Ministry of Transport and Communications of Ukraine
State Department of Communications
A.S.Popov Odessa National Academy
of Telecommunications

Department of Higher Mathematics

Gavdzinski V.N., Korobova L.N.

DIFFERENTIAL CALCULUS.
FUNCTIONS OF ONE VARIABLE
Textbook

For Students Studying a Course of Higher Mathematics in English

Odessa – 2011

УДК 510(083) План НМВ 2010/2011 уч.г.

Составители:
Доц.В.Н.Гавдзинский, ст.преп.Л.Н.Коробова

Методическое пособие содержит основные определения, формулы и теоремы дифференциального исчисления функции одной переменной и предназначено для студентов академии, изучающих математику на английском языке.

Основные теоремы и формулы приведены с доказательством, а также даны решения типовых задач, задания для самостоятельной работы. Кроме того прилагается вариант теста по данной теме и варианты комплексной контрольной работы.

Компьютерная верстка ***Корнейчук Е. С.***

Здано в набір 29.11.2011 Підписано до друку 16.12.2011.
Формат 60/88/16 Зам. № 4719.
Тираж 100 прим. Обсяг: 4,25 ум. друк. арк.
Віддруковано на видавничому устаткуванні фірми RISO
у друкарні редакційно-видавничого центру ОНАЗ ім. О.С. Попова
ОНАЗ, 2011

CONTENTS

DIFFERENTIAL CALCULUS

1. Derivative and its Geometrical and Physical Meaning

Definition. The limit of a quotient of an increment of a function $y = f(x)$ to an increment of a variable as the latter tends to zero is called a **derivative** of that function and denoted by $f'(x_0)$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = f'(x_0) \qquad (1.1)$$

The derivative of a function $y = f(x)$ admits of a simple geometrical interpretation.

Definition. A tangent line to a curve at a point P_0 is the limiting position of a chord P_0P as P moves along the curve to coincide with a point P_0

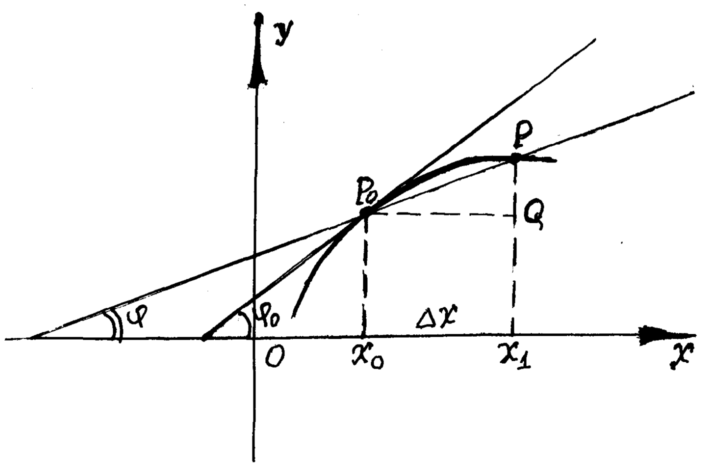


Fig. 1.1

We have $\Delta x = |P_0Q|$; $\Delta y = |PQ|$ and

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{P \to P_0} \frac{|PQ|}{|P_0Q|} = \lim_{\varphi \to \varphi_0} \tan\varphi = k_0.$$

Conclusion. A **derivative** $f'(x_0)$ **is** equal to **a slope** of a tangent line to a graph of a function $y = f(x)$ at a point with abscissa x_0.

Now we suppose to have a body moving along a coordinate line and we know that its position at time t is $s = f(t)$. As the body moves along a coordinate line, it has a velocity at each particular instant and we want to find out what that velocity is. To do it we reason like this:

In the interval from any time t to the slightly later time $t + \Delta t$, the body moves from position $s = f(t)$ to position $s + \Delta s$ and

$$s + \Delta s = f(t + \Delta t) \qquad (1.2)$$

The body's net change in position, or displacement, for this short time interval is

$$\Delta s = f(t + \Delta t) - f(t) \qquad (1.3)$$

The body's average velocity for the time interval is Δs divided by Δt, that is

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t} \qquad (1.4)$$

To find the body's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as the interval gets shorter and shorter and Δt strinks to zero. Here is where the derivative comes in. So we have

$$v = \lim_{\Delta x \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \qquad (1.5)$$

Thus, the physical meaning of a derivative is the **instantaneous velocity**.

There is a simple relationship between an existence of a derivative and continuity, which is stated in the following theorem.

Theorem. If a function $f(x)$ has a derivative an a point x_0, then $f(x)$ is continuous at x_0.

Proof. To show that $f(x)$ is continuous at x_0, it suffices to show that $\lim_{x \to x_0} f(x) = f(x_0)$, or equivalently $\lim_{\Delta x \to 0} \Delta y = 0$, where $\Delta y = f(x) - f(x_0)$ and $\Delta x = x - x_0$.

Notice that $f'(x_0)$ exists by hypothesis. Hence by the limit rules we have

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \Delta x = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta x = f'(x_0) \cdot 0 = 0.$$

The converse of this theorem is false. A function $f(x)$ continuous at a point x_0 may have no derivative at that point. For example, if $f(x) = |x|$ then $f(x)$ is continuous at 0, however, $f(x)$ does not have a derivative at 0, as we prove now.

Let $f(x) = |x|$. Show that $f'(0)$ does not exist. We find that

$$\lim_{x \to 0+0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+0} \frac{|x|}{x} = \lim_{x \to 0+0} \frac{x}{x} = 1,$$

and

$$\lim_{x \to 0-0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-0} \frac{|x|}{x} = \lim_{x \to 0-0} \frac{-x}{x} = -1.$$

Since the two one-sided limits are different, we conclude that $\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}$ does not exist, which is equivalent to $f'(0)$ not existing.

Since $|x|$ does not have a derivative at 0, it follows that the graph of $|x|$ does not have a tangent line at $(0,0)$. Notice that the graph of $f(x)=|x|$ is bent or pointed at $(0,0)$.

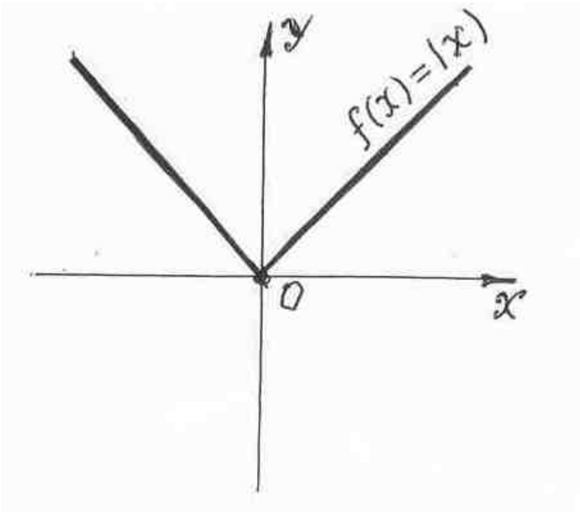


Fig. 1.2

2. The Chain Rule

Theorem .Let $f'(u_0)$ and $u'(x_0)$, where $u_0=u(x_0)$, exist. Then a composite function $y=f(u(x))$ has a derivative at x_0 and its value is $f'(u_0)\cdot u'(x_0)$.

Proof.

$$f'(u_0)=\lim_{\Delta u\to 0}\frac{\Delta y}{\Delta u}\Rightarrow\frac{\Delta y}{\Delta u}-f'(u_0)=o(\Delta u)\Rightarrow\frac{\Delta y}{\Delta u}=f'(u_0)+o(\Delta u)\Rightarrow$$

$$\Rightarrow\Delta y=f'(u_0)\cdot\Delta u+o(\Delta u)\cdot\Delta u,\qquad(2.1)$$

where $o(\Delta u)$ is an infinitesimal as $\Delta u\to 0$ that is $\lim\limits_{\Delta u\to 0}o(\Delta u)=0$.

Dividing (2.1) by Δx and passing to the limit as $\Delta x\to 0$ we get

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=f'(u_0)\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\lim_{\Delta x\to 0}o(\Delta u)\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}$$

and

$$y'(x_0)=f'(u_0)\cdot u'(x_0)$$

since a function $u(x)$ is continuous at a point x_0 and $\lim\limits_{\Delta x\to 0}o(\Delta u)=\lim\limits_{\Delta u\to 0}o(\Delta u)=0$.

Once we know what the functions are, as we usually do in any particular example, we can get by with writing the Chain Rule in a shorter way:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad (2.2)$$

3. Differentiation of Inverse Functions

Let $y = f(x)$ and $x = \varphi(y)$ be a pair of mutually inverse functions. We shall show that if the derivative of one of these functions is known it is easy to determine the derivative of the other. For definiteness, let the derivative $f'(x) = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ be known. We shall suppose that it does not turn into zero. In order to find the derivative $\varphi'(y)$we shall compute the limit $\lim\limits_{\Delta y \to 0} \frac{\Delta x}{\Delta y}$. Since $\Delta x \to 0$ when $\Delta y \to 0$ (because the inverse function is also continuous) the identity $\frac{\Delta x}{\Delta y} = \frac{1}{\frac{\Delta y}{\Delta x}}$ implies

$\lim\limits_{\Delta y \to 0} \frac{\Delta x}{\Delta y} = \frac{1}{\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$ that is $\varphi'(y) = \frac{1}{f'(x)}$.

Similarly, if $\varphi'(y) \neq 0$, then $f'(x) = \frac{1}{\varphi'(y)}$.

These can be written in the form

$$y'_x = \frac{1}{x'_y} \text{ , or } x'_y = \frac{1}{y'_x} \qquad (3.1)$$

4. The Table of Derivatives of Elementary Functions

1) $(C)' = 0$

2) $\left(x^n\right)' = n \cdot x^{n-1}$

3) $(\sin x)' = \cos x$

4) $(\cos x)' = -\sin x$

5) $(\tan x)' = \frac{1}{\cos^2 x}$

8) $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$

9) $(\arctan x)' = \frac{1}{x^2+1}$

10) $(\operatorname{arc\,cot} x)' = -\frac{1}{x^2+1}$

11) $(\log_a x)' = \frac{1}{x \ln a}$, $a > 0, a \neq 1, x > 0$

12) $(\ln x)' = \frac{1}{x}$

6) $(\cot x)' = -\dfrac{1}{\sin^2 x}$

13) $(a^x)' = a^x \ln a$

7) $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$

14) $(e^x)' = e^x$

• Let us prove that $(C)' = 0$. At every value of x, we find that

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{C-C}{\Delta x} = \lim_{\Delta x\to 0} 0 = 0.$$

• Now we find a derivative of the function $y = \arcsin x$. The derivatives of inverse trigonometric functions are obtained from the formulas (3.1). If $y = \arcsin x$, then $x = \sin y$ and $y'_x = \dfrac{1}{x'_y} = \dfrac{1}{\cos y}$.

To express the derivative as a function of the independent variable x we substitute into this formula the expression of $\cos y$ in terms of x, that is

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}.$$

Here we take the arithmetic root since the values of the function $y = \arcsin x$ lie in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and the cosine of y is positive in this interval. The substitution gives

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}.$$

• Let us consider the logarithmic function $y = \ln x$. If an increment of an argument is Δx, then the increment of the function is

$$\Delta y = \ln(x+\Delta x) - \ln x = \ln\frac{x+\Delta x}{x} = \ln\left(1+\frac{\Delta x}{x}\right).$$

Since

$$\frac{\Delta y}{\Delta x} = \frac{\ln\left(1+\dfrac{\Delta x}{x}\right)}{\Delta x} = \ln\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}$$

and

$$y' = (\ln x)' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\ln\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}}.$$

As a logarithmic function is continuous the symbols of the logarithm and of the limit can be interchanged. So we have

$$(\ln x)' = \ln \lim_{\Delta x \to 0}\left(1+\frac{\Delta x}{x}\right)^{\frac{1}{\Delta x}} = \ln e^{1/x} = \frac{1}{x}.$$

• To find the derivative of a power or exponential function we use, so called, logarithmic differentiation. Let a power function $y = x^n$ be given. Taking the logarithms we have

$$\ln y = \ln x^n = n \ln x .$$

Since $(\ln y)' = (n \ln x)'$. The function ln*y* is a composite function, let us use the chain rule: $\frac{1}{y}\cdot y' = \frac{n}{x}$. Solve this equation for y': $y' = \frac{n}{x}\cdot y = \frac{n}{x}\cdot x^n = nx^{n-1}$. Thus

$$\left(x^n\right)' = n \cdot x^{n-1}.$$

5. The Sum and Difference Rule

Let *y* be a function represented as the sum of given functions *u* and *v* of the same independent variable *x*: $y = u + v$. We have to prove that

$$y' = u' + v' \tag{5.1}$$

It is clear that

$$(u+v)' = \lim_{\Delta x \to 0}\frac{\Delta(u+v)}{\Delta x} = \lim_{\Delta x \to 0}\frac{u(x+\Delta x)+v(x+\Delta x)-(u(x)+v(x))}{\Delta x} =$$

$$= \lim_{\Delta x \to 0}\frac{u(x+\Delta x)-u(x)}{\Delta x} + \lim_{\Delta x \to 0}\frac{v(x+\Delta x)-v(x)}{\Delta x} = \lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} + \lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} =$$

$$= u' + v'.$$

By analogy we obtain

$$(u-v)' = u' - v'. \tag{5.2}$$

Example 5.1 Find the derivative of the function $y = 3^x - \tan x + x^3$.

Solution.

$$y' = \left(3^x - \tan x + x^3\right)' = \left[\begin{array}{l}\text{use the formulas } (5.1), (5.2) \text{ and} \\ 13), 5), 2) \text{ of the table of derivatives}\end{array}\right] =$$

$$= 3^x \ln 3 - \frac{1}{\cos^2 x} + 3x^2 .$$

Example 5.1 Find the derivative of the function $y = \ln(\cos(2x))$.

Solution.

Here is a composite function $y = \ln u$, where $u = \cos v$ and $v = 2x$. Use the formula (2.2.) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\cos x} \cdot (-\sin x) = -\frac{\sin x}{\cos x} = \tan x \text{ , as}$$

$$\frac{dy}{du} = (\ln u)' = \frac{1}{u} = \frac{1}{\cos x}; \; \frac{du}{dx} = (\cos x)' = -\sin x.$$

Answer: $y' = \tan x$.

6. The Product Rule

The product rule is understood to hold only at values of x where u and v have derivatives. At such a value of x, **the derivative of the product *uv* is *u* times the derivative of *v* plus *v* times the derivative of *u*.** That is

$$(u \cdot v)' = u'v + uv' \tag{6.1}$$

To prove this we have

$$(u \cdot v)' = \lim_{\Delta x \to 0} \frac{\Delta(u \cdot v)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(u + \Delta u)(v + \Delta v) - uv}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{uv + v\Delta u + u\Delta v + \Delta u \Delta v - uv}{\Delta x} = v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + u \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} +$$

$$+ \lim_{\Delta x \to 0} \Delta v \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \begin{bmatrix} \lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = u', \; \lim\limits_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = v', \\ \lim\limits_{\Delta x \to 0} \Delta v = 0 \end{bmatrix} = u'v + uv'$$

In this working we take into account that if $\Delta x \to 0$ then Δv as the increment of a continuous function also tends to zero.

If one of the factors is a constant, for instance $v = C$ we get

$$y' = (Cu)' = C'u + Cu' = Cu'.$$

Example 6.1. Find the derivative of the function $y = (x^2 + 1)(x^3 + 3)$.

Solution. From the product rule with $u = x^2 + 1, v = x^3 + 3$ we find

$$y' = 2x(x^3 + 3) + 3x^2(x^3 + 3) = 2x^4 + 6x + 3x^4 + 3x^2 = 5x^4 + 3x^2 + 6x.$$

Example 6.2. Find the derivative of the function $y = 2\sin x \cdot \ln(x^5 + 1)$.

Solution Here $u = 2\sin x \Rightarrow u' = 2\cos x$ and $v = \ln(x^5 + 1) \Rightarrow v' = \frac{1}{x^5 + 1} \cdot 5x^4$.

So $y' = 2\cos x \cdot \ln(x^5 + 1) + 2\sin x \cdot \frac{1}{x^5 + 1} \cdot 5x^4 = 2\left(\cos x \cdot \ln(x^5 + 1) + \frac{5x^4 \sin x}{x^5 + 1}\right)$.

The second, shorter way of solution: $y = 2(\sin x \cdot \ln(x^5 + 1))$. 2 as the constant factor can be taken out the sign of derivative,

$u = \sin x \Rightarrow u' = \cos x$, $v = \ln(x^5 + 1) \Rightarrow v' = \frac{5x^4}{x^5 + 1}$,

$$y' = 2\left(\cos x \cdot \ln(x^5 + 1) + \frac{5x^4 \sin x}{x^5 + 1}\right).$$

7. The Quotient Rule

Let y be a function equal to the quotient of two functions u and v: $y = \frac{u}{v}$. We should prove that

$$y' = \frac{u'v - uv'}{v^2} \tag{7.1}$$

The increment of the given function

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{uv + v\Delta u - uv - u\Delta v}{(v + \Delta v)v} = \frac{v\Delta u - u\Delta v}{(v + \Delta v)v}.$$

In this case $\frac{\Delta y}{\Delta x} = \frac{v\Delta u - u\Delta v}{(v + \Delta v)v\Delta x} = \frac{v\frac{\Delta u}{\Delta x} - u\frac{\Delta v}{\Delta x}}{(v + \Delta v)v}$. Passing to the limit as $\Delta x \to 0$ we obtain

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{v \lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} - u \lim\limits_{\Delta x \to 0} \frac{\Delta v}{\Delta x}}{\left(v + \lim\limits_{\Delta x \to 0} \Delta v\right)v} = \frac{u'v - uv'}{v^2}.$$

The formula (7.1) is proved.

Let $y = \frac{1}{v}$, where $v = v(x)$. The derivative of the numerator is zero and formula (7.1) becomes

$$y' = -\frac{v'}{v^2} \tag{7.2}$$

Example 7.1. Find the derivative of the function $y = \dfrac{\sin x}{1 - 2\cos x}$.

Solution We apply the quotient rule (7.1) with $u = \sin x$ and $v = 1 - 2\cos x$.

$$y' = \frac{\cos x \cdot (1 - 2\cos x) - \sin x \cdot (2\sin x)}{(1 - 2\cos x)^2}.$$

Simplifying the result

$$y' = \frac{\cos x - 2\left(\cos^2 x + \sin^2 x\right)}{(1 - 2\cos x)^2} = \frac{\cos x - 2}{(1 - 2\cos x)^2}.$$

Answer: $y' = \dfrac{\cos x - 2}{(1 - 2\cos x)^2}$.

Using the chain rule, the table of derivatives of elementary functions and rules of differentiation we can consider the following methods of differentiation.

