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Department of Higher Mathematics

Educational Aid on Elementary Mathematics

Modul № 1.Arithmetic, Algebra, Complex Numbers, Limits of Function Values

For students studying a course of higher mathematics in English

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Составители: доц. ***Гавдзинский В. Н***., ст. преподаватель ***Коробова Л. Н.***

В пособии в краткой форме представлены основные сведения по элементарной математике на английском языке для студентов академии, изучающих высшую математику на английском языке. Основные теоремы и формулы приведены с доказательством, а также даны решения типовых примеров и задания для самостоятельного решения.

Компьютерная верста ***Корнейчук Е. С.***

CONTENTS

§ 1.1 Arithmetic Operations and Properties

A. Addition:

$a + b = c$

a plus ***b*** equals ***c***;

b added to ***a*** is ***c***;

the sum of ***a*** and ***b*** is ***c***;

a, b are called the summands,

c is called the sum.

?? *Find the sum of **a** and **b**.*

*Add **b** to **a**.*

*How much is **a** plus **b**?*

addition – *сложение*
to add – *прибавлять*
summand – *слагаемое*
sum – *сумма*
how much – *сколько*

B. Subtraction: $a - b = c$

a minus ***b*** equals (is) ***c***;

the difference of ***a*** and ***b*** is ***c***;

a is called a minuend,

b is called a subtrahend;

c is called a difference.

?? *Find the difference between **a** and **b**.*

*Subtract **b** from **a**.*

*How much is **a** minus **b**?*

subtraction – *вычитание*
minuend – *уменьшаемое*
subtrahend – *вычитаемое*
difference – *разность*
to subtract – *вычитать*

C. Multiplication: $a \times b = c$,

$a \cdot b = c$;

a multiplied by ***b*** equals ***c***;

a by ***b*** is ***c***;

the product of ***a*** and ***b*** is ***c***;

a, b are called factors, ***c*** is called a product.

once – *один раз*
twice – *дважды*
three times – *трижды*

$4 \times 5 = 4 \cdot 5 = 20$ – **five times four (four times five) is twenty**

multiplication – *умножение*

multiply (by) – *умножать*
factor – *множитель*
product – *произведение*

?? *Find the product of **a** and **b**.*

*Multiply **a** by **b**.*

*How much is **a** multiplied by **b**?*

D. Division: $a : b = a / b = \frac{a}{b} = c$

division	– *деление*
to divide (by)	– *делить*
quotient	– *частное (от деления)*
dividend	– *делимое*
divisor	– *делитель*
remainder	– *остаток*
to be undefined	– *не определено*

a divided by ***b*** is ***c*****;**
a quotient of ***a*** and ***b*** is ***c*****;**
a is called a dividend;
b is called a divisor;
c is called a quotient.

?? *Find a quotient of **a** and **b**.*
*Divide **a** by **b**.*
*How much is **a** divided by **b**?*

<u>**Remember**</u>**:** ***Division by 0 is undefined!***

example	– *пример*	**increase**	– *увеличивать*
decrease	– *уменьшать*	**evaluate**	– *вычислять*

Exercise 1.1.1.

a) The product $a\cdot 25$ is given. Let ***a*** is increased at ***20***. What can you tell about the new product?

b) The difference $a - b$ is given. a) Let ***a*** is decreased at ***10***. What can you tell about the new difference? b) Let ***b*** is decreased at ***10.*** What can you tell about the new difference?

c) The sum $a + b$ is given. Let ***a*** is decreased at ***10.*** What can you tell about the new sum?

d) The quotient $a : b$ is given. What can you tell about the new quotient if a) ***a*** is decreased in 5 times? b) ***b*** is increased in 10 times?

e) Evaluate each of the following:
1) 684:9; 2) 504:8; 3) 1000 - 210·4; 4) 480:6 + 18.

E. Properties

1. **Commutative Property**

Let ***a*** and ***b*** be any numbers. Then

a) $a + b = b + a;$

b) $a \cdot b = b \cdot a.$

2. **Associative Property**

Let ***a***, ***b*** and ***c*** be any numbers. Then

a) $(a + b) + c = a + (b + c);$

b) $a(bc) = (ab)c.$

3. **Distributive Property**
   Let ***a***, ***b*** and ***c*** be any numbers. Then
   $$a(b \pm c) = ab \pm ac$$

4. **Identity Property**
   a) There is a unique number, namely ***0***, such that for any number ***a***,
   $$a + 0 = a = 0 + a.$$
   b) There is a unique whole number, namely **1**, such that for every number ***a***,
   $$a \cdot 1 = a = 1 \cdot a.$$

5. **Property of Zero**
   a) If $a \neq 0$, then $0 : a = 0;$ ·
   b) $0 \cdot a = 0 = a \cdot 0.$

6. **Whole-Number Exponent**
   Let ***m*** is the whole number where $m \neq 0.$ Then for any ***a***
   $$a^m = \underbrace{a \cdot a \cdot \ldots \cdot a}_{m}$$

For example $5^2 = 5 \cdot 5 = 25$, $2^3 = 2 \cdot 2 \cdot 2 = 8$.

F. Some Important Problems

known – *известный*
unknown – *неизвестный*
to (find) – *для того, чтобы (найти)*

1). $x + a = b \Rightarrow x = b - a$. To find unknown summand subtract the known summand from the sum.

2). $x - a = b \Rightarrow x = b + a.$ To find unknown minuend add the subtrahend to the difference.

3). $a - x = b \Rightarrow x = a - b$. To find unknown subtrahend subtract the difference from the minuend.

4). $x \cdot a = b \Rightarrow x = b : a$. To find unknown factor divide the product by the known factor.

5). $x : a = b \Rightarrow x = a \cdot b.$ ***T***o find unknown dividend multiply the divisor by the quotient.

6). a : x = b $\Rightarrow$ ***x = a : b***. To find known divisor divide the dividend by the divisor.

Exercise 1.1.2. Explain the finding of ***x*** if

1) $2 \cdot x = 120$	2) $x \cdot 7 = 49$	3) $5 + x = 12$	4) $x + 31 = 45$
5) $42 - x = 39$	6) $x - 17 = 34$	7) $x : 5 = 125$	8) $899 : x = 31$

§ 1.2. Counting Numbers

counting numbers – *натуральные числа*
composite – *составное число*
is divisible by – *делится на*
long division – *деление в столбик*
prime – *простое число*
multiple – *делимое*
divisibility – *делимость*
prime factorization – *разложение на простые множители*
Greatest Common Factor – *наибольший общий делитель*
Least Common Multiple – *наименьшее общее кратное*

Definition 1.2.1. Numbers 1, 2, 3, 4,… are called the **Counting Numbers**.

Definition 1.2.2. A counting number with exactly two different factors is called a **prime number**, or a **prime**. A counting number with more than two factors is called a **composite number**, or a **composite**.

For example, 2, 3, 5, 7, 11 are primes, since they have only themselves and 1 as factors. 4, 6, 8, 9, 10 are composites, since they each have more than two factors.

1 is neither prime nor composite, since 1 is its only factor.

Theorem 1.2.1. Fundamental Theorem of Arithmetic. Each composite number can be expressed as the product of primes in exactly one way (except for the order of the factors).

Definition 1.2.3. ***a*** divides ***b*** if and only if ***a*** is a factor of ***b***.
In this case

a is a **divisor** of ***b*** if ***a*** is a factor of ***b***;

b is a **multiple** of ***a*** if *b* is divisible by *a*.

Theorem 1.2.2. **Tests for Divisibility by 2, 3 and 5.**

A number is **divisible by 2** if and only if its ones digit is 0, 2, 4, 6, or 8.

A number is **divisible by 3** if and only if the sum of its digits is divisible by 3.

A number is **divisible by 5** if and only if its ones digit is 0 or 5.

The prime factor exponential form of a number.

$$72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2.$$

The form $2^3 \cdot 3^2$ is the prime factor exponential for of the number 72.

? *Try to give the definition of the prime factor exponential for any number.*

Problem Set 1.2.1

1. Find all primes less than 100.

2. Find the prime factorization for each of the following numbers.
   a) 36   b) 54   c) 102   d) 1000

3. Determine which of the following is true.
   a) 3 is a divisor of 21.   b) 6 is a factor of 3.
   c) 4 is a factor of 16.   d) 5 is divisible by 0
   e) 11 is divisible by 11   f) 48 is a multiple of 16.

Definition 1.2.3. **Greatest Common Factor (GCF)**
The greatest common factor of two (or more) nonzero whole numbers is the largest whole number that is a factor of both (all) of the numbers. The GCF of ***a*** and ***b*** is written **GCF(*a, b*)**.

Example 1.2.2. Find the GCF(24,36).
Solution.
Step 1:
Express the numbers 24 and 36 in their prime factor exponential form:

$$\begin{array}{r|l} 24 & 2 \\ 12 & 2 \\ 6 & 2 \\ 3 & 3 \\ 1 & \end{array} \qquad \begin{array}{r|l} 36 & 2 \\ 18 & 2 \\ 9 & 3 \\ 3 & 3 \\ 1 & \end{array} \qquad 24 = 2^3 \cdot 3, \quad 36 = 2^2 \cdot 3^2$$

Step 2:
The GSF will be the number $2^m \cdot 3^n$ where m is the smaller of the exponents of the 2s(twos) and n is the smaller of the exponents of the 3s (threes).

Thus GSF(24,36) = $2^2 \cdot 3 = 12$.

Definition 1.2.4. **Least Common Multiple (LCM)**

The least common multiple of two (or more) nonzero whole numbers is the smallest nonzero whole number that is the multiple of each (all) of the numbers. The LCM of ***a*** and ***b*** is written **LCM(*a, b*).**

Example 1.2.3. Find LCM (24, 36).

Solution. Use the Prime Factorization Method.

1) Express the numbers 24 and 36 in their prime factor exponential form: $24 = 2^3 \cdot 3$, $36 = 2^2 \cdot 3^2$.
2) The LSM will be the number $2^r \cdot 3^s$, where r is the larger of the exponents of the twos and s is the larger of the exponents of the threes. So LSM(24,36) = $2^3 \cdot 3^2 = 8 \cdot 9 = 72$.

Problem Set 1.2.2

1. Use the prime factorization method to find the GCFs.
   a) GCF (8,18) b) GCF (36,42)
   c) GCF (24,66) d) GCF (12,60,90)

2. Use the prime factorization method to find the LCMs.
   a) LCM (6,8) b) LCM (4,10)
   c) LCM (7,9) d) LCM (8,12,18)

§ 1.3. Common Fractions

fractiona – *дробь*
numerator – *числитель*
denominator – *знаменатель*
mixed number – *смешанное число*
proper fraction – *правильная дробь*
improper fraction – *неправильная дробь*
the least common denominator – *наименьший общий знаменатель*

A. Definitions

Definition 1.3.1. If ***a*** and ***b*** are whole numbers, where $b \neq 0$, then the fraction $\frac{a}{b}$, or ***a / b***, represents ***a*** of ***b*** equivalent parts.

a is called the **numerator**,
b is called the **denominator**.

Definition 1.3.2. If a numerator is less than a denominator the fraction is called **proper**, if a numerator is greater than or equal to a denominator the fraction is called **improper**.

For example the fractions 2/3, 17/21 are proper fractions, the fractions 3/2, 21/17, 15/15 are improper fractions.

Definition 1.3.3. The numbers of the kind $4\frac{3}{5}$, $1\frac{17}{18}$ and so on are called the **mixed numbers**.

Note. Any mixed number can be changed to improper fraction, and an improper fraction can be changed to mixed number. For example,

a) $4\frac{3}{5}=\frac{4\cdot 5+3}{5}=\frac{23}{5}$: **to changed a mixed number to improper fraction** multiple the whole part of this number by denominator and add to the numerator. This is the numerator of the improper fraction. The denominator is the same.

b) $\frac{15}{4}=3\frac{3}{4}$: **to change an improper fraction to a mixed number** divide its numerator by the denominator with remainder. The quotient is the whole part and the remainder is the numerator of the proper part.

Problem Set 1.3.1.

1. Change the following mixed numbers to improper fractions.

a) $3\frac{5}{6}$ b) $2\frac{7}{8}$ c) $5\frac{2}{5}$ d) $9\frac{1}{7}$

2. Change the following improper fractions to mixed numbers.

a) 241/9 b) 13/2 c) 56/8 d) 147/12

Definition 1.3.4. **Fraction Equality**

Let ***a/b*** and ***c/d*** be any fractions. Then ***a/b*** = ***c/d*** if and only if ***ad = bc***.

