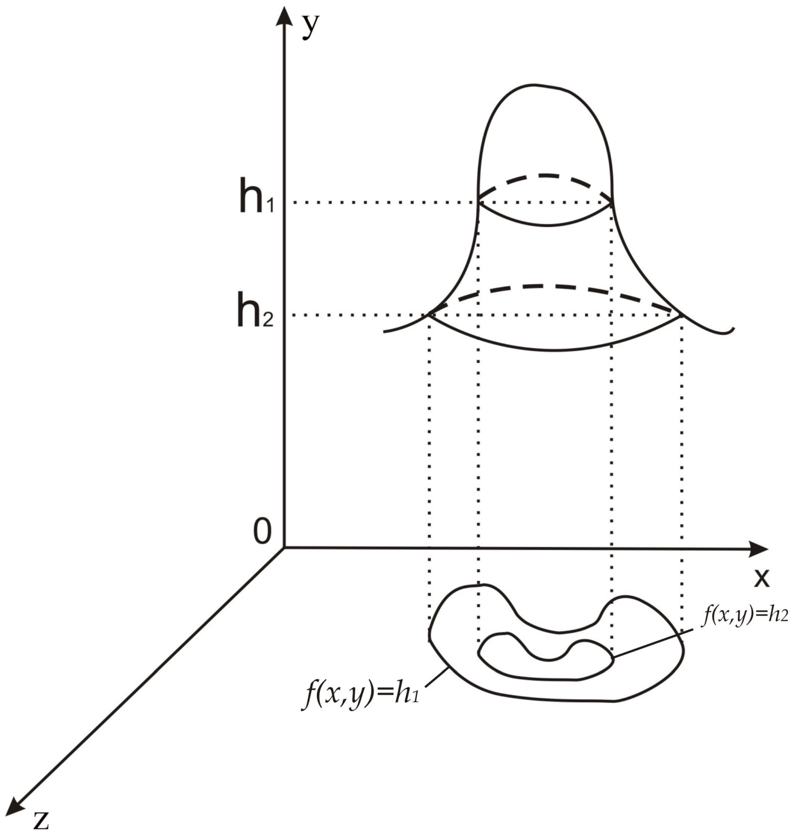


Fig. 3.1

Definition. The set of points in the *xy*-plane where function $f(x, y)$ has a constant value C is called a level line of this function $f(x, y) = C$.

Level curves are particularly useful in engineering applications. For instance equation $z = (x-2)^2 + (y+3)^2$ gave the celsius temperature at each point in a flat circular plate, then the level curves would be isotherms of the temperature distribution.

Example 3.1. Find the level lines of the function $z = (x-2)^2 + (y+3)^2$.

Solution. The equation of the level lines of the given function is

$$(x-2)^2 + (y+3)^2 = C$$

If C takes different values, for example, $C = 4$, $C = 9$, $C = 16, \ldots$ we will have a family of circles with origin at a point $O(2;-3)$ with corresponding radiuses $R = 2$, $R = 3$, $R = 4, \ldots$

Example3.2. Let $f(x, y) = 8 - 2x - 4y$. Sketch the graph of f, and determine the level curves.

Solution. If we let $z = f(x, y)$, then the given equation becomes

$$z = 8 - 2x - 4y$$

This is an equation of a plane with x intercept 4, y intercept 2, and z intercept 8.

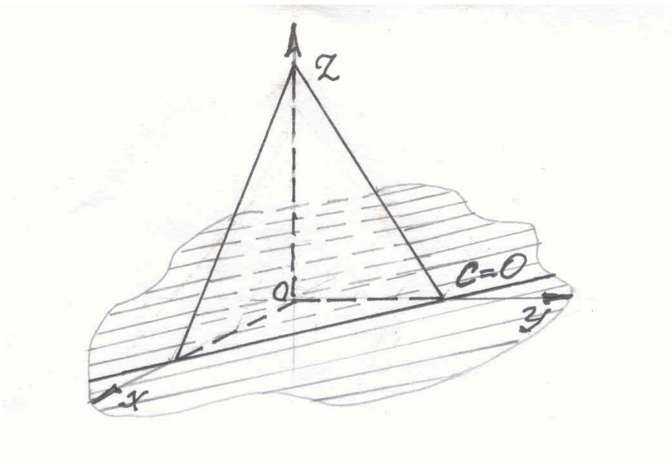


Fig.3.2

The portion of the plane in the first octant is sketched in *Fig.3.2*. For any value of C, the level curve $f(x, y) = C$ is the straight line in the xy plane with equation

$$2x + 4y - 8 + C.$$

The level curves are parallel lines.

4. Limits and Continuity

*Definition***.** The **limit** of a function $f(x, y)$ as $(x, y) \to (x_0, y_0)$ is a number A if for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that for all points $(x, y) \neq (x_0, y_o)$ in domain of $f(x, y)$, from $|x - x_o| < \delta$ and $|y - y_0| < \delta$ follows $|f(x, y) - A| < \varepsilon$.

Properties of Limits

Let $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = A$ $\lim\limits_{(x,y)\to(x_0,y_0)} g(x, y) = B$ be given, then the following rules hold	
Sum rule	$\lim\limits_{(x,y)\to(x_0,y_0)} [f(x, y) + g(x, y)] = A + B$
Difference rule	$\lim\limits_{(x,y)\to(x_0,y_0)} [f(x, y) - g(x, y)] = A - B$
Product rule	$\lim\limits_{(x,y)\to(x_0,y_0)} [f(x, y) \cdot g(x, y)] = A \cdot B$
Constant multiple rule	$\lim\limits_{(x,y)\to(x_0,y_0)} kg(x, y) = kB$, k is constant
Quotient rule	$\lim\limits_{(x,y)\to(x_0,y_0)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{A}{B}$ if $B \neq 0$

*Example 4.1***.** *Find limits of the functions*:

a) $\lim\limits_{(x,y)\to(3,4)} \left(x^2 + y^2\right)$,

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1}{x+y}$,

c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1}$.

*Solution***:**

a) $\lim\limits_{(x,y)\to(3,4)} \left(x^2 + y^2\right) = 3^2 + (-4)^2 = 9 + 16 = 25;$

b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1}{x+y} = \left[\dfrac{1}{0}\right] = \infty\,;$

c) $$\lim\limits_{(x,y)\to(0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2 + 1} - 1} = \left[\frac{0}{0}\right] = \lim\limits_{(x,y)\to(0,0)} \frac{\left(x^2 + y^2\right)\left(\sqrt{x^2 + y^2 + 1} + 1\right)}{x^2 + y^2 + 1 - 1} =$$
$$= \lim\limits_{(x,y)\to(0,0)} \sqrt{x^2 + y^2 + 1} + 1 = 2\,.$$

Definition. A function $f(x, y)$ is said to be **continuous** at the point (x_0, y_0) if:

1) $f(x, y)$ is defined at (x_0, y_0),

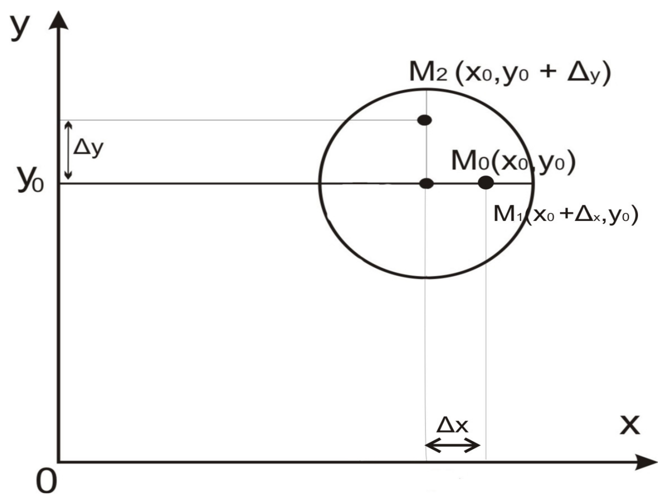
2) $\lim\limits_{(x,y)\to(x_0,y_0)} f(x.y)$ exists,

3) $\lim\limits_{(x,y)\to(x_0,y\ )} f(x, y) = f(x_0 y_0)$.

For a function $f(x, y)$ fair all properties of the continuous functions similar to functions of one variable. For example, the function $z = \dfrac{1}{x^2 + y^2}$ is continuous everywhere, except the point $(0,0)$. For the function $z = \dfrac{1}{x^2 - y^2}$ the points of discontinuity are all points for which $y = x$ or $y = -x$.

5. Partial Increments and Derivatives of the First Order

Definition. An open ***r*-neighborhood** of a given point $M_0(x_0, y_0)$ is the set of all points lying inside the circle of radius *r* and center at the point M_0.



Let a function $z = f(x, y)$ be defined in a neighborhood of a point $M_0(x_0, y_0)$ and points $M_1(x_0 + \Delta x, y_0)$, $M_2(x_0, y_0 + \Delta y)$ belong to this neighborhood too.

Fig. 4.1

Definition. A difference $f(x_0 + \Delta x, y_0) - f(x_0, y_0)$ is called the **partial increment** of the function $z = f(x, y)$ with respect to x and is denoted by $\Delta_x z$.

Similarly, $f(x_0, y_0 + \Delta y) - f(x_0, y_0) = \Delta_y z$.

It should be noted, that a ratio $\dfrac{\Delta_x z}{\Delta x}$ can be considered as a function of one variable with respect to $\boldsymbol{\Delta x}$ and a ratio $\dfrac{\Delta_y z}{\Delta y}$ - with respect to Δy.

Definition. The limit of a ratio of partial increment of the function $z = f(x, y)$ with respect to x to the increment $\boldsymbol{\Delta x}$, as $\boldsymbol{\Delta x}$ tends to zero, is called a partial derivative of function $z = f(x, y)$ with respect to $\boldsymbol{x}$

$$\lim_{\Delta x \to 0} \frac{\Delta_x z}{\Delta x}.$$

Partial derivative with respect to x of the function $z = f(x, y)$ can be denoted by one of the ways: $\dfrac{\partial z}{\partial x}$, z'_x, f'_x.

By analogy, $\lim\limits_{\Delta y \to 0} \dfrac{\Delta_y z}{\Delta y} = \dfrac{\partial z}{\partial y} = z'_y = f'_y$.

The partial derivatives z'_x and z'_y are functions of two variables. But when we were finding $\boldsymbol{z'_x}$ we assumed that the variable y is constant. It means that for finding partial derivatives we can use formulas and theorems for function of the one variable.