A. Implicit Differentiation is a special case of the chain rule. If y is differentiable function of x, given by an equation $F(x, y) = f(x)$, then

$$\frac{dF(x, y)}{dx} = f'(x) \Rightarrow F'_y \cdot y' = f' \Rightarrow y' = \frac{f'}{F'_y}.$$

Logorithmic differentiation. The expression $(f(x))^{g(x)}$ is differentiated by first simplifying using logarithms and then using implicit differentiation. Application of this procedure to the expression $(f(x))^{g(x)}$ leads to the following formula:

$$\left((f(x))^{g(x)}\right)' = g(x)(f(x))^{g(x)-1} f'(x) + (f(x))^{g(x)} \ln g(x) \cdot g'(x)$$

Example 7.2. Find y' if

$$y^2 + \sin y = 4x. \qquad (7.3)$$

Solution. In this case, the relation between y and x is not explicit but implicit. Use the following procedure, known as **implicit differentiation**:

1. Differentiate both sides of the (7.3) with respect to x.

$$\left(y^2 + \sin y\right)'_x = (4x)' \Rightarrow 2yy' + \cos y \cdot y' = 4.$$

2. Solve for y':

$$2yy' + \cos y \cdot y' = 4 \Rightarrow y'(2y + \cos y) = 4 \Rightarrow y' = \frac{4}{2y + \cos y}$$

Example 7.3. Find y' if

$$y = \left(x^{\sin x}\right) \qquad (7.4)$$

Solution. Take logarithms of both sides of (7.4):

$$y=\left(x^{\sin x}\right) \Rightarrow \ln y=\sin x\cdot\ln x \qquad (7.5)$$

Differentiate the equation (8.3) and find y':

$$(\ln y)'=(\sin x\cdot\ln x)' \Rightarrow \frac{y'}{y}=\cos x\cdot\ln x+\frac{\sin x}{x} \Rightarrow$$

$$\Rightarrow \frac{y'}{y}=\frac{x\cos x\cdot\ln x+\sin x}{x} \Rightarrow y'=\frac{y}{x}(x\cos x\cdot\ln x+\sin x).$$

Using relation (7.5) we get

$$\left(x^{\sin x}\right)'=x^{\sin x-1}(x\cos x\cdot\ln x+\sin x)$$

Finally write out the table of derivatives of composite functions

Table 7.1

I
$$\begin{cases} C'=0 \\ x'=1 \\ (u\pm v)'=u'\pm v' \\ (uv)'=u'v+uv' \\ (Cu)'=Cu' \\ \left(\dfrac{u}{v}\right)'=\dfrac{u'v-uv'}{v^2} \\ \left(\dfrac{C}{v}\right)'=-\dfrac{Cv'}{v^2} \end{cases}$$

II
$$\begin{cases} \left(u^a\right)'=au^{a-1}u' \\ \left(\sqrt{u}\right)'=\dfrac{1}{2\sqrt{u}}u' \\ \left(\sqrt[n]{u}\right)'=\dfrac{1}{n\cdot\sqrt[n]{u^{n-1}}}u' \end{cases}$$

III
$$\begin{cases} \left(a^u\right)'=a^u u'\ln a \\ \left(e^u\right)'=e^u u' \\ \left(u^v\right)'=u^v v'\ln u+vu^{v-1}u' \end{cases}$$

IY
$$\begin{cases} (\ell og_a u)'=\dfrac{u'}{u\ell na} \\ (\ell nu)'=\dfrac{u'}{u} \end{cases}$$

YI
$$\begin{cases} (\arcsin u)'=\dfrac{u'}{\sqrt{1-u^2}} \\ (\arccos u)'=-\dfrac{u'}{\sqrt{1-u^2}} \\ (\arctan u)'=\dfrac{u'}{1+u^2} \\ (arc\cot u)'=-\dfrac{u'}{1+u^2} \end{cases}$$

Y
$$\begin{cases} (\sin u)'=\cos u\cdot u' \\ (\cos u)'=-\sin u\cdot u' \\ (\tan u)'=\dfrac{u'}{\cos^2 u} \\ (\cot u)'=-\dfrac{u'}{\sin^2 u} \end{cases}$$

YII*
$$\begin{cases} (\sinh u)'=\cosh u\cdot u' \\ (\cosh u)'=\sinh u\cdot u' \\ (\tanh u)'=\dfrac{u'}{\cosh^2 u} \end{cases}$$

where C, e, a, α, n – are constants, $u = u(x), v = v(x)$ – are functions.

The functions of the seventh group are called the hyperbolic functions:

$y = \sinh x$ - hyperbolic sine, where $\sinh x = \dfrac{e^x - e^{-x}}{2}$;

$y = \cosh x$ – hyperbolic cosine, where $\cosh x = \dfrac{e^x + e^{-x}}{2}$;

$y = \tanh x$ – hyperbolic tangent, $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

8. Differential and its Geometrical Meaning

Definition. The function $y = f(x)$ is said to be **differentiable** at a point x_0 if an increment at this point represented in the form

$$\Delta y = A\Delta x + o(\Delta x) \tag{8.1}$$

A linear summand $A\Delta x$ is called a **differential** of a function and is denoted by *dy*, that is

$$dy = A\Delta x.$$

Theorem. A function $y = f(x)$ is differentiable at a point x_0 if and only if there $f'(x_0)$ exists at this point. When satisfying this condition $A = f'(x_0)$.

Proof. If $f(x)$ is differentiable at $x_0 \Leftrightarrow \Delta y = A\Delta x + o(\Delta x) \Leftrightarrow$

$$\Leftrightarrow \frac{\Delta y}{\Delta x} = A + \frac{o(\Delta x)}{\Delta x} \Leftrightarrow \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = A + \lim_{\Delta x \to 0} \frac{o(\Delta x)}{\Delta x} \Leftrightarrow A = f'(x_0).$$

The theorem has been proved.

For the function $y = x$ we have

$dy = x'\Delta x = \Delta x$, but $y = x$ so $dx = \Delta x$ and $dy = f'(x)dx$ or $\dfrac{dy}{dx} = f'(x)$.

Operation of finding a derivative is termed a differentiation.

The representation of a derivative as the ratio of differentials is important.

Let us explain a geometrical meaning of a differential of a function $y = f(x)$.

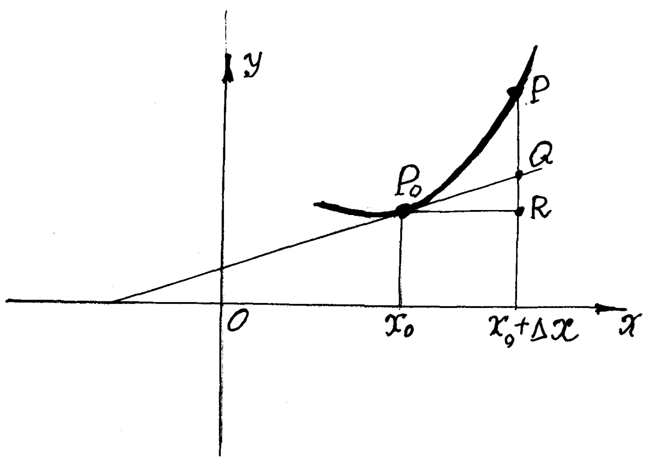


Fig.8.1.

We have $dy = f'(x_0)\Delta x = f'(x_0)(x - x_0)$ and $y_{\tan} - f'(x_0) = f'(x_0)(x - x_0)$.
Equating the right hand sides of these relations we get

$$dy = y_{\tan} - y_0 = QR.$$

So a differential is equal to an increment of the ordinate of a tangent line.

9. Applying Differential to Approximate Calculations

The application of the differential to approximate calculations is based on the replacement of the increment $\Delta y = f(x_0 + \Delta x) - f(x_0)$ of a given function $y = f(x)$ by the expression $dy = f'(x)dx$.

Thus, for small values of dx we write

$$\Delta y \approx f'(x)dx = dy \qquad (9.1)$$

Geometrically, this is equivalent to replacing the graph of the function $y = f(x)$ by its tangent line at the point $(x_0, f(x_0))$. In a sufficiently small neighborhood of the point x_0 this replacement leads to small errors.

Now we consider the function $y = \sqrt{x}$. Its differential is $dy = \dfrac{1}{2\sqrt{x}}dx$ and hence

$$\sqrt{x + dx} \approx \sqrt{x} + \frac{dx}{2\sqrt{x}}.$$

In particular, for $x = 1$ we obtain

$$\sqrt{1 + dx} \approx 1 + \frac{dx}{2}.$$

In the general case, for $x = a^2$ $(a > 0)$ we have

$$\sqrt{a^2 + dx} \approx a + \frac{dx}{2a}.$$

For instance, the application of these results yields

$$\sqrt{1.21} = \sqrt{1 + 0.21} \approx 1 + \frac{0.21}{2} = 1.105.$$

The exact value of the root equals 1.1, hence the relative error is about 0.5%.

10. Differentiating Functions Represented Parametrically

Let y as a function of x be represented parametrically by the equations

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \qquad (10.1)$$

Find a derivative $\dfrac{dy}{dx}$.

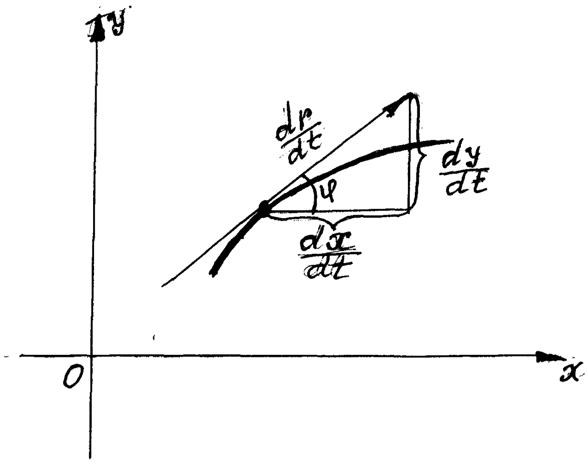


Fig.10.1

We have $\vec{r} = x(t)\vec{i} + y(t)\vec{j}$ and $\dfrac{d\vec{r}}{dt} = \dfrac{dx}{dt}\vec{i} + \dfrac{dy}{dt}\vec{j}$. Since $\dfrac{dy}{dx} = \tan\varphi = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{10.2}$$

Example 10.1. Find an equation of a tangent line and a normal line to a curve at a point corresponding to the value of parameter $t_0 = 1$, if

$$\begin{cases} x = \ln\left(1 + t^2\right), \\ y = t - \arctan t, \end{cases} \quad t \in (-\infty, +\infty).$$

Solution. First we find coordinates of a point $M_0(x_0, y_0)$ which is a point of contact and through which a normal line passes. If $t_0 = 1$ then $x_0 = \ln(1+1) = \ln 2,\ y_0 = 1 - \arctan 1 = 1 - \dfrac{\pi}{4} \Rightarrow M_0\left(\ln 2,\ 1 - \dfrac{\pi}{4}\right)$.

Now by the formula (10.2) we find $\dfrac{dy}{dx}$:

1) $\dfrac{dx}{dt} = \ln\left(1 + t^2\right)' = \dfrac{2t}{1 + t^2}$,

2) $\dfrac{dy}{dt} = (t - \arctan t)' = 1 - \dfrac{1}{1 + t^2} = \dfrac{t^2}{1 + t^2}$,

3) $\dfrac{dy}{dx} = \dfrac{t^2}{1 + t^2} : \dfrac{2t}{1 + t^2} := \dfrac{t^2 \cdot \left(1 + t^2\right)}{\left(1 + t^2\right) \cdot 2t} = \dfrac{t}{2}$.

If $t = 1$ then $y'_x(1) = \left.\dfrac{dy}{dx}\right|_{t=1} = \left.\dfrac{t}{2}\right|_{t=1} = \dfrac{1}{2}$. This value is the value of the derivative at the point x_0.

The equation of the tangent line can be written in the form

$$y - y_0 = y'(x_0)\cdot(x - x_0) \Rightarrow y - \left(1 - \frac{\pi}{4}\right) = \frac{1}{2}(x - \ln 2), \text{ or}$$

$$\underline{y = \frac{1}{2}x + \left(1 - \frac{\pi}{4} - \ln 2\right).}$$

For the equation of the normal we have $k_{nor.} = -k_{\tan.}$ as normal line is orthogonal with tangent line. Equation of normal is $\underline{y = -2x + \left(1 - \frac{\pi}{4} - \ln 2\right).}$

11. Second and Higher Order Derivatives

The derivative $y' = \frac{dy}{dx}$ is the first derivative of y with respect to x. The first derivative may also be a differentiable function of x. If so, its derivative

$$y'' = \frac{dy'}{dx} = \frac{dy}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

is called the **second derivative** of y with respect to x.

If y'' is differentiable, its derivative

$$y''' = \frac{dy''}{dx} = \frac{d^3y}{dx^3}$$

is the **third derivative** of y with respect to x.

The names continue as you imagine they would, with

$$y^{(n)} = \frac{dy^{(n-1)}}{dx} = \frac{d^ny}{dx^n}$$

denoting the nth derivative of y with respect to x.

Example 11.1. Find the first four derivatives of the function $y = x^3 - 3x^2 + 2$.

Solution.

First derivative: $y' = \left(x^3 - 3x^2 + 2\right)' = 3x^2 - 6x$.

Second derivative: $y'' = \left(3x^2 - 6x\right)' = 6x - 6$.

Third derivative: $y''' = (6x - 6)' = 6$.

Fourth derivative: $y^{(4)} = (6)' = 0$.

The given function has derivatives of all orders, the fifth and later derivatives all being zero.

12. Basic Theorems of Differential Calculus

Fermat's theorem. Let a function $f(x)$ be continuous in some interval and have the greatest (the least) value at an interior point ξ of this interval and if $f(x)$ has a derivative at this point, then $f'(\xi)=0$.

Proof. For definiteness, let the function $f(x)$ assume its greatest value at the point ξ. This means that $f(x)\le f(\xi)$ for all values of x near ξ.

Since ξ is an interior point of f's domain, the limit $\lim\limits_{x\to\xi}\dfrac{f(x)-f(\xi)}{x-\xi}$ defining $f'(\xi)$ is two-sided. This means that the right-hand and left-hand limits both exist at $x=\xi$, and both equal $f'(\xi)$. When we examine these limits separately, we find that

$$\lim_{x\to\xi+0}\frac{f(x)-f(\xi)}{x-\xi}\le 0 \qquad (12.1)$$

because, immediately to the right of ξ, $f(x)\le f(\xi)$ and $x-\xi>0$. Similarly,

$$\lim_{x\to\xi-0}\frac{f(x)-f(\xi)}{x-\xi}\ge 0 \qquad (12.2)$$

because, immediately to the left of ξ, $f(x)\le f(\xi)$ and $x-\xi<0$.

Inequality (12.1) says that $f'(\xi)$ cannot be greater than zero, whereas (12.2) says that $f'(\xi)$ cannot be less than zero. So, $f'(\xi)=0$.

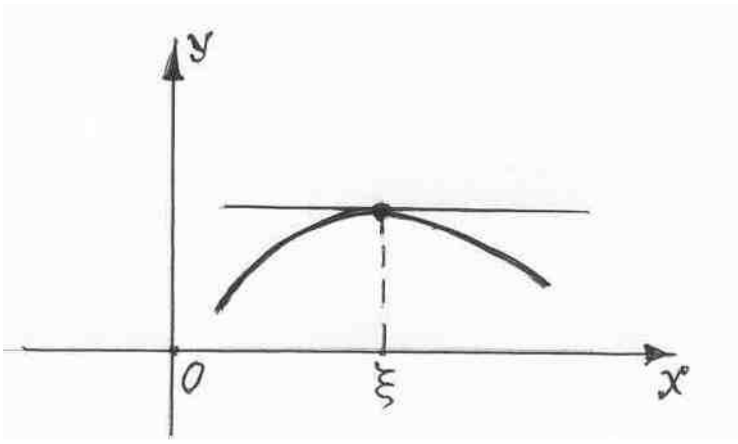


Fig. 12.1

The geometrical meaning of Fermat's theorem is clearly seen in *Fig.12.1:* the tangent line to the graph of a function at its interior highest (or lowest) point is parallel to the axis of absissas.

Rolle's theorem. Suppose that $y=f(x)$ is continuous in the closed interval $[a,b]$ and differentiable at every point of the open interval (a,b). If $f(a)=f(b)$, then there is at least one interior point ξ of (a,b) such that $f'(\xi)=0$.

Proof. If a function $y=f(x)$ is continuous in the closed interval $[a,b]$, then this function attains its greatest and least values in this interval. If at least one of these points is an interior point, then by Fermat's theorem a derivative at this point equals zero. If these points are end-points then by virtue of the condition $f(a)=f(b)$ this function is constant in this interval $[a,b]$ and $f'(x)=0$ everywhere. The theorem has been proved.

Example 12.1. The polynomial function $y=\frac{x^3}{3}-3x$ graphed in *Fig.12.2* is continuous at every point of $[-3,3]$ and differentiable at every point of $(-3,3)$. Since $f(-3)=f(3)$, Rolle's theorem says that $f'(x)$ must be zero at least once in the open interval between $a=-3$ and $b=3$. In fact, $f'(x)=\left(y=\frac{x^3}{3}-3x\right)=x^2-3$ is zero twice in this interval, once at $x=-\sqrt{3}$ and again at $x=\sqrt{3}$.

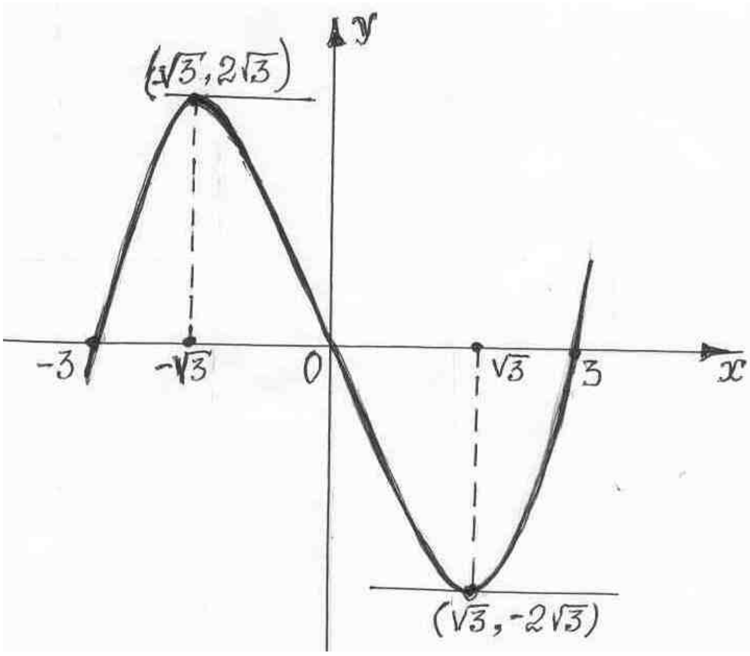


Fig. 12.2

Lagrange's theorem (The mean value theorem). If the function $y=f(x)$ is continuous at every point of the closed interval $[a,b]$ and differentiable at every point of the open interval (a,b), then there exists at least one interior point $\xi\in(a,b)$ such that

$$f'(x)=\frac{f(b)-f(a)}{b-a} \qquad (12.3)$$

Proof. Define a new function $F(x)$ as follows

$$F(x)=f(x)-\lambda x$$

This function is continuous in $[a,b]$ and has a derivative in (a,b). Choose λ to satisfy the third condition of Rolle's theorem, that is $F(a)=F(b)$. We have $f(a)-\lambda a=f(b)-\lambda b\Rightarrow\lambda=\frac{f(b)-f(a)}{b-a}$. By Rolle's theorem, there exists a point

$\xi \in (a, b)$ such that $F'(\xi) = 0$. But $F'(x) = f'(x) - \lambda$, and therefore $f'(\xi) - \lambda = 0 \Rightarrow f'(\xi) = \lambda \Rightarrow f'(\xi) = \dfrac{f(b) - f(a)}{b - a}$, which is what we had to prove.