In words, two fractions are equal if and only if their **cross-products**, that is, products ***ad*** and ***bc*** obtained by **cross-multiplication**, are equal.

For example, $\frac{3}{7}=\frac{12}{28}$, as 3·28 =7·12 = 84.

Theorem1.3.1. Let $\frac{a}{b}$ be any fraction and n a nonzero whole number. Then

$$\frac{a}{b}=\frac{an}{bn}=\frac{na}{nb} \tag{1.3.1}$$

This theorem can be used in two ways:

(1) to replace the fraction ***a / b*** with ***an / bn***. For example we can use (1.3.1) to compare fractions, to add or subtract fractions.

(2) to replace the fraction ***ab / bn*** with ***a / b.*** We use it to simplify a fraction.

Definition 1.3.5. **Less Than for Fractions**

Let $\frac{a}{b}$ and $\frac{c}{b}$ be any fractions. Then $\frac{a}{b} < \frac{c}{b}$ if and only if $a < c$

Note: Although the definition is stated for “less than”, a corresponding statement holds for “**greater than**”. Similar statements hold for “**less than or equal to**” and “**greater than or equal to**”. To compare fractions with unlike denominators can be compared by getting a common denominator.

Problem Set 1.3.2.

Prove the theorem **Cross-Multiplication of Fraction Inequality**.
Let ***a / b*** and ***c / d*** be any fractions. Then ***a/b < c/d*** if and only if ***ad < bc***.

B. Addition and Subtraction of Fractions and Mixed Numbers

Definition 1.3.6 **Addition of Fractions with Common Denominators**

Let $\frac{a}{b}$ and $\frac{c}{b}$ be any fractions. Then $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

To add fractions with unlike denominators, find equivalent fractions with the least common denominators. Then the sum will be represented by the sum of the numerators over the common denominator.

Note: The least common denominator of fractions is equal to their least common multiple.

Example 1.3.1. Find following sums and simplify.

a) 3/7 + 2/7 b) 5/9 + 3/7 c) 17/15 + 5/12 d) $2\frac{3}{4} + 3\frac{5}{6}$

Solution

a) $\frac{3}{7} + \frac{2}{7} = \frac{3+2}{7} = \frac{5}{7}$

b) $\frac{5}{9} + \frac{3}{7} = \frac{5 \cdot 7 + 3 \cdot 9}{9 \cdot 7} = \frac{62}{63}$

c) The first way of solution:

$$\frac{\overset{4}{\cancel{17}}}{15} + \frac{\overset{5}{\cancel{5}}}{12} = \frac{68+25}{60} = \frac{93}{60} = \frac{31}{20} = 1\frac{11}{20}.$$

The second way of solution: as $\frac{17}{15} = 1\frac{2}{15}$ we have

$$\frac{17}{15} + \frac{5}{12} = 1\,\frac{\overset{4}{\cancel{2}}}{15} + \frac{\overset{5}{\cancel{5}}}{12} = 1\frac{8+25}{60} = 1\frac{33}{60} = 1\frac{11}{20}.$$

d) $2\frac{3}{4} + 3\frac{5}{6} = (2+3) + \frac{9+10}{12} = 5\frac{19}{12} = 6\frac{7}{12}$.

So to add mixed numbers it is not necessary to change them to improper fractions.

Definition 1.3.7. **Subtraction of Fractions with Common Denominators**
Let a/b and c/b be any fractions with $a \geq c$. Then

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

If fractions have different denominators, subtraction is done by first finding the least common denominators, then subtracting as before.

Example 1.3.2. Find the difference of numbers

a) $2\frac{1}{4}$ and $1\frac{3}{4}$ b) $3\frac{1}{8}$ and $1\frac{2}{3}$

Solution.

a) $2\frac{1}{4} - 1\frac{3}{4} = 1\frac{5}{4} - 1\frac{3}{4} = \frac{5-3}{4} = \frac{2}{4} = \frac{1}{2}$; ;

b) $3\frac{1}{8} - 1\frac{2}{3} = 3\frac{3}{24} - 1\frac{16}{24} = 2\frac{27}{24} - 1\frac{16}{24} = 1\frac{11}{24}$.

Explain the solution of these examples in words!

Problem Set 1.3.3.

1. Perform the following subtractions.

   a) 8/15 – 4/15 b) 3/7 – 2/9 c) 4/5 – ¾
   d) 13/18 – 8/27 e) 21/51 – 7/39 f) 11/100 – 99/1000

2. Find the sum and difference (first minus second) for the following pairs of mixed numbers. Answers should be written as mixed numbers.
   a) $2\frac{2}{3}$ and $1\frac{1}{4}$; b) $7\frac{5}{7}$ and $5\frac{2}{3}$; c) $22\frac{1}{6}$ and $15\frac{11}{12}$.

3. Are the following statements correctly or not? Why?
   a) 3/7 < 5/7 b) ½ > 1/3 c) 1½ > 15/16

C. Multiplication and Division of Fractions

Definition 1.3.8 Multiplication of Fractions

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any fractions. Then

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$

Example 1.3.3. Compute the following products and express the answers in simplest form.

a) $\frac{2}{3}\cdot\frac{5}{13}$, b) $\frac{3}{4}\cdot\frac{28}{15}$, c) $2\frac{1}{3}\cdot 4$, d) $2\frac{1}{3}\cdot 7\frac{2}{7}$.

Solution

a) $\frac{2}{3}\cdot\frac{5}{13}=\frac{2\cdot 5}{3\cdot 13}=\frac{10}{39}$;

b) $\frac{3}{4}\cdot\frac{28}{15}=\frac{\overset{1}{\cancel{3}}\cdot\overset{7}{\cancel{28}}}{\underset{1}{\cancel{4}}\cdot\underset{5}{\cancel{15}}}=\frac{7}{5}=1\frac{2}{5}$;

c) $2\frac{1}{3}\cdot 4=\left[2\cdot 4+\frac{1}{3}\cdot 4\right]=8\frac{4}{3}=9\frac{1}{3}$.

To multiply a mixed number by a whole number you can do it separately with the whole and fraction parts. The answer is the mixed number.

d) $2\frac{1}{3}\cdot 7\frac{2}{7}=\frac{7}{3}\cdot\frac{51}{7}=\frac{\overset{1}{\cancel{7}}\cdot\overset{17}{\cancel{51}}}{\underset{1}{\cancel{3}}\cdot\underset{1}{\cancel{7}}}=\frac{17}{1}=17$.

To multiply two ore more mixed numbers you must change them to improper fractions. The product is a fraction (proper or improper).

Theorem 1.3.2. **Division of Fractions**

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any fractions with $\boldsymbol{c}\neq 0$. Then

$$\frac{a}{b}:\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$$

Example 1.3.4. Find the following quotients.

a) $\frac{8}{15}:4$; b) $\frac{7}{8}:\frac{3}{5}$; c) $\frac{12}{13}:\frac{4}{13}$; d) $1\frac{19}{21}:1\frac{5}{6}$.

Solution

a) $\frac{8}{15} : 4 = \left[\frac{8}{15} : \frac{4}{1}\right] = \frac{\overset{2}{\cancel{8}}}{15 \cdot \underset{1}{\cancel{4}}} = \frac{2}{15}$;

b) $\frac{7}{8} : \frac{3}{5} = \frac{7 \cdot 5}{8 \cdot 3} = \frac{35}{24} = 1\frac{11}{24}$;

c) $\frac{12}{13} : \frac{4}{13} = \frac{\overset{3}{\cancel{12}} \cdot \cancel{13}}{\cancel{13} \cdot \cancel{4}} = \frac{3}{1} = 3$;

d) $1\frac{19}{21} : 1\frac{5}{6} = \frac{40}{21} : \frac{11}{6} = \frac{40 \cdot \overset{2}{\cancel{6}}}{\underset{7}{\cancel{21}} \cdot 11} = \frac{80}{77} = 1\frac{3}{77}$.

Problem Set 1.3.4.

1. Explain the solution of the example 1.3.4 in words.
2. Solve the following equations involving fractions.

   a) $\frac{2}{5}x = \frac{3}{7}$; b) $\frac{x}{6} = \frac{5}{12}$; c) $2\frac{3}{7}x = 1\frac{16}{35}$.

§ 1.4. Decimals, Ratio, Proportion, and Percent

decimal – *десятичная дробь*
terminating decimal – *конечная десятичная дробь*
repeating decimal – *периодическая дробь*

Definition 1.4.1. If a denominator of a fraction is 10, 100, … then this fraction is called a **decimal**.

For example

$\frac{123}{10000} = 0.0123$, read "zero point, zero, one, two, three".

$1\frac{875}{1000} = 1.875$, read or "one point, eight, seven, five".

Theorem 1.4.1. Let $\frac{a}{b}$ be a fraction in simplest form. Then $\frac{a}{b}$ has a **terminating decimal** representation if and only if b contains only 2s and (or) 5s in its prime factorization.

Thus $\frac{3}{50} = \frac{3}{2\cdot5\cdot5} = \frac{3\cdot5\cdot2\cdot2}{(2\cdot5)\cdot(5\cdot2)\cdot(5\cdot2)} = \frac{60}{1000} = \frac{6}{100} = 0.06$ – zero point, zero, six.

Theorem 1.4.2. Let $\frac{a}{b}$ be a fraction in simplest form. Then $\frac{a}{b}$ has a **repeating decimal** representation that does **not terminate** if and only if b has a prime factor other than 2 or 5.

Thus , division of 34 by 99 gives us

$$\frac{34}{99} = 0.343434... = 0.\overline{34}.$$

Definition 1.4.3. A **ratio** is the ordered pair of numbers, written $a:b = \frac{a}{b}$, with $b \neq 0$.

Definition 1.4.4. Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two ratios. Then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

Definition 1.4.5. A **proportion** is a statement that two given ratios are equal.

Exercise 1.4.1. Find x from the proportion $\frac{5}{6} = \frac{x}{15}$.

Solution. $x = \frac{5\cdot15}{6} = \frac{5\cdot5}{2} = \frac{25}{2} = 12.5$.

Definition 1.4.6. One hundredth of a number is **1 percent** (1%) of this number.

Exercise 1.4.2.

a) Find the number that is 15% of 30 ;

b) find the number 10% of that is 24.

Solution.

a) $30 \sim 100\%$

$x \sim 15\%$.

So we have the proportion $\frac{30}{x} = \frac{100}{15} \Rightarrow x = \frac{30\cdot15}{100} = \frac{9}{2} = 4.5$

b) $10\% \sim 24$

$100\% \sim x$

Find x from the proportion $\frac{10}{100} = \frac{24}{x} \Rightarrow x = \frac{100\cdot24}{10} = 240.$

II. ALGEBRA

§ 2.1. Fundamental Concepts

1.1. Let A and B be sets. Then

- $A = \varnothing$ the **empty set**
- $A \cup B$ is a **union** of A and B
- $A \cap B$ is an **intersection** of A and B
- $A \subset B$ A is a **subset** of B
- $x \in A$ x belongs to A, x is an **element** of the set A
- $\Rightarrow$ to follow
- $\Leftrightarrow$ if and only if
- $\forall$ any, for any
- $\exists$ to exist
- **:** such that, for example, $[a,b] = \{x : a \le x \le b\}$ means "a closed interval is a set of x such that $a \le x \le b$".

2.1.2. The Real Number System

Integers (whole numbers)**:** $\boldsymbol{Z} = \{\ldots,-3,-2,-1,0,1,2,3\ldots.\}$

Rational numbers – $\{$all terminating or repeating decimals$\}$ **:** $\boldsymbol{Q} = \left\{\frac{m}{n}\right\}$, where $m \in Z, n \ne 0$

Irrational numbers: $\{$all nonterminating, nonrrepeating decimals$\}$

Real numbers $\boldsymbol{R}$**:** $\{$all rational and irrational numbers$\}$

- Let a and b be real numbers, then:

a **equals** b denoted by $a = b$ if $a - b = 0$.
a is **greater than** b (denoted by $a > b$) if $a - b$ is positive.
a is **less than** b (denoted by $a < b$) if $a - b$ is negative.

2.1.3. Intervals, Absolute Value, and Distance

The interval notations:

$(a,b) = \{x : a < x < b\}$ represents all real numbers between a and b, not including a and not including b. This is an **open interval.**

$[a,b] = \{x : a \le x \le b\}$ represents all real numbers between a and b, including a and including b. This is a **closed interval.**

$(a,b] = \{x : a < x \le b\}$ represents all real numbers between a and b, not including a and including b.

$[a,b) = \{x : a \le x < b\}$ represents all real numbers between a and b, including a and not including b.