Example 5.1. Find partial derivatives of the functions:

a) $z = x^3 + y^5 - 6$, b) $z = x^2 y^3$,

c) $z = \ln\left(x^2 y - y^3 x^5 + 7\right)$.

***Solution*:**

a) For finding z'_x we regard y as a constant and differentiate with respect to $\boldsymbol{x}$:

$$z'_x = \left(x^3 + y^5 - 6\right)'_x \Big|_{y=const} = 3x^2;$$

$$z'_y = \left(x^3 + y^5 - 6\right)'_y \Big|_{x=const} = 5x^4.$$

b) $z = x^2 y^3$.

$$z'_x = \left(x^2 y^3\right)'_x \Big|_{y=conwt} = y^3 \left(x^2\right)'_x = y^3 2x = 2xy^3;$$

$$z'_y = \left(x^2 y^3\right)'_y \Big|_{x=const} = x^2 \left(y^3\right)'_y = x^2 3y^2 = 3x^2 y^2.$$

c) $z = \ln\left(x^2 y - y^3 x^5 + 7\right)$.

$$z'_x = \left(\ln\left(x^2 y - y^3 x^5 + 7\right)\right)'_x \Big|_{y=const} = \frac{2xy - 5y^3 x^4}{x^2 y - y^3 x^5 + 7};$$

$$z'_y = \left(\ln\left(x^2 y - y^3 x^5 + 7\right)\right)'_y \Big|_{x=const} = \frac{x^2 + 3y^2 x^5}{x^2 y - y^3 x^5 + 7}.$$

6. Geometric Interpretation of Partial Derivatives

The plane $y = y_0$ cuts the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$. At each x, the slope of this curve is $f'_x(x, y_0)$.

Similarly, the plane $x = x_0$ cuts the surface in a curve whose slope is $f'_y(x_0, y)$.

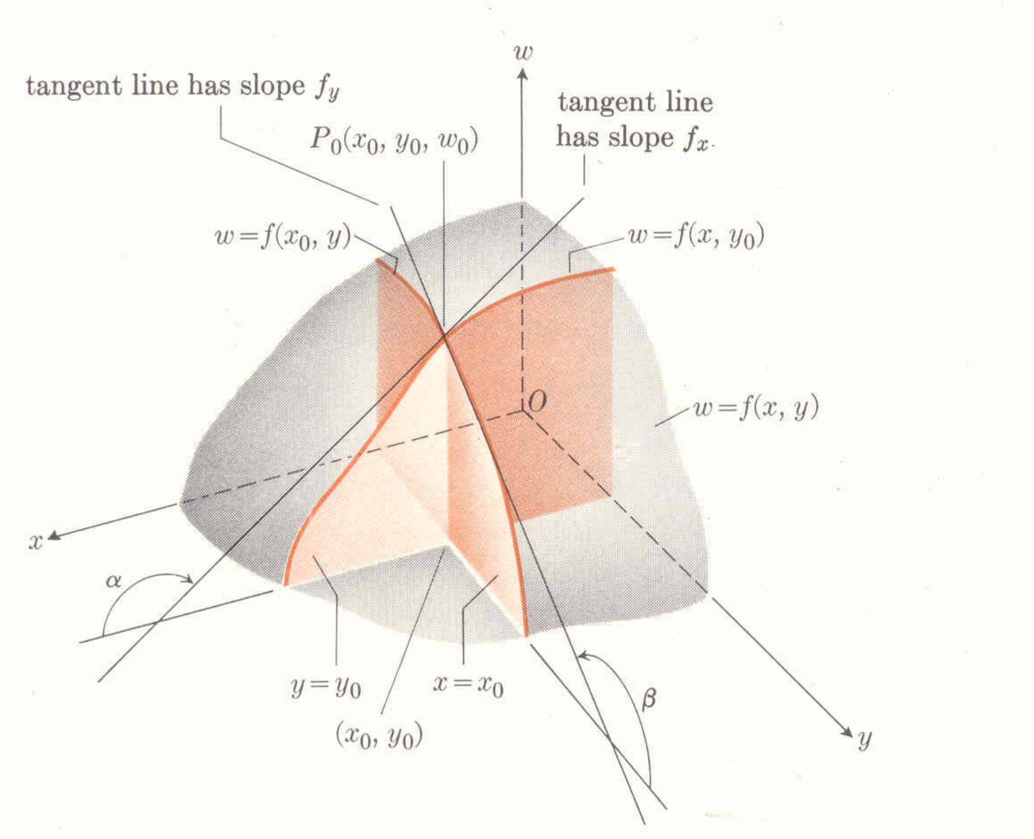


Fig. 6.1

The geometric interpretation of partial derivatives of the function $z = f(x, y)$ is the that:

$f'_x(x_0, y_0)$ is equal to the slope at the point $P_0(x_0, y_0, z_0)$ of the curve $z = f(x, y_0)$ in which the plane $y = y_0$ cuts the surface $z = f(x, y)$.

Thus, in *Fig.6.1*, if x, y and z are measured in the same units

$$\tan \alpha = \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)} = f'_x(x_0, y_0).$$

Similarly,

$$\tan \beta = \left.\frac{\partial z}{\partial y}\right|_{(x_0, y_0)} = f'_y(x_0, y_0).$$

7. Level Surfaces. Quadric Surfaces

Definition. Let a function of three variables $f(x, y, z)$ be given. For any number C the set of points (x, y, z) for which $f(x, y, z) = C$ is called a **level surface** of f, and we identify a level surface with the corresponding equation $f(x, y, z) = C$.

Level surfaces of functions of three are analogous to level curves of functions of two variables.

We observe that the graph of any function f of two variables is a level surface. We need only let

$g(x, y, z) = z - f(x, y)$

and notice that $g(x, y, z) = 0$ if and only if $z = f(x, y)$

Thus the level surface $g(x, y, z) = 0$ is the graph of f, or equivalently, the graph of the equation $z = f(x, y)$. This is why we call the graph of a function of two variables a **surface.**

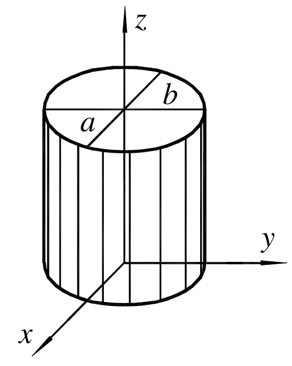
In sketching a level surface we will use the intersections of level surface with planes of the form $x = c$ or $y = c$, as well as those of the form $z = c$. In each case the intersection of the level surface with the plane is called the **trace** of the level surface.

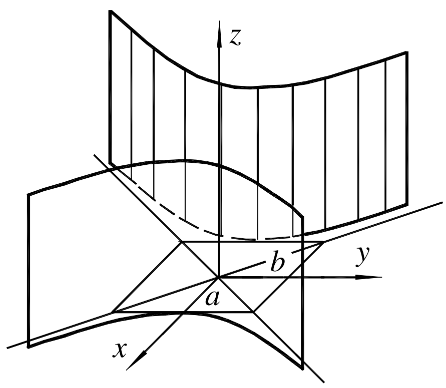
a) **Elliptic Cylinder**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

b) **Hyperbolic Cylinder**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

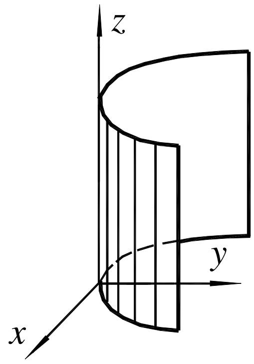




a) The trace of the elliptic cylinder in any plane parallel to the *xy* plane is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. If $a = b$, the surface is a **circular cylinder.**
b) The trace of the hyperbolic cylinder in any plane parallel to the *xy* plane is the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

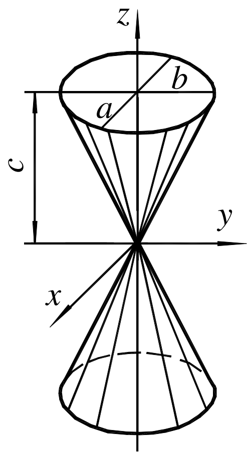
Parabolic Cylinder $x^2 = 2py$

The trace of the parabolic cylinder in any plane parallel to the *xy* plane is the parabola $x^2 = 2py$.



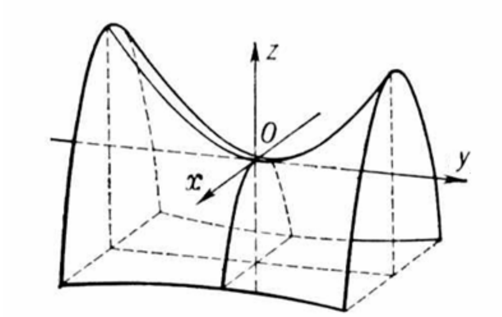
d) **Elliptic Double Cone** $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$

The trace of the cone in any plane parallel to the *xy* plane is either an ellipse (a circle if $a=b$) or a point. The traces in the *yz* and *xz* planes consist of two straight lines through the origin.
If $a=b$, the surface is called a **circular double cone.**



e) **Hyperbolic Paraboloid** $\frac{\acute{o}^2}{b^2}-\frac{\tilde{o}^2}{\grave{a}^2}=\frac{z}{c}$

The traces in the *yz* and *xz* planes are parabolas. The trace in the *xy* plane consists of two intersecting lines. The trace in any other plane parallel to the *xy* plane is a hyperbola. The surface has the appearance of a saddle.

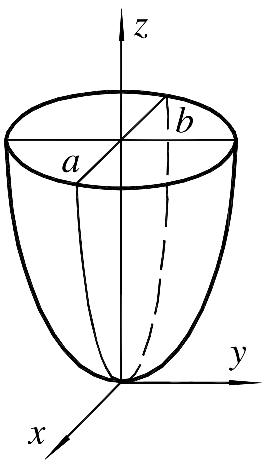


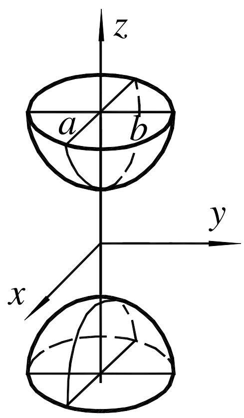
f) **Elliptic Paraboloid**

$$\frac{\tilde{o}^2}{\grave{a}^2}+\frac{\acute{o}^2}{b^2}=\frac{z}{c}$$

g) **Two-sheeted Hyperboloid**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$





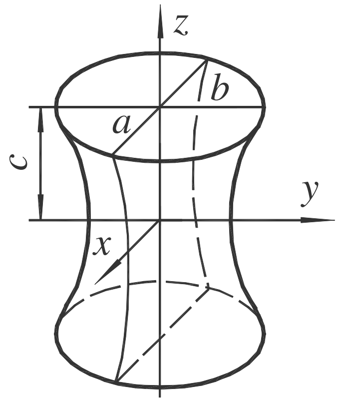
The trace of the paraboloid in any plane parallel to the *xy* plane is either an ellipse (a circle if $a = b$), or empty. The traces in the *yz* and *xz* planes are parabolas. If $a = b$, the surface is called a **circular paraboloid**.
g) *Problem 1*. Denote the traces of the two-sheeted hyperboloid.