It follows from Lagrange's theorem that

$$f(b) - f(a) = f'(\xi)(b - a), \ a < \xi < b \qquad (12.4)$$

The relation (12.4) is known as the **formula of finite increments**.

Physical Interpretation

When the renowned physicist Andre′ Ampère (1775– 1836) first stated this theorem around 200 years ago, the terms "average" and "mean" were synonymous. If $f(t)$ denotes the position of an object on the *x*-axis at time *t*, then the average (or mean) velocity during the interval $[a, b]$ is $\dfrac{f(b) - f(a)}{b - a}$. Thus by the Mean Value Theorem the mean velocity during an interval $[a, b]$ is equal to the velocity $f'(\xi)$ at some instant ξ in (a, b).

Example 12.2. If a car accelerating from zero takes 8 *sec*. to go 352 *ft*., its average velocity for the 8-second interval is 352/8 = 44 *ft/sec*. At some point during the acceleration, Lagrange's theorem says, the speedometer must read exactly 30 *mph* (44 *ft/sec*).

Caushy's Theorem. Suppose, that functions $f(x)$ and $g(x)$ satisfy the following conditions

1) $f(x)$ and $g(x)$ are continuous at every point of the closed interval $[a, b]$;
2) $f(x)$ and $g(x)$ are differentiable at every point of the open interval (a, b), and also $g'(x) \neq 0$.

Then there at least one interior point $\xi \in (a, b)$ exists such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)} \qquad (12.5)$$

Proof. Let us take an auxiliary function similar to the one used in the proof of Lagrange's theorem

$$F(x) = f(x) - \lambda g(x) \qquad (12.6)$$

This function is continuous in $[a, b]$ and differentiable in (a, b). Choose λ to satisfy the third condition of Rolle's theorem, that is $F(a) = F(b)$. We have $f(a) - \lambda g(a) = f(b) - \lambda g(b) \Rightarrow \lambda = \dfrac{f(b) - f(a)}{g(b) - g(a)}$. By Rolle's theorem, there a point

$\xi \in (a, b)$ exists such that $F'(\xi) = 0$. But $F'(x) = f'(x) - \lambda g'(x)$, and therefore $f'(\xi) - \lambda g'(x) = 0$ whence $\dfrac{f'(\xi)}{g'(\xi)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$. The theorem has been proved.

Remark 1. Note, that , when $g(x) = x$ Cauchy's theorem reduces to Lagrange's theorem.

Remark 2. Note that Cauchy's theorem cannot be proved by a simple term-by-term division of the relations expressing Lagragne's theorem for the functions *f* and *g* since the values ξ of the argument do not necessarily coincide in these relations.

Exercises 12.1.

a) For the functions and intervals in Exercises 1 – 4, find the value ξ satisfying the equation $f'(\xi) = \dfrac{f(b) - f(a)}{b - a}$ in the conclusion of Lagrange's theorem.

1. $f(x) = x^2 + 2x - 1, \quad 0 \le x \le 1$;
2. $f(x) = x^{2/3}, \quad 0 \le x \le 1$;
3. $f(x) = x + \dfrac{1}{x}, \quad \dfrac{1}{2} \le x \le 2$;
4. $f(x) = \sqrt{x - 1}, \quad 1 \le x \le 3$.

b) Show that at some instant during a 2-h automobile trip the car's speedometer reading will equal the average speed for the trip.

c) With the aid of Lagrange's formula prove the inequalities $\dfrac{a - b}{a} \le \ln\dfrac{a}{b} \le \dfrac{a - b}{b}$,

for the condition $0 < b \le a$.

d) Two functions $f(x) = 4x^3 - 1$ and $g(x) = 4x - 1$ are given in the interval $[-2, 2]$. Check the validity of Caushy's theorem for these functions.

13. L'Hopital's Rule

In the late 1600s, John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero. Today the rule is known as L'Hopital's rule, after de L'Hopital (1661 – 1704). L'Hopital's rule gives fast results and often applies when other methods fail. If functions $f(x)$ and $g(x)$ are continuous at $x = x_0$, but $f(x_0) = g(x_0) = 0$, the limit $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)}$ can not be evaluated by substituting $x = x_0$. The substitution produces $\frac{0}{0}$, meaningless expression known as an **indeterminate form**.

L'Hopital's Rule. Let functions $f(x)$ and $g(x)$ be continuous functions in some neighborhood of a point $x = x_0$ and have derivatives in the deleted neighborhood of this point, and $g'(x) \neq 0$. Assume also that $f(x_0) = g(x_0) = 0$. Then the limit of their ratio as $x \to x_0$ equals the limit of the derivatives provided the latter limit exists, that is

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \tag{13.1}$$

Proof. We have

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \begin{bmatrix} \text{apply Cauchy's} \\ \text{theorem} \end{bmatrix} = \lim_{\xi \to x_0} \frac{f'(\xi)}{g'(\xi)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Example 13.1. Find the limit $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 6x + 8}$.

Solution. $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 6x + 8} = \left[\frac{0}{0}\right] = \lim_{x \to 2} \frac{\left(x^2 - 5x + 6\right)'}{\left(x^2 - 6x + 8\right)'} = \lim_{x \to 2} \frac{2x - 5}{2x - 6} = \frac{1}{2}$.

In some cases one must apply L'Hopital's rule several times before arriving at a limit which can be conveniently evaluated.

Example 13.2. Find the limit $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

Solution.

$$\lim_{x \to 0} \frac{x - \sin x}{x^3} = \left[\frac{0}{0}\right] = \lim_{x \to 0} \frac{(x - \sin x)'}{\left(x^3\right)'} = \lim_{x \to 0} \frac{1 - \cos x}{3x^2} = \left[\frac{0}{0}\right] = \lim_{x \to 0} \frac{(1 - \cos x)'}{\left(3x^2\right)'} =$$

$$= \lim_{x \to 0} \frac{\sin x}{6x} = \left[\frac{0}{0}\right] = \begin{bmatrix} \text{use the first remarcable} \\ \text{limit} \end{bmatrix} = \frac{1}{6}.$$

However, caution should always be exercised in any application of the rule, lest it be used blindly in a situation where it is not valid.

Example 13.3. $\lim_{x\to\pi}\frac{\sin x}{x} \neq \lim_{x\to\pi}\frac{\cos x}{1}$. Here, the denominator on the left-hand side fails to approach zero as x approaches π, and thus the use of L'Hopital's rule would lead to erroneous results.

It can be proved that L'Hopital's rule applies to the indeterminate form $\frac{\infty}{\infty}$ as well as $\frac{0}{0}$. If $f(x)$ and $g(x)$ both approach infinity as x approaches x_0, then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to x_0}\frac{f'(x)}{g'(x)},$$

provided the limit on the right exists. In the notation $x \to x_0$, x_0 may be either finite or infinite.

Example 13.4. Find limits

$a)$ $\lim_{x\to\frac{\pi}{2}-0}\frac{\tan x}{1+\tan x}$ $\qquad$ $b)$ $\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5}$

Solution.

a) $\lim_{x\to\frac{\pi}{2}-0}\frac{\tan x}{1+\tan x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to\frac{\pi}{2}-0}\frac{\sec^2 x}{\sec^2 x} = 1$, where $\sec x = \frac{1}{\cos x}$;

b) $\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5} = \left[\frac{-\infty}{\infty}\right] = \lim_{x\to\infty}\frac{1-4x}{6x} = \lim_{x\to\infty}\frac{-4}{6} = -\frac{2}{3}$.

We can sometimes handle the forms $0\cdot\infty$ and $\infty-\infty$ by using algebra to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$ instead. Here again, we do not mean to suggest that there is a number $0\cdot\infty$ or $\infty-\infty$ any more than we mean to suggest that there is a number $\frac{0}{0}$ or $\frac{\infty}{\infty}$. These forms are not numbers but descriptions of limits.

14. Evaluation of Some Indeterminate Forms

The case [0•∞]. Let it be necessary to find the limit $\lim_{x\to x_0} f(x)\cdot g(x)$ on condition that $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} g(x) = \infty$.

We have $\lim\limits_{x\to x_0} f(x)\cdot g(x)$= [0•∞] = $\lim\limits_{x\to x_0}\dfrac{f(x)}{\dfrac{1}{g(x)}}=\left[\dfrac{0}{0}\right]$, or

$$\lim_{x\to x_0} f(x)\cdot g(x) = [\,\mathbf{0\bullet}\infty\,] = \lim_{x\to x_0}\frac{g(x)}{\dfrac{1}{f(x)}}=\left[\frac{\infty}{\infty}\right].$$

So it is possible to use L'Hopital's rule in both cases.

Example 14.1. Find limit $\lim\limits_{x\to+\infty} x\cdot\left(e^{1/x}-1\right)$.

$$\lim_{x\to+\infty} x\cdot\left(e^{1/x}-1\right)=[\infty\cdot 0]=\lim_{x\to+\infty}\frac{e^{1/x}-1}{\dfrac{1}{x}}=\left[\frac{0}{0}\right]=$$

Solution.

$$=\lim_{x\to+\infty}\frac{-e^{1/x}\cdot\dfrac{1}{x^2}}{-\dfrac{1}{x^2}}\lim_{x\to+\infty} e^{1/x}=1.$$

The case [∞ – ∞]. Let $\lim\limits_{x\to x_0} f(x)=\infty$ and $\lim\limits_{x\to x_0} g(x)=\infty$. We find the limit $\lim\limits_{x\to x_0}(f(x)-g(x))=[\infty-\infty]$.

Let $f(x)$ and $g(x)$ be fractions, then reducing the expression $f(x)-g(x)$ to the common denominator we arrive at indeterminate forms of the type $\left[\dfrac{0}{0}\right]$ or $\left[\dfrac{\infty}{\infty}\right]$.

Example 14.2. Find the limit $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$.

Solution.

If $x\to+0$, then $\sin x\to+0$ and

$$\frac{1}{\sin x}-\frac{1}{x}\to\infty-\infty.$$

Similarly, if $x \to -0$, then $\sin x \to -0$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty + \infty .$$

Neither form reveals what happens in the limit. To find out, we combine the original fractions,

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$$

and apply L'Hopital's rule to the single fraction on the right:

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\sin x} = \left[\frac{0}{0}\right] = \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\cos x} = \left[\text{still } \frac{0}{0}\right] =$$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.$$

Example 14.3. Find the limit $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1} - \sqrt{x^2+1}\right) = [\infty - \infty]$.

Solution. In this case let us transform the expression $\sqrt{x^2+1} - \sqrt{x^2+1}$ in such a way:

$$\sqrt{x^2+1} - \sqrt{x^2-1} = \frac{\sqrt{x^2+1} - \sqrt{x^2-1}}{1} = \frac{\left(\sqrt{x^2+1} - \sqrt{x^2-1}\right)}{1} \cdot \frac{\left(\sqrt{x^2+1} + \sqrt{x^2-1}\right)}{\sqrt{x^2+1} + \sqrt{x^2-1}} =$$

$$= \frac{x^2+1-\left(x^2-1\right)}{\sqrt{x^2+1} + \sqrt{x^2-1}} = \frac{2}{\sqrt{x^2+1} + \sqrt{x^2-1}}.$$

So $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1} - \sqrt{x^2+1}\right) = [\infty - \infty] = \lim\limits_{x\to\infty}\frac{2}{\sqrt{x^2+1} + \sqrt{x^2-1}} = 0.$

The case [0°].

Let $\lim\limits_{x\to x_0} f(x) = 0$ and $g(x) = 0$. Find the limit $\lim\limits_{x\to x_0}(f(x))^{g(x)} = \left[0^0\right]$. On taking the logarithm of both sides of the equality $y = (f(x))^{g(x)}$ we obtain

$$\ln y = g(x)\ln f(x),\ (f(x)) > 0.$$

$$\lim_{x\to x_0}\ln y = \lim_{x\to x_0} g(x)\ln(f(x)) = [0\cdot\infty] = \lim_{x\to x_0}\frac{\ln(f(x))}{\dfrac{1}{g(x)}} = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x\to x_0}\frac{(\ln(f(x)))'}{\left(\dfrac{1}{g(x)}\right)'} = -\lim_{x\to x_0}\frac{\dfrac{f'(x)}{f(x)}}{\dfrac{g'(x)}{g^2(x)}} = -\lim_{x\to x_0}\frac{f'(x)g^2(x)}{f(x)g'(x)}$$

and $\lim\limits_{x\to x_0}(f(x))^{g(x)} = \exp\left(-\lim\limits_{x\to x_0}\dfrac{f'(x)g^2(x)}{f(x)g'(x)}\right).$

Example 14.3. Find $\lim\limits_{x\to +0} x^{\sin x}$.

Solution. Let us denote

$A = \lim\limits_{x\to +0} x^{\sin x} = \left[0^0\right]$, $y = x^{\sin x}$, $\ln y = \ln x^{\sin x} = \sin x \cdot \ln x$

and $\lim\limits_{x\to +0} \ln y = \ln A.$

Hence

$$\underline{\ln A} = \lim_{x\to +0} \ln y = \lim_{x\to +0} \sin x \cdot \ln x = [0\cdot\infty] = \lim_{x\to +0}\frac{\ln x}{\frac{1}{\sin x}} = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{x\to +0}\frac{\frac{1}{x}}{-\frac{\cos x}{\sin^2 x}} = -\lim_{x\to +0}\frac{\sin^2 x}{x\cos x} = -\lim_{x\to +0}\frac{\sin x}{x}\cdot\lim_{x\to +0}\frac{\sin x}{\cos x} = -1\cdot 0 = \underline{0}.$$

Thus $\ln A = 0 \Rightarrow A = e^0 = 1$, but $A = \lim\limits_{x\to +0} x^{\sin x} = 1$.

Answer: $\lim\limits_{x\to +0} x^{\sin x} = 1$.

The case [1^∞].

Let $\lim\limits_{x\to x_0} f(x) = 1$ and $g(x) = \infty$. Find the limit $\lim\limits_{x\to x_0}(f(x))^{g(x)} = \left[1^\infty\right]$. On taking the logarithm of both sides of the equality

$y = (f(x))^{g(x)}$ we get $\ln y = g(x)\ln f(x)$, $(f(x)) > 0$.

Working in the same way as in the previous case we arrive at the following result

$$\lim_{x\to x_0}(f(x))^{g(x)} = \exp\left(-\lim_{x\to x_0}\frac{f'(x)g^2(x)}{f(x)g'(x)}\right).$$

Example 14.4. . Find $\lim\limits_{x\to 3}(4-x)^{1/(x-3)}$.

Solution. Taking the logarithm of both sides of the equality $y = (4-x)^{1/(x-3)}$ and passing to the limit as $x \to 3$ we have

$$\lim_{x\to 3}\ln y = \lim_{x\to 3}\frac{\ln(4-x)}{x-3} = \left[\frac{0}{0}\right] = \lim_{x\to 3}\frac{\frac{-1}{4-x}}{1} = -\lim_{x\to 3}\frac{1}{4-x} = -1.$$

Hence $\lim\limits_{x\to 3} y = e^{-1}$.

Answer: $\lim\limits_{x\to 3}(4-x)^{1/(x-3)} = e^{-1}$.

Exercises 14.1. Find the limits.

1. $\lim\limits_{x\to 1}\dfrac{x^3-1}{4x^3-x-3}$

2. $\lim\limits_{x\to\infty}\dfrac{6x+5}{3x-8}$

3. $\lim\limits_{x\to\pi/2}\dfrac{2x-\pi}{\cos x}$

4. $\lim\limits_{x\to 0}\dfrac{x\sin x}{1-\cos x}$

5. $\lim\limits_{x\to 0}\dfrac{x(1-\cos x)}{x-\sin x}$

6. $\lim\limits_{x\to\infty} 3\left(x-\sqrt{x^2+x}\right)$

7. $\lim\limits_{x\to\infty}\dfrac{\ln(1+x)}{x}$

8. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^{\alpha}}\ (\alpha>0)$

9. $\lim\limits_{x\to\infty} 3\left(\pi-2\arctan\sqrt{x}\right)\sqrt{x}$

10. $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\cot^2 x\right)$

11. $\lim\limits_{x\to +0}(\arcsin x)^{\tan x}$

12. $\lim\limits_{x\to +\infty}\left(x+\sqrt{x^2+x}\right)^{\frac{1}{\ln x}}$

13. $\lim\limits_{x\to a}\left(3-\dfrac{2x}{a}\right)^{\tan\frac{\pi x}{2a}}$

14. L'Hospital rule does not seem to help with $\lim\limits_{x\to\infty}\dfrac{\sqrt{9x+1}}{\sqrt{x+1}}$**.** Find this limit some other way.

15. Asymptotes of Curves

Definition. A straight line ℓ is called an **asymptote of a curve** K if the distance PQ between the moving point of the curve and the line ℓ tends to zero as the distance from this point to the origin increases indefinitly.

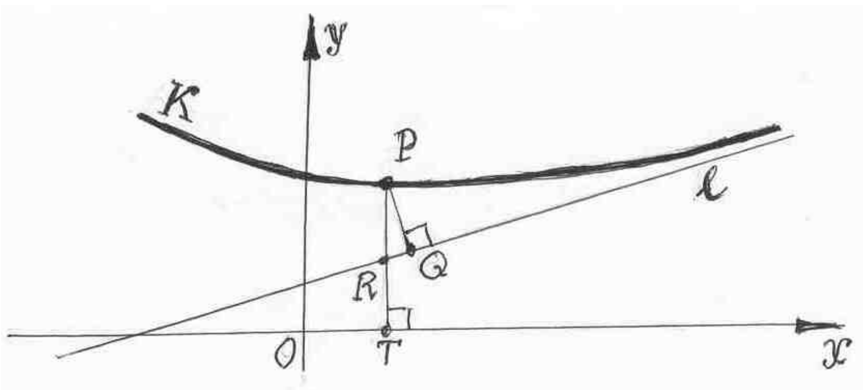


Fig 15.1

Let a cure $y = f(x)$ have a vertical asymptote. The equation of such an asymptote is of the form $x = x_0$ and hence, according to the definition of an asymptote, there must be $f(x) \to \pm\infty$ for $x \to x_0 \neq 0$. Conversely, if x_0 is a point of infinite discontinuity of $y = f(x)$ the straight line $x = x_0$ is a vertical asymptote.