$(-\infty, b) = \{x : x < b\}$ represents all real numbers less than b.

$(-\infty, b] = \{x : x \le b\}$ represents all real numbers less than or equal to b.

$(a, \infty) = \{x : x > a\}$ represents all real numbers greater than a.

$[a, \infty) = \{x : x \ge a\}$ represents all real numbers greater than or equal to a.

$(-\infty, \infty) = R$ represents all real numbers.

• The **absolute value** of the real number a is defined by

$$|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a & \text{if } a < 0 \end{cases}$$

Absolute Value Theorems

For all real numbers a and b,

1. Nonnegative: $|a| \ge 0$
2. Product: $|ab| = |a| \cdot |b|$
3. Quotient: $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, if $b \ne 0$
4. Triangle inequality: $|a+b| \le |a| + |b|$
5. Difference: $|a-b| = |b-a|$

• The **distance** between points a and b on a real number line is denoted $d(a,b)$ and $d(a,b) = |a-b|$

2.1.4. Exponents

• If b is any real number and n is any natural number, then

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ factors of } b}$$

In the expression b^n, b is the **base**, n is the **exponent**, and b^n is the ***n*th power of** b

• Let $a \ge 0$ be a real number and n be a positive integer, then

$\sqrt[n]{a} = b \Leftrightarrow b^n = a$. If $a<0$, then $\sqrt[n]{a}$ is a real number if and only if n is odd. If $a<0$ and n is even, then $\sqrt[n]{a}$ is not a real number.

The number a in $\sqrt[n]{a}$ is called the **radicand** and n is called the index. The symbol $\sqrt[n]{a}$ is read **the *n*th root of *a*** and is called the **radical.**

Notation. 1. $\sqrt[2]{a} = \sqrt{a}$ is read a square root of a, $\sqrt[3]{a}$ is read the cube root of a;

$$2.\ \sqrt[n]{a^n} = \begin{cases} a, & \text{if } n = 2k+1 \,(n \text{ is odd}), \\ |a|, & \text{if } n = 2k \,(n \text{ is even}). \end{cases} \tag{2.1.1}$$

For example,

a) $\sqrt[4]{16} = 2$ as $2^4 = 16$; b) $\sqrt[3]{-27} = -3$ as $(-3)^3 = -27$

c) $\sqrt{-9}$ is not a real number as $a = -9 < 0$, $n = 2$ is even

d) $\sqrt[5]{(1-x)^5} = 1-x$, as $n = 5$ is odd

e) $\sqrt[4]{(1-x)^4} = |1-x| = \begin{cases} 1-x, x \le 1, \\ x-1, x > 1. \end{cases}$

- For any nonzero real number b,

$$b^0 = 1. \tag{2.1.2}$$

- If $b \neq 0$ and $n \neq 0$,

$$b^{-n} = \frac{1}{b^n} \text{ and } \frac{1}{b^{-n}} = b^n. \tag{2.1.3}$$

- If n is a positive integer and b ia a real number such that $b^{1/n}$ is a real number, then

$$\sqrt[n]{b} = b^{1/n} \tag{2.1.4}$$

Properties of Exponents

$$\begin{cases} b^m \cdot b^n = b^{m+n} \\ \dfrac{b^m}{b^n} = b^{m-n} \text{ if } b \neq 0 \\ \left(b^m\right)^n = b^{mn} \end{cases} \tag{2.1.5}$$

$$\begin{cases} (ab)^n = a^n b^n \\ \left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n} \end{cases} \qquad (2.1.6)$$

Properties of Radicals

If m and n are natural numbers greater than or equal to 2, and a and b are nonnegative real numbers, then

Product: $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ (2.1.7)

Quotient: $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}} \quad (b \neq 0)$ (2.1.8)

Index : $\begin{cases} \sqrt[m]{\sqrt[n]{b}} = \sqrt[mn]{b} \\ \left(\sqrt[n]{b}\right)^n = \sqrt[n]{b^n} = b \\ \sqrt[km]{b^{kn}} = \sqrt[m]{b^n} \end{cases}$ (2.1.9)

Example2.1.1. **Simplify radicals**

a) $\sqrt[9]{64}$ b) $\sqrt{12y^7}, y \geq 0$ c) $\sqrt[3]{\sqrt{x^8 y}}$

d) $\sqrt[3]{2} \cdot \sqrt[3]{4}$ e) $\sqrt[3]{\dfrac{8}{54}}$ f) $\sqrt[6]{x^2 y^9}, x \geq 0, y \geq 0$

Solution

a) $\sqrt[9]{64} = \sqrt[9]{2^6} =$ [use $(2.1.9)$, k = 3] $= \sqrt[3]{2^2} = \sqrt[3]{4}$

b) $\sqrt{12y^7} = \sqrt{3 \cdot 2^2 y^6 y} =$ [use (2.1.7), (1.1) and (2.1.9)]$= \sqrt{3} \cdot \sqrt{2^2} \cdot \sqrt{y^6} \cdot \sqrt{y} =$
$= \sqrt{3} \cdot 2 \cdot y^3 \cdot \sqrt{y} = 2y^3\sqrt{3y}$

c) $\sqrt[3]{\sqrt{x^8 y}} = \sqrt[6]{x^6 \cdot x^2 y} = x\sqrt[6]{x^2 y}$

d) $\sqrt[3]{2} \cdot \sqrt[3]{4} =$ [use (2.1.7)] $= \sqrt[3]{8} = \sqrt[3]{2^3} = 2$

e) $\sqrt[3]{\dfrac{8}{54}} = \sqrt[3]{\dfrac{4}{27}} = \sqrt[3]{\dfrac{4}{3^3}} = \dfrac{\sqrt[3]{4}}{3}$

f) $\sqrt[6]{x^2y^9} = \sqrt[6]{x^2} \cdot \sqrt[6]{y^9} = \sqrt[3]{x} \cdot \sqrt{y^3} = y \cdot \sqrt{y} \cdot \sqrt[3]{x}$

EXERCISE SET 2.1.1.

In Exercises 1 to 16, evaluate each expression.

1. -4^3 **2.** $(-4)^3$ **3.** -4^2 **4.** $(-4)^2$

5. -4^0 **6.** $(-4)^0$ **7.** 4^{-2} **8.** $4^{1/2}$

9. $2^7 \cdot 2^{-4} \cdot 2^{-1}$ **10.** $3^{-12} \cdot 3^{10}$ **11.** $\dfrac{5^6}{5^4}$ **12.** $\dfrac{4^{-4}}{4^{-6}}$

13. $(-27)^{-1/3}$ **14.** $\dfrac{5^{-1}}{5^2}$ **15.** $\dfrac{(2 \cdot 5)^2}{(2^{-1} \cdot 5)^3}$ **16.** $\left(\dfrac{-3^6 \cdot 2^{-4}}{-4^{-14}}\right)^0$

In Exercises 17 to 28, simplify each expression.

17. $(2x^2y^3) \cdot (3x^5y)$ **18.** $(2x^{-3}y^0) \cdot (3^{-1}xy)^2$ **19.** $(x^3y^{-2})^{-3}$

20. $\dfrac{(5ab^{-2})^2}{(-3a^2b)^3}$ **21.** $\left(\dfrac{3pq^2}{-2p^{-1}q^3r}\right)^2$ **22.** $\left(\dfrac{2x}{5y}\right)^{-2}$

23. $(3x^{-1}y)^{-2}(3xy^2)$ **24.** $\dfrac{6abc^{-2}}{(-3)^{-1}ab^{-2}c^{-3}}$ **25.** $\left(\dfrac{x^{-2}+x^3}{x^{-3}}\right)^0$

26. $\sqrt{\dfrac{20a^5}{9b^3}}$ **27.** $\dfrac{\sqrt[3]{24x^2y}}{\sqrt[3]{3x^2y^4}}$ **28.** $\dfrac{\sqrt{2xy}}{\sqrt[3]{2xy}}$

§ 2.2. Polynomials

- A **monomial** is a product of a constant and the variables having only nonnegative integer exponents. The constant is called the **coefficient** of the monomial. The **degree** of the monomial is the sum of the exponents of the variables. For example, $-5xy^2$ is a monomial with coefficient -5 and degree 3.

- The **general form of a polynomial** of degree n in the variable x is

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \qquad (2.2.1)$$

where $a_n \neq 0$ and n is nonnegative integer. The coefficient a_n is the **leading coefficient**, and a_0 is the **constant term.**

• To **add** (subtract) polynomials, we combine like terms.

Example 2.2.1

$$(3x^2 + 7x - 5) - (4x^2 - 2x + 1) = (3x^2 - 4x^2) + (7x - (-2x)) + (-5 - 1) =$$
$$= -x^2 + 9x - 6$$

• To **multiply** polynomials, we multiply every monomial of the first polynomial by every monomial of the second one and combine like terms.

Example 2.2.2.

$$(3x - 4)\cdot(2x^2 + 5x + 1) = 3x\cdot 2x^2 - 4\cdot 2x^2 + 3x\cdot 5x - 4\cdot 5x + 3x\cdot 1 - 4\cdot 1 =$$
$$= 6x^3 - \underline{8x^2} + \underline{15x^2} - \underline{\underline{20x}} + \underline{\underline{3x}} - 4 = 6x^3 + 7x^2 - 17x - 4.$$

In the following, a vertical format has been used to find the product of $(x^2 + 6x - 7)$ and $(5x - 2)$. Note that like terms are arranged in the same vertical column.

$$\begin{array}{r} x^2 + 6x - 7 \\ 5x - 2 \\ \hline -2x^2 - 12x + 14 \\ 5x^3 + 30x^2 - 35x \phantom{{}+14} \\ \hline 5x^3 + 28x^2 - 47x + 14 \end{array}$$

• **Special Product Formulas**

$$x^2 - y^2 = (x - y)\cdot(x + y) \qquad (2.2.2)$$

$$x^3 \pm y^3 = (x \pm y)\cdot(x^2 \mp xy + y^2) \qquad (2.2.3)$$

$$(x \pm y)^2 = x^2 \pm 2xy + y^2 \qquad (2.2.4)$$

$$\begin{aligned}(x \pm y)^3 &= x^3 \pm 3x^2y + 3xy^2 \pm y^3 = \\ &= x^3 \pm y^3 \pm 3xy(x \pm y)\end{aligned} \qquad (2.2.5)$$

Note that it is possible to **add** and to **multiply irrational (radical) expressions as polynomials.**

Example 2.2.3. Find the indicated product. Express each term in simplest form.

a) $\left(\sqrt{7}+4\right)\left(\sqrt{7}-1\right)$ b) $\left(\sqrt{2x}+a\right)\left(\sqrt{2x}-a\right)$

c) $\left(3\sqrt{2x}+4\right)^2$ d) $\left(\sqrt[3]{x-1}+\sqrt[3]{2}\right)^3$

Solution

a) $\left(\sqrt{7}+4\right)\cdot\left(\sqrt{7}-1\right) \quad =\sqrt{7}\cdot\sqrt{7}+\underline{4\sqrt{7}}-\underline{\sqrt{7}}+4\cdot(-1)=7+3\sqrt{7}-4=3-3\sqrt{7}$;

b) $\left(\sqrt{2x}+a\right)\left(\sqrt{2x}-a\right)=[\text{use } 2.2.2]=\left(\sqrt{2x}\right)^2-a^2=2x-a^2$;

c) $\left(3\sqrt{2x}+4\right)^2=[\text{use } (2.2.3)\ ]=\left(3\sqrt{2x}\right)^2+2\cdot 3\sqrt{2x}\cdot 4+4^2=9\cdot 2x+24\sqrt{2x}+16=$
$=18x+24\sqrt{2x}+16;$

d) $\left(\sqrt[3]{x-1}+\sqrt[3]{2}\right)^3=[\text{use } (2.2.5)]=(x-1)+3\sqrt[3]{2(x-1)^2}+3\sqrt[3]{2^2(x-1)}+2=x-1+$
$+3\sqrt[3]{2(x-1)^2}+3\sqrt[3]{4(x-1)}.$

Example 2.2.4.