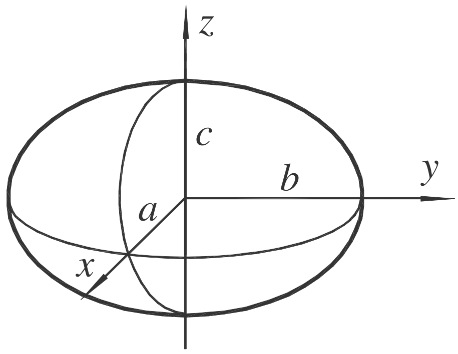
h) **One-sheeted Hyperboloid**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

i) **Ellipsoid**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$





Problem 7.1. Denote the traces of the one-sheeted hyperboloid and the ellipsoid.

8.Tangent Plane and Normal Line

Let $z = f(x,y)$ be a differentiable function at a point $(x_0, y_0,)$. Consider the sections of the surface *S* representing this function by the planes $x = x_0$ and $y = y_0$. Let M_0T_x and M_0T_y be the tangent lines at the point $M_0(x_0, y_0, z_0)$ (see Fig.8.1) to the plane curves obtained in the sections. The plane *T* passing through these lines which meet at the point M_0 is called the **tangent plane** to the surface *S* at the point M_0. The point M_0 is called the **point of tangency** (the point of contact) of the plane *T* and the surface *S*.

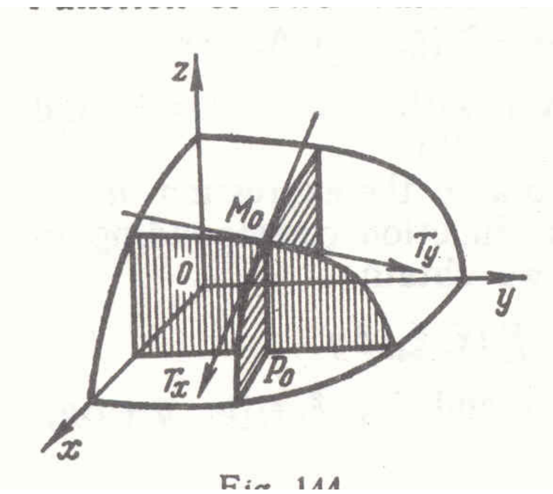


Fig.8.1

The straight line through M_0 normal to the tangent plane is called the **normal line** to the surface *S* at the point M_0.

Let us find equations of the tangent plane and the normal line.

The straight line M_0T_x is in the plane $y = y_0$ parallel to the plane *Oxz*. Its slope relative to the *x*-axes is equal to f'_x. Therefore the equations of the line M_0T_x are

$$z - z_0 = f'_x(x_0, y_0) \cdot (x - x_0), \quad y = y_0 \tag{8.1}$$

The equations of the straight line M_0T_y are found similarly:

$$z - z_0 = f'_y(x_0, y_0) \cdot (y - y_0), \quad x = x_0 \tag{8.2}$$

An equation of the plane *T* through $M_0(x_0, y_0, z_0)$ can be written as

$$z - z_0 = A(x - x_0) + B(y - y_0) \tag{8.3}$$

The straight lines M_0T_x and M_0T_y are in the plane *T*, so the coordinates of the points of these lines satisfy the equation of the plane. Substituting the expressions of $z - z_0$ and $y - y_0$ from (8.1) into the equation (8.3) we obtain

$$f'_x(x_0, y_0) \cdot (x - x_0) = A(x - x_0) \Rightarrow A = f'_x(x_0, y_0)$$

Furthermore, we similarly find that

$$B = f'_y(x_0, y_0)$$

Thus, the **equation of the tangent plane** takes the form

$$z - z_0 = f'_x(x_0, y_0) \cdot (x - x_0) + f'_y(x_0, y_0) \cdot (y - y_0) \tag{8.4}$$

It is clear that **equations of the normal line** are

$$z - z_0 = \frac{x - x_0}{f'_x(x_0, y_0)} = \frac{y - y_0}{f'_y(x_0, y_0)} \tag{8.5}$$

9. Total increment and Total Differential of a Function of Two Variables

Let a function $z = f(x, y)$ be continuous on some set $\boldsymbol{D}$ and have z'_x, z'_y in this set. Let us take arbitrary point $M_0(x_0, y_0) \in D$, then

$$\Delta z(M_0) = z'_x(x_0, y_0)\Delta x + z'_y(x_0 y_0)\Delta y + \alpha x + \beta y, \tag{9.1}$$

will be the total increment at this point, where α, β are infinitesimal values as $\Delta x \to 0$ and $\Delta y \to 0$.

***Definition*.** A function $f(x, y)$ is differentiable at (x_0, y_0) if $f'_x(x_0, y_0), f'_y(x_0, y_0)$ exist and the equation (9.1) holds for $f(x, y)$ at (x_0, y_0).

We call $f(x, y)$ differentiable if it is differentiable in its domain.

*Definition***.** The principal linear part with respect to Δx and Δy, of a total increment Δz is called **total differential** of the function z

$$dz(M_0) = z'_x(x_0, y_0)\Delta x + z'_y(x_0 y_0)\Delta y. \qquad (9.2)$$

*Example***.** The factory manufactures right circular cylindrical storage tanks that are 25m high with radius of 5m. How is sensitive tank's volume to small variations in height and radius?

Solution. A tank's volume is

$$V = \pi \cdot r^2 \cdot h.$$

So V is a function of two variables r and h. Then the change in volume caused by small changes dr and dh in radius and height is approximately

$$dV = V'_r(5;25)dr + V'_h(5;25)dh = (2\pi\, r\, h)\Big|_{(5;25)} dr + \left(\pi\, r^2\right)\Big|_{(5;25)} dh =$$

$$= 250\pi\, dr + 25\pi\, dh$$

The volume of cylinder is more sensitive to the small change in r than it is to an equally small change in h.

In contrast, if the values of r and h are reserved to make $r = 25$ and $h = 5$, then full differential in $\boldsymbol{V}$ becomes

$$dV = V'_r(25{,}5)dr + V'_h(25{,}5)dh = (2\pi rh)\Big|_{(25,5)} dr + \left(\pi r^2\right)\Big|_{(25,5)} dh = 250\pi dr + 625\pi dh$$

Now the volume is more sensitive to changes in h than to changes in r.

From this example we can make the conclusion**:**

Functions are most sensitive to small changes in the variables that generate the largest partial derivatives.

10. Applying a Total Differential to Approximate Calculations

A total differential and a total increment of a function $z = f(x, y)$ at a point $M_0(x_0, y_0)$ are:

$$dz(M_0) = z'_x(x_0, y_0)\Delta x + z'_y(x_0 y_0)\Delta y$$

$$\Delta z(M_0) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

Because $dz(M_0) = \Delta z(M_0)$ when $\Delta x \to 0$ and $\Delta y \to 0$, than

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0, y_0) + z'_x(x_0, y_0)\Delta x + z'_y(x_0 y_0)\Delta y. \quad (10.1)$$

The formula (10.1) allows us to find approximate value of the function of two variables.

Example. Find approximate value of the function $z = x^2 + 2xy + y$ at the point $M(1.03;1.97)$.

Solution.

Let $M_0(1,2)$, than $\Delta x = 0{,}03$ and $\Delta y = -0{,}03$. Find partial derivatives and calculate their values at the point $M_0(1,2)$.

$$\frac{\partial z}{\partial x} = 2x + 2y \Rightarrow \frac{\partial z(M_0)}{\partial x} = 2 \cdot 1 + 2 \cdot 2 = 6,$$

$$\frac{\partial z}{\partial y} = 2x + 3y^2 \Rightarrow \frac{\partial z(M_0)}{\partial y} = 2 \cdot 1 + 3 \cdot 2^2 = 14,$$

Using the formula (10.1), we obtain

$$z(M) \approx 13 + 6 \cdot 0{,}003 + 14 \cdot (-0{,}03) = 13 + 0{,}18 - 0{,}42 = 12{,}76.$$

11. Partial Derivatives of the Higher Order

Partial derivatives of the first order z'_x and z'_y of a function $z = z(x, y)$, in general, are functions of two variables. From them we can find derivatives of the second order with respect to x and y

$$z''_{xx},\ z''_{xy},\ z''_{yy},\ z''_{yx}.$$

We can differentiate a function as long as derivatives involved exist.

The derivatives z''_{xy} and z''_{yx} are called the mixed partial derivatives of the second order and when they are continuous they are equal $z''_{xy} = z''_{yx}$.

Example. Show, that $z''_{xy} = z''_{yx}$ for given function

$$z = x^4 y^3 - 2x^2 y^5 + 8.$$

Solution.

$$z'_x = 4x^3y^3 - 4xy^5 \Rightarrow \begin{cases} z''_{xx} = 12x^2y^3 - 4y^5 \\ z''_{xy} = 12x^3y^2 - 20xy^4 \end{cases},$$

$$z'_y = 3x^4y^2 - 10x^2y^4 \Rightarrow \begin{cases} z''_{yy} = 6x^4y - 40x^2y^3 \\ z''_{yx} = 12x^3y^2 - 20xy^4 \end{cases}.$$

It is clear that $z''_{xy} = z''_{yx}$.

Note that we can continue calculation of partial derivative of the third order, et cetera:

$$z = z(x, y) \Rightarrow \begin{cases} z'_x \Rightarrow \begin{cases} z''_{xx} \Rightarrow \begin{cases} z'''_{xxx} \\ z'''_{xxy} \end{cases} \\ z''_{xy} \Rightarrow \begin{cases} z'''_{xyx} \\ z'''_{xyy} \end{cases} \end{cases} \\ z'_y \Rightarrow \begin{cases} z''_{yx} \Rightarrow \begin{cases} z'''_{yxx} \\ z'''_{yxy} \end{cases} \\ z''_{yy} \Rightarrow \begin{cases} z'''_{yyx} \\ z'''_{yyy} \end{cases} \end{cases} \end{cases}$$

In similar way we can find partial derivatives of the higher order of functions of three, four, …, n variables.