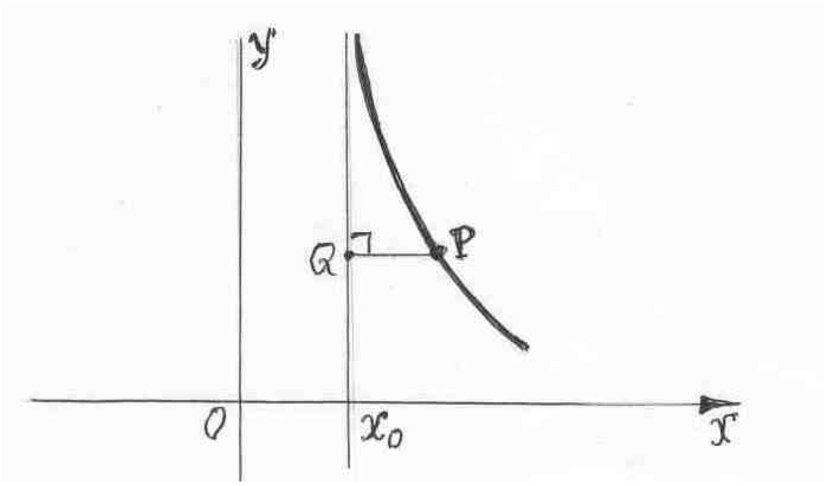


Fig.15.2

So, we have the following definition of a vertical asymptote:

Definition. A line $x = x_0$ is a **vertical asymptote** of the graph if ether $\lim\limits_{x \to +0} f(x) = \pm\infty$ or $\lim\limits_{x \to -0} f(x) = \pm\infty$.

Let a curve *K* having an equation $y = f(x)$ has an oblique asymptote ℓ with an equation of the form $y = kx + b$. By the definition of an asymptote, the distance *PQ* tends to zero as $x \to \pm\infty$. It is more convenient to take the line segment *PR* instead of the distance *PQ*.

We have

$PQ \to 0 \; (x \to \pm\infty) \Rightarrow PR = PT - RT \to 0 \; (x \to \pm\infty)$

(see *Fig 15.1*)$\Rightarrow PR = y_{cur.} - y_{as.} \to 0 \; (x \to \pm\infty) \Rightarrow f(x) - (kx + b) \to 0 \; (x \to \pm\infty) \Rightarrow$ $\Rightarrow f(x) - kx = b + \alpha(x)$, where $\alpha(x)$ is an infinitesimal as $x \to \pm\infty$. On dividing both sides of this relation by *x* and passing to the limit as $x \to \pm\infty$ we get

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \left(k + \frac{b}{x} + \frac{\alpha(x)}{x} \right),$$

and

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x}. \qquad (15.1)$$

After *k* has been determined the number *b* is defined by the limit

$$k = \lim_{x \to \pm\infty} \left(f(x) - kx \right) \qquad (15.2)$$

Conversely, if the limits (15.1) and (15.2) exist and numbers k and b can be found, then the graph of the function $f(x)$ has an oblique asymptote.

Example 15.1. Find the asymptotes of the curve $y = \frac{1-x^3}{x^2}$.

Solution. The function is discontinuous at $x = 0$, and this point is the point of infinite discontinuity. So, the vertical asymptote of this curve is defined by the equation $\boldsymbol{x = 0}$.

To determine the oblique asymptotes we calculate the limits

$$k = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{1-x^3}{x^2} \cdot \frac{1}{x} = -1,$$

$$b = \lim_{x\to+\infty} (f(x) - kx) = \lim_{x\to+\infty} \left(\frac{1-x^3}{x^2} + x \right) = \lim_{x\to+\infty} \frac{1}{x^2} = 0.$$

Hence, the oblique asymptote is defined by the equation $\boldsymbol{y = -x}$.

If $x \to -\infty$ we obtain the same asymptote.

Now we sketch a graph of the function $y = \frac{1-x^3}{x^2}$.

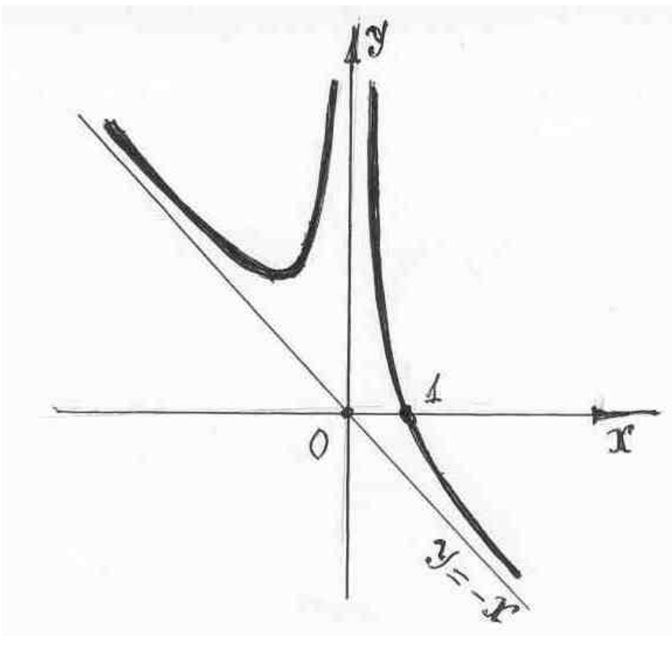


Fig. 15.3

Example 15.1. Find the asymptotes of the curve $y = x + \frac{\sin x}{x}$.

Solution. The function is discontinuous at $x = 0$, and this point is the point of removed discontinuity since $\lim_{x\to+0} \frac{\sin x}{x} = \lim_{x\to-0} \frac{\sin x}{x} = 1$. So there is no vertical asymptote.

To find the oblique asymptotes we compute the limits

$$k = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty}\left(1 + \frac{\sin x}{x^2}\right) = 1,$$

$$b = \lim_{x\to+\infty} \left(f(x) - kx\right) = \lim_{x\to+\infty}\left(x + \frac{\sin x}{x} - x\right) = 0.$$

Hence, the oblique asymptote is defined by the equation $\boldsymbol{y = x}$.

If $x \to -\infty$ we obtain the same asymptote.

Combining these results we have the sketch as shown in *Fig.15.4.*

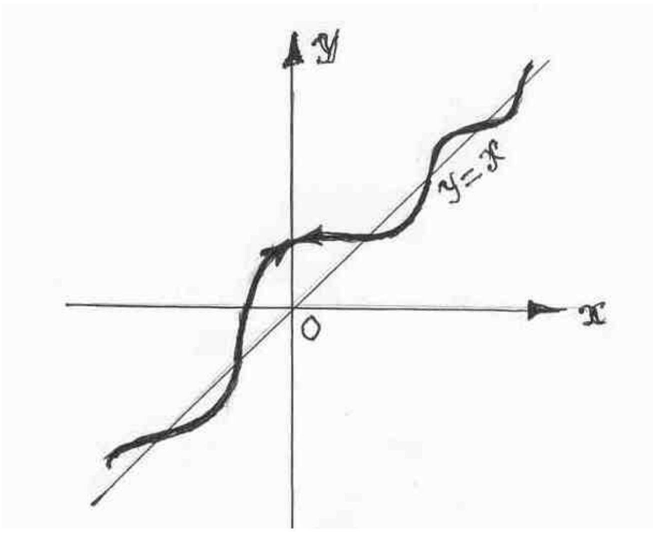


Fig.15.4.

16. Increasing and Decreasing Functions

Definition. A function $f(x)$ defined throughout an interval *I* is said to **increase** on *I* if, for any two points x_1 and x_2 in *I*,

$$x_2 > x_1 \Rightarrow f(x_2) > f(x_1).$$

Similarly, $f(x)$ is said to **decrease** on *I* if, for any two points x_1 and x_2 in *I*

$$x_2 > x_1 \Rightarrow f(x_2) < f(x_1).$$

Theorem. Suppose that $f(x)$ is continuous at each point of the closed interval $[a, b]$, and differentiable at each point of its interior (a, b).If $f'(x) > 0$ at each point of (a, b), then $f(x)$ increases throughout $[a, b]$. If $f'(x) < 0$ at each point of (a, b), then $f(x)$ decreases throughout $[a, b]$. In other case, $f(x)$ is one-to-one on $[a, b]$.

Proof. Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. Apply Lagrange's theorem to $f(x)$ on $[x_1, x_2]$:

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \qquad (16.1)$$

for some ξ between x_1 and x_2. The sign of the right-hand side of the equation (16.1) is the same as the sign of $f'(\xi)$ because $x_2 - x_1$ is positive. Therefore $f(x_2) > f(x_1)$

if $f'(x)$ is positive on (a,b) ($f(x)$ is increasing) and $f(x_2)<f(x_1)$ if $f'(x)$ is negative on (a,b) ($f(x)$ is decreasing). In either case, $x_1 \neq x_2$ implies that $f(x_1)\neq f(x_2)$, so $f(x)$ is one-to-one.

Corollary 1. When we solve equations numerically, we want to know beforehand how many solutions to look for in a given interval. With the help of Lagrange's theorem we can find out.

Suppose, that

1. $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b);
2. $f(a)$ and $f(b)$ have opposite signs;
3. $f'(x)>0$ or $f'(x)<0$ throughout (a,b).

Then $f(x)$ has exactly one zero between *a* and *b*.

It cannot have more than one because $f(x)$ is one-to-one, by Lagrange's theorem..

Example 16.1. The function $f(x)=x^3+3x+1$ is continuous and differentiable on $[-1,1]$, $f(-1)=-3$ and $f(1)=5$ have opposite signs, and $f'(x)=3x^2+3$ is always positive. The equation $x^3+3x+1=0$ has exactly one solution in the interval $[-1,1]$.

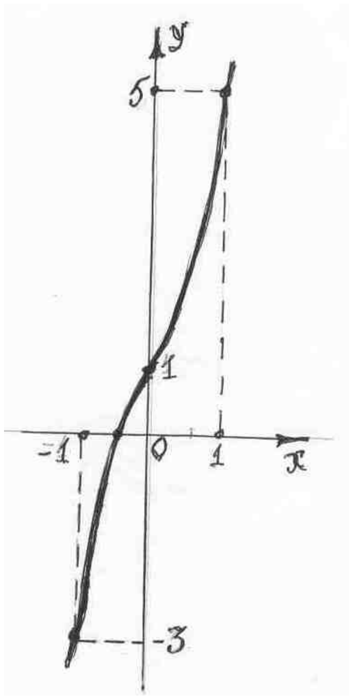


Fig.16.1

In this case we also know that the zero in $[-1,1]$ is *f*'s only zero because $f(x)$ is one-to-one throughout its entire domain.

Example 16.2. Let $f(x)=x^2e^x$. Determine the intervals on which $f(x)$ is increasing and those on which it is decreasing.

Solution. We begin by taking the derivative of $f(x)$:

$$f'(x)=\left(x^2e^x\right)' = 2xe^x + x^2e^x = x(2+x)e^x.$$

Now we determine the sign of $f'(x)$. To do it we divide the interval $(-\infty,+\infty)$ by the zeros of the given function into the intervals $(-\infty,-2)$, $(-2,0)$ and $(0,+\infty)$. Define the sings of $f'(x)$ on each of them.

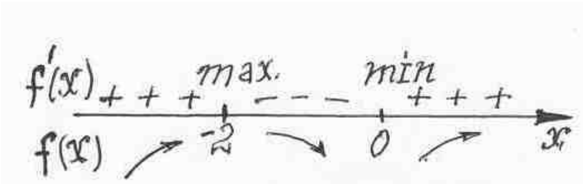


Fig.16.2

It follows from *Fig.16.2* that $f(x)$ is increasing on $(-\infty,-2)$ and $(0,+\infty)$, and decreasing on $(-2,0)$.The graph in *Fig.16.3* confirms these assertions.

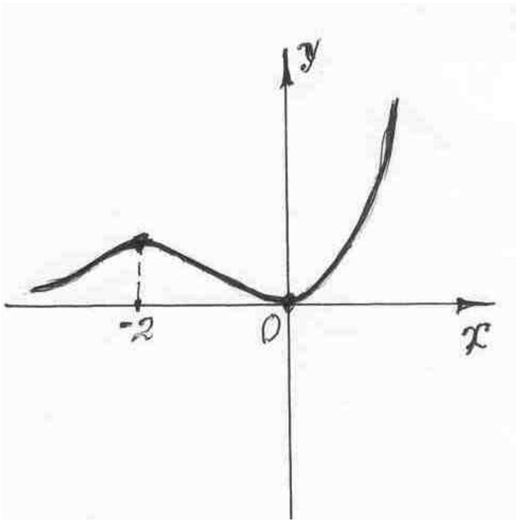


Fig.16.3

Example 16.3. The acting force of an electric current upon a small magnet an axis of which is perpendicular to the plane of a disk and passes through its center is given by the formula

$$F(x)=\frac{x}{\sqrt{\left(a^2+x^2\right)^3}},$$

where a is the center of the disk; x is the distance from the center to the magnet. Find the values of x at which the force $F(x)$ is rising.

*Solution.*The domain of the function $F(x)$ is $[0,+\infty)$. Show that $F'>0$ for all $x\in[0,+\infty)$.

First we find the derivative $F'(x)$:

$$F'(x)=\left(\frac{x}{\sqrt{\left(a^2+x^2\right)^3}}\right)'=\frac{\left(a^2+x^2\right)^{3/2}-\frac{3}{2}\left(a^2+x^2\right)^{1/2}\cdot 2x^2}{\left(a^2+x^2\right)^3}=$$

$$=\frac{\left(a^2+x^2\right)^{1/2}\left(a^2+x^2-3x^2\right)}{\left(a^2+x^2\right)^3}=\frac{a^2-2x^2}{\left(a^2+x^2\right)^{5/2}}.$$

Solving the inequality

$$\frac{a^2-2x^2}{\left(a^2+x^2\right)^{5/2}}>0 \Leftrightarrow a^2-2x^2>0 \Leftrightarrow \left(x-\frac{a}{\sqrt{2}}\right)\left(x-\frac{a}{\sqrt{2}}\right)>0$$

We get $F'(x)>0$ in the interval $\left(0, \frac{a}{\sqrt{2}}\right)$. Hence the function $F(x)$ is increasing in the interval $\left(0, \frac{a}{\sqrt{2}}\right)$.

17. Extremum of Functions

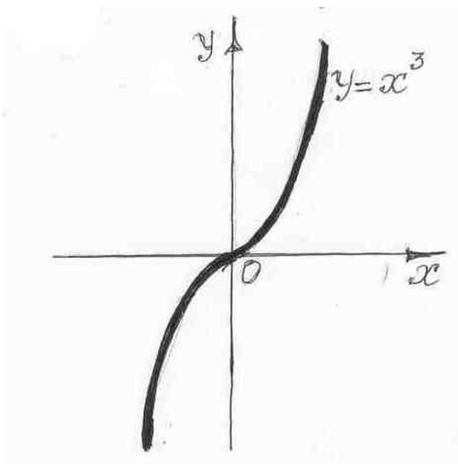
Definition. A point x_0 is called a point of **maximum** (**minimum**) of a function $f(x)$ if $f(x_0)$ is the greatest (least) value of the function $f(x)$ in a neighborhood of the point x_0.

Points of maximum and minimum are called points of extremum of the function.

Theorem. (Necessary condition for extremum)

If a function $f(x)$ has an extremum an am interior point x_0 of an interval, where it is defined, and if $f'(x)$ is defined at x_0, then $f'(x_0)=0$.

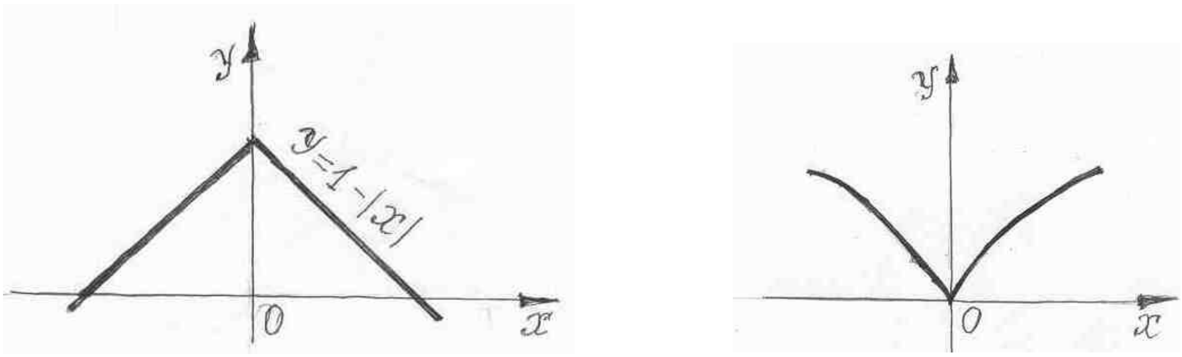
Proof. If the function attains an extremum at the point x_0 its value at this point is the greatest (least) in a neighborhood of the point x_0. By Fermat's theorem $f'(x_0)=0$ since x_0 is an interior point of f's domain. Geometrically, this means that the tangent to the graph of a function is parallel to x – axis at its "tops" and "cavities". A function can also have extrema at some of the points, where it is nondifferentiable. There are two things to watch out, however. A curve may have a horizontal tangent without having a maximum or minimum (*Fig.17.1*)



(*Fig.17.1*)

The curve $y = x^3$ has a horizontal tangent at the origin without having maxima or minima there.

Also, a curve may have an extremum without having a horizontal tangent (*Fig.17.2*)



(*Fig.17.2*)

(a) The graph of $y = 1 - |x|$ has a **corner** at $x = 0$, where the function's derivative is undefined. The right-hand and left-hand derivatives exist there but they have different values.

(b) The graph of $y = x^{2/3}$ has a **cusp** at $x = 0$. The derivative $y' = \left(x^{2/3}\right)' = \frac{2}{3}x^{-1/3}$ approaches $+\infty$ as $x \to +0$ and approaches $-\infty$ as $x \to -0$. The curve does not have a horizontal tangent at the origin, but the tangent is vertical.

Definition. The graph of a continuous function $f(x)$ is said to have a cusp at a point $x = x_0$, if $f'(x) \to +\infty$ as x approaches x_0 from one side and $f'(x) \to -\infty$ as x approaches x_0 from the other side.

The tangent to a graph at a cusp is vertical. Taking this into account we give the following definition.

Definition. A point x_0, at which $f'(x) = 0$ or does not exist is called a **critical point**.

Now we can conclude, that a function $f(x)$ can assume an extremum only at critical points.