Using the Special Product Formulas rationalize the Denominator

a) $\dfrac{2}{\sqrt{3}+\sqrt{5}}$ b) $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$ c) $\dfrac{1}{\sqrt[3]{a}+b}$

Solution

a) $$\frac{2}{\sqrt{3}+\sqrt{5}}=\left[\begin{array}{l}\text{multiply the numerator and the}\\ \text{dinominator by congugate of}\\ \text{dinominator}:\left(\sqrt{3}-\sqrt{5}\right)\end{array}\right]=\frac{2\left(\sqrt{3}-\sqrt{5}\right)}{\left(\sqrt{3}+\sqrt{5}\right)\left(\sqrt{3}-\sqrt{5}\right)}=$$

$$[\text{use } (2.2.2)]=\frac{2\left(\sqrt{3}-\sqrt{5}\right)}{\left(\sqrt{3}\right)^2-\left(\sqrt{5}\right)^2}=[\text{use}(2.1.9)]=\frac{2\left(\sqrt{3}-\sqrt{5}\right)}{3-5}=-\left(\sqrt{3}-\sqrt{5}\right)=\sqrt{5}-\sqrt{3}$$

b) $$\frac{a+\sqrt{b}}{a-\sqrt{b}}=\frac{\left(a+\sqrt{b}\right)\left(a+\sqrt{b}\right)}{\left(a-\sqrt{b}\right)\left(a+\sqrt{b}\right)}=\frac{\left(a+\sqrt{b}\right)^2}{a-b}$$

c) $$\frac{1}{\sqrt[3]{a}+b}=\left[\begin{array}{l}\text{multiply the numerator and the dinominator by}\\ \left(\sqrt[3]{a^2}-b\sqrt[3]{a}+b^2\right)\text{ and use}(2.2.3)\end{array}\right]=\frac{\sqrt[3]{a^2}-b\cdot\sqrt[3]{a}+b^2}{a+b^3}.$$

- **The Division Algorithm for Polynomials**

If $P(x)$ and $D(x) \neq 0$ are polynomials, then there exist unique polynomials $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$, where either $R(x) = 0$ or the degree of $R(x)$ is less than the degree of $D(x)$. To find the polynomials $Q(x)$ and $R(x)$ we can use the **long division** of the polynomial $P(x)$ by $D(x)$.

Example 2.2.5. Perform the indicated division. $\dfrac{2x^3 + 2x^2 + 5x - 6}{x^2 - 3x + 5}$

Solution

$$
\begin{array}{rl|l}
 & 2x^3 + 2x^2 + 5x - 6 & x^2 - 3x + 5 \\ \cline{3-3}
 - & \underline{\left(2x^3 - 6x^2 + 10x\right)} & 2x + 8 \\
 & 8x^2 - 5x - 6 & \\
 - & \underline{\left(8x^2 - 24x + 40\right)} & \\
 & 19x - 46 &
\end{array}
$$

Thus we have

$$\frac{2x^3 + 2x^2 + 5x - 6}{x^2 - 3x + 5} = 2x + 8 + \frac{19x - 46}{x^2 - 3x + 5}.$$

§ 2.3. Functions and Graphs

- Let two sets $D = \{x\}$ and $E = \{y\}$ be given. A **function** f from a set D to a set E is a correspondence of these sets such that for any $x \in D$ there is exactly one element $y \in E$. A function is denoted by $y = f(x)$.
- The set D is called the **domain** of f,(for any $x \in D$ $f(x)$ exists) and the set E is called the **range** of f.
- A meaning of $x \in D$ such that $f(x) = 0$ is called a **root** or a **zero** of this function. It is x-intersect point. Roots of a function divide its domain onto intervals where a function has one and the same sign (is positive (negative)).
- A function $f(x)$ is said to have a relative, or local, **maximum** at $x = a$ if

$$f(a) \geq f(a + \varepsilon)$$

for all positive and negative values of ε close to zero. For a local **minimum** at $x = b$

$$f(b) \leq f(b + \varepsilon)$$

for all values of ε close to zero

- A set of the points in the coordinate plane with the coordinates $(x, f(x))$ is called a **graph of a function** $y = f(x)$.

• If x_1 and x_2 are elements of any interval I that is a subset of domain of a function $f(x)$, then

1) f is **increasing** on I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$;
2) f is **decreasing** on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$;
3) f is **constant** if $f(x_1) = f(x_2)$ for all x_1 and x_2.

• The function $f(x)$ is called an **even function** if for any $x \in D$, $-x \in D$ and $f(-x) = f(x)$. if $f(-x) = -f(x)$ then the function $f(x)$ is called an **odd function.**

• Let P be a constant. If $f(x+P) = f(x)$ for all x in the domain of $f(x)$, then the function $f(x)$ is called a **periodic function**. The smallest positive value of P for which $f(x+P) = f(x)$ is called a **period** of the given function.

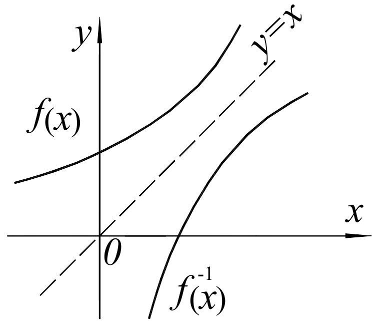
• A function $f(g(x))$ is called a **composite function**. Let us denote it by symbol $(f \circ g)(x)$.

• If $f(x)$ is **one-to-one** function with domain D and range E and $g(x)$ is a function with domain E and range D, then $g(x)$ is the inverse function of $f(x)$ if and only if

$(f \circ g)(x) = x$ for all $x \in D(g)$

and

$(g \circ f)(x) = x$ for all $x \in D(f)$.



The inverse function of a function $f(x)$ is denoted as $f^{-1}(x)$.

• The graphs of the given function $f(x)$ and its inverse $f^{-1}(x)$ are symmetric one to other with respect to the line $y = x$.

Example 2.3.1.

Verify that $g(x) = \dfrac{1}{2}x - 4$ is the inverse for $f(x) = 2x + 8$.

Solution.

We need to show that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$.

1) $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{2}x - 4\right) = 2\left(\frac{1}{2}x - 4\right) + 8 = x;$

2) $(g \circ f)(x) = g(f(x)) = g(2x + 8) = \frac{1}{2}(2x + 8) - 4 = x\,.$

• **To find the inverse function** of the given one we must

1) solve the equation of the given function for x;
2) interchange x and y;
3) verify that $D(f) \to E\left(f^{-1}\right)$ and $E(f) = D\left(f^{-1}\right)$.

Example 2.3.2.

Find the inverse of $f(x) = 2x - 6$.

Solution.

1) Solve the equation of the given function for x :

$$f(x) = 2x - 6 \Leftrightarrow y = 2x - 6 \Rightarrow 2x = y + 6 \Rightarrow x = \frac{1}{2}y + 3$$

2) Interchange x and y: $y = \frac{1}{2}x + 3$. So we have $f^{-1} = \frac{1}{2}x + 3$.
3) It is obviously that $\left(f \circ f^{-1}\right)(x)=\left(f^{-1} \circ f\right)(x) = x$, and $D(f) = E\left(f^{-1}\right)$, $D\left(f^{-1}\right) = E(f)$.

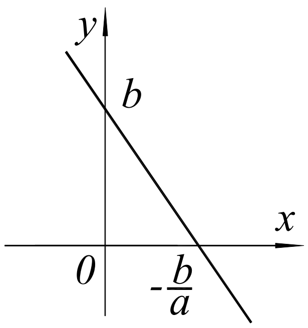
Answer: $f^{-1} = \frac{1}{2}x + 3$.

2.3.1. Linear Function

• A **linear function** is a function that can be represented by an equation of the form

$$y = ax + b \tag{2.3.1}$$

a is a **slope** $(a = \tan\alpha)$,
b is a y-**intercept point**.



A graph of a linear function is a **straight line.**

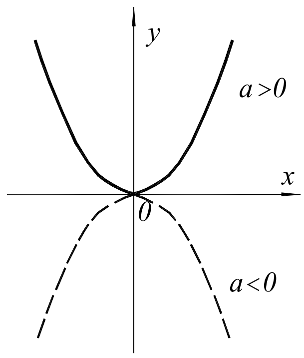
$D = (-\infty, \infty)$

$E = (-\infty . \infty)$

$x = -\frac{b}{a}$ is a **root**.

2.3.2. Quadratic Function

• A **quadratic function** is a function that can be represented by an equation of the form



$$y = ax^2 + bx + c \qquad (2.3.2)$$

where a, b, c are real numbers and $a \neq 0$.
The graph of a quadratic function is **parabola**.
If b and c are both zero, then (2.3.2) simplifies to $y = ax^2$

The graph of this function is a parabola that

1. opens up if $a > 0$, $D(y) = (-\infty;\infty)$, $E(y) = [0;\infty)$

2. opens down if $a < 0$, $D(y) = (-\infty;\infty)$, $E(y) = (-\infty;0]$.

• **Note** that the graph of a function $y = ax^{2n}$, $n \in N$ looks like the graph of $y = ax^2$.

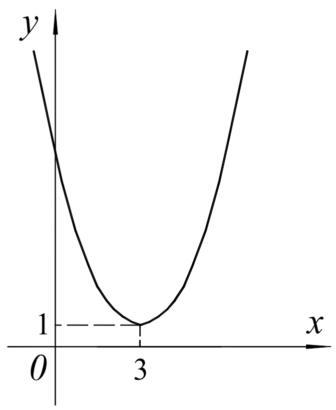
Every quadratic function given by (2.3.2) can be written in the **standard form**

$$y = a(x - x_0)^2 + y_0, \qquad a \neq 0$$

The graph of this function is a parabola with vertex (x_0, y_0).

Example 2.3.1. Use the technique of completing the square to find the standard form of the quadratic function $y = 2x^2 - 12x + 19$. Sketch the graph. Define $D(y)$ and $E(y)$.

Solution.



$$y = 2x^2 - 12x + 19 = \left[\begin{array}{l}\text{factor 2 from the}\\ \text{variable terms}\end{array}\right] =$$

$$= 2(x^2 - 6x) + 19 = [\text{complete the square}] =$$

$$= 2\left(x^2 - 2\cdot 3x + 3^2 - 9\right) + 19 = [\text{regroup}] =$$

$$= 2\left(x^2 - 6x + 3^2\right) - 18 + 19 = 2(x-3)^2 + 1.$$

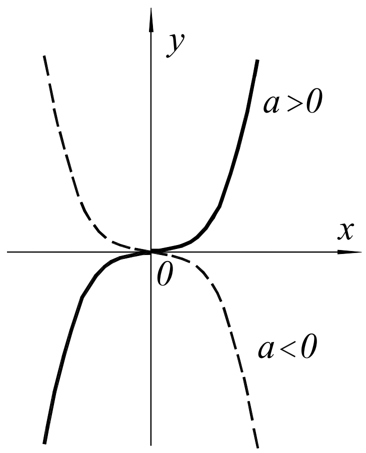
Exercise 2.3.2.

Prove that the vertex of the graph of $y = ax^2 + bx + c$ is a point

$$C\left(-\frac{b}{2a};\frac{4ac - b^2}{4a}\right).$$

2.3.3. Cube Parabola

Graph of a function $y = ax^3$ is called a **cube parabola**.

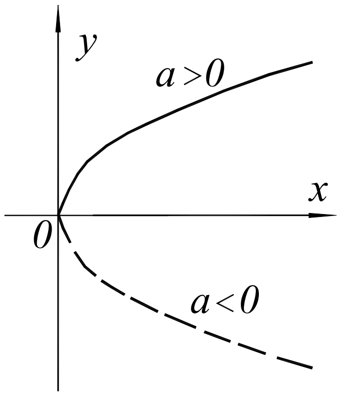


$D(y) = (-\infty, +\infty)$
$E(y) = (-\infty, \infty)$

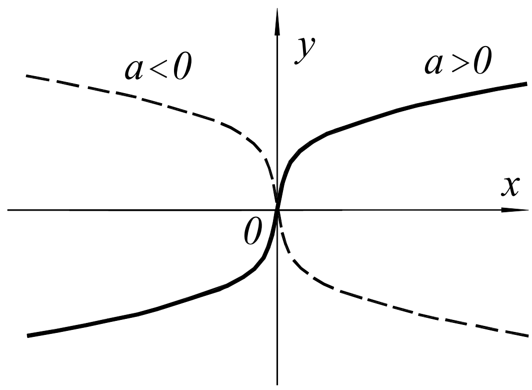
• The graph of the function $y = ax^{2n+1}, n \in N$ is the same as the one given above.