12. Differentiating Composite Functions

Let us consider a function $z = z(x, y)$, where $x = x(t)$ and $y = y(t)$ are functions of one independent variable. Then formula for finding the derivative of composite function $z = z(x, y)$ will be:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \qquad (12.1)$$

Example. Find the derivative of the function

$z = \sqrt{x}\ln y$, where $x = \sin^2 t$ and $y = 5^t$.

Solution: Using the formula (12.1)

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}$$

we have

$$\frac{dz}{dt}=\left(\sqrt{x}\ln y\right)'_x\left(\sin^2 t\right)'_t+\left(\sqrt{x}\ln y\right)'_y\left(5^t\right)'_t=$$
$$=\frac{1}{2\sqrt{x}}\ln y 2\sin t\cos t+\frac{\sqrt{x}}{y}5^t\ln 5.$$

13. Directional Derivative and Gradient of a Function of Several Variables

An important characteristic of a function $u=f(x,y,z)$ is the rate of its change in the given direction of a vector $\overline{a}=(a_x;a_y;a_z)$.

It is possible to prove that a **directional derivative** of the function $u=f(x,y,z)$ at a point $M_0(x_0,y_0,z_0)$ can be calculated by the formula

$$\frac{\partial u(M_0)}{\partial \overline{a}}=u'_x(M_0)\cos\alpha+u'_y(M_0)\cos\beta+u'_z(M_0)\cos\gamma \qquad (13.1)$$

Here

$$\begin{cases}\cos\alpha=\dfrac{a_x}{\sqrt{a_x^{\,2}+a_y^{\,2}+a_z^{\,2}}},\\ \cos\beta=\dfrac{a_y}{\sqrt{a_x^{\,2}+a_y^{\,2}+a_z^{\,2}}},\\ \cos\gamma=\dfrac{a_z}{\sqrt{a_x^{\,2}+a_y^{\,2}+a_z^{\,2}}}.\end{cases} \qquad (13.2)$$

Example 13.1. Find the directional derivative of the function $z=5xy-x^2y^2$ at a point $M_0(3;2)$ in the direction of the vector $\overline{M_0M_1}$, when the point M_1 has coordinates $(6;6)$.

*Solution***.** Find the coordinates of the vector $\overline{M_0 M_1}$:

$$\bar{a} = \overline{M_0 M_1} = \left(x_{M_1} - x_{M_0}; y_{M_1} - y_{M_0}\right) = (6-3; 6-2) = (3; 4)$$

Find partial derivatives of the given function:

$$\frac{\partial z}{\partial x} = \left(5xy - x^2 y^2\right)'_x = 5y - 2xy^2;$$

$$\frac{\partial z}{\partial y} = \left(5xy - x^2 y^2\right)'_y = 5x - 2x^2 y.$$

Using the formulas (13.1) and (13.2) we have

$$\frac{\partial z(M_0)}{\partial \bar{a}} = \left(5y - 2xy^2\right)\Big|_{\substack{x=3\\y=2}} \cdot \frac{3}{\sqrt{9+16}} + \left(5x - 2x^2 y\right)\Big|_{\substack{x=3\\y=2}} \cdot \frac{4}{\sqrt{9+16}} =$$
$$= -14 \cdot 0.6 + (-21) \cdot 0{,}8 = -25.2.$$

The vector which coordinates are the values of partial derivatives of the first order at a point $M_0(x_0, y_0, z_0)$ is called **gradient** vector of a function $u = f(x, y, z)$ at a point $M_0(x_0, y_0, z_0)$

$$\overline{grad}\, z(M_0) = \left(u'_x(M_0); u'_y(M_0); u'_z(M_0)\right) \tag{13.3}$$

Gradient vector points in the direction of the greatest rate of the change of the function and whose magnitude is the greatest rate of change

$$\left|\overline{grad}\, z(M_0)\right| = \sqrt{\left(\frac{\partial u(M_0)}{\partial x}\right)^2 + \left(\frac{\partial u(M_0)}{\partial y}\right)^2 + \left(\frac{\partial u(M_0)}{\partial z}\right)^2} \tag{13.4}$$

Example 13.2. Find the direction of the greatest rate of the change of the function $u = x^2 y^3$ and magnitude of the greatest rate of the change at the point $M_0(1;2)$.

*Solution***.** In this case the formula (13.4) has the form

$$\left|\overline{grad}\, z(M_0)\right| = \sqrt{\left(\frac{\partial u(M_0)}{\partial x}\right)^2 + \left(\frac{\partial u(M_0)}{\partial y}\right)^2}.$$

As

$$\frac{\partial u(M_0)}{\partial x} = \left(x^2 y^3\right)'_x\Big|_{\substack{x=1\\y=2}} = \left(2xy^3\right)\Big|_{\substack{x=1\\y=2}} = 16,$$

$$\frac{\partial u(M_0)}{\partial y}=\left(x^2y^3\right)_y'\Big|_{\substack{x=1\\y=2}}=\left(3x^2y^2\right)\Big|_{\substack{x=1\\y=2}}=12,$$

Then $\overline{grad}\,z(M_0)=(16;12)$

and $\left|\overline{grad}\,z(M_0)\right|=\sqrt{16^2+12^2}=\sqrt{400}=20$.

14. Extreme of a Functions of Two Variables

Definition 14.1. A function $z=f(x,y)$ has **minimum (maximum)** at a point $M_0(x_0,y_0)$ if $f(x_0,y_0)<f(x,y)$ $(f(x_0,y_0)>f(x,y))$ for all points which belong to sufficiently small neighborhood of a point $M_0(x_0,y_0)$.

Definition 14.2. The points at which a function $z=f(x,y)$ has maximum or minimum are called the **points of extreme** of the function.

Definition 14.3. The points at which partial derivatives of the first order do not exist or equal to zero are called the critical points or points suspicious on extreme.

Definition 14.4. The points at which all partial derivatives of the first order exist and are equal to zero are called the **critical** (**stationary**) **points**.

Example 14.1. Find critical points of the function

$$z=x^2-xy+y^2+3x-2y+1.$$

*Solution***:** Find partial derivatives

$$z_x'=2x-y+3;$$
$$z_y'=-x+2y-2.$$

These derivatives exist for all $\boldsymbol{x}$ and y. It means that function have stationary points if $z_x'=0$ and $z_y'=0$.

Make up the system of the equations

$$\begin{cases}z_x'=0\\ z_y'=0\end{cases}\Rightarrow\begin{cases}2x-y+3=0\\ -x+2y-2=0\end{cases}\Rightarrow\begin{cases}2x-y=-3\\ -x+2y=2\end{cases}\Rightarrow\begin{cases}x=-\dfrac{4}{3}\\ y=\dfrac{1}{3}\end{cases}.$$

Hence the point $M_0\left(-\frac{4}{3};\frac{1}{3}\right)$ is the stationary point of the given function.

Theorem 14.1. (**the necessary conditions for extreme**).

If a differentiable function $z = f(x, y)$ attains an extremum at a point $M_0(x_0, y_0)$ its partial derivatives turn into zero at that point:

$$z = f(x, y) \text{ has extremum at } M_0(x_0, y_0) \Rightarrow \begin{cases} \left(\dfrac{\partial z}{\partial x}\right)\Bigg|_{\substack{x=x_0 \\ y=y_0}} = 0, \\ \left(\dfrac{\partial z}{\partial y}\right)\Bigg|_{\substack{x=x_0 \\ y=y_0}} = 0. \end{cases}$$

In other words a point $M_0(x_0, y_0)$ is a critical point of the given function.

Proof.

Let the function $z = f(x, y)$ have an extremum at the point $M_0(x_0, y_0)$. Let y be fixed: $y = y_0$. It means that the function $z = f(x, y_0)$ is the function of one variable.It follows from the condition of the theorem, that this function has extremum at the point $x = x_0$. Hence, its derivative with respect to x should be equal to zero at the point $x = x_0$. That is $\left(\dfrac{\partial z}{\partial x}\right)\Bigg|_{\substack{x=x_0 \\ y=y_0}} = 0$. Similarly we get $\left(\dfrac{\partial z}{\partial y}\right)\Bigg|_{\substack{x=x_0 \\ y=y_0}} = 0$. The theorem is proved.

As we know the equation of the tangent plane to the surface $z = f(x, y)$ is

$$z - z_0 = f'_x(x_0, y_0) \cdot (x - x_0) + f'_y(x_0, y_0) \cdot (y - y_0).$$

It is clear that it turns to $z = z_0$ for a stationary point $M_0(x_0, y_0)$.

Thus, **geometric meaning** of this theorem is:

If a differentiable function $z = f(x, y)$ attains an exstremum at a point $M_0(x_0, y_0)$, then the tangent plane to the surface at the corresponding point should be parallel to the coordinate plane of the independent variables.

• • Let the function $z = f(x, y)$ be continuous together with its partial derivatives of the first and the second orders and let a point $M_0(x_0, y_0)$ be a stationary point of this function. Denote

$$\frac{\partial^2 z(M_0)}{\partial x^2} = A; \quad \frac{\partial^2 z(M_0)}{\partial x \partial y} = B; \quad \frac{\partial^2 z(M_0)}{\partial y^2} = C, \text{ and form}$$

$$\Delta\left(M_0\right)=\begin{vmatrix}\dfrac{\partial^2 z(M_0)}{\partial x^2} & \dfrac{\partial^2 z(M_0)}{\partial x\partial y}\\ \dfrac{\partial^2 z(M_0)}{\partial x\partial y} & \dfrac{\partial^2 z(M_0)}{\partial y^2}\end{vmatrix}=\begin{vmatrix}A & B\\ B & C\end{vmatrix}=AC-B^2.$$

Theorem 14.2. (**the sufficient conditions for extreme**).

If $\Delta\left(M_0\right)>0$ the function $z=f(x,y)$ has extreme at a point $M_0(x_0,y_0)$ which is maximum if $A<0$ and minimum if $A>0$.

If $AC-B^2<0$ there is no extreme of the given function at the point $M_0(x_0,y_0)$. Note, that the point $M_0(x_0,y_0)$ is called a **saddle point**.

If $AC-B^2=0$ the properties of the second derivatives do not provide any answer to the question of existence of an extreme and further investigation will be needed.