Theorem. (Sufficient condition for extremum). Let $f(x)$ be a continuous function in some neighborhood of a point x_0 and differentiable in a deleted neighborhood of this point. Then:

1. If the derivative $f'(x)$ is positive for $x < x_0$ and negative for $x > x_0$ the point x_0 is a point of maximum.
2. If the derivative $f'(x)$ is negative for $x < x_0$ and positive for $x > x_0$ the point x_0 is a point of minimum.
3. If the derivative $f'(x)$ does not change sign as x passes through the point x_0, there is no extremum at this point.

Proof. Let $f'(x)$ is positive for $x < x_0$. This means that on the left of the point x_0 the function increases. If $f'(x) < 0$ for $x > x_0$, then on the right of the point x_0 the function decreases. According to the definition of a point of maximum the point x_0 is the point of maximum.

By analogy we can prove the second case.

As far as the third case we assume for definiteness that $f'(x) > 0$ both for $x < x_0$ and $x > x_0$. Then the function increases both on the left and on the right of the point x_0 and therefore an extremum at this point does not exist.

Example 17.1. Examine the function $y = (x-1)^2(x+1)^3$ for the points at which it reaches its maximum and minimum.

Solution. The domain of this function is the interval $(-\infty, +\infty)$. To obtain the critical points we find $f'(x)$ and set it equal to zero, and solve for x.

$$f'(x) = \left((x-1)^2(x+1)^3\right)' = 2(x-1)(x+1)^3 + 3(x-1)^2(x+1)^2 =$$

$$= (x-1)(x+1)^2(2(x+1) + 3(x-1)) = (x-1)(x+1)^2(5x-1) = 0.$$

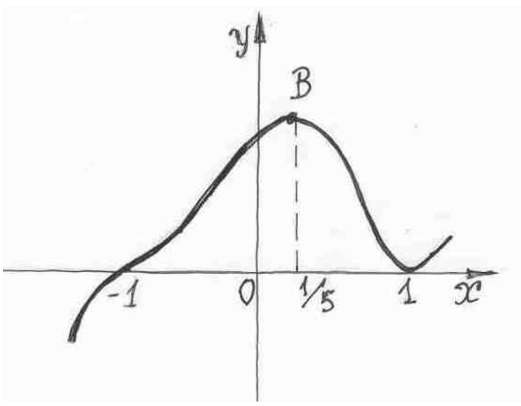


Fig.17.3

Solving, we get $x_1 = -1$, $x_2 = \frac{1}{5}$, $x_3 = 1$, the critical points.

We know, that a function increases where $f'(x) > 0$ and decreases where $f'(x) < 0$, and that maximum and minimum occurs at the points, where $f'(x)$ changes its sign. Therefore, let us look at $f'(x)$ in the following intervals:

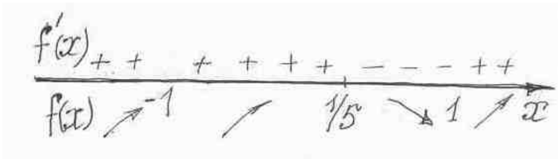


Fig.17.4

This sign pattern for $f'(x)$ tells that the curve rises as it comes in from the left toward $x = \frac{1}{5}$, falls from $x = \frac{1}{5}$ to $x = 1$, and rises again to the right of $x = 1$. Therefore, when $x = 1$ the function has a minimum value $f(1) = 0$ (the ordinate of the point *C*).

When $x = \frac{1}{5}$ the function has a maximum value $f\left(\frac{1}{5}\right) = 1.11$ (the ordinate of the point *B*).

When $x = -1$ the function has neither a maximum nor a minimum.

*Example 17.2.*Two lamps of intensities *a* and *b*, respectively, are *d* feet apart. If the intensity of illumination at any point due to a given point source is directly proportional to the intensity of the source and inversely proportional to the square of the distance from the source, find the darkest point on the line joining the two sources.

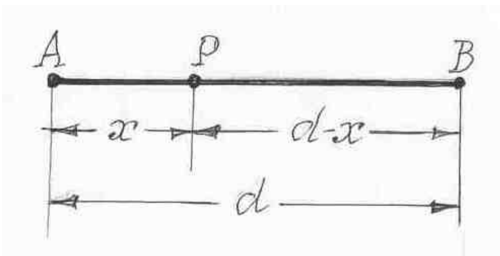


Fig.17.5

Solution. Considering the lamps, for simplicity, to be point-sources, let *A* be a source of intensity *a*, *B* be a source of intensity *b*, *P* be a point on *AB*, *x* be a distance from *A* to *P* and $(d - x)$be the distance from *B* to *P* with *d*, the distance between the two sources. This arrangement can be seen in the accompanying diagram (*Fig.17.5*). Then from the definition of intensity of illumination, at point *P*, the intensity of illumination due to A is $\frac{ka}{x^2}$, and that due to B is $\frac{kb}{(d-x)^2}$. If I is total illumination of P due to both sources,

$$I = \frac{ka}{x^2} + \frac{kb}{(d-x)^2} \tag{17.1}$$

It is clear that at a point very close to either source the value of I is great and decreases as the point recedes. Thus, if P is near either source, one or the other of the denominators is small, and the corresponding fraction is large. Since I decreases as P moves away from either source, it must reach a minimum somewhere between them. Hence, we expect a minimum, and we solve for its location. To obtain this minimum value we must find $\frac{dI}{dx}$, equate it to zero and solve for x. Differentiating, we find

$$\frac{dI}{dx} = k\left(\frac{-2a}{x^3} + \frac{2b}{(d-x)^3}\right),$$

$$k\left(\frac{-2a}{x^3} + \frac{2b}{(d-x)^3}\right) = 0,$$

$$\frac{b}{(d-x)^3} = \frac{a}{x^3}.$$

Clearing fractions,

$$bx^3 = a(d-x)^3.$$

Taking the cube roots of both sides,

$$\sqrt[3]{b}\, x = \sqrt[3]{a}(d-x).$$

Multiplying out and transposing,

$$x\left(\sqrt[3]{a} + \sqrt[3]{b}\right) = d\sqrt[3]{a}.$$

Therefore,

$$x = \frac{d\sqrt[3]{a}}{\sqrt[3]{a}+\sqrt[3]{b}}, \text{ and } d - x = d - \frac{d\sqrt[3]{a}}{\sqrt[3]{a}+\sqrt[3]{b}} = \frac{d\sqrt[3]{b}}{\sqrt[3]{a}+\sqrt[3]{b}}.$$

We can now find the ratio of the distances $\frac{x}{d-x}$. Doing this we have

$$\frac{x}{d-x}=\frac{d\sqrt[3]{a}}{\sqrt[3]{a}+\sqrt[3]{b}}:\frac{d\sqrt[3]{b}}{\sqrt[3]{a}+\sqrt[3]{b}}=\frac{\sqrt[3]{a}}{\sqrt[3]{b}}.$$

Therefore, the ration of the distances $\frac{x}{d-x}$ for the minimum, is $\frac{\sqrt[3]{a}}{\sqrt[3]{b}}$.

18. The Second Derivative Test for Extremum

Theorem. If a function $f(x)$ satisfies the conditions $f'(x_0)=0$ and $f''(x_0)>0$ ($f''(x_0)<0$) then x_0 is a point of minimum (maximum) of the function $f(x)$.

Proof. The existence of the second derivative at the point x_0 implies the existence of the first derivative $f'(x)$ in the neighborhood of the point x_0 and, of course, the continuity of $f(x)$ in this neighborhood. The condition $f''(x_0)>0$ ($f''(x_0)<0$) means that $f'(x)$ increases (decreases) at the point x_0 and, since $f'(x_0)=0$, we have $f'(x)<0$ $(f'(x)>0)$ to the left of x_0 and $f'(x)>0$ $(f'(x)<0)$ to the right of x_0. The assertion of the theorem now follows from the foregoing test for extremum in terms of the first derivative.

Example 18.1. The iron core, which fills the interior part of a cylindrical coil of a transformer of alternating current with radius *R*, has a crosslike section in the form of a square with small squares cut out of the corners. Find the angle φ in such a way that the area of a section should be maximal.

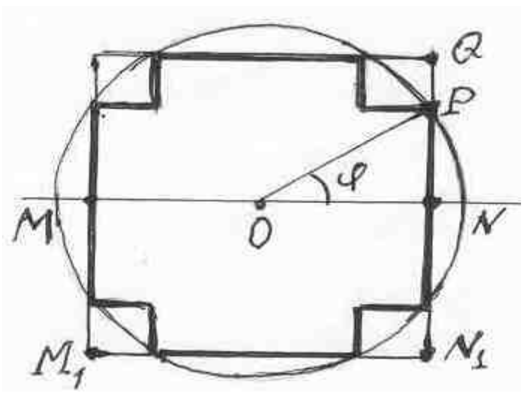


Fig.18.1

Solution. The area *S* of the section will be found as a difference of the area of the square with the side *MN* and the area of four small squares with the side *PQ*: $S=|MN|^2-4|PQ|^2$.

Let the point *O* be a center of symmetry of the square. Then $|MN|=2|NO|$. From the right triangle we obtain

$|ON| = |OP|\cos\varphi$, then $|ON| = R\cos\varphi$, and $|MN| = 2R\cos\varphi$,
$|MN| = 2R\cos\varphi$ $|NP| = R\sin\varphi$, $|PQ| = R\cos\varphi - R\sin\varphi$, whence

$$S = |2R\cos\varphi|^2 - 4|R\cos\varphi - R\sin\varphi|^2.$$

On simplifying this relation we have

$$S = S(\varphi) = 2R^2(2\sin 2\varphi + \cos 2\varphi - 1).$$

Examine the function for an extremum.

$$S' = \left(2R^2(2\sin 2\varphi + \cos 2\varphi - 1)\right)' = 2R^2(4\cos 2\varphi - 2\sin 2\varphi) \qquad (18.1)$$

Equating the derivative to zero we obtain the following equation

$$2\cos 2\varphi = \sin 2\varphi, \text{ or } \tan 2\varphi = 2.$$

Solving this equation we get such critical point

$$\varphi = \frac{1}{2}\arctan 2 \approx 31^\circ 43'.$$

Now we find the second derivative

$$S'' = \left(2R^2(4\cos 2\varphi - 2\sin 2\varphi)\right)' = -8R^2(\cos 2\varphi + 2\sin 2\varphi) \qquad (18.2)$$

Substituting in (18.2) the value $\varphi \approx 31^\circ 43'$ we have

$$S''(31^\circ 43') = -8R^2(\cos 63^\circ 26' + 2\sin 63^\circ 26') < 0$$

This means, that the critical point $\varphi \cong 31^\circ 43'$ is a point of maximum. The value of the function $S(\varphi)$ at this point is

$$S_{\max} \approx 2.472R^2.$$

19. The Greatest and the Least Values of a Function

In order to find the greatest and the least values of a function in the closed interval $[a,b]$, we calculate values of this function at critical points and at the end-points of the interval $[a,b]$. These values should be compared with each other. For the greatest (the least) value of the function in the interval $[a,b]$ is either one of its maximum (minimum) values or an end-point value.

Example 19.1. A source of an electromotive force with internal resistance r is thrown on the load with resistance R. Find the greatest value of the power $P_r(R)$ of the load if the resistance R varies in the interval $[0, 2r]$.

Solution. It is known, that a power $P_r(R)$ is given by the expression

$$P_r(R) = \frac{\varepsilon^2 r}{(R+r)^2}.$$

The problem is reduced to finding the greatest value of this function in the interval $[0, 2r]$. We have

$$P_r'(R)=\left(\frac{\varepsilon^2 r}{(R+r)^2}\right)'=-\frac{2\varepsilon^2 r}{(R+r)^3}.$$

It is clear, that $P_r'(R)\neq 0$ for $R\in(0,2r)$ and $P_r'(R)=\infty$ if $R=-r$, but $-r\notin[0,2r]$.

The values of the function $P_r(R)$ at the end-points of the interval $[0,2r]$ are:

$$P_r(0)=\frac{\varepsilon^2}{r};\ P_r(2r)=\frac{\varepsilon^2}{9r}.$$

Therefore the greatest value of the power is $P_{gr.}(R)=P_r(0)=\dfrac{\varepsilon^2}{r}$.

Thus the function $P_r(R)$ attains the greatest value at the point $R=0$, and this value is equal to $\dfrac{\varepsilon^2}{r}$.

20. Convexity and Concavity of a Curve. Points of Inflection

Definition. An arc of a curve is said to be **convex**, (**concave**) if it lies entirely below (above) the tangent line, drawn through each point of the arc.

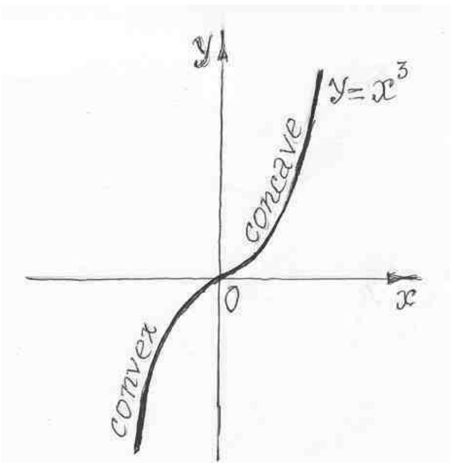


Fig.20.1

The curve $y=x^3$ is convex on $(-\infty,0)$ and concave on $(0,+\infty)$.

Theorem. If a second derivative $f''(x)$ is positive (negative) in some interval, then a graph of the function $y=f(x)$ is concave (convex) in this interval.

Proof. Suppose, that $f''(x)>0$ in some interval.

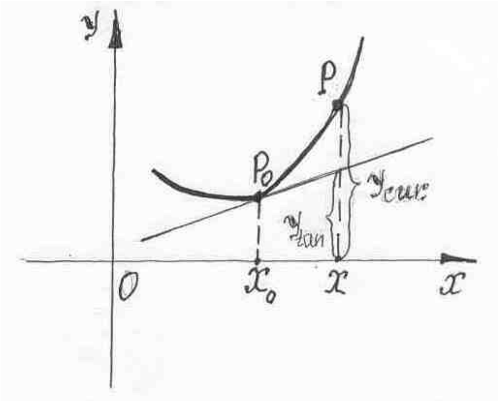


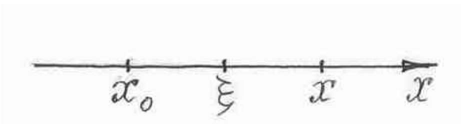
Fig.20.2

Prove, that the graph of this function is concave, that is $y_{cur.} - y_{\tan.} > 0$.
We have

$$y_{cur.} - y_{\tan.} = f(x) - f(x_0) - f'(x_0)(x - x_0) = f'(\xi)(x - x_0) - f'(x_0)(x - x_0) =$$
$$= (f'(\xi) - f'(x_0))(x - x_0) = f''(\eta)(\xi - x_0)(x - x_0).$$

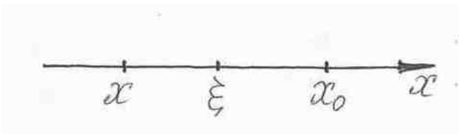
There are two cases of situation of the points x, ξ and x_0:

a)



$(\xi - x_0)(x - x_0) > 0$;

b)



$(\xi - x_0)(x - x_0) > 0$.

Taking into account that $f''(\eta) > 0$, we come to the conclusion, that $y_{cur.} - y_{\tan.} > 0$ in both cases *a)* and *b*).

This means, that the graph of the function $f(x)$ is concave.

The theorem has been proved.

Definition. A point of a curve, separating its convex arc from its concave arc, is called a **point of inflection**.

In *Fig.20.1* the point $x = 0$ is a point of inflection of the function $y = x^3$.

Example 20.1. Let $f(x) = 3x^4 - 4x^3$. Find the intervals, on which the graph of $f(x)$ is concave and those, on which it is convex. Then sketch the graph of $f(x)$.

Solution. Differentiating, we have

$$f'(x) = \left(3x^4 - 4x^3\right)' = 12x^3 - 12x^2 \Rightarrow f''(x) = 36x^2 - 24x.$$

The sign of $f''(x)$ is constant between the zeros of $f''(x)$.

Find zeroes of $f''(x)$:

$$f''(x)=0 \Rightarrow 36x^2-24x=0 \Rightarrow \left[\begin{array}{l} x_1=0, \\ x_2=\dfrac{2}{3}. \end{array}\right.$$

Now we determine the sign of $f''(x)$ from *Fig.20.3*:

sign of $f''(x)$

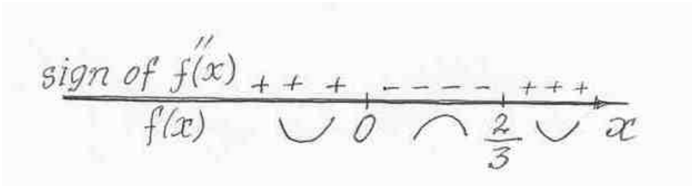


Fig.20.3

Using the sign of $f''(x)$ along with the previous theorem, we deduce, that the graph of $f(x)$ is concave on $(-\infty,0)$ and on $\left(\dfrac{2}{3},+\infty\right)$ and is convex on $\left(0,\dfrac{2}{3}\right)$. From this information we conclude, that the graph of $f(x)$ is as shown in *Fig.20.4*

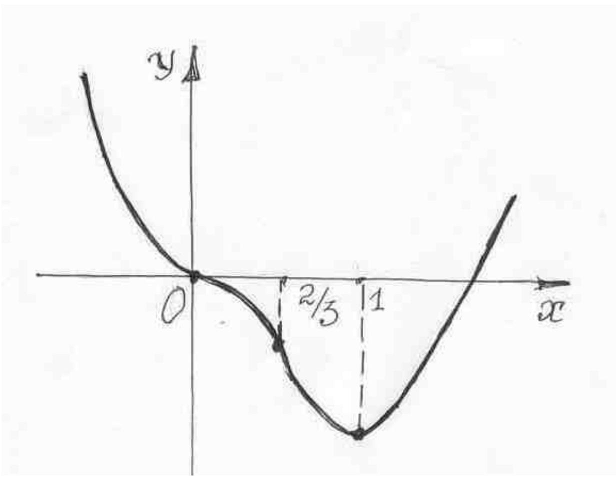


Fig.20.4

Example 20.2. Consider the function $f(x)=\dfrac{a}{1+be^{-kax}}$ for $x \geq 0$, where a and k are positive numbers and $b>1$. Find the inflection point of the graph of $f(x)$ and show, that its y-coordinate is $\dfrac{a}{2}$.