2.3.4. Other Power Functions

1). $y = a\sqrt[2n]{x}, n \in N$



2). $y = a\sqrt[2n+1]{x}, n \in N$



$D(y) = [0,+\infty)$

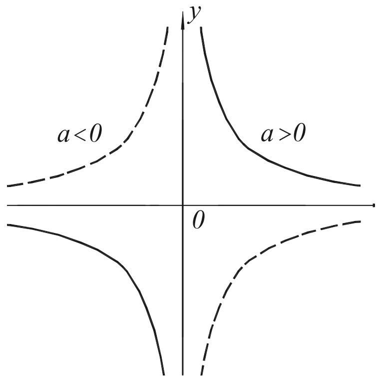
$E(y) = \begin{cases} [0,+\infty), a > 0 \\ (-\infty,0], a < 0 \end{cases}$

$D(y) = (-\infty,+\infty)$

$E(y) = (-\infty,+\infty)$

3). Hyperbolas

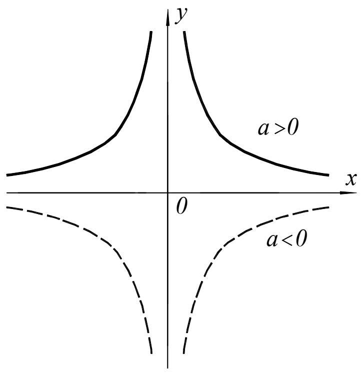
a). $y = \dfrac{a}{x^{2n-1}}, n \in N$



$D(y) = (-\infty,0) \cup (0,+\infty)$

$E(y) = (-\infty,0) \cup (0,+\infty)$

b). $y = \dfrac{a}{x^{2n}}, n \in N$



$D(y) = (-\infty,0) \cup (0,+\infty)$

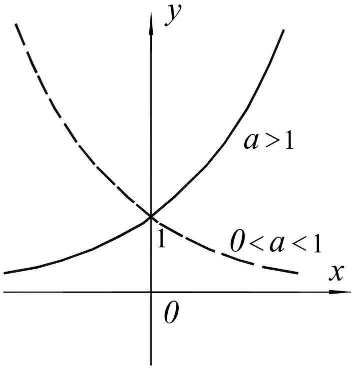
$E(y) = \begin{cases} (0,+\infty), a > 0 \\ (-\infty,0), a < 0 \end{cases}$

2.3.5. Exponential Function

The exponential function *y* with base *a* is defined by

$$y = a^x$$

where *a* is a positive constant other than 1 and *x* is any real number.



1. $D(y) = (-\infty,+\infty)$, $E(y) = (0,+\infty)$
2. ***y*-intercept point** is $(0;1)$
3. *y* has a graph **asymptotic** to the *x*-axis
4. *y* is **one-to-one** function
5. *y* is an **increasing** function if $a > 1$ and *y* is a **decreasing** function if $a < 1$.

• We can verify that as n increases without bound, $\left(1+\frac{1}{n}\right)^n$ approaches an irrational number that is denoted by e $\left(e \approx 2.71828183\right)$.

• For all real numbers x, the function defined by

$$y = e^x$$

is called the **natural exponential function**.

2.3.6. Logarithms, Logarithmic Properties, Logarithmic Functions and Their Graphs

Every exponential function is a one-to-one function and therefore has an inverse function. We can determine the inverse of a function represented by an equation by interchanging the variables and then solving for the dependent variable. If we attempt to use this procedure for , we get

$$y = a^x \Rightarrow \left[\text{interchange the variables}\right] \Rightarrow x = a^y$$

Sometimes it is possible to find this equation with respect to y, for example

$$8 = 2^y \Rightarrow y = 3.$$

In fact, $2^3 = 8$. But to find y if $2^y = 7$ we must develop a new procedure. We use the notation in the following definition.

• If $x > 0$ and a is a positive constant $\left(a \neq 1\right)$, then

$$y = \log_a x \text{ if and only if } a^y = x \qquad (3.6.1)$$

in the equation $y = \log_a x$, y is referred to as the **logarithm**, a is the **base**, and x is the **argument**. The notation $\log_a x$ is read **"the logarithm of x to the base a "**. The definition of a logarithm indicates that **a logarithm is an exponent**. Now it is possible to solve the equation $2^y = 7$:

$$2^y = 7 \Rightarrow y = \log_2 7.$$

Example 3.6.1.

Evaluate each logarithm.

a) $\log_2 32 = x$ **b)** $\log_5 125 = x$ **c)** $\log_7 \frac{1}{49} = x$

Solution

a) $\log_2 32 = x$ **if and only if** $2^x = 32 \Rightarrow 2^x = 2^5 \Rightarrow x = 5.$ **Thus** $\log_2 32 = 5.$

b) $\log_5 125 = x \Leftrightarrow 5^x = 125 \Rightarrow 5^x = 5^3 \Rightarrow x = 3$**. Thus** $\log_5 125 = 3.$

d) $\log_7 \frac{1}{49} = -2$ **as** $7^{-2} = \frac{1}{49}.$

Properties of Logarithms

In the following properties, b, M, and N are positive real numbers $(b \neq 1)$, and p is any real number.

$$\log_b b = 1 \tag{3.6.2}$$
$$\log_b 1 = 0 \tag{3.6.3}$$

$$\left.\begin{array}{l}\log_b\left(b^p\right) = p \\ b^{\log_b p} = p\,(\text{for } p > 0)\end{array}\right\} \quad \bullet\ \textit{An inverse property} \tag{3.6.4}$$

$$\log_b MN = \log_b M + \log_b N \quad \bullet\ \textit{Product property} \tag{3.6.5}$$

$$\log_b \frac{M}{N} = \log_b M - \log_b N \quad \bullet\ \textit{Quotient property} \tag{3.6.6}$$

$$\log_b\left(M^p\right) = p \log_b M \quad \bullet\ \textit{Power property} \tag{3.6.7}$$

$$\log_b M = \log_b N \Rightarrow M = N \quad \bullet\ \textit{One-to-one property} \tag{3.6.8}$$

$$M = N \Rightarrow \log_b M = \log_b N \quad \bullet \begin{cases}\textit{Logarithm of each side} \\ \textit{property}\end{cases} \tag{3.6.9}$$

Change-of-Base Formula

If $a, x,$ and b are positive real numbers with $a \neq 1$ and $b \neq 1$**, then**

$$\log_b x = \frac{\log_a x}{\log_a b} \tag{3.6.10}$$

• Logarithms with a base of 10 are called **common logarithm**. It is customary to write $\log_{10} x$ as $\log x$ or $\lg x$.

• Logarithms with a base of e are called **natural logarithm**. It is customary to write $\log_e x$ as $\ln x$.

Example 3.6.2. Rewrite Logarithmic Expressions

1. Use the properties of logarithms to express the following logarithms in terms of logarithms of *x, y,* and *z*.

a) $\log_b xy^2$ b) $\log_b \frac{xy^2}{\sqrt[5]{z}}$

Solution

a) $\log_b xy^2 =$[use product property (3.6.5)] $= \log_b x + \log_b y^2 =$
$=$ [use power property (3.6.7)] $= \log_b x + 2\log_b y$

b) $\log_b \frac{xy^2}{\sqrt[5]{z}} =$ [use quotient formula (3.6.6)] $= \log_b xy^2 - \log_b z^{1/5} =$

$$= \log_b x + 2\log_b y - \frac{1}{5}\log_b z$$

2. Use the properties of logarithms to rewrite the following logarithms as a single logarithm.

a) $2\log_a x + \frac{1}{2}\log_a(x+4)$ b) $4\log_b(x+2) - 3\log_b(x-5)$

Solution

a) $2\log_a x + \frac{1}{2}\log_a(x+4) = \log_a x^2 + \log_a(x+4)^{1/2} = \log_a x^2 + \log_a \sqrt{x+4} =$
$= \log_a x^2\sqrt{x+4}$

b) $4\log_b(x+2) - 3\log_b(x-5) = \log_b(x+2)^4 - \log_b(x-5)^3 = \log_b \frac{(x+2)^4}{(x-5)^3}$

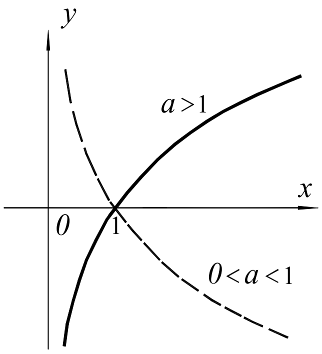
Logarithmic Functions and Their Graphs

• The **logarithmic functions with base** a is defined by

$$y = \log_a x \qquad (3.6.11)$$

where b is positive constant $b \neq 1$, and x is any positive real number.

The logarithmic function $y = \log_a x$ is the inverse of the exponential function $y = a^x$. That is why the graph of $y = \log_a x$ is symmetric to the graph of $y = a^x$ with respect to the line $y = x$.



You can be sure that it is correct. To do it compare the graph of the function $y = a^x$ (see page 28) and the graph of the function $y = \log_a x$.

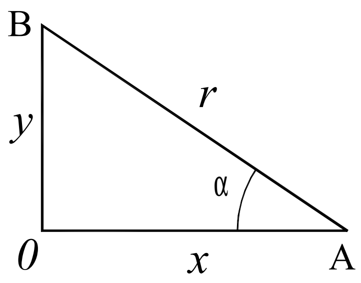
It is obvious that

$D(y) = (0, +\infty)$

$E(y) = (-\infty, +\infty)$

2.3.7. Trigonometric Functions of an Acute Angle

Let α be an acute angle of a right triangle. The values of four trigonometric functions of α are



$$\sin\alpha = \frac{\text{length of opposite side}}{\text{length of hypotenuse}} = \frac{y}{r}$$

$$\cos\alpha = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}} = \frac{x}{r}$$

$$\tan\alpha = \frac{\text{length of opposite side}}{\text{length of adjacent side}} = \frac{y}{x}$$

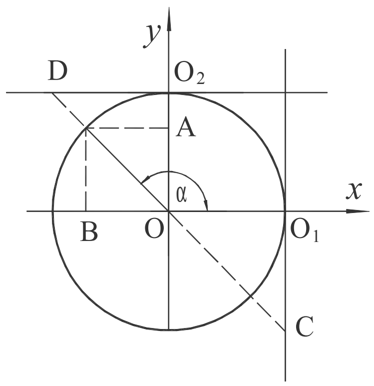
$$\cot\alpha = \frac{\text{length of adjacent side}}{\text{length of opposite side}} = \frac{x}{y}$$

Using these definitions we get the table of trigonometric functions of special angles

α	$30^0 = \frac{\pi}{6}$	$45^0 = \frac{\pi}{4}$	$60^0 = \frac{\pi}{3}$
$\sin\alpha$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\alpha$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\alpha$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
$\cot\alpha$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

2.3.8. Trigonometric Functions of any Angle and their Graphs

In general case if $P(x; y)$ be given, then



$$\sin\alpha = \frac{y}{r},\ \cos\alpha = \frac{x}{r},$$

$$\tan\alpha = \frac{y}{x},\cot\alpha = \frac{x}{y},$$

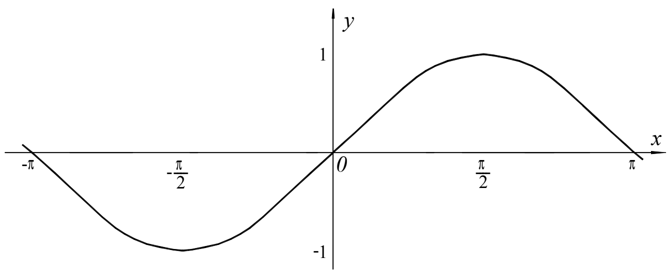
where $r = OP = \sqrt{x^2 + y^2}$.