Example. Find the extrema of the function $z=x^2-2xy+\frac{1}{3}y^3-3y.$

Solution.

The given function is continuous and differentiable everywhere in the *xy*-plane.

1). Using the conditions $\begin{cases}\dfrac{\partial z}{\partial x}=0,\\ \dfrac{\partial z}{\partial y}=0.\end{cases}$ find the critical points of the given function.

$$\begin{cases}\dfrac{\partial z}{\partial x}=\left(x^2-2xy+\dfrac{1}{3}y^3-3y\right)_x'=2x-2y=0\\ \dfrac{\partial z}{\partial y}=\left(x^2-2xy+\dfrac{1}{3}y^3-3y\right)_y'=-2x+y^2-3=0\end{cases}\Rightarrow\begin{cases}x-y=0\\ -2x+y^2-3=0\end{cases}$$

The solution of this system gives two stationary points $M_1(3,3)$ and $M_2(-1,-1)$.

2). To use the sufficient conditions for extreme evaluate the second derivatives of the given function.

$$\frac{\partial^2 z}{\partial x^2}=\left(\left(x^2-2xy+\frac{1}{3}y^3-3y\right)_x'\right)_x'=(2x-2y)_x'=2,$$

$$\frac{\partial^2 z}{\partial y^2}=\left(-2x+y^2-3\right)_y'=2y,$$

$$\frac{\partial^2 z}{\partial x\partial y}=(2x-2y)_y'=-2.$$

a) Consider the point $M_1(3,3)$.

Hence $A = \dfrac{\partial^2 z(M_1)}{\partial x^2} = 2;\quad B = \dfrac{\partial^2 z(M_1)}{\partial x \partial y} = -2;\quad C = \dfrac{\partial^2 z(M_1)}{\partial y^2} = 2y\Big|_{\substack{x=3\\y=3}} = 6$, then

$\Delta\left(M_1\right) = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ -2 & 6 \end{vmatrix} = 12 - 4 = 8 > 0$, $A = 2 > 0$ and the function has minimum at the point $M_1(3,3)$.

b) Consider the point $M_2(-1,-1)$.

Hence $A = \dfrac{\partial^2 z(M_2)}{\partial x^2} = 2;\quad B = \dfrac{\partial^2 z(M_2)}{\partial x \partial y} = -2;\quad C = \dfrac{\partial^2 z(M2)}{\partial y^2} = 2y\Big|_{\substack{x=-1\\y=-1}} = -2$, then

$\Delta\left(M_2\right) = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ -2 & -2 \end{vmatrix} = -4 - 4 = -8 < 0$, so the function has not extremum at the point $M_2(-1,-1)$.

15. The Conditional Extreme of Functions of Two Variables

*Definition***.** An extreme of function $z = f(x, y)$ under condition $\varphi(x, y) = 0$ is called a **conditional extreme** of the function.

There are two methods of finding conditional extreme.

I. To find a conditional extreme by **Lagrange method.** To do it we need:

1). Write the Lagrange function

$$L(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y),$$

where λ is an arbitrary constant parameter.

2). Find critical points $M_k(x_k, y_k, \lambda_k)$ of Lagrange's function, using necessary condition of existence of the extreme

$$\begin{cases} \dfrac{\partial L}{\partial x} = 0 \\ \dfrac{\partial L}{\partial y} = 0 \\ \dfrac{\partial L}{\partial \lambda} = 0 \end{cases} \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial \varphi}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial \varphi}{\partial y} = 0 \\ \varphi(x, y) = 0 \end{cases}$$

3). Check sufficient conditions at every critical points of the function for existence of an extreme :

a) if at a point $M_k(x_k, y_k, \lambda_k)$ the determinant of the third order

$$\Delta(M_k)=\begin{vmatrix} 0 & \varphi'_x(M_k) & \varphi'_y(M_k) \\ \varphi'_x(M_k) & L''_{xx}(M_k) & L''_{xy}(M_k) \\ \varphi'_y(M_k) & L''_{yx}(M_k) & L''_{yy}(M_k) \end{vmatrix}$$

is positive, then a point M_k is a point of maximum and

$$z_{\max} = f(M_k) = f(x_k, y_k):$$

b) if determinant $\Delta(M_k)$ is negative, a point M_k is a point of minimum

$$z_{\min} = f(M_k) = f(x_k, y_k).$$

II. To find a conditional extreme by **the substitution method.** To do it we need:

a) solve equation $\varphi(x, y) = 0$ for x;

b) substitute this variable into the equation $z = f(x, y)$;

c) examine the function of one variable for extreme.

Example. Let us determine the greatest value of the function $z = x^2 y$ on condition that x and y are positive and satisfy the equation $4x + 5y = 60$.

Solution.

I. Let us use the Lagrange method.

1) write the Lagrange function

$$L(x, y, \lambda) = x^2 y + \lambda(4x + 5y - 60);$$

2) find critical points

$$\begin{cases} \dfrac{2x(60-4x)}{5} + 4\lambda = 0 & \Big|\cdot 5 \\ x^2 + 5\lambda = 0 & \Big|\cdot 4 \end{cases} \Rightarrow \begin{cases} 2x(60-4x) + 20\lambda = 0 \\ 4x^2 + 20\lambda = 0 \end{cases},$$

$$120x - 8x^2 - 4x^2 = 0$$

$$120x - 12x^2 = 0$$

$$10x - x^2 = 0$$

$$x(10 - x) = 0 \Rightarrow x = 0, \quad x = 10$$

$x=0$ does not make sense,

$$y=\frac{60-40}{5}=4 \qquad \lambda=\frac{-2xy}{4}=-\frac{xy}{2}=-\frac{10\cdot 4}{2}=-20$$

So, critical point is $M_0(10;4;-20)$.

3) find $\Delta(M_0)$

$$\frac{\partial^2 L}{\partial x^2}=2y,\quad \frac{\partial^2 L}{\partial y^2}=0,\quad \frac{\partial^2 L}{\partial x\partial y}=2x$$

$$\frac{\partial^2 L(M_o)}{\partial x^2}=8,\quad \frac{\partial^2 L(M_0)}{\partial y^2}=0,\quad \frac{\partial^2 L(M_0)}{\partial x\partial y}=20$$

$$\frac{\partial\varphi}{\partial x}=4,\quad \frac{\partial\varphi}{\partial y}=5$$

$$\Delta(M_0)=\begin{vmatrix}0 & 4 & 5\\ 4 & 8 & 20\\ 5 & 20 & 0\end{vmatrix}=400+400-200=600>0$$

4) $x=10$ and $y=4$ maximized the utility function which have maximum value

$$z_{\max}=10^2\cdot 4=400.$$

II. Let us use the substitution method.

1) Solve equation $4x+5y=60$ for $\boldsymbol{x}$

$$y=\frac{(60-4x)}{5},$$

2) Substitute $y=\dfrac{(60-4x)}{5}$ in $z=x^2y$

$$z=\frac{x^2(60-4x)}{5},$$

3) examine $z=\dfrac{x^2(60-4x)}{5}$ for extreme

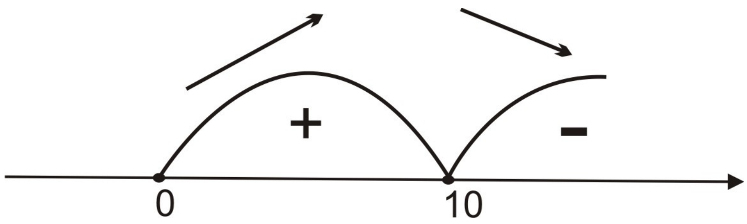
Domain of the function z: $D(z)=R$

$$z'=\frac{2x(60-4x)+x^2(-4)}{5}=\frac{120x-8x^2-4x^2}{5}=\frac{120x-12x^2}{5}$$

$$z'=0\Rightarrow 12x(10-x)=0\Rightarrow\left[\begin{array}{l}x_1=0,\\ x_2=10.\end{array}\right.$$

$$\text{As } y=\frac{(60-4x)}{5}\Rightarrow\left[\begin{array}{l}y_1=12,\\ x_2=4,\end{array}\right.$$

then we have two critical points $M_1(0;12)$ and $M_2(10;4)$.



$$z''=\frac{120-24}{5}\Big|_{x=10}=-24<0\text{,then}$$

$$z_{\max}=10^2\cdot 4=400.$$

16. The Greatest and the Least Values of a Function of Two Variables

To find the greatest $f_{\max D}(x,y)$ and the least $f_{\min D}(x,y)$ values of the function $z=f(x,y)$ in the closed region D we need to find the extreme values of the function at the points that lie inside of D, and on the boundary of the region. From these values choose the greatest and the least values. This numbers will be the greatest and the least values of the function $z=f(x,y)$ in the closed region D.

Example. Find the greatest and the least values of the function $z = x^2 y(4 - x - y)$ in the triangle bounded by the lines

$x = 0, \qquad y = 0, \qquad x + y = 6$.

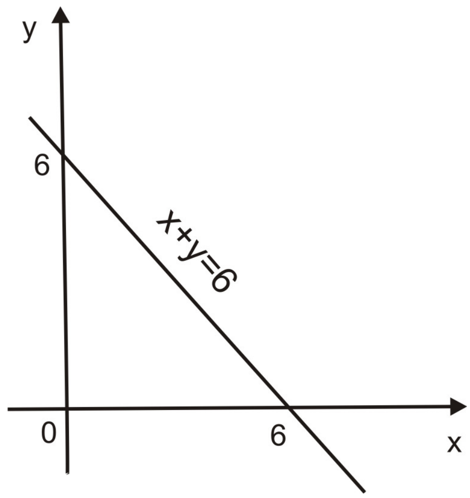


Fig. 16.1.

*Solution***.**

Find the critical points inside the region

$$\begin{cases} z'_x = 2xy(4 - x - y) - x^2 y = xy(8 - 3x - 2y) \\ z'_y = x^2(4 - x - y) - x^2 y = x^2(4 - x - 2y) \end{cases}.$$

According to the necessary condition for the existence of the extreme of the function of two variables have a system of equations

$$\begin{cases} xy(8 - 3x - 2y) = 0 \\ x^2(4 - x - 2y) = 0 \end{cases}$$

Inside of the region $x \neq 0$ and $y \neq 0$, then

$$\begin{cases} 3x + 2y = 8 \\ x + 2y = 4 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 1 \end{cases}.$$

At the critical point $M_1(2,1)$ we have $z(2,1) = 4$.

Now we examine the function on the boundary of the triangle. On the straight line $x + y = 6$ variable $y = 6 - x$ and function z takes the form

$$z = x^2(6 - x)\cdot(4 - x + x - 6) = 2x^2(x - 6), \qquad x \in [0,6].$$

Let us find the least and the greatest volumes of this function of one variable on the closed interval $[0,6]$.

$z' = 6x^2 - 24x; \qquad z' = 0, \qquad 6x(x-4) = 0 \Rightarrow x_1 = 0, \qquad x_2 = 4$

Find the values of the function at the points $x = 0, \quad x = 4, \quad x = 6$.

$$z(4) = -64, \quad z(0) = 0, \quad z(6) = 0.$$

On the straight line $y = 0$ we have $z = 0$.

On the straight line $x = 0$ we have $z = 0$.

At the point $(6,0)$ $z = 0$.

At the point $(0,6)$ z=o.

So, the given function z have the greatest value at the point $M_1(2,1)$ inside the region and the least value at the point $M_2(4,2)$ on the boundary of the region.

$$z_{\max D} = 4$$
$$z_{\min D} = -64^{\cdot}$$

17. Solution of Problems

Problem 1. *Let the function* $z = \dfrac{y^2}{52\,x}$ *be given. Find*:

a) *the total differential of the first order*;

b) *all partial derivatives of the second order.*

***Silution*.**

• **a)** The total differential of the first order can be calculated using the formula

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Calculate the partial derivatives of the first order:

$$\frac{\partial z}{\partial x} = \left(\frac{y^2}{52x}\right)'_x\Bigg|_{y=const} = \frac{y^2}{52}\left(\frac{1}{x}\right)' = -\frac{y^2}{52x^2};$$

$$\frac{\partial z}{\partial y} = \left(\frac{y^2}{52x}\right)'_y\Bigg|_{x=const} = \frac{1}{52x}\left(y^2\right)' = \frac{y}{26x}.$$

So,

$$\underline{dz = -\frac{y^2}{52\,{}^2x}dx + \frac{y}{26x}dy}.$$

• **b)** Calculate the partial derivatives of the second order:

$$\underline{\frac{\partial^2 z}{\partial x^2}} = \left(-\frac{y^2}{52x^2}\right)'_x = -\frac{y^2}{52}\left(x^{-2}\right)' = -\frac{y^2}{52}\left(-2x^{-3}\right)' = \underline{\frac{y^2}{26x^2}};$$

$$\underline{\frac{\partial^2 z}{\partial x\partial y}}=\left(-\frac{y^2}{52x^2}\right)'_y=-\frac{1}{52x^2}\left(y^2\right)'=-\frac{2y}{52x^2}=\underline{-\frac{y}{26x^2}};$$

$$\underline{\frac{\partial^2 z}{\partial y^2}}=\left(\frac{y}{26x}\right)'_y=\frac{1}{26x}y'=\underline{\frac{1}{26x}};$$

$$\underline{\frac{\partial^2 z}{\partial y\partial x}}=\left(\frac{y}{26x}\right)'_x=\frac{y}{26}\left(\frac{1}{x}\right)'=\underline{-\frac{y}{26x^2}}.$$

Note that: $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$.

Problem 2. *Let the function* $u=e^{52xz+2y}$*, the point* $A(-1;1;-1)$*, and the vector* $\bar{a}=\left(2;2\sqrt{52};52\right)$ *be given. Find:*

a) *a gradient of the given function at the given point: (* $\overline{grad\ f(A)}$ *);*

b) *a derivative of the function into direction of the vector* $\bar{a}$ *at the point A:* $\left(\frac{\partial f(A)}{\partial \bar{a}}\right)$.

Solution.

• **a)** Gradient of the given function at a given point we'll find using the formula:

$$\overline{grad\ u(A)}=\left(\frac{\partial u(A)}{\partial x};\frac{\partial u(A)}{\partial y};\frac{\partial u(A)}{\partial x}\right).$$

1) $\frac{\partial u}{\partial x}=\left(e^{52xz+2y}\right)'_x=e^{52xz+2y}\cdot 52z\Rightarrow$

$$\Rightarrow\frac{\partial u(A)}{\partial x}=\frac{\partial u(1;-1;1)}{\partial x}=e^{52xz+2y}\cdot 52z\Big|_{\substack{x=-1\\y=1\\z=-1}}=-52e^{54};$$

2) $\frac{\partial u}{\partial y}=\left(e^{52xz+y^2}\right)'_y=e^{52xz+2y}\cdot 2y\Rightarrow$

$$\Rightarrow\frac{\partial u(A)}{\partial y}=\frac{\partial u(1;-1;1)}{\partial y}=2e^{52xz+2y}\Big|_{\substack{x=-1\\y=1\\z=-1}}=2e^{54};$$

3) $\frac{\partial u}{\partial z}=\left(e^{52xz+y^2}\right)'_z=e^{52xz+2y}\cdot 52z\Rightarrow$

$$\Rightarrow\frac{\partial u(A)}{\partial z}=\frac{\partial u(1;-1;1)}{\partial z}=e^{52xz+2y}\cdot 52z\Big|_{\substack{x=-1\\y=1\\z=-1}}=-52e^{54}.$$

So

$$\overline{grad\ u(A)} = \left(-52e^{54}; 2e^{54}; -52e^{54}\right) \Rightarrow \overline{grad\ u(A)} = 2e^{54}(-26; 1; -26).$$

• **b)** Using the formula for finding a derivative of the function into direction of the vector $\overline{a}$ at the point A we have:

$$\underline{\frac{\partial u(A)}{\partial \overline{a}}} = \frac{1}{\sqrt{a_x^2 + a_y^2 + a_z^2}} \left(\frac{\partial u(A)}{\partial x} a_x + \frac{\partial u(A)}{\partial y} a_y + \frac{\partial u(A)}{\partial z} a_z \right) =$$

$$\left[\begin{array}{l} \begin{cases} a_x = 2 \\ a_y = 2\sqrt{52} \Rightarrow \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{2^2 + 4 \cdot 52 + 52^2} = \sqrt{(2+52)^2} = 54 \\ a_z = 52 \end{cases} \\ \dfrac{\partial u(A)}{\partial x} = -52e^{54}, \dfrac{\partial u(A)}{\partial y} = 2e^{54}, \dfrac{\partial u(A)}{\partial z} = -52e^{54} \end{array} \right]$$

$$= \frac{1}{54}\left(-52e^{54} \cdot 2 + 2e^{54} \cdot 2\sqrt{52} - 52e^{54} \cdot 52\right) = \frac{2e^{54}}{27}\left(-26 + \sqrt{52} - 52 \cdot 13\right) =$$

$$\underline{= \frac{2e^{54}}{27}\left(\sqrt{52} - 702\right).}$$

The answer: $\overline{grad\ u(A)} = 2e^{54}(-26; 1; -26)$; $\dfrac{\partial u(A)}{\partial \overline{a}} = \dfrac{2e^{54}}{27}\left(\sqrt{52} - 702\right).$

Problem 3. Investigate the function $z = x^3 + y^3 - 3xy$ for extremum.

Solution.

1. Find the critical points using the conditions $\begin{cases} \dfrac{\partial z}{\partial x} = 0, \\ \dfrac{\partial z}{\partial y} = 0. \end{cases}$.

$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\partial\left(x^3 + y^3 - 3xy\right)}{\partial x} = 3x^2 - 3y \\ \dfrac{\partial z}{\partial y} = \dfrac{\partial\left(x^3 + y^3 - 3xy\right)}{\partial y} = 3y^2 - 3x \end{cases}$$. So we have the system:

$$\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \Rightarrow \begin{cases} x^2 - y = 0 & (\ ) \\ y^2 - x = 0 & (\mathbf{2}) \end{cases}$$

1). **(1)** $\Rightarrow$ $y = x^2$ **(3)**

2). **(3)** $\to$ **(2)** $\Rightarrow x^4 - x = 0 \Rightarrow x\left(x^3 - 1\right) = 0 \Rightarrow \left[\begin{array}{l} x_1 = 0, \\ x_2 = 1, \end{array}\right.$ **(4)**

3). **(4)** $\to$ **(3)** $\Rightarrow \left[\begin{array}{l} y_1 = 0 \\ y_2 = 1 \end{array}\right.$.

Thus there are two critical points: $P_1(0;0)$ and $P_2(1;1)$.

2. To answer the question if the point $P_0(x_0, y_0)$ is the point of extremum or not we must consider $\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2$ where

$$A = \frac{\partial^2 z(P_0)}{\partial x^2}, \qquad B = \frac{\partial^2 z(P_0)}{\partial x \partial y}, \qquad \text{and } C = \frac{\partial^2 z(P_0)}{\partial y^2}:$$

1) if $\Delta > 0$ then extremum exists, and for <u>$A > 0$</u> it is minimum,for <u>$A < 0$</u> it is maximum,

2) if $\Delta < 0$ extremum doesn't exist,

3) $\Delta = 0$ we can say nothing.
The second derivatives are

$$\frac{\partial^2 z}{\partial x^2} = \left(3x^2 - 3y\right)'_x \Big|_{y=const} = 6x,$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left(3x^2 - 3y\right)'_y \Big|_{x=const} = -3,$$

$$\frac{\partial^2 z}{\partial y^2} = 6y.$$

a) Consider the point $P_1(0;0)$:

$$A = \frac{\partial^2 z(0,0)}{\partial x^2} = 6x \Big|_{\substack{x=0 \\ y=0}} = 0, \qquad C = \frac{\partial^2 z(0,0)}{\partial y^2} = 6y \Big|_{\substack{x=0 \\ y=0}} = 0,$$

$$B = \frac{\partial^2 z(0,0)}{\partial x \partial y} = -3 \Big|_{\substack{x=0 \\ y=0}} = -3.$$

So $\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0 \Rightarrow$ extremum doesn't exist at the point $P_1(0;0)$.

b) Consider the point $P_2(1,1): A = \dfrac{\partial^2 z(1,1)}{\partial x^2} = 6x\Big|_{\substack{x=1\\y=1}} = 6,$

$$C = \frac{\partial^2 z(1,1)}{\partial y^2} = 6y\Big|_{\substack{x=1\\y=1}} = 6, \qquad B = \frac{\partial^2 z(1,1)}{\partial x \partial y} = -3\Big|_{\substack{x=1\\y=1}} = -3.$$

Thus we have $\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 27 > 0 \Rightarrow$ extremum exists. As $A = 6 > 0$ it is minimum.

$z_{\min}(1,1) = \left(x^3 + y^3 - 3xy\right)\Big|_{\substack{x=1\\y=1}} = 1 + 1 - 3 = -1.$

The answer: $z_{\min}(P_2) = -1$.

Problem 4. Find the domain of the function:

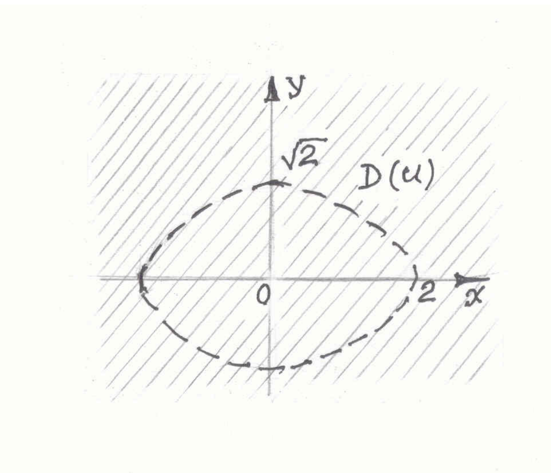
a) $u = \dfrac{1}{x^2 + 2y^2 - 4}$ **b)** $u = \lg\left(x^2 - y^2 + 2x + 4y\right)$

Solution.

• **a)** $u = \dfrac{1}{x^2 + 2y^2 - 4} \Rightarrow D(u): x^2 + 2y^2 - 4 \neq 0.$

First find the boundary whose equation is $x^2 + 2y^2 - 4 = 0.$

$$x^2 + 2y^2 - 4 = 0 \Rightarrow x^2 + 2y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1 \Rightarrow \frac{x^2}{2^2} + \frac{y^2}{\left(\sqrt{2}\right)^2} = 1$$

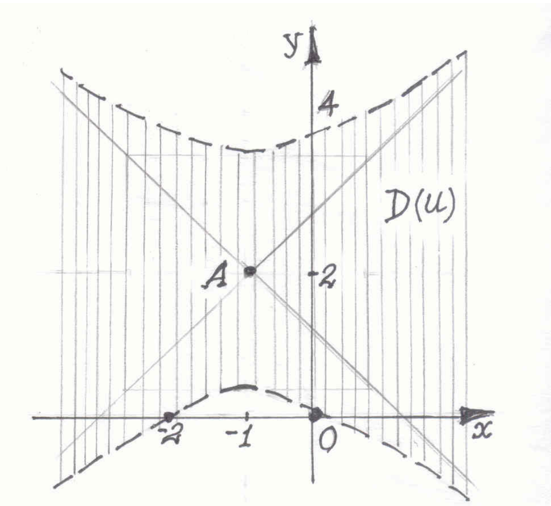


The boundary of the domain is the ellipse with the center in the original and semi axes $a = 2$, $b = \sqrt{2}$. Hence the domain of the given function is the *xy*-plane except the points of the ellipse.

• **b)** $u=\lg\left(x^2-y^2+2x+4y\right)\Rightarrow D(u): x^2-y^2+2x+4y>0$.

The boundary of the domain of the given function is the curve $x^2-y^2+2x+4y=0$. Completing the squares gives us

$$x^2-y^2+2x+4y=0\Rightarrow (x+1)^2-(y-2)^2=-3\Rightarrow \frac{(y-2)^2}{\left(\sqrt{3}\right)^2}-\frac{(x+1)^2}{\left(\sqrt{3}\right)^2}=1$$



The boundary of the domain is the hyperbola with the center at the point $A(-1,2)$. Hence the domain of the given function is the outer part of the plane $0xy$ with respect to hyperbola. The hyperbola does not belong to the domain.

***Problem* 5.** The function $\frac{y}{x^2}e^{y/x}$ is given. Does this function satisfy the equation $\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial y^2}=0$?

Solution.

1) $\frac{\partial z}{\partial x}=\left(e^{y/x}\right)_x'=-\frac{y}{x^2}e^{y/x}$;

2) $\frac{\partial z}{\partial y}=\left(e^{y/x}\right)_y'=\frac{1}{x}e^{y/x}\Rightarrow\frac{\partial^2 z}{\partial y^2}=\left(\frac{1}{x}e^{y/x}\right)_y'=\frac{1}{x^2}e^{y/x}$;

3) $\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial y^2}=-\frac{y}{x^2}e^{y/x}+y\left(\frac{1}{x^2}e^{y/x}\right)=0\Rightarrow 0=0$.

Thus the function $\frac{y}{x^2}e^{y/x}$ satisfies the equation $\frac{\partial z}{\partial x}+y\frac{\partial^2 z}{\partial y^2}=0$.

***Problem* 6.** Using the total differential of the function of two variables calculate approximately $\sqrt[3]{3.01^2 - 2^{0.02}}$, knowing that $\ell n 2 \approx 0.69$.

Solution.

Let us consider the function $z = f(x, y) = \sqrt[3]{x^2 - 2^y}$.
Assume

$$x_0 = 3,\ x_1 = 3.01 \Rightarrow \Delta x = 0.01,$$
$$y_0 = 0,\ y_1 = 0.02 \Rightarrow \Delta y = 0.02,$$
$$z_0 = f(3; 0.02) = \sqrt[3]{3^2 - 2^0} = 2,$$

then the desired value is $z_1 = f(3; 0.02) = \sqrt[3]{3.01^2 - 2^{0.02}}$.
Using the formula (10.1) we have

$$z_1 \approx z_0 + d\,z,$$

where $dz = \left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} \cdot \Delta x + \left.\frac{\partial z}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}} \cdot \Delta y$.

As

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}} = \left.\left(\sqrt[3]{x^2 - 2^y}\right)_x'\right|_{\substack{x=3\\ y=0}} = \left.\frac{2x}{3 \cdot \sqrt[3]{\left(x^2 - 2^y\right)^2}}\right|_{\substack{x=3\\ y=0}} = \frac{4}{3 \cdot 4} \approx 0.333,$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}} = \left.\left(\sqrt[3]{x^2 - 2^y}\right)_y'\right|_{\substack{x=3\\ y=0}} = \left.\frac{-2^y \ln 2}{3 \cdot \sqrt[3]{\left(x^2 - 2^y\right)^2}}\right|_{\substack{x=3\\ y=0}} \approx \frac{-0.69}{3 \cdot 4} = -0.058,$$

then

$$\sqrt[3]{3.01^2 - 2^{0.02}} = [z_1 \approx z_0 + d\,z] = 2 + 0.333 \cdot 0.01 - 0.058 \cdot 0.02 = 2.004.$$

Answer: $\sqrt[3]{3.01^2 - 2^{0.02}} \approx= 2.004$.

***Problem* 7.** Find equations of the tangent plane and normal of the surface $z = x^2 + xy + y^2$ at the point $M(1, 2, 7)$.

Solution.

1) To find the equation of the tangent plane use the equation (8.5)

$$z - z_0 = f'_x(x_0, y_0) \cdot (x - x_0) + f'_y(x_0, y_0) \cdot (y - y_0),$$

where $x_0 = 1,\ y_0 = 2,\ z_0 = 7;$

$$f'_x(x_0, y_0) = \left(x^2 + xy + y^2\right)'_x \Big|_{\substack{x=1 \\ y=2}} = (2x + y)\Big|_{\substack{x=1 \\ y=2}} = 4,$$

$$f'_y(x_0, y_0) = \left(x^2 + xy + y^2\right)'_y \Big|_{\substack{x=1 \\ y=2}} = (x + 2y)\Big|_{\substack{x=1 \\ y=2}} = 5.$$

So the equation of the tangent plane is

$$z = 7 + 4(x - 4) + 5(y - 5) \Rightarrow z = 4x + 5y - 34.$$

2) To find the equations of the normal line use the equation (8.4)

$$z - z_0 = \frac{x - x_0}{f'_x(x_0, y_0)} = \frac{y - y_0}{f'_y(x_0, y_0)} \Rightarrow \frac{z - 7}{1} = \frac{x - 1}{4} = \frac{y - 2}{5}.$$

Answer: $z = 4x + 5y - 34,\ \dfrac{z - 7}{1} = \dfrac{x - 1}{4} = \dfrac{y - 2}{5}.$

18. REVISION EXERSISES

The values of the parameters *a, b, c* in the conditions of variants are:

***a* – the first letter of your surname**
***b* – the first letter of your name**
***c* – the first letter of your patronymic**

1	2	3	4	5	6	7	8	9
A	B	C	D	E	F	G	H	I
J	K	L	M	N	O	P	Q	R
S	T	U	V	W	X	Y	Z	

Variant 1.

1. Let the function $z = \dfrac{x^2}{(b+1)\,y}$ be given**.** Find:

a) differential of the first order);
b) partial derivatives of the second order.

2. Let the function $u = x^2 + (c+2)xy^3 + z^2$, the point $A(1;2;-1)$ and the vector $\bar{a} = \left(2;\, 2\sqrt{a+1};\, a+1\right)$ be given. Find:

a) the gradient of the given function at a given point ($\overline{grad\ f(A)}$)

b) derivative of the function into direction of the vector $\bar{a}$ at the point A $\left(\dfrac{\partial f(A)}{\partial \bar{a}}\right)$.

3. Investigate the function $z = x^3 + 3xy^2 - 15x - 12y$ for extremum.

4. Find the domain of the function :

a) $z = \sqrt{9y^2 - x^2y^2}$;

b) $z = \dfrac{1}{x^2 + y^2}$.

5. The function $z = \dfrac{y}{\left(x^2 - y^2\right)^5}$ is given. Does this function satisfy the equation

$\dfrac{1}{x} \cdot \dfrac{\partial z}{\partial x} + \dfrac{1}{y} \cdot \dfrac{\partial z}{\partial y} - \dfrac{z}{y^2} = 0$?

6. Using the total differential of the function of two variables calculate approximately $(1.02)^{4.05}$

7. Find equations of the tangent plane and normal of the surface $z = \dfrac{x^2}{4} + \dfrac{y^2}{9}$ at the point $M(2, 3, 2)$.

Variant 2.

1. Let the function $z = \ln\dfrac{x}{(b+2)\,y}$ be given**.** Find:

a) differential of the first order

b) partial derivatives of the second order.