Solution. Taking derivatives, we find that

$$f'(x)=\left(\frac{a}{1+be^{-kax}}\right)'=\frac{-a\left(be^{-kax}\right)(-ka)}{\left(1+be^{-kax}\right)^2}=a^2bk\frac{e^{-kax}}{\left(1+be^{-kax}\right)^2}$$

and

$$f''(x)=\left(a^2bk\frac{e^{-kax}}{\left(1+be^{-kax}\right)^2}\right)'=$$

$$=a^2bk\frac{-ake^{-kax}\left(1+be^{-kax}\right)^2+2e^{-kax}\left(1+be^{-kax}\right)\left(abke^{-kax}\right)}{\left(1+be^{-kax}\right)^4}=$$

$$=a^2bk\frac{ake^{-kax}\left(-1+be^{-kax}\right)}{\left(1+be^{-kax}\right)^3}.$$

It follows, that $f''(x)$ changes from positive to negative at the number x such that $be^{-kax}=1$, or equivalently, $b=e^{kax}$. Let x_0 be this value. Then

$$b=e^{kax_0}\Rightarrow \ln b=kax_0,$$

So that

$$x_0=\frac{\ln b}{ka}.$$

Consequently there is an inflection point at $(x_0, f(x))$. Substituting x_0 for x in the equation for $f(x)$, we obtain

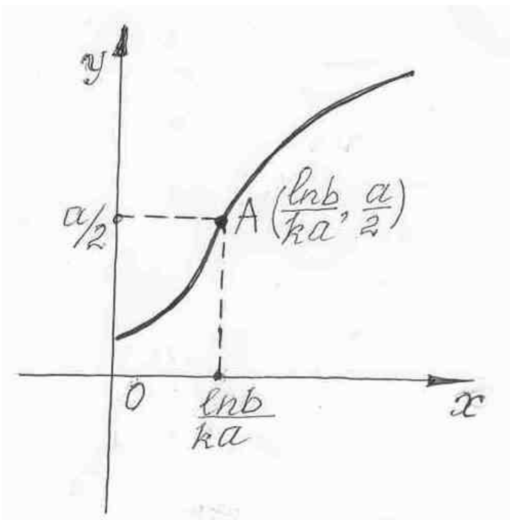


Fig.20.5

$$f(x_0)=f\left(\frac{\ln b}{ka}\right)=\frac{a}{1+be^{-ka\left(\frac{\ln b}{ka}\right)}}=\frac{a}{1+\frac{b}{b}}=\frac{a}{2}.$$

Therefore the y-coordinate of the inflection point is $\frac{a}{2}$.

The graph of $f(x)$ is shown in *Fig.20.5.*

The function $f(x)$ in this example was introduced in 1838 by the Belgian mathematician P.F.Verhulst in order to model population growth of paramecia. The graph of $f(x)$ is known as the **logistic curve**. Notice, that since $f'(x)>0$ and

$f''(x) > 0$ for $0 < x < x_0$, it follows, that if x represents time, the population grows at an ever-increasing rate for $0 < x < x_0$. By contrast, if $x > x_0$, then $f'(x) > 0$ and $f''(x) < 0$. So that the population continues to grow, but at an ever-decreasing rate. Thus, x_0 represents the time, when the population is growing the fastest.

One could also model an epidemic of, say, influenza, in a city with a logistic curve, where the carrying capacity denotes the number of people in the city, who are susceptible. Health officials would be very much interested in the point at which the rate of inflection begins decreasing. That point corresponds to the inflection point of the logistic curve.

21. Test

1. What is the first derivative of the following function?

$$y = e^{-x} \ln 2x$$

(A) $y' = e^{-x}\left(\frac{1}{x} + \ln 2x\right)$ (B) $y' = e^{-x}\left(\frac{1}{x} - \ln 2x\right)$

(C) $y' = e^{-x}\left(\frac{1}{\ln 2x} + \ln 2x\right)$ (D) $y' = -e^{-x} \ln 2x + \frac{e^{-x}}{2x}$

2. Evaluate the following limit.

$$\lim_{x \to 1} \frac{3x^2 + 2x - 5}{x^4 + 3x^2 - 4}$$

(A) 0 (B) $\frac{2}{5}$ (C) $\frac{4}{5}$ (D) ∞

3. Using the formula $k = \left| \frac{y''}{\left(1 + (y')^2\right)^{3/2}} \right|$ find the curvature k of the curve

$y = f(x) = \frac{1}{x}$ at the point (1, 1)?

(A) -2 (B) $-\frac{1}{\sqrt{2}}$ (C) 0 (D) $\frac{1}{\sqrt{2}}$

4. What are the minimum and maximum values, respectively, of the function $f(x) = x^3 + 3x^2 - 9x + 5$ on the interval $[-6, 25]$?

(A) – 49; 32 (B) 0; 32 (C) 7; 35 (D) 11; 49

5. If x and y are related by the equation $x e^{yx} = 3$ and $x > 0$, what is $\frac{dy}{dx}$?

(A) $\frac{-(y+\ln x)-1}{x}$ (B) $\frac{-(xy-1)}{y^2}$

(C) $\frac{-(xy+1)}{x^2}$ (D) $\frac{3}{y}$

6. The value of constant α such that the function $f(x)=\alpha x^2+4x+13$ has a maximum at $x=1$ is

(A) – 5 (B) – 4 (C) – 3 (D) – 2

7. Evaluate the following limit $\lim\limits_{x\to\infty}\frac{\ln x^{99}}{x^2}$

(A) – 2 (B) – 1 (C) 0 (D) 3

8. What is the minimum of the function $f(x)=-3x^5+5x^2$?

(A) (0, 0) (B) $(1,-1)$ (C) $(0,-1)$ (D) $(-1,-1)$

9. The asymptote of the function $y=x+\frac{4}{x^2+1}$

(A) $y=-x$ (B) $y=-x+1$ (C) $x=1$ (D) $x=-1$

10. The function $f(x)=3x^4+4x^3$ is convex on the interval

(A) $\left(-\frac{2}{3},0\right)$ (B) $\left(-\frac{3}{4},0\right)$ (C) $(-1,0)$ (D) $(0,+\infty)$

22. Miscellaneous problems

I. Find the domain of a function.

1.1. $y=\arcsin(x^2-1)$ **1.2.** $y=\tan\frac{1}{x}$

1.3. $y=\frac{1}{\sqrt{x^3-5x^2+6x}}$ **1.4.** $y=\frac{x-1}{\sqrt[3]{x^3-8}}$

II. Find the limits.

2.1. $\lim\limits_{x\to2}\frac{x^2+5}{x^2-3}$ **2.2.** $\lim\limits_{x\to0}\left(\frac{x^3-3x+1}{x-1}\right)$ **2.3.** $\lim\limits_{x\to1}\frac{x}{1-x}$

2.4. $\lim\limits_{x\to\sqrt{3}}\frac{x+3}{x-\sqrt{3}}$ **2.5.** $\lim\limits_{x\to-2}\frac{x^3+3x^2+2x}{x^2-x-6}$ **2.6.** $\lim\limits_{x\to1}\frac{x^2-2x+1}{x^3-x}$

2.7. $\lim\limits_{x\to1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right)$ **2.8.** $\lim\limits_{x\to\infty}\frac{x^3+x}{x^4-4x^2-3}$ **2.9.** $\lim\limits_{x\to\infty}\frac{x^4-5x}{x^2-3x+1}$

2.10. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+1}+\sqrt{x}}{\sqrt[4]{x^3-4x^2}-x}$ **2.11.** $\lim\limits_{x\to 1}\dfrac{x^2-x}{\sqrt{x}-1}$ **2.12.** $\lim\limits_{x\to 5}\dfrac{\sqrt{x-1}-2}{x-5}$

2.13. $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{7+x}-\sqrt[3]{9-x}}{1-x}$ **2.14.** $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x^2}-1}{x^2}$ **2.15.** $\lim\limits_{x\to 0}\dfrac{2\arcsin x}{\tan 3x}$

2.16. $\lim\limits_{x\to 1}\dfrac{\arctan(x-1)}{5\sin(2(x-1))}$ **2.17.** $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin x}-\dfrac{1}{\tan x}\right)$ **2.18.** $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x\sin 2x}$

2.19. $\lim\limits_{x\to\infty}\left(\dfrac{1+x}{x}\right)^{x}$ **2.20.** $\lim\limits_{x\to\infty}\left(\dfrac{1+x}{x+2}\right)^{2x-1}$ **2.21.** $\lim\limits_{x\to 0}\dfrac{e^x-1}{\sin x}$

2.22. $\lim\limits_{x\to 0}\dfrac{\ln(\cos x)}{x^2}$ **2.23.** $\lim\limits_{x\to 0}\dfrac{x-\arctan x}{x^2}$ **2.24.** $\lim\limits_{x\to 0}\dfrac{\ln(\sin 2x)}{\ln(\sin x)}$

2.25. $\lim\limits_{x\to 1}\left(\dfrac{1}{x-1}-\dfrac{1}{\ln x}\right)$ **2.26.** $\lim\limits_{x\to 0}\left(\cot x-\dfrac{1}{x}\right)$ **2.27.** $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)^{\tan x}$

2.28. $\lim\limits_{x\to 0} x^{\sin x}$ **2.29.** $\lim\limits_{x\to 0} x^{1/\ln\left(e^x-1\right)}$ **2.30.** $\lim\limits_{x\to 0}\left(e^x+x\right)^{1/x}$

III. Find the increment Δy if

3.1. $\begin{cases} y=x^3, x_0=2, \\ \Delta x=0.1; \end{cases}$ **3.2.** $\begin{cases} y=\sin x, x_0=\dfrac{\pi}{3}, \\ \Delta x=0.1; \end{cases}$ **3.3.** $\begin{cases} y=e^x, x_0=0, \\ \Delta x=0.2; \end{cases}$

IV. Find the ratio $\dfrac{\Delta y}{\Delta x}$ if

4.1. $\begin{cases} y=2x^3-x^2+1, \\ x_0=1, \Delta x=0.1; \end{cases}$ **4.2.** $\begin{cases} y=\dfrac{1}{x}, x_0=2, \\ \Delta x=0.4; \end{cases}$ **4.3.** $\begin{cases} y=\sqrt{x}, x_0=4, \\ \Delta x=0.4; \end{cases}$

V. Find $f'(x)$ using the definition of derivative if

5.1. $f(x)=x^2$ **5.2.** $f(x)=\ln x$ **5.3.** $f(x)=\sin x$

VI. Find y' at a point $x=x_0$ if

6.1. $\begin{cases} y=\cos^2 2x\cdot\sin^3\dfrac{x}{2} \\ x_0=0 \end{cases}$ **6.2.** $\begin{cases} y=\dfrac{(2x-1)^3}{\sqrt{x^2+1}} \\ x_0=0 \end{cases}$

VII. Find derivatives of the functions

7.1. $y = \arcsin 2^x + \lg(7 - 2x)$ **7.2.** $y = (\sin x)^{\sqrt{x}}$ **7.3.** $y = \arctan \frac{x}{\sin x}$

4. $\sqrt{1 - x^2} \cdot arc\cot \sqrt{1 - x^2}$ 5. $y = \tanh^3 2^{\sqrt{x}}$ 6. $y = (\log_2 x)^x$

7. $y = \sqrt{5^x - x^5} - \cosh^{10} \frac{1}{4 - x}$ 8. $y = \sinh^2(5 - x^4) \cdot \sqrt{1 - 7x^5}$

9. $y = \cot\left(\frac{5}{\sqrt{3}} x - x^5\right) + \sqrt[3]{\frac{2}{x^5}}$ 10. $y = \exp\left(\tan \frac{x}{x^2 + 7x}\right)$

VIII.Find the differentials of the functions

1. $y = \ln \tan \frac{x}{7}$ 2. $y = \arccos\ e^x$ 3. $y = \cos^2\left(3 - \frac{5}{x}\right)$

4. $y = \lg\left(e - \frac{\pi}{x}\right)$ 5. $y = \coth \frac{2^x}{x^2}$ 6. $y = \sinh \sqrt[5]{7x^5}$

IX. Find the derivatives of y with respect to *x*

1. $\begin{cases} y = arc\cot 2^t \\ x = \arctan 3^t \end{cases}$ 2. $\begin{cases} x = t - 1 \\ y = \log(t - 1) \end{cases}$

3. $\begin{cases} x = \sin \sqrt{t} \\ y = \cos \sqrt{t} \end{cases}$ 4. $\begin{cases} x = 2\cos^3 t \\ y = 3\sin^3 t \end{cases}$

X. Find the second derivatives of the giving functions at the point x_0 if

1. $\begin{cases} y = \arcsin \sqrt{2x} \\ x_0 = 0 \end{cases}$ 2. $\begin{cases} y = e^x \sin x \\ x_0 = 0 \end{cases}$ 3. $\begin{cases} y = \cot x \\ x_0 = \frac{\pi}{2} \end{cases}$

XI. Solve the problems

1. A law of motion gives $s = 0.25t^4 - 4t^3 + 16t^2$ as a function of *t*.
   a) At what time its velocity equals 0?
   b) Find the acceleration at the moment *t* = 2s.

2. A body which mass is 2 kg performs rectilinear motion according to the formula $s = t^2 - 2t + 1$, where *s* is measured in centimeters and t in seconds. Find the kinetic energy $\left(E = \frac{mv^2}{2}\right)$ of the body in 3s. after the start.

3. The angle θ through which a wheel turns with time t is given by the function

$\theta = t^2 + 3t - 5$. Find the angular velocity for $t = 5$s.

4. Find the force of current for $t = 3$s. if the quantity of electricity is given by function $Q = \frac{1}{3}t^3 - 5t^2 - 2t$.

5. Find the slope of the tangent line of the parabola $y = x^2$

a) at the origin of coordinates, b)at the point $(3;9)$

b) at the points of intersection of the parabola with the straight line $y = 3x - 2$.

6. Find the equations of tangent and normal to the curve $y = x^3$ at the point whose abscissa is equal to 2.

7. Determine the angle of intersection of the hyperbola $y = \frac{1}{x}$ and the parabola $y = x^2$.

X. Investigation of a function

1. Determine asymptotes of a function $y = f(x)$, if

a) $f(x) = \frac{x}{\sqrt{x^2 + 2}}$ b) $f(x) = \frac{x + 2}{x - 2}$

c) $f(x) = \frac{x^2 + 2}{x + 2}$ d) $f(x) = \frac{\sqrt{x^2 + 2}}{x}$

2. Find intervals of monotonicity and extremums of the following functions

a) $f(x) = \frac{1}{\sqrt{x^2 + 2}}$ b) $f(x) = x^2 e^x$

c) $f(x) = \frac{x^2 - 2}{x + 2}$ c) $f(x) = x \ln x$

3. Find the intervals on which the graph of the function is concave and those on which it is convex. Then sketch the graph of the function.

a) $f(x) = x^4 - 4x$ b) $f(x) = x + \frac{1}{x}$

c) $f(x) = 3x^5 + 5x^3$ d) $f(x) = x\sqrt{x^2 - 4}$

4.Find all inflection points of the graph of the function

a) $f(x) = x^4 - 2x^3$ b) $f(x) = x^3 + 3$

23. Revision Exersises

Variant 1.

1. Find limits:

a) $\lim\limits_{x\to 1}\dfrac{x^3-(a+2)x+a+1}{x^3-1}$; **b)** $\lim\limits_{x\to 0}\dfrac{\arcsin bx^2}{2x\tan(b+1)x}$;

c) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+cx^2}-1}{(c+2)x^2+x^3}$; **d)** $\lim\limits_{x\to\infty}\left(\dfrac{ax-1}{ax+3}\right)^{4x}$;

e) $\lim\limits_{x\to\infty}\dfrac{(b+2)x^3+7x+4}{8+(3b+1)x^2}$; **f)** $\lim\limits_{x\to\infty}\dfrac{\sqrt{(c+1)x}}{\sqrt{2x+\sqrt{cx}}}$.

2. Investigate if the given function $y=\begin{cases}\sin x, & x<0\\ x^2-(a+1)x, & 0\le x\le 1\\ 2x+a, & x>1\end{cases}$ is continuous.

Denote the types of discontinuity points. Graph this function.

3. Find a derivative $\dfrac{dy}{dx}$:

a) $y=\left(\sqrt{x}-\sqrt{5}\right)^{b+1}$; **b)** $y=\arcsin\dfrac{c+1}{x^3}$; **c)** $\begin{cases}x=\arctan n(a+1)\,t+a\\ y=arc\cot(a+1)\,t-a\end{cases}$.

4. Calculate $y'(x_0)$ if $\begin{cases}y=\log_{c+1}\left(b+2+5x^2\right)\\ x_0=0\end{cases}$.

5. Find dy if $y=\dfrac{\arccos(c+1)\,x}{\sin x}+\ln ctg\dfrac{x}{c+1}$.

6. Find the derivative of the second order of the function $y=e^{(a+1)x}\cdot\cos x$.

7. Find the domain of the function $y=\lg\left(x^2-(b+1)^2\right)$.

8. Find the asymptotes of the function $y=x+\dfrac{c+1}{x}$.

9. Find intervals of concavity (convexity) and inflection points of the function $y=x^3-3(a+1)x^2+ax-1$.

10. Investigate the function $y=\dfrac{x^2-ax}{x+1}$ and graph it.

Variant 2.

1. Find limits:

a) $\lim\limits_{x\to 2}\dfrac{x^3+4bx^2-(8+16b)}{x^3-b^3}$; **b)** $\lim\limits_{x\to 0}\dfrac{\tan(c+1)x-\sin 2cx}{5x^3}$;

c) $\lim\limits_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}}{x^2}$; **d)** $\lim\limits_{x\to\infty}\dfrac{3x^3-bx^5+ax^6}{4+bx^6+5x^2}$;

e) $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{x^6+ax^4+7}}{\sqrt{a^2x^4-5x+2}}$; **f)** $\lim\limits_{x\to\infty}\left(\dfrac{x+c}{x-c}\right)^{3x+1}$.

2. Investigate if the given function $y=\begin{cases}2x, & x\le 0\\ \sqrt{x+a}, & 0<x<a\\ 1, & x\ge a\end{cases}$ is continuous. Denote the types of discontinuity points. Graph this function.

3. Find a derivative $\dfrac{dy}{dx}$:

a) $y=\cot\dfrac{(b+1)x}{(b+1)x+2}$ **b)** $y=\arccos\dfrac{c+1}{x}+\lg\left(\dfrac{x^3}{c+1}+1\right)$

c) $\begin{cases}x=\cos(a+1)t+t\sin(a+1)t\\ y=\sin(a+1)t-t\cos(a+1)t\end{cases}$.