Let the radius of circle be equal to 1, then the vertical diameter is called the **sines-line**, the horizontal diameter – the **cosines-line**, the upper tangent – **line of cotangent**, the right tangent – **line of tangent**. In this case circle is called the trigonometric circle, and

$$\sin\alpha = y = OA,$$
$$\cos\alpha = x = OB,$$
$$\tan\alpha = O_1C,$$
$$\cot\alpha = O_2D.$$

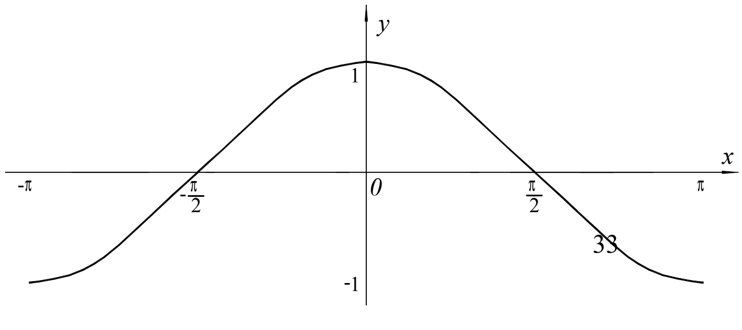
Using this circle it is possible to draw the graphs of trigonometric functions.

a) **Sine curve** $y = \sin x$



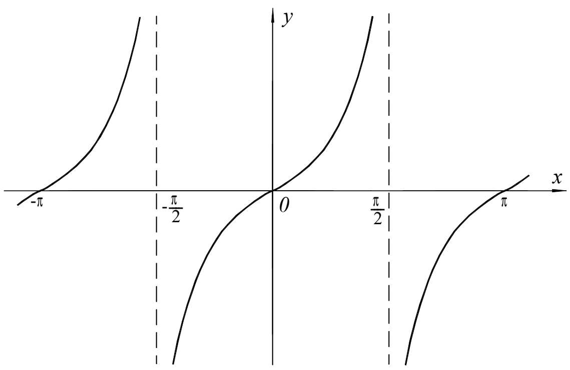
$D(y) = (-\infty, +\infty)$,
$E(y) = [-1, +1]$

b) **Cosine curve** $y = \cos x$



$D(y) = (-\infty, +\infty)$,
$E(y) = [-1, +1]$

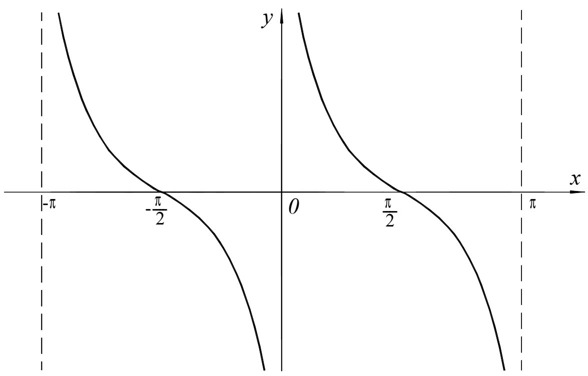
c) **Tangent curve** $y = \tan x$



$$D(y) = \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right), k = 0, \pm 1, \pm 2, \ldots$$

$$E(y) = (-\infty, +\infty)$$

d) **Cotangent curve** $y = \cot x$

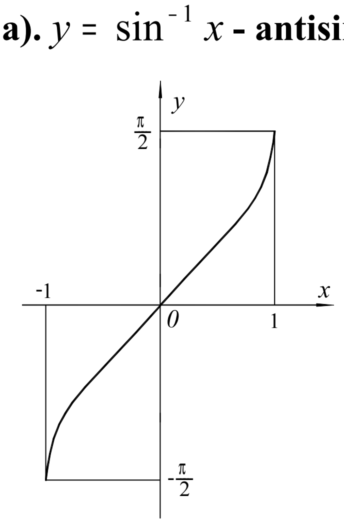


$$D(y) = (k\pi, (k+1)\pi), k = 0, \pm 1, \pm 2, \ldots$$

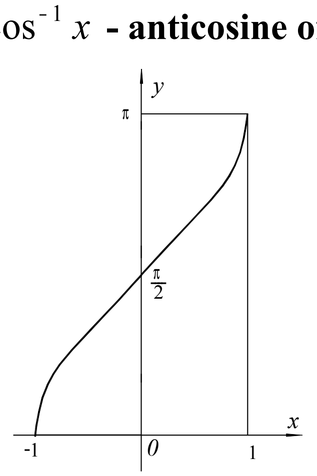
$$E(y) = (-\infty, +\infty)$$

2.3.9. Inverse Trigonometric Functions

a). $y = \sin^{-1} x$ **- antisine of** x



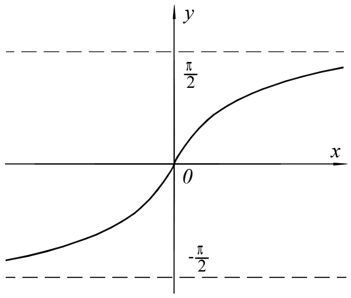
b). $y = \cos^{-1} x$ **- anticosine of** x



$D(y) = [-1;1]$, $E(y) = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

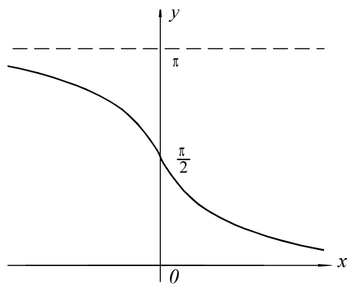
$D(y) = [-1;1]$, $E(y) = [0;\pi]$

c). $y = \tan^{-1} x$ **- antitangent of** $\boldsymbol{x}$



$D(y) = (-\infty, +\infty)$, $E(y) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

d). $y = \cot^{-1} x$ **- anticotangent of** $\boldsymbol{x}$



$D(y) = (-\infty, +\infty)$, $E(y) = (0, \pi)$

- To find the domain of composite function we can use the next table

$y =$	$\frac{f(x)}{g(x)}$	$\sqrt[2k]{f(x)}$	$\log_a f(x)$	$\tan f(x)$	$\cot f(x)$	$\sin^{-1} f(x)$, $\cos^{-1} f(x)$
$D(y)$:	$g(x) \neq 0$	$f(x) \geq 0$	$f(x) > 0$ $a > 0, a \neq 1$	$f(x) \neq \frac{\pi}{2} + \pi n$ $n \in Z$	$f(x) \neq \pi n$ $n \in z$	$\lvert f(x) \rvert \leq 1$

2.3.10. The Fundamental Trigonometric Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$1 + \tan^2 x = \frac{1}{\cos^2 x} \qquad 1 + \cot^2 x = \frac{1}{\sin^2 x}$$

2.3.11. Sum and Difference Identities

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$
$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}$$

2.3.12. Cofunction Identities

$$\sin(90^0 - x) = \cos x \qquad \cos(90^0 - x) = \sin x$$
$$\tan(90^0 - x) = \cot x \qquad \cot(90^0 - x) = \tan x$$

2.3.13. Double- and Half-Angle Identities

$$\sin 2x = 2\sin x \cos x \qquad \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$$
$$\cos 2x = \cos^2 x - \sin^2 x \qquad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x} \qquad \tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

2.3.14. Product-to-Sum Identities

$$\cos\alpha \cos\beta = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$$
$$\sin\alpha \sin\beta = -\frac{1}{2}(\cos(\alpha+\beta) - \cos(\alpha-\beta))$$
$$\cos\alpha \sin\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))$$
$$\sin\alpha \cos\beta = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$$

2.3.15. Sum-to-Product Identities

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$
$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$$

$$a\sin x + b\cos x = k\sin(x+\alpha),$$

where $$k = \sqrt{a^2+b^2}, \quad \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}, \quad \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}.$$

III. COMPLEX NUMBERS

§ 3.1. The Fundamental Operations

The square of a real number is never negative. Thus, for example, the elementary quadratic equation $x^2 = -1$ has no solution among the real numbers. New types of numbers, called complex numbers, have been introduced to provide solutions to such equations.

Definition. By a **complex number** we mean an ordered pair of real numbers which we denote by (x, y).
The first member, *x,* is called the **real part** of the complex number; the second member, *y*, is called the **imaginary part**. We write

$$z = (x, y).$$

The equality relation and the arithmetical operations are defined according to the following rules:

1. equality $(x_1, y_1) = (x_2, y_2)$ takes place if and only if $x_1 = x_2,\ y_1 = y_2$;
2. $(x_1, y_1) \pm (x_2, y_2) = (x_1 \pm x_2, y_1 \pm y_2)$;
3. $(x_1, y_1)\cdot(x_2, y_2)\,(x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)$.

If the fundamental operations are thus defined, we easily see that the fundamental laws of algebra are all satisfied.

1. The commutative and associative laws of addition hold:
$$z_1 + z_2 = z_2 + z_1;$$
$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 = z_1 + z_2 + z_3.$$

2. The same laws of multiplication hold:
$$z_1z_2 = z_2z_1;$$
$$z_1(z_2z_3) = (z_1z_2)\,z_3 = z_1z_2z_3.$$

3. The distributive law holds:
$$(z_1 + z_2)z_3 = z_1z_3 + z_2z_3.$$

§ 3.2. Geometrical Representation of Complex Numbers

Just as real numbers are represented geometrically by points on a line, so complex numbers are represented by points in a plane. The complex number $z = (x, y)$ can be thought of as the point with coordinates (x, y). When this is done, the definition of addition amounts to addition by the parallelogram law.

The idea of expressing complex numbers geometrically as points on a plane was formulated by Gauss in his dissertation in 1799 and, independently, by Argand in 1806. Gauss later coined the somewhat unfortunate phrase "complex number".

§ 3.3. The Imaginary Unit

It is convenient to think of the complex number $(x.y)$ as a two-dimensional vector with components *x* and *y*. Adding two complex numbers is the same as adding two vectors component by component. The complex number $1 = (1,0)$ plays the same role as a unit vector in the horizontal direction. The analog of a unit vector in the vertical direction will now be introduced.

Definition. The complex number $(0,1)$ is defined by i and is called the **imaginary unit**.

Theorem. Every complex number $z = (x, y)$ can be represented in the form $z = x + yi$ which is called **standard** or **rectangular form** of complex numbers.

Proof.

$z = (x.y) = (x.0) + (0, y) = x(1,0) + y(0,1) = x + yi.$

Let us now prove that $i^2 = -1$. In fact,

$i^2 = (0,1)(0,1) = (-1,0) = -1.$

Example 3.3.1.

Find the product of $z_1 = 2 + 3i$ and $z_2 = 5 - 4i.$

Solution.

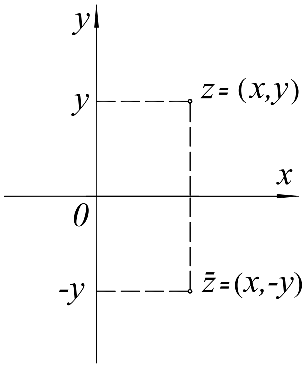
$z_1 z_2 = (2 + 3i)(5 - 4i) = 10 - 8i + 15i - 12i^2 = 22 + 7i.$

Exercise 3.3.1. Prove that $$i^n = \begin{cases} 1, \text{if } n = 4k \\ i, \text{if } n = 4k + 1 \\ -1, \text{if } n = 4k + 2 \\ -i, \text{if } n = 4k + 3 \end{cases}$$

§ 3.4. Absolute Value of a Complex Number and Conjugate Complex Number

Definition. If $z=(x,y)$, we define the **modulus**, or **absolute value**, of z to be the non-negative real number $|z|$ given by

$$|z|=\sqrt{x^2+y^2}.$$



Geometrically, $|z|$ represents the length of the segment joining the origin to the point $z=(x,y)$.

Definition. The number $x-yi$ is said to be **conjugate** to z and is denoted by $\bar{z}$

Let us calculate $z\bar{z}$.

$$z\bar{z}=(x+yi)\cdot(x-yi)=x^2-(yi)^2=x^2+y^2=|z|^2.$$

§ 3.5. Definition of Division

The division is an operation inverse to the multiplication.
The number z is called the quotient of z_1 and z_2 if $z_1=z\cdot z_2$. If $z_2\neq 0$ then on multiplying both parts of the relation $z_1=z\cdot z_2$ by $\overline{z_2}$ we get

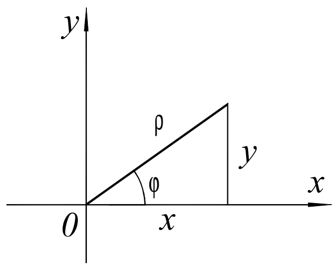
$$z_1\overline{z_2}=z\left(z_1\overline{z_2}\right) \text{ and } z=\frac{z_1}{z_2}=\frac{z_1\overline{z_2}}{z_2\overline{z_2}}.$$

Example 3.5.1 Find the quotient of $z_1=2+3i$ and $z_2=1+4i$.

Solution.

$$\frac{z_1}{z_2}=\frac{2+3i}{1+4i}=\frac{(2+3i)\cdot(1-4i)}{(1+4i)\cdot(1-4i)}=\frac{14-5i}{1+16}=\frac{14}{17}-\frac{5}{17}i.$$

§ 3.6. The Trigonometric Form of a Complex Number



If the point $z=(x,y)=x+yi$ is represented by polar coordinates ρ and φ, we can write $x=\rho\cos\varphi$ and $y=\rho\sin\varphi$ then $z=\rho(\cos\varphi+i\sin\varphi)$. This form is called the **trigonometric form** of a complex number .
The x-axis along which x is reckoned is called **real axis** and the y-axis along which y is reckoned is the **imaginary axis**.

The two numbers ρ and φ uniquely determine z. Conversely, the positive number ρ is uniquely determined by z. In fact, $\rho=|z|$

$$\rho = \sqrt{x^2 + y^2} \tag{3.6.1}$$

However, z determines the angle φ only up to multiples of 2π. There are infinitely many values of φ which satisfy the equations $x = |z|\cos\varphi$, $y = |z|\sin\varphi$.

The unique real number φ which satisfies the condition $-\pi < \varphi \le \pi$ is called the principal argument of z and is denoted by $\arg z$:

$$\varphi = \arg z.$$

$$\cos\varphi = \frac{x}{\sqrt{x^2 + y^2}}, \ \sin\varphi = \frac{y}{\sqrt{x^2 + y^2}}, \tag{3.6.2}$$

Let z_1 and z_2 be two complex numbers written in trigonometric form. The product of z_1 and z_2 can be found by using several trigonometric identities.

If $z_1 = \rho_1(\cos\varphi_1 + i\sin\varphi_2)$, and $z_2 = \rho_2(\cos\varphi_2 + i\sin\varphi_2)$, then

$$z_1 z_2 = \rho_1\rho_2\left(\cos\varphi_1 \cdot \cos\varphi_2 + i\cos\varphi_1\sin\varphi_2 + i\sin\varphi_1\cos\varphi_2 + i^2\sin\varphi_1\sin\kappa_2\right) =$$
$$= \rho_1\rho_2((\cos\varphi_1 co\varphi_2 - \sin\varphi_1\sin\varphi_2) + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2)) =$$
$$= \rho_1\rho_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)) \Rightarrow$$
$$\rho_1\rho_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)) \tag{3.6.3}$$

The modulus for **the product** of two complex numbers **in trigonometric form** is the product of moduli of the two complex numbers, and the argument of the product is the sum of the arguments of these numbers.

Similarly,

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)) \tag{3.6.4}$$

The modulus for **the quotient** of two complex numbers **in trigonometric form** is the quotient of moduli of the two complex numbers, and the argument of the quotient is the difference of the arguments of these numbers.

Example 3.6.1.

Find the product of $z_1 = -1 + i\sqrt{3}$ and $z_2 = -\sqrt{3} + i$.

Solution.

1) Using (3.6.1) and (3.6.2) write z_1 and z_2 in trigonometric form:

$$z_1 = 2\left(\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}\right); \quad z_2 = 2\left(\cos\frac{5\pi}{6} + \sin\frac{5\pi}{6}\right)$$

2) Use (3.6.3)

$$z_1 z_2 = 4\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right) = 4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 4(0 - i) = -4i.$$

§ 3.7. Integral Powers and Roots of Complex Numbers

Let $z = \rho(\cos\varphi + i\sin\varphi)$. Then z^2 can be written as

$$z\cdot z = \rho(\cos\varphi + i\sin\varphi)\,\rho(\cos\varphi + i\sin\varphi) = \rho^2(\cos 2\varphi + i\sin 2\varphi).$$

This formula can be extended for raising a complex number to the *n*th power:

$$z^n = \rho^n(\cos n\varphi + i\sin n\varphi) \qquad (3.7.1)$$

The formula

$$(\cos\varphi + i\sin\varphi)^n = \rho^n(\cos n\varphi + i\sin n\varphi)$$

is called **De Moivre's** formula.

Definition. A number *w* is called the *n*th root of *z* if $w^n = z$ and is denoted by

$$w = \sqrt[n]{z}.$$

Let $w = r(\cos\theta + i\sin\theta)$ and $z = \rho(\cos\varphi + i\sin\varphi)$. Then as $w^n = z$ we have

$$r^n(\cos n\theta + i\sin n\theta) = \rho(\cos\varphi + i\sin\varphi).$$

Two complex numbers written in trigonometric form are equal if and only if their moduli are equal and their arguments are equal up to multiples of 2π. Thus

$$\begin{cases} r^n = \rho \\ n\theta = \varphi + 2k\pi \end{cases} \Rightarrow \begin{cases} r = \sqrt[n]{\rho} \\ \theta = \dfrac{\varphi + 2k\pi}{n} \end{cases}.$$

If $z = \rho(\cos\varphi + i\sin\varphi)$ is a complex number, then there are *n* distinct *n*th roots of *z* given by the formula

$$w_k = \sqrt[n]{\rho}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right) \qquad (3.7.2)$$

for $k = 0,1,2,\ldots,n-1$.

Example 3.7.1. Find the three cube roots of 27.

Solution. Write 27 on trigonometric form:

$$27 = 27(\cos 0 + i\sin 0).$$

Then, using formula (3.7.2), the cube roots of 27 are

$$w_k = \sqrt[3]{27}\left(\cos\frac{0 + 2k\pi}{3} + i\sin\frac{0 + 2k\pi}{3}\right) \text{ for } k = 0,1,2.$$

Substitute for *k* to find the cube roots of 27 :

$$w_0 = 3(\cos 0 + i\sin 0) = 3$$

$$w_1 = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i,$$

$$w_2 = 3\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i.$$

For $k = 3$ cosines and sines of the angles start repeating, thus there are only three cube roots of 27.

§ 3.8. Complex Exponentials

Let us write a complex number in trigonometric form

$$z = \rho\left(a\cos\varphi + i\sin\varphi\right)$$

Using Euler's formula

$$e^{i\varphi} = \cos\varphi + \sin\varphi \qquad (3.8.1)$$

we obtain $z = \rho e^{i\varphi}$ in the so-called **exponential form**.

Representing complex numbers in exponential form is particularly useful in connection with multiplication and division since we have

$$z_1 z_2 = \rho_1 e^{i\varphi_1} \rho_2 e^{i\varphi_2} = \rho_1 \rho_2 e^{i(\varphi_1 + \varphi_2)}$$

and

$$\frac{z_1}{z_2} = \rho_1 e^{i\varphi_1} : \rho_2 e^{i\varphi_2} = \frac{\rho_1}{\rho_2} e^{i(\varphi_1 - \varphi_2)}$$

If $z = \rho e^{i\varphi}$ then

$$z^n = \left(\rho e^{i\varphi}\right)^n = \rho^n e^{in\varphi}$$

This is De Moivre's formula in exponential form.

On replacing φ for $-\varphi$ we get such formula

$$e^{-i\varphi} = \cos\varphi - i\sin\varphi \qquad (3.8.2)$$

On adding and subtracting formulas (3.8.1) and (3.8.2) we have

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}.$$

The product of a complex number $z = \rho e^{i\varphi}$ by the factor $e^{i\alpha}$ is

$$z e^{i\alpha} = \rho e^{i(\varphi + \alpha)}$$

The geometrical interpretation of this fact is that the multiplication by $e^{i\alpha}$ makes the vector representing the complex number z rotate about the origin through the angle α. In particular, putting $\alpha = \frac{\pi}{2}$ we see that the multiplication by $e^{\frac{i\pi}{2}} = i$ results in the rotation of the representing vector of the number z through 90^0 in counterclockwise direction.

Example 3.8.1.
Calculate the product $\left(1 - i\sqrt{3}\right)^3 (1 + i)^2$.

Solution.

Expressing complex numbers in the exponential form, we get

$$\left(1 - i\sqrt{3}\right)^3 (1 + i)^2 = \left(2e^{-\frac{\pi i}{3}}\right)^3 \left(\sqrt{2}e^{\frac{\pi i}{4}}\right)^2 = 2^3 e^{-\pi i} \cdot 2e^{\frac{\pi i}{2}} = 8(-1) \cdot 2i = -16i.$$

IY. LIMITS

§ 4.1. Limits of Function Values

Sometimes we want the outputs of a function $y = f(x)$ to lie near a particular target value y_0. This need come about in different ways. A gas station attendant, asked for \$ 5.00 worth of gas, will try to pump the gas to the nearest cent. A mechanic griding a 3.385 – inch cylinder bore will not let the bore exceed this value by more than 0.002 in. A pharmacist making ointments will measure the ingredients to the nearest milligram. So the question becomes: How accurate do our machines and instruments have to be to keep the outputs within useful bounds? When we express this question with mathematical symbols, we ask: How closely must we control *x* to keep $y = f(x)$ within an acceptable interval about some particular target value y_0? The following example shows how to answer this question.

Example 4.1.1.
Controlling a Linear Function. How close to $x_0 = 4$ must we hold *x* to be sure that $y = 2x + 1$ lies within 2 units of $y_0 = 7$?

Solution.
We are asked: For what values of *x* is $|y - 7| < 2$? To find the answer, we first express $|y - 7|$ in terms of *x*:

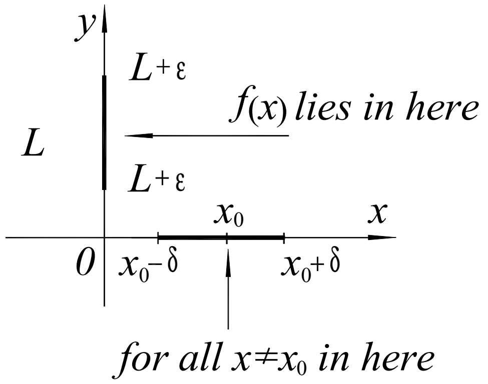
$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: What values of *x* satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality

$$|2x - 8| < 2 \Leftrightarrow -2 < 2x - 8 < 2 \Leftrightarrow 6 < 2x < 10 \Leftrightarrow 3 < x < 5.$$

To keep y within 2 units of $y_0 = 7$, we must keep x within 1 unit of $x_0 = 4$.

Suppose we are watching the values of a function $f(x)$ as x approaches x_0 (without taking on the value x_0 itself). What do we have to know about the values of f to say that they have a particular number L as their limit? What observable pattern in their behavior would guarantee their eventual approach to L? We need to require that for every interval about L, no matter how small, we can find an interval of numbers about x_0 whose f – values all lie within that interval about L. In other words, given any positive radius ε about L , there should exist some positive radius δ about x_0 such that for all x within δ units of x_0 (except x_0 itself) the values $y = f(x)$ lie within ε units of L.



If *f (x)* satisfies these requirements, we will say that

$$\lim_{x \to x_0} f(x) = L$$

Here, at last, is a mathematical way to say "the closer x to x_0, the closer $y = f(x)$ must get L."

The limit of $f(x)$ as x approaches x_0 is the number L if the following criterion holds:

Definition. Given any radius $\varepsilon > 0$ about L there exists a radius $\delta > 0$ about x_0 such that for all x, $0 < |x - x_0| < \delta$ implies $|f(x) - L| < \varepsilon$.

These are the letters that Cauchy and Weierstrass used in their pioneering work on continuity in the nineteenth century. In their arguments, δ meant "difference" (French for difference) and ε meant "erreur" (French for error).

To return to the notions of error and difference, we might think of machining something like a generator shaft to a close tolerance. We try for diameter L, but since nothing is perfect we must be satisfied to get the diameter $f(x)$ somewhere between $L - \varepsilon$ and $L + \varepsilon$. The δ is measure of how accurate our control setting for x must to guarantee this degree of accuracy in the diameter of the shaft.

§ 4.2. Algebra of Limits

The following rules hold if $\lim_{x \to x_0} f(x) = L_1$ and $\lim_{x \to x_0} g(x) = L_2$

1. *Sum Rule:* $\lim_{x \to x_0} (f(x) + g(x)) = L_1 + L_2$

2. *Difference Rule:* $\lim_{x \to x_0} (f(x) - g(x)) = L_1 - L_2$

3. *Product Rule:* $\lim_{x \to x_0} (f(x) \cdot g(x)) = L_1 \cdot L_2$

4. *Constant Multiple Rule:* $\lim_{x \to x_0} (kf(x)) = kL_1$ for any number k

5. *Quotient Rule:* $\lim_{x \to x_0} \left(\frac{f(x)}{g(x)} \right) = \frac{L_1}{L_2}$ if $L_2 \neq 0$

Let us prove the first rule.