2. Let the function $u = (c+1)x\ln y + z^3$, $A(1;1;2)$the point $A(1;2;-1)$ and the vector $\bar{a} = \left(2; 2\sqrt{c+1}; c+1\right)$ be given. Find:

a) the gradient of the given function at a given point ($\overline{grad\ f(A)}$),

b) derivative of the function into direction of the vector $\bar{a}$ at the point A $\left(\dfrac{\partial f(A)}{\partial \bar{a}}\right)$.

3. Investigate the function $z = x^3 + y^3 - 9xy$ for extremum.

4. Find the domain of the function:

a) $z = \arcsin\dfrac{y}{x^2}$;

b) $z = \dfrac{1}{1 + x^2 + y^2}$.

5. The function $z = x^y$ is given. Does this function satisfy the equation

$$y \cdot \frac{\partial^2 z}{\partial x \partial y} - (1 + y\ell nx) \cdot \frac{\partial z}{\partial x} = 0 \ ?$$

6. Using the total differential of the function of two variables calculate approximately

$$\sqrt{8.04^2 + 6.03^2}\ .$$

7. Find equations of the tangent plane and normal of the surface $z = \sqrt{9 - x^2 - y^2}$ at the point $M(1, 2, 2)$.

Variant 3.

1. Let the function $z = e^{(a+2)xy^2}$ be given**.** Find:

a) differential of the first order;

b) partial derivatives of the second order.

2. Let the function $u = \frac{(b+2)y^3}{x} + b\,xz,$ the point $A(1;0;-1)$ and the vector $\bar{a} = \left(2;\, 2\sqrt{b+1};b+1\right)$ be given. Find:

a) the gradient of the given function at a given point ($\overline{grad\ f(A)}$);

b) derivative of the function into direction of the vector $\bar{a}$ at the point A $\left(\frac{\partial f(A)}{\partial \bar{a}}\right)$.

3. Investigate the function $z = x^2 + y^2 - 8x - 2$ for extremum.

4. Find the domain of the function:

a) $z = \ell n\,(xy-1)$;

b) $z = \frac{1}{x+y}$.

5. The function $z = \frac{y^2}{3x} + \arcsin xy$ is given. Does this function satisfy the equation $x^2 \cdot \frac{\partial z}{\partial x} - xy \cdot \frac{\partial z}{\partial y} + y^2 = 0$?

6. Using the total differential of the function of two variables calculate approximately

$$\sin 32^\circ \cdot \cos 59^\circ.$$

7. Find equations of the tangent plane and normal of the surface $z = \sqrt{26 - x^2 - y^2}$ at the point $M(3,4,1)$.

Variant 4.

1. Let the function $z=\sin\left(a\,x^2\,y\right)$ be given**.** Find:

a) differential of the first order;

b) partial derivatives of the second order.

2. Let the function $u=e^{(b+1)xz+2y}$, the point $A(-1;1;-1)$ and the vector $\bar{a}=\left(2;2\sqrt{a+1};a+1\right)$ be given. Find:

a) the gradient of the given function at a given point ($\overline{grad\ f(A)}$);

b) derivative of the function into direction of the vector $\bar{a}$ at the point A $\left(\dfrac{\partial f(A)}{\partial \bar{a}}\right)$.

3. Investigate the function $z=3x-x^2-xy-y^2+6y$ for extremum.

4. Find the domain of the function:

a) $z=\arccos\dfrac{x}{y^2}$;

b) $z=\dfrac{1}{x-y}$.

5. The function $z=\ell n\left(x^2+y^2+2x+1\right)$ is given. Does this function satisfy the equation $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$.

6. Using the total differential of the function of two variables calculate approximately $(4.05)^{0.98}$.

7. Find equations of the tangent plane and normal of the surface $z=x^2-y^2$ at the point $M(5,4,9)$.

Variant 5.

1. Let the function $z = \frac{y^2}{b\,x}$ be given**.** Find:

a) differential of the first order;

b) partial derivatives of the second order.

2. Let the function $u = y^{(c+1)x} + x\cos z,$ the point $A\left(0;-2;\frac{\pi}{2}\right)$ and the vector $\bar{a} = \left(2;2\sqrt{b+1};b+1\right)$ be given. Find

a) the gradient of the given function at a given point ($\overline{grad\ f(A)}$);

b) derivative of the function into direction of the vector $\bar{a}$ at the point A $\left(\frac{\partial f(A)}{\partial \bar{a}}\right)$.

3. Investigate the function $z = 2x^3 + 2y^3 - 36xy + 430$ for extremum.

4. Find the domain of the function:

a) $z = \sqrt[4]{9 - 3x^2 - y^2}$;

b). $z = \frac{1}{x^2 - y^2 - 1}$.

5. The function $z = e^{xy}$ is given. Does this function satisfy the equation $x \cdot \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial y} - xy^2 z = 0$?

6. Using the total differential of the function of two variables calculate approximately $z = \ell n\left(0.9^3 + 0.99^2\right)$, knowing that $\ell n2 \approx 0.69$.

7. Find equations of the tangent plane and normal of the surface $z = x^2 + y^2$ at the point $M(1,2,5)$.

Variant 6.

1. Let the function $z = \cos\dfrac{x}{ay}$ be given**.** Find:

a) differential of the first order;

b)partial derivatives of the second order.

2. Let the function $u = z\arcsin x + bxy$, the point $A\left(\dfrac{1}{2}; b; 0\right)$ and the vector $\bar{a} = \left(2; 2\sqrt{a+1}; a+1\right)$ be given. Find:

a) the gradient of the given function at a given point ($\overline{grad\ f(A)}$);

b) derivative of the function into direction of the vector $\bar{a}$ at the point A $\left(\dfrac{\partial f(A)}{\partial \bar{a}}\right)$.

3. Investigate the function $z = 2xy - 4x - 2y$ for extremum.

4. Find the domain of the function:

a) $z = \ell g\left(x^2 - y^2\right)$;

b) $z = \dfrac{1}{x^2 + y^2 + 9y}$.

5. The function $z = \dfrac{x}{y}$ is given. Does this function satisfy the equation

$x \cdot \dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial z}{\partial y} = 0$?

6. Using the total differential of the function of two variables calculate approximately $\sqrt[4]{2.99^2 + 7 \cdot e^{0.01}}$.

7. Find equations of the tangent plane and normal of the surface $z = xy$ at the point $M(3, 4, 12)$.

Variant 7.

1. Let the function $z = \cos\dfrac{x}{ay}$ be given**.** Find:

a) differential of the first order;

b)partial derivatives of the second order.

2. Let the function $u = z\arcsin x + bxy,$ the point $A\left(\dfrac{1}{2};b;0\right)$ and the vector $\bar{a} = \left(2;2\sqrt{a+1};a+1\right)$ be given. Find:

a) the gradient of the given function at a given point ($\overline{grad\ f(A)}$);

b) derivative of the function into direction of the vector $\bar{a}$ at the point A $\left(\dfrac{\partial f(A)}{\partial \bar{a}}\right)$

.

3. Investigate the function $z = x^2 + 2y^2 + xy - x + 3y$ for extremum.

4. Find the domain of the function:

a) $z = \dfrac{1}{\sqrt{x^2 + 4y - 1}}$;

b) $z = \dfrac{1}{x^2 + y^2 + 2x}$.

5. The function $z = \ell n\left(x + e^{-y}\right)$ is given. Does this function satisfy the equation $\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial^2 z}{\partial x \partial y} - \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial^2 z}{\partial x^2} = 0$?

6. Using the total differential of the function of two variables calculate approximately $\sqrt{2.03^2 + 5\cdot e^{0.02}}$.

7. Find equations of the tangent plane and normal of the surface $z = \sqrt{x^2 - 1 + y^2}$ at the point $M(1,2,2)$.

Variant 8.

1. Let the function $z = ax^2 \ln y$ be given**.** Find:

a) differential of the first order;

b)partial derivatives of the second order.

2. Let the function $u = tg\left(x^2 + by\right) + \frac{1}{z}$, the point $A\left(0; \frac{\pi}{b}; 2\right)$ and the vector $\bar{a} = \left(2; 2\sqrt{b+1}; b+1\right)$ be given. Find:

a) the gradient of the given function at a given point ($\overline{grad\ f(A)}$);

b) derivative of the function into direction of the vector $\bar{a}$ at the point A $\left(\frac{\partial f(A)}{\partial \bar{a}}\right)$.

3. Investigate the function $z = x^3 + xy^2 + 6xy$ for extremum.

4. Find the domain of the function:

a) $z = \frac{1}{\ell og_2\left(4 - x^2 + y^2\right)}$;

b) $z = \frac{1}{x^2 - 2y}$.

5. The function $z = \frac{y}{\left(x^2 - y^2\right)^5}$ is given. Does this function satisfy the equation $\frac{1}{x} \cdot \frac{\partial z}{\partial x} + \frac{1}{y} \cdot \frac{\partial z}{\partial xy} - \frac{z}{y^2} = 0$?

6. Using the total differential of the function of two variables calculate approximately $\ell n\left(0.01^4 + 1.1^2\right)$.

7. Find equations of the tangent plane and normal of the surface $z = 3x^2 - xy + 2y^2$ at the point $M(-1, 3, 24)$.

Variant 9.

1. Let the function $u = c^3 yx^2 + \frac{z}{x}$ be given**.** Find:

a) differential of the first order;

b)partial derivatives of the second order.

2. Let the function $u = z^2 - axy^3$, the point $A(-1;-b;1)$ and the vector $\bar{a} = \left(2; 2\sqrt{c+1}; c+1\right)$ be given. Find:

a) the gradient of the given function at a given point ($\overline{grad\ f(A)}$);

b) derivative of the function into direction of the vector $\bar{a}$ at the point A $\left(\frac{\partial f(A)}{\partial \bar{a}}\right)$.

3. Investigate the function $z = x^3 + 8y^3 + 6xy - 1$ for extremum.

4. Find the domain of the function:

a) $z = \sin(x + 2y)\frac{1}{\sqrt[6]{x^2 - 4x + y^2 + 2y}}$;

b) $z = \frac{1}{y^2 - 4x}$.

5. The function $z = z = xe^{y/x}$ is given. Does this function satisfy the equation $\frac{\partial^2 z}{\partial y^2} - 4\frac{\partial^2 z}{\partial x^2} = 0$?

6. Using the total differential of the function of two variables calculate approximately $\sqrt[3]{3.01^2 - 2^{0.02}}$, knowing that $\ell n2 \approx 0.69$.

7. Find equations of the tangent plane and normal of the surface $z = x^2 + xy + y^2$ at the point $M(1, 2, 7)$.

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Textbook

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