4. Calculate $y'(x_0)$, if $\begin{cases}y=5(b+1)\arctan\dfrac{x}{b+1}.\\ x_0=0\end{cases}$

5. Find dy if $y=\tan((c+1)x+1)-3^{(c+1)x}$.

6. Find the derivative of the second order of the function $y=\arcsin(a+3)\sqrt{x}$.

7. Find the domain of the function $y=\sqrt{x^2-(b+2)^2}$.

8. Find the asymptotes of the function $y=\dfrac{x^2}{x-2(c+1)}$.

9. Find intervals of concavity (convexity) and inflection points of the function $y=x^4-6(a+1)x^2+2ax$.

10. Investigate the function $y=\dfrac{x^2+x-1}{x^2-2x+1}$ and graph it.

Variant 3

1. Find limits:

a) $\lim\limits_{x\to -2}\dfrac{x^3-3cx+6c+8}{x^3+8}$ **b)** $\lim\limits_{x\to 0}\dfrac{1-\cos(a+1)x}{1-\cos 2(a+1)x}$

c) $\lim\limits_{x\to b+1}\dfrac{\sqrt{x}-\sqrt{b+1}}{x^2-(b+1)^2}$ **d)** $\lim\limits_{x\to\infty}\dfrac{\sqrt[4]{(b+1)x^8-5x^4+3x}}{(x+3)^2+(2x+1)^2}$

e) $\lim\limits_{x\to\infty}\dfrac{(a+2)x^3+4}{4x^2+ax+3}$ **f)** $\lim\limits_{x\to\infty}\left(\dfrac{x+c+1}{x-c}\right)^{2cx}$

2. Investigate if the given function $y=\begin{cases}0, & x\le 0\\ x^2+c, & 0<x\le 1\\ \dfrac{1}{x-1}, & x>1\end{cases}$ is continuous. Denote the types of discontinuity points. Graph this function.

3. Find a derivative $\dfrac{dy}{dx}$

a) $y=\arccos\dfrac{(a+3)x}{x^2+a+1}$ **b)** $\begin{cases}x=e^{(c+1)\,t}\cos t\\ y=e^{(c+1)\,t}\sin t\end{cases}$ **c** $y=\log_3\tan((b+1)x+5)$

4. Calculate $y'(x_0)$, if $\begin{cases}y=(a+1)^x-\sqrt[a]{x}\\ x_0=1\end{cases}$.

5. Find dy if $y=\cot(b+1)x-\dfrac{1}{arc\cot x}$.

6. Find the derivative of the second order of the function $y=\ln\left(x^2+(c+1)x+1\right)$.

7. Find the domain of the function $y=\dfrac{1}{\sqrt{x-a-3}}$ **.**

8. Find the asymptotes of the function $y=\dfrac{x^2+5x}{x-b}$**.**

9. Find intervals of concavity (convexity) and inflection points of the function $y=x^3-\dfrac{3x^2}{c+1}+2x-c$**.**

10. Investigate the function $y=x-\dfrac{a^2}{x}$ and graph it.

Variant 4.

1. Find limits:

a) $\lim\limits_{x\to -1}\dfrac{x^3-2ax+1+2a}{x^2-x-2}$ **b)** $\lim\limits_{x\to 0}\dfrac{bx}{\arctan 7x}$

c) $\lim\limits_{x\to\infty}\left(\dfrac{x^2+a}{x^2}\right)^{3x}$ **d)** $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{x+c}+\sqrt[4]{c^2x^2+3}}{7x+9}$

e) $\lim\limits_{x\to\infty}\dfrac{10x^6+bx^5+7x^4-5}{bx^3+6x+1}$ **f)** $\lim\limits_{x\to 0}\dfrac{cx}{\sqrt[3]{1+x}-1}$.

2. Investigate if the given function $y=\begin{cases}\dfrac{x^2-x}{a+1}, & x\le 0\\ \sqrt{x+1}, & 0<x\le a+1\\ \dfrac{1}{x}, & x>a+1\end{cases}$ is continuous.

Denote the types of discontinuity points. Graph this function.

3. Find a derivative $\dfrac{dy}{dx}$:

a) $y=arctg\left(x^3-ax^2\right)+tg5x$ **b)** $\begin{cases}x=\ln\left(t^{c+1}+c\right)\\ y=2^t\end{cases}$ **c)** $y=\arcsin\dfrac{b\,x}{1-x^2}$.

4. Calculate $y'(x_0)$, if $\begin{cases}y=\dfrac{a}{2\sqrt{x}}+\sqrt{a}\\ x_0=1\end{cases}$.

5. Find dy if $y=\cos\log_3(bx+2)$.

6. Find the derivative of the second order of the function $y=e^{-cx}\sin x$.

7. Find the domain of the function $y=\arcsin(x-a)$.

8. Find the asymptotes of the function $y=x+\dfrac{b+1}{\sqrt{x}}$.

9. Find intervals of concavity (convexity) and inflection points of the function $y=x^3-\dfrac{6x^2}{c+1}+5x-c$.

10. Investigate the function $y=\dfrac{(x-c)^2}{x^2}$ and graph it.

Variant 5

1. Find limits:

a) $\lim\limits_{x\to 1}\dfrac{2x^3+ax^2-b}{x^3-1}$ **b)** $\lim\limits_{x\to 0}\dfrac{\arctan^2 6x}{4x\sin(c+2)x}$

c) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+a}-\sqrt{a}}{3x^2}$ **d)** $\lim\limits_{x\to\infty}\left(\dfrac{x+2b+4}{x+2b}\right)^{(b+2)x}$

e) $\lim\limits_{x\to\infty}\dfrac{ax^3+x^2-2}{4x-bx^3}$ **f)** $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{c^3x^6-5x^4+2x}}{(x+2)^2+(2x+3)^2}$

2. Investigate if the given function $y=\begin{cases}\cos(x-1), & x\le\pi+1\\ a, & \pi+1<x\le 8\\ a(x-8), & x>8\end{cases}$ is continuous.

Denote the types of discontinuity points. Graph this function.

3. Find a derivative $\dfrac{dy}{dx}$:

a) $y=\log_7\left(bx+\sqrt[4]{x}\right)$ **b)** $y=\dfrac{x^2+cx}{\arctan ax}$ **c)** $\begin{cases}x=\sin^2(bt)\\ y=\cos(bt)\end{cases}$

4. Calculate $y'(x_0)$, if $\begin{cases}y=e^{cx}\cdot\arccos\dfrac{x}{c}.\\ x_0=0\end{cases}$

5. Find dy if $y=\dfrac{1}{1+x^a}-b\tan x$.

6. Find the derivative of the second order of the function $y=2^{cx}+\cot(ax)$.

7. Find the domain of the function $y=\ln\left(b^2-x^2\right)$.

8. Find the asymptotes of the function $y=x-\dfrac{c}{x}$.

9. Find intervals of concavity (convexity) and inflection points of the function $y=\dfrac{6a^2x^2-x^4}{9}$.

10. Investigate the function $y=\dfrac{x-a}{x^2-b^2}$ and graph it.

Variant 6.

1. Find limits:

a) $\lim\limits_{x\to 3}\dfrac{x^3-3ax+9a-27}{x^2+3x-18}$ **b)** $\lim\limits_{x\to 0}\dfrac{\sqrt{1-\cos bx}}{x}$

c) $\lim\limits_{x\to\infty}\dfrac{(c+3)x^2-3x+1}{x-5+ax^2}$ **d)** $\lim\limits_{x\to\infty}\dfrac{\sqrt{16+b^2x^2}+\sqrt{4x^2+13}}{\sqrt{80+9x^2}+\sqrt{x^2+1}}$

e) $\lim\limits_{x\to 0}\left(1+(b+2)x\right)^{b/x}$ **f)** $\lim\limits_{x\to 1}\dfrac{\left(\sqrt[3]{x}-1\right)(a+1)}{\sqrt{x-1}}$

2. Investigate if the given function $y=\begin{cases}0, & x\le 0\\ x^3, & 0<x\le b\\ \dfrac{1}{x}, & x>b\end{cases}$ is continuous. Denote the types of discontinuity points. Graph this function.

3. Find a derivative $\dfrac{dy}{dx}$:

a) $y=\ln\left(2+\sqrt{2-x^a}\right)$ **b)** $y=\cot bx\cdot\arctan\dfrac{x}{c}$ **c)** $\begin{cases}y=\sin^3 at\\ x=\cos^3 at\end{cases}$.

4. Calculate $y'(x_0)$ if $y=\lg\dfrac{x+b+1}{2c+2}+x^{c+2}, x_0=0$.

5. Find dy if $y=\arccos(a/x)$.

6. Find the derivative of the second order of the function $y=\dfrac{\tan(b-x)}{2^{c-x}}$.

6. Find the domain of the function $y=\sqrt[4]{a^2-x^2}$.

8. Find the asymptotes of the function $y=-2x-\dfrac{b}{x}$.

9. Find intervals of concavity (convexity) and inflection points of the function $y=x^2+\dfrac{c^3}{x}$.

10. Investigate the function $y=cx+\dfrac{1}{cx}$ and graph it.

Variant 7.

1. Find limits:

a) $\lim\limits_{x\to -1}\dfrac{x^4-2ax-2a-1}{x^2-5x-6}$ **b)** $\lim\limits_{x\to\infty}\dfrac{9x-\sqrt{x^2-b}}{(3b+1)x}$

c) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+c^2}-c}{\sqrt{x^2+4}-2}$ **d)** $\lim\limits_{x\to\infty}\dfrac{x^2+ax-5}{30bx+1000}$

e) $\lim\limits_{x\to 0}\dfrac{\tan(c-1)x}{27x}$ **f)** $\lim\limits_{x\to\infty}\left(\dfrac{ax-4}{ax}\right)^{\frac{x}{3}+1}$.

2. Investigate if the given function is continuous. Denote the types of discontinuity points. Graph this function. $y=\begin{cases}\sin 2x, & x<0\\ x^2-b, & 0\le x\le 2\\ \log_2 x, & x>2\end{cases}$.

3. Find a derivative $\dfrac{dy}{dx}$:

a) $y=\arctan^a\dfrac{x}{2}+e^{\cos bx}$; **b)** $y=\sqrt{\log_2(cx+3)}$; **c)** $\begin{cases}x=at-b\\ y=t^a\end{cases}$.

4. Calculate $y'(x_0)$ if $y=\dfrac{x}{\tan(x+1)x}$, $x_0=\dfrac{\pi}{4c+4}$.

5. Find dy if $y=\arccos\dfrac{a+2}{x-1}$.

6. Find the derivative of the second order of the function $y=\ln(b+2x)\cdot\sin x$.

7. Find the domain of the function $y=\dfrac{c+1}{\sqrt[4]{a-x}}$.

8. Find the asymptotes of the function $y=\dfrac{x^2-5x}{x+b}$.

9. Find intervals of concavity (convexity) and inflection points of the function $y=x^2-\dfrac{c^3}{x}$.

10. Investigate the function $y=\dfrac{-2x^2+x+4}{(x-2)^2}$ and graph it.

Variant 8

1. Find limits:

a) $\lim\limits_{x\to 2}\dfrac{x^3+5ax-10a-8}{x^4-16}$ **b)** $\lim\limits_{x\to\infty}\dfrac{5\sqrt{x^2+b}+\sqrt[3]{x^2+x}}{\sqrt[5]{x+b}+4bx}$

c) $\lim\limits_{x\to\infty}\left(\dfrac{2x+c+1}{2x-a}\right)^{3x-1}$ **d)** $\lim\limits_{x\to 0}\dfrac{x(a+1)-\sqrt{(a+1)x}}{2\sqrt{x}+x}$

e) $\lim\limits_{x\to\infty}\dfrac{\left(bx^2+2\right)^2+4x^4+3}{(6x+1)^4}$ **f)** $\lim\limits_{x\to 0}\dfrac{\sin^2\frac{cx}{2}}{tg^2 6x}$

2. Investigate if the given function is continuous. Denote the types of discontinuity points. Graph this function.

$$y=\begin{cases}\dfrac{1}{x^4}, & x<0\\ x-a-2, & 0\le x<a+2\\ \ln(x-a-1), & x\ge a+2\end{cases}.$$

3. Find a derivative $\dfrac{dy}{dx}$**:**

a) $y=arc\cot\dfrac{1}{2^{bx}+c}$ **b)** $y=\dfrac{5^x-x^{a+1}}{b+\ln x}$ **c)** $\begin{cases}x=2+e^{a\,t}\\ y=t-e^{-a\,t}\end{cases}$**.**

4. Calculate $y'(x_0)$ if $y=\cot\dfrac{2}{\sqrt{bx}}$, $x_0=\dfrac{\pi^2}{b}$.

5. Find dy if $y=\sin x^{c+1}-\arccos^a x$.

6. Find the derivative of the second order of the function $y=\cos bx\cdot\lg(cx-3)$.

7. Find the domain of the function $y=\arccos(a-x)$**.**

8. Find the asymptotes of the function $y=x-\dfrac{b+4}{\sqrt{x}}$**.**

9. Find intervals of concavity (convexity) and inflection points of the function $y=c^3x^2-\dfrac{1}{x}$**.**

11. Investigate the function $y=\dfrac{x-1}{x^2-2x+2}$ and graph it.

Variant 9

1. Find limits:

a) $\lim\limits_{x\to 1}\dfrac{2x^3+ax-a-2}{x^4-1}$;

b) $\lim\limits_{x\to\frac{\pi}{4b+4}}\dfrac{\cos(b+1)x-\sin(b-1)x}{(b+3)\cos(2b+2)x}$;

c) $\lim\limits_{x\to 0}\dfrac{\sqrt{a+x}-\sqrt{a-x}}{x}$;

d) $\lim\limits_{x\to 0}(1+b\tan x)^{b\cot x}$;

e) $\lim\limits_{x\to\infty}\dfrac{(x+1)^6+(x+2)^6}{cx^6+1000}$;

f) $\lim\limits_{x\to\infty}\dfrac{2x^2-ax+b}{\sqrt{x^4+20}}$.

2. Investigate if the given function is continuous. Denote the types of discontinuity points. Graph this function. $y=\begin{cases} c, & x\le 0\\ 2\sin x, & 0<x\le\pi/6\\ a, & x>\pi/6\end{cases}$.

3. Find a derivative $\dfrac{dy}{dx}$:

a) $y=\dfrac{3^{b-x}}{arcctg^2 x}$

b) $y=\dfrac{c}{e^x+a}+\sqrt{x^2-bx+c}$

c) $\begin{cases} x=\sin at\\ y=\arccos bt\end{cases}$

4. Calculate $y'(x_0)$, if $y=\ln(x+c)\cdot tgax,\quad x_0=0$.

5. Find dy if $y=\arcsin^5 cx$.

6. Find the derivative of the second order of the function $y=a\log_b\left(x+\sqrt{x}\right)$.

7. Find the domain of the function $y=\log_{x+c}(a+2)$.

8. Find the asymptotes of the function $y=\dfrac{x^2}{x+2b}$.

9. Find intervals of concavity (convexity) and inflection points of the function $y=c^3x^2+\dfrac{1}{x}$.

10. Investigate the function $y=x-b\sqrt{x}$ and graph it.

The values of the parameters *a, b, c*:

a **– the first letter of your surname**
b **– the first letter of your name**
c **– the first letter of your patronymic**

1	2	3	4	5	6	7	8	9
A	B	C	D	E	F	G	H	I
J	K	L	M	N	O	P	Q	R
S	T	U	V	W	X	Y	Z	

24. Model of Solution of Assignment on Theme Differential Calculus

Task 1. Calculate limits without using of l'Hopital's rule:

a) $\lim\limits_{x\to 1}\frac{x^3-4x+3}{x^3-1}=\left[\frac{0}{0}\right]$ **b)** $\lim\limits_{x\to 0}\frac{ar\cot 7x}{14x}$ **c)** $\lim\limits_{x\to 0}\frac{\sqrt{1+3x^2}-1}{5x^2+x^3}$

d) $\lim\limits_{x\to\infty}\left(\frac{4x-1}{4x+3}\right)^{6x+2}$ **e)** $\lim\limits_{x\to 0}\frac{\sqrt{1+3x^2}-1}{5x^2+x^3}$ **f)** $\lim\limits_{x\to\infty}\frac{\sqrt[4]{10x^2+3}}{7x+9}$

Solution.

a). To calculate $\lim\limits_{x\to 1}\frac{x^3-4x+3}{x^3-1}=\left[\frac{0}{0}\right]$ divide the numerator and the denominator of the given fraction by $(x-1)$.

As $x^3-1=(x-1)\cdot(x^2+x+1)$, then $(x^3-1):(x-1)=x^2+x+1$;

$$
\begin{array}{l|l}
x^3-4x+3 & x-1 \\
\underline{-(x^3-x^2)} & x^2+x-3 \\
x^2-4x+3 & \\
\underline{-(x^2-x)} & \\
-3x+3 & \\
\underline{-(-3x+3)} & \\
0 &
\end{array}
$$

Thus, $\lim\limits_{x\to 1}\dfrac{x^3-4x+3}{x^3-1}=\lim\limits_{x\to 1}\dfrac{x^2+x-3}{x^2+x+1}=\dfrac{1+1-3}{1+1+1}=-\dfrac{1}{3}$.

b). $\lim\limits_{x\to 0}\dfrac{\arctan 7x}{14x}=\left[\dfrac{0}{0}\right]=\left[\begin{matrix}\arctan 7x\sim 7x\\ \text{if } x\to 0\end{matrix}\right]=\lim\limits_{x\to 0}\dfrac{7x}{14x}=\lim\limits_{x\to 0}\dfrac{7}{14}=\dfrac{1}{2}$**.**

c). $\lim\limits_{x\to 0}\dfrac{\sqrt{1+3x^2}-1}{5x^2+x^3}=\left[\dfrac{0}{0}\right]=\left[\begin{matrix}\text{multiply the numerator and the denominator by}\\ \left(\sqrt{1+3x^2}+1\right)-\text{conjugate of the numerator}\end{matrix}\right]=$

$=\lim\limits_{x\to 0}\dfrac{3}{(5+x)\left(\sqrt{1+3x^2}+1\right)}=\lim\limits_{x\to 0}\dfrac{\left(\sqrt{1+3x^2}-1\right)\left(\sqrt{1+3x^2}+1\right)}{x^2(5+x)\left(\sqrt{1+3x^2}+1\right)}=$

$\lim\limits_{x\to 0}\dfrac{3x^2}{x^2(5+x)\left(\sqrt{1+3x^2}+1\right)}=\lim\limits_{x\to 0}\dfrac{3}{(5+x)\left(\sqrt{1+3x^2}+1\right)}=\dfrac{3}{5\cdot 2}=\dfrac{3}{10}=0{,}3.$

d). $\lim\limits_{x\to\infty}\left(\dfrac{4x-1}{4x+3}\right)^{6x+2}=\left[1^{\infty}\right]$**.**

To use the second remarkable limit transform the fraction $\dfrac{4x-1}{4x+3}$:

$\dfrac{4x-1}{4x+3}=\dfrac{(4x+3)-3-1}{4x+3}=\dfrac{(4x+3)-4}{4x+3}=\dfrac{(4x+3)}{4x+3}+\dfrac{-4}{4x+3}=1+\dfrac{-4}{4x+3}$.