To show that $\lim_{x \to x_0} (f(x) + g(x)) = L_1 + L_2$, we must show that for any $\varepsilon > o$ there exists a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow |f(x) + g(x) - (L_1 + L_2)| < \varepsilon \qquad (4.2.1)$$

Suppose, then, that ε is a positive number. The number $\frac{\varepsilon}{2}$ is positive too, and because $\lim_{x \to x_0} f(x) = L_1$ we know that there is a $\delta_1 > 0$ such that for all x,

$$0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\varepsilon}{2} \qquad (4.2.2)$$

Because $\lim_{x \to x_0} g(x) = L_2$, there is also a $\delta_2 > 0$ such that for all x,

$$0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - L_2| < \frac{\varepsilon}{2} \qquad (4.2.3)$$

Let δ be smaller of δ_1 and δ_2. The implications in (4.2.2) and (4.2.3) then both hold for all x such that

$$0 < |x - x_0| < \delta$$

and we get

$$|f(x) + g(x) - (L_1 + L_2)| = |(f(x) - L_1) + (g(x) - L_2)| \leq$$

$$\leq |f(x_1) - L_1| + |f(x_2) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

According to the $\varepsilon - \delta$ definition of limit, then,

$$\lim_{x \to x_0} (f(x) + g(x)) = L_1 + L_2.$$

§ 4.3. One-sided Limits

Sometimes the values of a function $f(x)$ tend to different limits as x approaches a number x_0 from different sides. When this happens, we call the limit of $f(x)$ as x approaches x_0 from the right the **right-hand limit** of f at x_0, and the limit, as x approaches x_0 from the left the **left-hand limit** of f at x_0.

The notation for the right-hand limit is $\lim\limits_{x \to x_0 + 0} f(x)$ or symbolically $f(x_0 + 0)$.

The notation for the lift-hand limit is $\lim\limits_{x \to x_0 - 0} f(x)$ or symbolically $f(x_0 - 0)$.

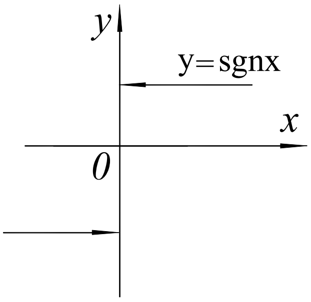
We sometimes call $\lim\limits_{x \to x_0} f(x)$ the **two-sided limit** of f at x_0. How to distinguish it from the one-sided right-hand and left-hand limits of f at x_0?

If the two one-sided limits of f exist at x_0 and are equal, their common value is the two-sided limit of f at x_0. Conversely, if the two-sided limit of f at x_0 exists, the two one-sided limits exist and have the same value as the two-sided limit. In symbols

$$\lim_{x \to x_0} f(x) = L \Leftrightarrow \lim_{x \to x_0 + 0} f(x) = L \text{ and } \lim_{x \to x_0 - 0} f(x) = L.$$

Example 4.3.1. Let the function $y = \operatorname{sgn} x = \begin{cases} 1, \text{if } x > 0 \\ -1. \text{if } x < 0 \end{cases}$ be given. Find limit of this function as x tends to 0.

Solution.



In this case $f(0 + 0) = 1$, and

$$f(0 - 0) = -1,$$

and these one-sided limits are not equal.

So

$\lim\limits_{x \to 0} \operatorname{sgn} x$ does not exist.

§ 4.4. Infinitesimal Functions

Definition. F function $\alpha(x)$ is called an **infinitesimal** as $x \to x_0$ if

$$\lim_{x \to x_0} \alpha(x) = 0.$$

This means that, given any $\varepsilon > 0$ (however small), there is $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \text{ implies } |f(x)| < \varepsilon .$$

Definition. The reciprocal of infinitesimal, that is $f(x) = \dfrac{1}{\alpha(x)}$ is an **infinitely large function**.

In this case we write

$$\lim_{x \to x_0} f(x) = \infty .$$

This means that given any positive number M (however large) there is a number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \text{ implies } |f(x)| > M.$$

Now let us prove the theorem to be important for applications.

Theorem. If a function has a limit it is representable as a sum of a constant, equal to that limit and, an infinitesimal.

Proof.

Let $\lim\limits_{x \to x_0} f(x) = A$. Then, given an arbitrarily number $\varepsilon > 0$, we have $|f(x) - A| < \varepsilon$ for all $x \neq x_0$ lying sufficiently close to x_0, which, in accordance with the definition, implies that $f(x) - A$ is an infinitesimal. Consequently

$f(x) - A = \alpha(x)$, i.e. $f(x) = A + \alpha(x)$,

where $\alpha(x)$ is an infinitesimal as $x \to x_0$.

It is possible to draw from limit rules for functions the corresponding propositions for infinitesimal functions.

1. A sum of any finite number of infinitesimals is an infinitesimal.
2. The product of a bounded function and an infinitesimal is an infinitesimal.
3. A product of any finite number of infinitesimals is an infinitesimal.

§ 4.5. Sandwich Theorem, and $\lim\limits_{x \to +0} \dfrac{\sin x}{x}$

The sandwich theorem.

Suppose that

$$g(x) \le f(x) \le h(x)$$

for all $x \neq x_0$ in some interval about x_0 and that

$$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = L .$$

Then $\lim\limits_{x\to x_0} f(x) = L$.

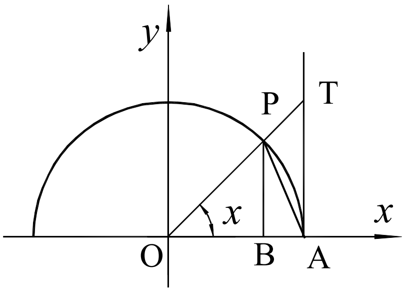
We do not include a proof of this theorem.

Example 4.5.1.

Show that if x is measured in radians then

Solution.

Our plan is to show that the right-hand and left-hand limits are both 1.



Take a circle of unit radius and suppose the central angle x is expressed in radians. To show that the right-hand limit is 1, we begin with values of x that are positive and less than $\frac{\pi}{2}$. We compare the areas of $\Delta\, OAP$, sector OAP, and $\Delta\, OAT$, and note that

Area $\Delta\, OAP <$ Area sector $OAP <$ Area $\Delta\, OAT$. These areas being respectively equal to $\frac{\sin x}{2}, \frac{1}{2}x, \frac{1}{2}\tan x$.

We have

$$\sin x < x < \tan x.$$

On dividing all the terms of these inequalities by $\sin x$ we obtain

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

that is

$$\cos x < \frac{\sin x}{x} < 1.$$

Because $\cos x$ approaches 1 as x approaches 0, the Sandwich Theorem tells us that

$$\lim_{x\to +0} \frac{\sin x}{x} = 1.$$

Since $\frac{\sin x}{x}$ is an even function, then

$$\lim_{x\to -0} \frac{\sin x}{x} = 1.$$

Together these equations imply that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Example 4.5.2.

Prove that $\lim\limits_{x\to 0} \frac{\tan x}{x} = 1$.

Solution.

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x} = 1\cdot 1 = 1.$$

§ 4.6. Comparison of Infinitesimals

Let $\alpha(x)$ and $\beta(x)$ be infinitesimals ax $x\to x_0$. To compare the infinitesimals $\alpha(x)$ and $\beta(x)$ means to determine the limit of their ratio

$$\lim_{x\to x_0}\frac{\alpha(x)}{\beta(x)}$$

provided that it exists.

In what follows we consider different cases occurring when infinitisemals are compared.

1. Let $\lim_{x\to x_0}\frac{\alpha(x)}{\beta(x)} = 0$, then $\alpha(x)$ is called an **infinitesimal of higher order**, than $\beta(x)$. In this case we write $\alpha(x)=o(\beta(x))$ for $x\to x_0$ (read "$\alpha(x)$ is small o of $\beta(x)$ for $x\to x_0$"). Here are examples: $x^2 = o(x)$ for $x\to 0$ since $\lim_{x\to 0}\frac{x^2}{x} = \lim_{x\to 0} x = 0$.

2. Let $\lim_{x\to x_0}\frac{\alpha(x)}{\beta(x)} = A$, where $A\neq 0, A\neq 1$, then $\alpha(x)$ and $\beta(x)$ are called **infinitesimals of the same order**. In this case we write $\alpha(x)=O(\beta(x))$ for $x\to x_0$.

For example. $x^2-4 = O(x^2-5x+6)$ $x\to 2$ since

$$\lim_{x\to 2}\frac{x^2-4}{x^2-5x+6} = \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(x-3)} = \lim_{x\to 2}\frac{x+2}{x-3} = \frac{4}{-1} = -4.$$

4. Let $\lim_{x\to x_0}\frac{\alpha(x)}{\beta(x)} = 1$, then $\alpha(x)$ and $\beta(x)$ are said to be **equivalent infinitesimals**. For equivalent infinitesimals $\alpha(x)$ and $\beta(x)$ we write $\alpha(x)\sim\beta(x)$ for $x\to x_0$. Important examples of equivalent infinitesimals are sin x and x, tan x and x, $\sin^{-1}x$ and x, $\tan^{-1}x$ and x for $x\to 0$.

Example 4.6.1. Calculate $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$.

Solution.

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2} = \lim_{x\to 0}\frac{2\sin^2 x}{x^2} = [\sin x\sim x \text{ then } \sin^2 x\sim x^2]=$$

$$= 2\lim_{x \to 0} \frac{x^2}{x^2} = 2.$$

§ 4.7. Limit of a Sequence

Let us consider a function of an integral argument. Usually such an argument is denoted by the letter n and the values of the function by some other letter supplied with the subscript indicating the value of the integral argument. For instance, if $y = f(n)$ is a function of the integral argument n we write $y_n = f(n)$. Given such a function, we say that the values

$$y_1 = f(1),\ y_2 = f(2),\ \dots,\ y_n = f(n), \dots$$

assumed by the function form a sequence. It may occur that, as n increases, the values $y_n = f(n)$ become arbitrarily close to a number L. Then we say that the number L is the limit of the function $f(n)$ of the integral argument n or that the sequence $y_1, y_2, \dots y_n, \dots$ has the limit L, as $n \to \infty$ and write

$$\lim_{n \to \infty} f(n) = L \text{ or } \lim_{n \to \infty} y_n = L$$

The definition of the limit of a sequence can be regarded as a special case of the definition of the limit of a function, as its argument becomes positively infinite and assumes only integral values. Hence, if L is the limit of a sequence $y_1, y_2, \dots, y_n, \dots$, then, given an arbitrary positive number ε, there is an integer N such that the inequality $|y_n - L| < \varepsilon$ holds for all $n > N$.

Example 4.7.1. The limit of the sequence $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$ exists and is equal to 0. To prove this, we must show that for any $\varepsilon > 0$, there exists an integer N such that for all n,

$$\text{n} > \text{N} \Rightarrow \left| \frac{1}{n} - 0 \right| < \varepsilon \qquad (4.7.1)$$

This implication will hold for all n for which $\frac{1}{n} < \varepsilon$ or, equivalently, $n > \frac{1}{\varepsilon}$. Pick an integer N greater than $\frac{1}{\varepsilon}$. Then any n greater than N will automatically be greater than $\frac{1}{\varepsilon}$ and the implication in (4.7.1) will hold.

§ 4.8. Test for the Existence of the Limit of a Sequence. The Limit of the Sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$

Not every sequence has a limit. It often happens that it is necessary to find out whether a given sequence possesses a limit. The theorem below provides a simple existence criterion for the limit of a sequence.

Theorem. Any monotone and bounded sequence has a limit.

We shall apply this theorem to prove the existence of a limit which plays an extremely important role in mathematical analysis. This result is expressed by the following theorem.

Theorem. The sequence $a_n = \left(1 + \frac{1}{n}\right)^n$ possesses a limit as $n \to \infty$.

Proof. By Newton's Binomial Formula, we have

$$a_n = \left(1 + \frac{1}{n}\right)^n = 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots +$$
$$+ \frac{n(n-1)\ldots(n-n+1)}{n!} \cdot \frac{1}{n^n} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \ldots +$$
$$+ \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right).$$

Replacing n by $n+1$ we obtain an analogous expression for a_{n+1}. Next, comparing these expressions we conclude that $a_{n+1} > a_n$, and a_n is increasing sequence.

Let us show that it is bounded. To do it, we replace the proper fractions in the parentheses by units and thus receive

$$a_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} < 2 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n} = 2 + \frac{1/2 - 1/2^n}{1 - 1/2} =$$
$$= 2 + 1 - \frac{1}{2^{n-1}} < 3.$$

Thus, the increasing sequence is bounded above, and hence, according to the above Theorem, it has a finite limit. This limit is denoted by *e*.

So, we have

$$\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e. \qquad (4.8.1)$$

The number *e* is irrational and its approximate value is

$$e \approx 2.718281828459045.$$

It can be also shown, that

$$\lim_{x \to \infty} \left(1 + \frac{1}{x} \right)^{x} = e \tag{4.8.2}$$

$$\lim_{x \to \infty} \left(1 + \frac{k}{mx + n} \right)^{ax + b} = e^{\frac{ak}{m}} \tag{4.8.3}$$