So we have

$$\lim\limits_{x\to\infty}\left(\dfrac{4x-1}{4x+3}\right)^{6x+2}=\left[1^{\infty}\right]=\lim\limits_{s\to\infty}\left(1+\dfrac{-4}{4x+3}\right)^{6x+2}=e^{\frac{-4\cdot 6}{4}}=e^{-6}=\dfrac{1}{e^6}.$$

e). $\lim\limits_{x\to\infty}\dfrac{12x^3+5x+3}{8+31x^3}=\left[\dfrac{\infty}{\infty}\right]=\left[\begin{matrix}12x^3+5x+3\sim 12x^3\\ 8+31x^3\sim 31x^3,\text{ if } x\to\infty\end{matrix}\right]=$

$$=\lim\limits_{x\to\infty}\dfrac{12x^3}{31x^3}=\lim\limits_{x\to\infty}\dfrac{12}{31}=\dfrac{12}{31}.$$

f). $\lim\limits_{x\to\infty}\dfrac{\sqrt[4]{10x^2+3}}{7x+9}=\left[\dfrac{\infty}{\infty}\right]=\left[\begin{matrix}10x^2+3\sim 10x^2\\ 7x+9\sim 7x,\text{ if } x\to\infty\end{matrix}\right]=$

$$= \lim_{x\to\infty} \frac{10x^{2/4}}{7x} = \lim_{x\to\infty} \frac{10}{7x^{1/2}} = 0.$$

Task 2. *Investigate if the given function* $f(x) = \begin{cases} e^x, & x < 0, \\ x+3, & 0 \le x \le 3, \\ \dfrac{1}{x-3}, & x > 3. \end{cases}$ *is continuous. Denote the types of discontinuity points if they are. Sketch this function.*

Solution.

Domain of the given function is:
$D(y) = (-\infty, 0) \cup [0, 3] \cup (3, +\infty)$. Thus the function can have discontinuity only at the points $x = 0$ and $x = 3$.

As we know,

1. a function $f(x)$ is continuous at a point x_0 if

a) the function is defined at the point $x = x_0$ and has finite value : $f(x_0) < \infty$,,

b) the right-hand limit and left-hand limit of $f(x)$ exist and are finite:
$\lim\limits_{x\to x_0-0} f(x) = f(x_0 - 0) < \infty$, $\lim\limits_{x\to x_0+0} f(x) = f(x_0 + 0) < \infty$,

c) $f(x_0 - 0) = f(x_0 + 0) = f(x_0)$.

2. A point x_0 is the point of discontinuity of the first kind if conditions a),b) take place but not condition c).

3. If a point x_0 is not the discontinuity of the first kind it is the point of discontinuity of the second kind.

Let us consider the point $x = 0$.

a) $f(0) = (x+3)\big|_{x=0} = 0 + 3 \Rightarrow f(0) = 3 < \infty$,

b) $\lim\limits_{x\to -0} f(x) = \lim\limits_{x\to -0} e^x = e^0 = 1 < \infty$, $\lim\limits_{x\to +0} f(x) = \lim\limits_{x\to +0} (x+3) = 3 < \infty$,

c) $1 \ne 3$.

The point $x = 0-$ is the point of discontinuity of the first kind.

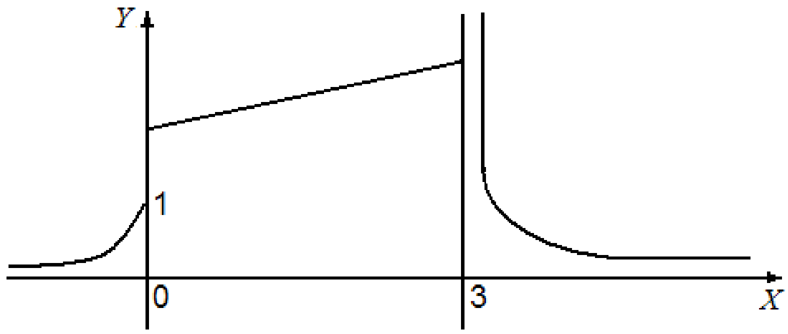
Now let us consider the point $x = 3$:

a) $f(3) = (x+3)\big|_{x=3} = 3 + 3 = 6 < \infty$,

b) $\lim\limits_{x\to 3-0} f(x) = \lim\limits_{x\to 3-0} (x+3) = 6 < \infty$, $\lim\limits_{x\to 3+0} f(x) = \lim\limits_{x\to 3+0} \dfrac{1}{x-3} = \infty$.

The point $x = 3-$ is the point of discontinuity of the second kind.

Sketch of the given function:



Task 3. *Find a derivative* $\frac{dy}{dx}$:

a) $y=\cot\frac{52x}{52x+2}$ **b)** $y=\arccos\frac{52}{x}+\lg\left(\frac{x^3}{52}+1\right)$ **c)** $\begin{cases} x=\cos 52t+t\sin 52t \\ y=\sin 52t-t\cos 52t \end{cases}$

Solution.

a) $\left(\cot\frac{52x}{52x+2}\right)'=\left[\begin{array}{l}\text{use the formulas}:(\cot u)'=-\frac{u'}{\sin^2 u} \\ \left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}\end{array}\right]=$

$$=-\frac{1}{\sin^2\frac{52x}{52x+2}}\cdot\frac{(52x)'(52x+2)-52x(52x+2)'}{(52x+2)^2}=$$

$$=-\frac{1}{\sin^2\frac{52x}{52x+2}}\cdot\frac{52\cdot(52x+2)-52x\cdot 52}{(52x+2)^2}==-\frac{1}{\sin^2\frac{52}{52x+2}}\cdot\frac{104}{(52x+2)^2}=$$

$$=-\frac{1}{\sin^2\frac{52x}{52x+2}}\cdot\frac{52\cdot(52x+2-52x)}{(52x+2)^2}=-\frac{104}{(52x+2)^2\sin^2\frac{52x}{52x+2}}.$$

b) $\left(\arccos\frac{52}{x}+\lg\left(\frac{x^3}{52}+1\right)\right)'=\left[\begin{array}{l}\text{use the formulas}:(u+v)'=u'+v',\\ (\arccos u)'=-\frac{u'}{\sqrt{1-u^2}},\left(\frac{C}{v}\right)'=-\frac{Cv'}{v^2},\\ (\lg u)'=\frac{u'}{u\ln 10},\left(u^n\right)'=n\cdot u^{n-1}\cdot u'\end{array}\right]=$

$$=-\frac{1}{\sqrt{1-\frac{2704}{x^2}}}\left(-\frac{52}{x^2}\right)+\frac{1}{\left(\frac{x^3}{52}+1\right)\ln 10}\cdot\frac{3x^2}{52}=\underline{\frac{52}{x\sqrt{x^2-2704}}+\frac{3x^2}{\left(x^3+52\right)\ln 10}}.$$

c) $\begin{cases}x=\cos 52t+t\sin 52t\\ y=\sin 52t-t\cos 52t\end{cases}.$

The function is given in the parametric form. Use the formula: $\frac{dy}{dx}=\frac{y'_t}{x'_t}$. As

$x'_t=-52\sin 52t+t'\sin 52t+t(\sin 52t)'=-52\sin 52t+\sin 52t+52t\cos 52t=$
$=-51\sin 52t+52t\cos 52t,$

$y'_t=52\cos 52t-\cos 52t+52t\sin 52t=51\cos 52t+52t\sin 52t,$

then

$$\underline{\frac{dy}{dx}=\frac{51\cos 52t+52t\sin 52t}{52t\cos 52t-51\sin 52t}}.$$

d). *Prove that the derivative of the function* $y=(u)^v$ *can be calculated by the formula:*

$$\left((u)^v\right)'=v\cdot u^{v-1}\cdot u'+u^v\cdot v'\cdot\ln u. \qquad (1)$$

Solution.

1). Take the logarithm of the both parts of the equality $y=(u)^v$:

$$\ln y=\ln\left((u)^v\right)\Rightarrow \ln y=v\ln u.$$

2). After differentiation of the last equality, knowing that *y, u* and *v* are the functions of *x,* we have:

$$\frac{y'}{y}=v'\ln u+v\frac{u'}{u}\Rightarrow \quad y'=y\left(v'\ln u+v\frac{u'}{u}\right)\Rightarrow\left[y=(u)^v\right]\Rightarrow y'=u^v\left(v'\ln u+v\frac{u'}{u}\right)\Rightarrow$$

$$y'=u^v\left(v'\ln u+v\frac{u'}{u}\right)\Rightarrow y'=u^v v'\ln u+vu^{v-1}u',$$

or $\left((u)^v\right)'=v\cdot u^{v-1}\cdot u'+u^v\cdot v'\cdot\ln u.$

So the formula (1) has been proved.

Task 4. Calculate $y'(x_0)$, if $y = 260\arctan\frac{x}{52}$, $x_0 = 0$.

Solution.

1) $$y' = \left(260\arctan\frac{x}{52}\right)' = \frac{260}{1+\left(\frac{x}{52}\right)^2}\cdot\frac{1}{52} = \frac{5\cdot 52^2}{52^2 + x^2}$$

2) $$y'(0) = \frac{5\cdot 52^2}{52^2 + x^2}\,\Big|_{x=0} = \frac{5\cdot 52^2}{52^2} = 5.$$

Task 5. *Find* dy, if $y = \tan(52x+1) - 3^{52x}$.

Solution.

Use the formula: $dy = y'dx$.

$$y' = \left(\tan(52x+1) - 3^{52x}\right)' = \frac{52}{\cos^2(52x+1)} - 3^{52x}\ell n3\cdot 52 \Rightarrow$$

$$\Rightarrow dy = 52\cdot\left(\frac{1}{\cos^2(52x+1)} - 3^{52x}\ell n3\right)dx.$$

Task 6. *Find* y'', if $y = \arcsin 54\sqrt{x}$.

1) $$y' = \left(\arcsin 54\sqrt{x}\right)' = \frac{1}{\sqrt{1-\left(54\sqrt{x}\right)^2}}\cdot 54\cdot\frac{1}{2\sqrt{x}} = \frac{27}{\sqrt{1-2916x}\cdot\sqrt{x}} =$$

$$= \frac{27}{\sqrt{x-2916x^2}} \Rightarrow$$

2) $$y'' = (y')' = \left(\frac{27}{\sqrt{x-2916x^2}}\right)' = \left(27\left(x-2916x^2\right)^{-1/2}\right)' =$$

$$= -\frac{27}{2}\left(\left(x-2916x^2\right)^{-3/2}\right)\cdot(1-2916\cdot 2x) = \frac{-27(1-5832x)}{2\sqrt{\left(x-2916x^2\right)^3}}.$$

Task 7. *Find the domain of the function* $y = \ln\left(53^2 - x^2\right)$.

Solution.

As a logarithm exists for positive volume of its argument (compare with the corresponding table) we have:

$D(y): 53^2 - x^2 > 0 \Rightarrow (53-x)(53+x) > 0 \Rightarrow x\in(-53; 53)$. Thus

$D(y) = (-53; 53)$.

Task 8. *Find equations of asymptotes of the function* $y = x - \dfrac{54}{x}$.

Solution.

1) $D(y) = (-\infty; 0) \cup (0; +\infty)$. The straight line $\underline{x = 0}$ is the vertical asymptote. In fact,

$$\lim_{x \to -0} f(x) = \lim_{x \to -0}\left(x - \frac{54}{x}\right) = \lim_{x \to -0} \frac{x^2 - 54}{x} = +\infty,$$

$$\lim_{x \to +0} f(x) = \lim_{x \to +0}\left(x - \frac{54}{x}\right) = \lim_{x \to +0} \frac{x^2 - 54}{x} = -\infty.$$

2) As we know, an equation of nonvertical asymptote is: $y = kx + b$, where

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^2 - 54}{x^2} = \left[\frac{\infty}{\infty}\right] = \text{[use L'Hopital's rule]} = \lim_{x \to \infty} \frac{2x}{2x} = 1,$$

$$b = \lim_{x \to \pm\infty}(f(x) - kx) = \lim_{x \to \pm\infty}\left(\frac{x^2 - 54}{x} - x\right) = \lim_{x \to \pm\infty}\left(\frac{x^2 - 54 - x^2}{x}\right) = \lim_{x \to \pm\infty}\left(\frac{-54}{x}\right) = 0$$

. So the equation of inclined asymptote is $\underline{y = x}$.

The answer: the equations of the asymptotes are : $x = 0$ and $y = x$.

Task 9*. Find intervals of concavity (convexity) and inflection points of the function* $y = \dfrac{6 \cdot 52^2 x^2 - x^4}{9}$.

Solution.

It is known that if $y'' > 0$, then the function is concave, if $y'' < 0$, then the function is convex. To find the intervals, where $y'' > 0$, or $y'' < 0$ we denote domain of the given function and the points in which y'' is equal to zero or does not exist.

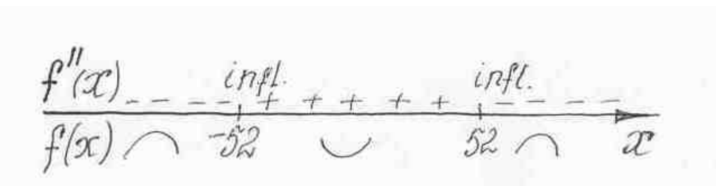
$D(y) = (-\infty; +\infty)$;

$$y' = \left(\frac{6 \cdot 52^2 x^2 - x^4}{9}\right)' = \frac{12 \cdot 52^2 x - 4x^3}{9}, \text{ then}$$

$$y'' = \left(\frac{12 \cdot 52^2 x - 4x^3}{9}\right)' = \frac{1}{9}\left(12 \cdot 52^2 - 12x^2\right) = \frac{4}{3}\left(52^2 - x^2\right).$$

Obviously, y'' exists for any value of an argument. Let us find its roots:

$$y'' = 0 \Rightarrow \frac{4}{3}\left(52^2 - x^2\right) \Rightarrow (52 - x)(52 + x) = 0 \Rightarrow x_{1,2} = \pm 52$$



So, inflection points are: $x_1 = -52,\ x_2 = 52,$
the function is convex at $x \in (-\infty;-52) \cup (52:\infty)$, as here $y'' < 0$, and
the function is concave at $x \in (-52; 52)$, as here $y'' > 0$.

Task 10. *Carry out complete investigation of the given function and graph it*:

$$y = x^3 e^{-x}.$$

Solution.

1. a) The given function is defined everywhere. So, $D(y) = (-\infty, +\infty)$. There are not vertical asymptotes.

Let us find inclined asymptotes, using the equation: $\underline{y = kx + b}$, where

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x},\ b_1 = \lim_{x \to \pm\infty} (f(x) - kx).$$

$$k_1 = \lim_{x \to +\infty} \frac{x^3 e^{-x}}{x} = \lim_{x \to +\infty} \frac{x^2}{e^x} = \begin{bmatrix} \text{use L'Hospital} \\ \text{rule twice} \end{bmatrix} = \lim_{x \to +\infty} \frac{2x}{e^x} = \lim_{x \to +\infty} \frac{2}{e^x} = 0;$$

$$b_1 = \lim_{x \to \infty} (f(x) - kx) = \lim_{x \to \infty} \left(\frac{x^2}{e^x} - 0 \cdot x \right) = 0.$$

$$k_2 = \lim_{x \to -\infty} \frac{f(x)}{x} = \lim_{x \to -\infty} \frac{x^3 e^{-x}}{x} == \lim_{x \to -\infty} x^2 e^{-x} = \infty.$$ In this case an asymptote does not exists.

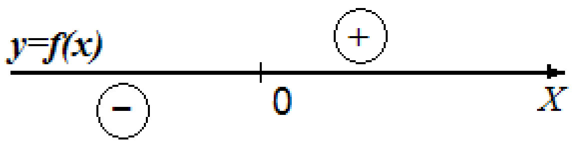
So the inclined asymptote is the right-hand part of the straight line $\underline{y = 0}$.

b) The given function is neither even nor odd:

$$f(-x) = (-x)^3 e^x = -x^3 e^x \neq \begin{cases} f(x) \\ -f(x) \end{cases}.$$

c) This function is not a periodic function.

c) Roots of the function: $f(x) = 0 \Rightarrow x^3 e^{-x} = 0 \Rightarrow x = 0$. The intervals of the constant sign are: $\begin{cases} f(x) > 0 \ \text{if } x > 0, \\ f(x) < 0 \ \text{if } x < 0 \end{cases}$



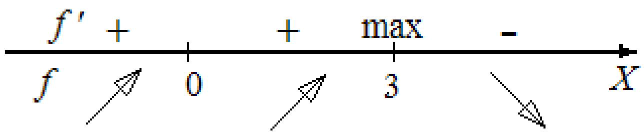
2. To define the intervals of monotonicity and the points of extremum find the critical points, using the corresponding definition: $\left[\begin{array}{l} f'(x)=0 \\ f'(x)=\infty \end{array}\right.$

$$f'(x)=\left(x^3e^{-x}\right)' = 3x^2e^{-x} - x^3e^{-x} = x^2e^{-x}(3-x).$$

The first derivative exists everywhere.

$$f'(x)=0 \text{ if } \left[\begin{array}{l} x=0 \\ x=3 \end{array}\right..$$

Here are two critical points $x=0$ and $x=3$



$y\uparrow$ when $x\in(-\infty,3)$, $\quad y\downarrow$ when $x\in(3,+\infty)$.

It is clear, that the given function has one point of exstremum:

$y_{\max}(3)=27\cdot e^{-3} \approx 1.34$

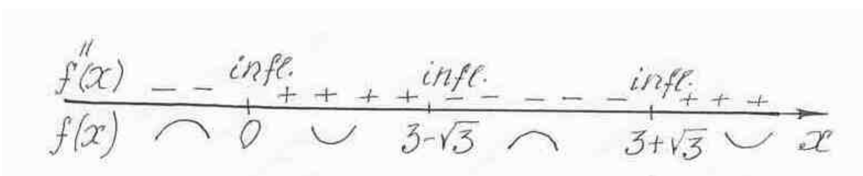
2. To find the intervals of convexity and concavity and the inflection points we must consider the second derivative of the function.

$$f''(x)=\left(e^{-x}\left(3x^2-x^3\right)\right)' =$$

$$-e^{-x}\left(3x^2-x^3\right)+e^{-x}\left(6x-3x^2\right)=e^{-x}\left(x^3-6x^2+6x\right)=$$

$$=e^{-x}x\left(x^2-6x+6\right)=\left[f''(x)=0 \Rightarrow \left[\begin{array}{l} x_1=3-\sqrt{3}, \\ x_2=0, \\ x_3=3+\sqrt{3}. \end{array}\right]\right]$$

$$=e^{-x}x\left(x-\left(3-\sqrt{3}\right)\right)\left(x-\left(3+\sqrt{3}\right)\right).$$



$$\begin{cases} \text{concavity intervals}: \left(0, 3-\sqrt{3}\right)\cup\left(3+\sqrt{3},+\infty\right), \\ \text{convexity intervals}: (-\infty,0)\cup\left(3-\sqrt{3},3+\sqrt{3}\right). \end{cases}$$

The graph of the given function has two points of inflection:
$y_{\inf}\left(3-\sqrt{3}\right)\approx 0.58$ and $y_{\inf}\left(3+\sqrt{3}\right)\approx 1.33$.

Using the results of the investigation of the given function we can draw the graph of this function: