Ministry of Ukraine Transport and Communication
State Department of Communication and Informatization

Odessa National Academy of Communication after A.Popov

Department of Higher Mathematics

Textbook on Sections

COMPLEX NUMBERS AND FUNCTIONS

For Students Studying a Course of Higher Mathematics in English

Odessa – 2008

Authors: Gavdzinski V.N., Korobova L.N.

Методичний посібник міситить основні розділи по теорії комплексного аргументу, англійською мовою для студентів академії, що вивчають вищу математику англійською мовою. Основні теореми і формули приведені з доказом, а також дани вирішення типових прикладів і завдання для самостійного вирішення.

CONTENTS

I. COMPLEX NUMBERS

1.1. The Fundamental Operations

The square of a real number is never negative. Thus, for example, the elementary quadratic equation $x^2 = -1$ has no solution among the real numbers. New types of numbers, called complex numbers, have been introduced to provide solutions to such equations.

Definition. By a **complex number** we mean an ordered pair of real numbers which we denote by (x, y).
The first member, *x,* is called the **real part** of the complex number; the second member, *y*, is called the **imaginary part**. We write

$$z = (x, y).$$

The equality relation and the arithmetical operations are defined according to the following rules:

1. equality $(x_1, y_1) = (x_2, y_2)$ takes place if and only if $x_1 = x_2,\ y_1 = y_2$;

2. $(x_1, y_1) \pm (x_2, y_2) = (x_1 \pm x_2, y_1 \pm y_2)$;

3. $(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$.

If the fundamental operations are thus defined, we easily see that the fundamental laws of algebra are all satisfied.

1. The commutative and associative laws of addition hold:

$$z_1 + z_2 = z_2 + z_1;$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 = z_1 + z_2 + z_3.$$

2. The same laws of multiplication hold:

3. $$z_1 z_2 = z_2 z_1;$$
$$z_1 (z_2 z_3) = (z_1 z_2) z_3 = z_1 z_2 z_3.$$

4. The distributive law holds:

$$(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3.$$

1.2. Geometrical Representation of Complex Numbers

Just as real numbers are represented geometrically by points on a line, so complex numbers are represented by points in a plane. The complex number $z = (x, y)$ can be thought of as the point with coordinates (x, y). When this is done, the definition of addition amounts to addition by the parallelogram law.

The idea of expressing complex numbers geometrically as points on a plane was formulated by Gauss in his dissertation in 1799 and, independently, by Argand in 1806. Gauss later coined the somewhat unfortunate phrase "complex number".

1.3. The Imaginary Unit

It is convenient to think of the complex number $(x.y)$ as a two-dimensional vector with components x and y. Adding two complex numbers is the same as adding two vectors component by component. The complex number $1 = (1,0)$ plays the same role as a unit vector in the horizontal direction. The analog of a unit vector in the vertical direction will now be introduced.

Definition. The complex number $(0,1)$ is defined by i and is called the **imaginary unit**.

Theorem. Every complex number $z = (x, y)$ can be represented in the form $z = x + yi$ which is called **standard** or **rectangular form** of complex numbers.

Proof.

$$z = (x.y) = (x.0) + (0, y) = x(1,0) + y(0,1) = x + yi.$$

Let us now prove that $i^2 = -1$. In fact,

$$i^2 = (0,1)(0,1) = (-1,0) = -1.$$

Example 1.3.1.

Find the product of $z_1 = 2 + 3i$ and $z_2 = 5 - 4i$.

Solution.

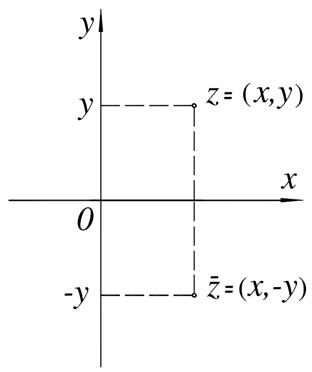
$$z_1 z_2 = (2 + 3i)(5 - 4i) = 10 - 8i + 15i - 12i^2 = 22 + 7i.$$

Exercise 1.3.1. Prove that $i^n = \begin{cases} 1, \text{if } n = 4k \\ i, \text{if } n = 4k+1 \\ -1, \text{if } n = 4k+2 \\ -i, \text{if } n = 4k+3 \end{cases}$

1.4. Absolute Value of a Complex Number and Conjugate Complex Number

Definition. If $z = (x, y)$, we define the **modulus**, or **absolute value**, of z to be the non-negative real number $|z|$ given by

$$|z| = \sqrt{x^2 + y^2} \ .$$



Geometrically, $|z|$ represents the length of the segment joining the origin and the point $z = (x, y)$.

Definition. The number $x - yi$ is said to be **conjugate** to z and is denoted by $\bar{z}$.

Let us calculate $z\bar{z}$.

$$z\bar{z} = (x + yi)\cdot(x - yi) = x^2 - (yi)^2 = x^2 + y^2 = |z|^2.$$

1.5. Definition of Division

The division is an operation inverse to the multiplication.
The number z is called the quotient of z_1 and z_2 if $z_1 = z \cdot z_2$. If $z_2 \neq 0$ then on multiplying both parts of the relation $z_1 = z \cdot z_2$ by $\overline{z_2}$ we get

$$z_1\overline{z_2} = z\left(z_2\overline{z_2}\right) \text{ and } z = \frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}}.$$

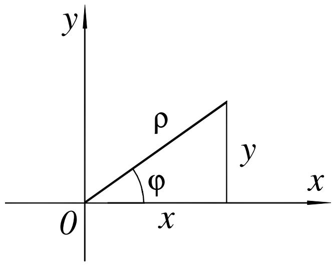
Example .5.1 Find the quotient of $z_1 = 2 + 3i$ and $z_2 = 1 + 4i$.

Solution.

$$\frac{z_1}{z_2}=\frac{2+3i}{1+4i}=\frac{(2+3i)\cdot(1-4i)}{(1+4i)\cdot(1-4i)}=\frac{14-5i}{1+16}=\frac{14}{17}-\frac{5}{17}i.$$

1.6. The Trigonometric Form of a Complex Number

If the point $z=(x,y)=x+yi$ is represented by polar coordinates ρ and φ,we can write $x=\rho\cos\varphi$ and $y=\rho\sin\varphi$ then $z=\rho(\cos\varphi+i\sin\varphi)$. This form is called the **trigonometric form** of a complex number .



The x-axis along which x is reckoned is called **real axis** and the y-axis along which y is reckoned is the **imaginary axis**.

The two numbers ρ and φ uniquely determine z. Conversely, the positive number ρ is uniquely determined by z. In fact, $\rho=|z|$

$$\rho=\sqrt{x^2+y^2} \tag{1.6.1}$$

However, z determines the angle φ only up to multiples of 2π. There are infinitely many values of φ which satisfy the equations $x=|z|\cos\varphi$, $y=|z|\sin\varphi$.

The unique real number φ which satisfies the condition $-\pi<\varphi\le\pi$ is called the principal argument of z and is denoted by $\arg z$: $\varphi=\arg z$, then

$$\cos\varphi=\frac{x}{\sqrt{x^2+y^2}},\ \sin\varphi=\frac{y}{\sqrt{x^2+y^2}}, \tag{1.6.2}$$

Let z_1 and z_2 be two complex numbers written in trigonometric form. The product of z_1 and z_2 can be found by using several trigonometric identities.

If $z_1=\rho_1(\cos\varphi_1+i\sin\varphi_2)$, and $z_2=\rho_2(\cos\varphi_2+i\sin\varphi_2)$, then

$$\begin{aligned} z_1z_2 &=\rho_1\rho_2\left(\cos\varphi_1\cdot\cos\varphi_2+i\cos\varphi_1\sin\varphi_2+i\sin\varphi_1\cos\varphi_2+i^2\sin\varphi_1\sin\kappa_2\right)= \\ &=\rho_1\rho_2((\cos\varphi_1 co\varphi_2-\sin\varphi_1\sin\varphi_2)+i(\sin\varphi_1\cos\varphi_2+\cos\varphi_1\sin\varphi_2))= \\ &=\rho_1\rho_2(\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2))\Rightarrow \end{aligned}$$

$$\rho_1\rho_2(\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)) \tag{1.6.3}$$

The modulus for **the product** of two complex numbers **in trigonometric form** is the product of moduli of the two complex numbers, and the argument of the product is the sum of the arguments of these numbers.

Similarly,

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}\left(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right) \qquad (1.6.4)$$

The modulus for **the quotient** of two complex numbers **in trigonometric form** is the quotient of moduli of the two complex numbers, and the argument of the quotient is the difference of the arguments of these numbers.

Example 1.6.1.

Find the product of $z_1 = -1 + i\sqrt{3}$ and $z_2 = -\sqrt{3+i}$.

Solution.

1) Using (1.6.1) and (1.6.2) write z_1 and z_2 in trigonometric form:

$$z_1 = 2\left(\cos\frac{2\pi}{3} + \sin\frac{2\pi}{3}\right); \quad z_2 = 2\left(\cos\frac{5\pi}{6} + \sin\frac{5\pi}{6}\right)$$

2) Use (1.6.3)

$$z_1 z_2 = 4\left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right) = 4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 4(0 - i) = -4i.$$

1.7. Integral Powers and Roots of Complex Numbers

Let $z = \rho(\cos\varphi + i\sin\varphi)$. Then z^2 can be written as

$$z \cdot z = \rho(\cos\varphi + i\sin\varphi)\,\rho(\cos\varphi + i\sin\varphi) = \rho^2(\cos 2\varphi + i\sin 2\varphi).$$

This formula can be extended for raising a complex number to the *n*th power:

$$z^n = \rho^n(\cos n\varphi + i\sin n\varphi) \qquad (1.7.1)$$

The formula

$$(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i\sin n\varphi$$

is called **De Moivre's** formula.

Definition. A number w is called the *n*th root of z if $w^n = z$ and is denoted by

$$w = \sqrt[n]{z}.$$

Let $w = r(\cos\theta + i\sin\theta)$ and $z = \rho(\cos\varphi + i\sin\varphi)$. Then as $w^n = z$ we have

$$r^n(\cos n\theta + i\sin n\theta) = \rho(\cos\varphi + i\sin\varphi).$$

Two complex numbers written in trigonometric form are equal if and only if their moduli are equal and their arguments are equal up to multiples of 2π. Thus

$$\begin{cases} r^n = \rho \\ n\theta = \varphi + 2k\pi \end{cases} \Rightarrow \begin{cases} r = \sqrt[n]{\rho} \\ \theta = \dfrac{\varphi + 2k\pi}{n} \end{cases}.$$

If $z = \rho(\cos\varphi + i\sin\varphi)$ is a complex number, then there are n distinct nth roots of z given by the formula

$$w_k = \sqrt[n]{\rho}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right) \tag{1.7.2}$$

for $k = 0,1,2,\dots,n-1$.

Example 1.7.1. Find the three cube roots of 27.

Solution. Write 27 on trigonometric form:

$$27 = 27(\cos 0 + i\sin 0).$$

Then, using formula (1.7.2), the cube roots of 27 are

$$w_k = \sqrt[3]{27}\left(\cos\frac{0 + 2k\pi}{3} + i\sin\frac{0 + 2k\pi}{3}\right) \text{ for } k = 0,1,2.$$

Substitute for k to find the cube roots of 27 :

$$w_0 = 3(\cos 0 + i\sin 0) = 3$$

$$w_1 = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i,$$

$$w_2 = 3\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i.$$

For $k = 3$ cosines and sines of the angles start repeating, thus there are only three cube roots of 27.

1.8. Complex Exponentials

Let us write a complex number in trigonometric form

$$z = \rho(a\cos\varphi + i\sin\varphi)$$

Using Euler's formula

$$e^{i\varphi} = \cos\varphi + \sin\varphi \tag{1.8.1}$$

we obtain $z = \rho e^{i\varphi}$ in the so-called **exponential form**.

Representing complex numbers in exponential form is particularly useful in connection with multiplication and division since we have

$$z_1 z_2 = \rho_1 e^{i\varphi_1} \rho_2 e^{i\varphi_2} = \rho_1 \rho_2 e^{i(\varphi_1+\varphi_2)}$$

and

$$\frac{z_1}{z_2} = \rho_1 e^{i\varphi_1} : \rho_2 e^{i\varphi_2} = \frac{\rho_1}{\rho_2} e^{i(\varphi_1-\varphi_2)}$$

If $z = \rho e^{i\varphi}$ then

$$z^n = \left(\rho e^{i\varphi}\right)^n = \rho^n e^{in\varphi}$$

This is De Moivre's formula in exponential form.

On replacing φ for $-\varphi$ we get such formula

$$e^{-i\varphi} = \cos\varphi - i\sin\varphi \tag{1.8.2}$$

On adding and subtracting formulas (1.8.1) and (1.8.2) we have

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad\qquad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}.$$

The product of a complex number $z = \rho e^{i\varphi}$ by the factor $e^{i\alpha}$ is

$$ze^{i\alpha} = \rho e^{i(\varphi+\alpha)}$$

The geometrical interpretation of this fact is that the multiplication by $e^{i\alpha}$ makes the vector representing the complex number z rotate about the origin through the angle α. In particular, putting $\alpha = \frac{\pi}{2}$ we see that the multiplication by $e^{\frac{i\pi}{2}} = i$

results in the rotation of the representing vector of the number z through 90^0 in counterclockwise direction.

Example 1.8.1.
Calculate the product $\left(1-i\sqrt{3}\right)^3(1+i)^2$.

Solution.

Expressing complex numbers in the exponential form, we get
$$\left(1-i\sqrt{3}\right)^3(1+i)^2=\left(2e^{-\frac{\pi i}{3}}\right)^3\left(\sqrt{2}e^{\frac{\pi i}{4}}\right)^2=2^3e^{-\pi i}\cdot 2e^{\frac{\pi i}{2}}=8(-1)\cdot 2i=-16i.$$

II. Functions of a Complex Variable

2.1. Definitions. Continuity

If $w=u+iv$ and $z=x+iy$ are any two complex numbers, we might say that w **is a function of** z**,** $w=f(z)$, if, to every value of z in a certain domain D, there correspond one or more values of w.

This definition, similar to that given for real variables. On this definition, a function of the complex variable z is exactly the same thing as a complex function $u(x,y)+i\,v(x,y)$ of two real variables x and y.

For functions defined in this way, the definition of continuity is exactly the same as that for functions of a real variable.

Definition. The function $f(z)$ is **continuous** at the point z_0 if given any $\boldsymbol{\varepsilon}>0$, we can find a number $\boldsymbol{\delta}>0$ such that
$$|f(z)-f(z_0)|<\boldsymbol{\varepsilon}$$
for all points z of D satisfying $|z-z_0|<\boldsymbol{\delta}$.
The number $\boldsymbol{\delta}$ depends on $\boldsymbol{\varepsilon}$ and also, in general, upon z_0. In this case we write

$$\lim_{z\to z_0} f(z)=f(z_0) \tag{2.1.1}$$

A difference $z-z_0=\Delta z$ is called **increment or change of an argument** z.
$f(z)-f(z_0)=\Delta w$ is called **the increment or the change of a function** $f(z)$.

The condition of a continuity of a function $f(z)$ can be stated in such a way

$$\lim_{\Delta z\to 0}\Delta w=0 \tag{2.1.2}$$

It easy to show that this definition of continuity is equivalent to the statement that a continuous function of z is merely a continuous complex function of the two variables x and y $f(z)=u(x,y)+iv(x,y)$.

2.2. Differentiability

Let $w=f(z)$ be one-valued function defined in a domain D, then $f(z)$ is differentiable at a point z_0 of D if $\dfrac{\Delta w}{\Delta z}$ tends to a unique limit as $\Delta z \to 0$, provided that z is also a point of D, that is

$$\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\lim_{\Delta z\to 0}\frac{f(z)-f(z_0)}{z-z_0}=f'(z_0) \qquad (2.2.1)$$

That continuity does not imply differentiability is seen from the following example:

Let $f(z)=|z|^2$. This continuous function is differentiable at the origin, but nowhere else. For if $z_0 \neq 0$ we have

$$\frac{|z|^2-\left|z_0{}^2\right|}{z-z_0}=\frac{z\bar{z}-z_0\overline{z_0}}{z-z_0}=\frac{z\bar{z}-z_0\bar{z}+z_0\bar{z}-z_0\overline{z_0}}{z-z_0}=$$

$$\bar{z}+z_0\frac{\bar{z}-\bar{z}_0}{z-z_0}=\bar{z}+z_0(\cos 2\varphi - i\sin 2\varphi),$$

where $\varphi=\arg(z-z_0)$.

It is clear that this expression does not tend to a unique limit as $z\to z_0$.

If $z_0=0$ the incrementary ratio is $\bar{z}$, which tends to zero as $z\to 0$.

2.3. Analytic Functions

A function of z which is one-valued and differentiable an every point of a domain D is said to be **analytic** in the domain D.

A function may be differentiable in a domain D save possible for a finite number of points. These points are called **singularities** of $f(z)$. We next discuss the necessary and sufficient conditions for a function to be analytic.

1. The necessary conditions for *f(z)* to be analytic.

If $f(z)=u(x,y)+iv(x,y)$ is differentiable at a given point *z*, the ratio $\dfrac{f(z+\Delta z)-f(z)}{\Delta z}$ must tend to a definite limit as $\Delta z\to 0$ in any manner. Now $\Delta z=\Delta x+i\Delta y$. Take Δz to be wholly real, so that $\Delta y=0$, then

$$\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}+i\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}$$

must tend to a definite limit as $\Delta x\to 0$. It follows that partial derivatives $\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$ must exist at the point (*x, y*) and the limit is $\dfrac{\partial u}{\partial x}+i\dfrac{\partial v}{\partial x}$. Similarly, if we take Δz to be wholly imaginary, so that $\Delta x=0$, we find that $\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial y}$ must exist at the point (*x, y*) and the limit in this case is $\dfrac{\partial v}{\partial y}-i\dfrac{\partial u}{\partial y}$.

Since the two limits obtained must be identical, on equating real and imaginary parts we get

$$\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y},\\[2mm] \dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}\end{cases} \tag{2.3.1}$$

These two relations are called the **Cauchy-Riemann differential equations.**

2. Sufficient conditions for $f(z)$ to be analytic.

It is possible to prove that, the continuous one- sided function $f(z)$ is analytic in a domain *D* if four partial derivatives u_x,v_x,u_y,v_y exist, are continuous and satisfy the Cauchy-Riemann differential equations at each point of *D*.

If $u+iv=f(x+iy)$, where $f(z)$ is an analytic function, then the real functions *u* and *v* of the two real variables *x* and *y* are called **conjugate functions.** Let these functions satisfy the relation

$$\boldsymbol{\theta}_{xy}=\boldsymbol{\theta}_{yx}$$

then by partial differentiation of Cauchy-Riemann equations (2.3.1) we have

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}.$$

Hence both u and v satisfy Laplace's equation in two dimensions

$$\nabla^2 \boldsymbol{\theta} = \frac{\partial^2 \boldsymbol{\theta}}{\partial x^2} + \frac{\partial^2 \boldsymbol{\theta}}{\partial y^2} = 0 \tag{2.3.2}$$

This equation occurs frequently in mathematical physics. It is satisfied by the potential at a point not occupied by matter in a two-dimensional gravitational field.

2.4. Complex Integration

Let $f(z)$ be any complex function of z, continuous along a piece-wise smooth curve C with end-points a and b, and write $f(z) = u(x, y) + iv(x, y)$. We divide the arc C into n subarcs by points

$z_0 = a,\ z_1, z_2, \dots, z_{n-1},\ z_n = b$ and denote $z_k - z_{k-1} = \Delta z_k = \Delta x_k + i\Delta y_k$.

Next we take in each subarc an arbitrary point $\boldsymbol{\varsigma}_k = \boldsymbol{\xi}_k + i\boldsymbol{\eta}_k$ and form a sum

$$\sum_{k=1}^{n} f(\boldsymbol{\varsigma}_k)\Delta z_k = \sum_{k=1}^{n} \left(u(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)\Delta x_k - v(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)\Delta y_k\right) +$$

$$+\, i\sum_{k=1}^{n} \left(v(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)\Delta x_k + u(\boldsymbol{\xi}_k, \boldsymbol{\eta}_k)\Delta y_k\right) \tag{2.4.1}$$

This sum is called an **integral sum**.

Let's denote $\max(|\Delta z_1|, |\Delta z_2|, \dots |\Delta z_n|) = \boldsymbol{\delta}_n \to 0$ as $n \to \infty$.

Definition. The limit of integral sums (2.4.1) (provided it exists) as $n \to \infty$ is termed a complex integral along a curve C and is written as

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(\boldsymbol{\varsigma}_k)\Delta z_k = \int_C f(z)dz \tag{2.4.2}$$

In addition

$$\int_C f(z)dz = \int_C u(x,y)dx - v(x,y)dy + i\int_C v(x,y)dx + u(x,y)dy \qquad (2.4.3)$$

Example. Compute the integral

$$\oint_{|z|=2} z \operatorname{Im} z^2 dz$$

Solution. Taking into account that for the function $f(z) = z \operatorname{Im} z^2$, $u(x,y) = 2x^2 y$, $v(x,y) = 2xy^2$ and using the formula (2.4.3) we obtain

$$\oint_{|z|=2} z \operatorname{Im} z^2 dz = \int_{x^2+y^2=4} 2x^2 ydx - 2xy^2 dy + i \int_{x^2+y^2=4} 2xy^2 dx + 2x^2 ydy.$$

The equation of the contour $|z| = 2$ we write in such a way

$$x = 2\cos t,\ y = 2\sin t \ \ (0 \le t \le 2\pi).$$

So, we have

$$\oint_{|z|=2} z \operatorname{Im} z^2 dz = 32(-2\int_0^{2\pi} \cos^2 t \sin^2 t dt + i\int_0^{2\pi} \left(\cos^3 t \sin t - \cos t \sin^3 t\right) dt) = -16\pi.$$

2.5. Cauchy's Theorem

If $f(z)$ is an analytic function and if $f'(z)$ is continuous at each point within and on a closed contour *C*, then

$$\int_C f(z)dz = 0 \qquad (2.5.1)$$

Let *D* be the closed domain which consists of all points within and on *C*. Then by the formula (2.4.1) we can write the integral (2.5.1) as a combination of curvilinear integrals

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy.$$

We transform each of these integrals by Green's theorem, which states that, if $P(x,y)$, $Q(x,y)$, $\dfrac{\partial Q}{\partial x}$, $\dfrac{\partial P}{\partial y}$ are all continuous functions of *x* and *y* in *D*, then

$$\int_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Since $f'(z) = u_x + iv_x = v_y - iu_y$ and, by hypothesis, $f'(z)$ is continuous in *D*, the conditions of Green's theorem are satisfied and so

$$\int_C f(z)dz = -\iint_D \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dxdy + i \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dxdy = 0$$

by virtue of the Cauchy-Riemann equations.

2.6. Cauchy's Integral Formula

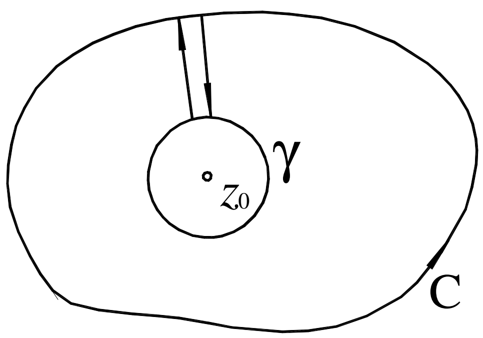
Theorem. If $f(z)$ is analytic within and on a closed contour *C* and z_0 be a point within *C*, then

$$\frac{1}{2\pi i} \int_C \frac{f(z)dz}{z - z_0} = f(z_0) \qquad (2.6.1)$$

Describe about $z = z_0$ a small circle γ of radius δ lying entirely within *C*. In the region between *C* and γ the function $\varphi(z) = \dfrac{f(z)}{z - z_0}$ is analytic. By making a cross-cut joining any point of γ to any point of *C* we form a closed contour Γ within which $\varphi(z)$ is analytic, so that, by Cauchy's theorem,

$$\int_\Gamma \varphi(z)dz = 0.$$

In traversing the contour Γ in the positive (counterclockwise) sense, the cross-cut is traversed twice, once in each sense, and so it follows that



$$\int_C \varphi(z)dz - \int_\gamma \varphi(z)dz = 0.$$

Now

$$\frac{1}{2\pi i} \int_\gamma \varphi(z)dz =$$

$$=\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)dz}{z-z_0}=\frac{1}{2\pi i}\int_{\gamma}\frac{f(z_0)dz}{z-z_0}+\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)-f(z_0)}{z-z_0}dz. \qquad (2.6.2)$$

Now on γ $z-z_0=\delta e^{i\varphi}$, and so the first of the two terms on the right becomes

$$\frac{f(z)}{2\pi i}\int_0^{2\pi}\frac{\delta i e^{i\theta}d\theta}{\delta e^{i\theta}}=f(z_0)$$

On using the statement that if $f(z)$ is continuous on a contour *L*, of length *l*, on which it satisfies the inequality $|f(z)|\le M$, then $\left|\int_L f(z)dz\right|\le Ml,$ we get, that the modulus of the second term on the right of (2.6.2) cannot exceed $\frac{1}{2\pi\delta}\max_{\gamma}|f(z)-f(z_0)|\cdot 2\pi\delta$.

Since $f(z)$ is continuous at $z=z_0$ this expression tends to zero as $\delta\to 0$. This proves the theorem.

Complex integral in the formula (2.6.1) is called a **Cauchy type integral**.

2.7. Complex Series. Power Series

Let an infinite sequence of complex numbers be given:

$a_n=\alpha_n+i\beta_n,\ n=1,2,...$

Definition. An expression $a_1+a_2+...+a_n+...$ is called a **complex series**.

A series is briefly written as $\sum_{n=1}^{\infty}a_n$.

The sum $S_n=a_1+a_2+...+a_n$ is termed the *n*th partial sum of the series. Making *n* take on the values 1, 2, … we obtain the sequence of partial sums of the series.

As *n* increases indefinitely a greater and still greater number of terms of the series is involved in the sum S_n.

Definition. If the sequence of partial sums of the given series has a definite limit an $n\to\infty$ i.e. $\lim_{n\to\infty}S_n=S$ the series is said to be **convergent** and the number S is called the **sum of the series**. If the sequence S_n does not tend to any limit or tends to infinity the series is said to be **divergent**.

The following theorem helps to reduce the study of complex series to that of real series.

Theorem. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if two real series $\sum_{n=1}^{\infty} \alpha_n$ and $\sum_{n=1}^{\infty} \beta_n$ are convergent.

Proof. To prove this we denote $\sigma_n = \alpha_1 + \alpha_2 + ... + \alpha_n$, $\tau_n = \beta_1 + \beta_2 + ... + \beta_n$; then $S_n = \sigma_n + i\tau_n$, hence

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sigma_n + i \lim_{n\to\infty} \tau_n .$$

The theorem has been proved.

Definition. A series $\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if the series $|a_1| + |a_2| + ... + |a_n| + ...$ is convergent.

Consider the series $\sum_{n=0}^{\infty} c_n z^n$ or $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, where the coefficients c_n and z, z_0 may be complex. Since the latter series may be obtained from the former by a simple change of origin, the former may be regarded as a typical power series.

So far as absolute convergence is concerned, everything that has been proved for absolutely convergent series of real terms extends at once to complex series, for the series of moduli $|c_0| + |c_1||z| + |c_2||z|^2 + ...$ is a series of positive terms.

The most useful convergence test for power series is Cauchy’s root test, which states that that a series of positive terms $\sum_{n=0}^{\infty} u_n$ is convergent or divergent according as $\overline{\lim_{n\to\infty}} \sqrt[n]{u_n}$ is less than or grater than unity. If we write $\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} = \frac{1}{R}$, then we easily see that the power series $\sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent if $|z| < R$, divergent if $|z| > R$, and if $|z| = 0$ we can give no general verdict and the behavior of the series may be of the most diverse nature.

The number R is called the **radius of convergence**, and the circle, center the origin, and radius R, is the **circle of convergence** of the power series.

Clearly there are three cases to consider

1) $R = 0$,

2) R finite,
3) R infinite.

The first case is trivial, since the series is then convergent only when $z = 0$.

In the third case the series converges for all values of z.

In the second case the radius of the circle of convergence is finite and the power series is absolutely convergent at all points within this circle, and divergent at all points outside it.

Example. Find the redii of convergence of the following power series.

1) $\sum_{n=0}^{\infty} (\sin in) z^n$; 2) $\sum_{n=1}^{\infty} \left(3 + (-1)^n\right)^n z^n$; 3) $\sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$.

Solution

.

1) Making use of the ratio test, we find

$$R = \frac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|\sin in|}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{e^n - e^{-n}}{2}}} = \frac{1}{e}.$$

2) $R = \dfrac{1}{\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}}$, but $\sqrt[n]{|c_n|} = 3 + (-1)^n$.

Since the upper limit of a sequence is the greatest point of accumulation, then $\lim_{n\to\infty} \sqrt[n]{|c_n|} = 4$. Hence $R = \dfrac{1}{4}$.

3) In this case we use D'Alembert's test

$$R = \lim_{n\to\infty} \frac{|c_n|}{|c_{n+1}|} = \lim_{n\to\infty} \frac{n!(n+1)^{n+1}}{n^n (n+1)!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

2.8. Taylor's and Laurent's Theorems

Theorem. If $f(z)$ is analytic in $|z - z_0| \le R$, and if z is a point such that $|z - z_0| = r \ (\mathrm{r} < \mathrm{R})$, then

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \tag{2.8.1}$$

where $c_n = \dfrac{f^{(n)}(z_0)}{n!}$.

Let C be a circle of radius ρ, centre $z = z_0$, where $r < \rho < R$, and consider the identity

$$\frac{1}{t-z} = \frac{1}{t-z_0} + \frac{z-z_0}{(t-z_0)^2} + \ldots + \frac{(z-z_0)^{n-1}}{(t-z_0)^n} + \frac{(z-z_0)^n}{(t-z_0)^n} \cdot \frac{1}{t-z}.$$

Multiply each term by $\dfrac{f(t)}{2\pi i}$ and integrate round C we clearly obtain

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \ldots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + R_n$$

where

$$R_n = \frac{(z-z_0)^n}{2\pi i} \int_C \frac{f(t)dt}{(t-z_0)^n (t-z)}.$$

This is Taylor's theorem with remainder R_n.

Since $|f(z)| \le M$ on C we readily see that

$$|R_n| = \frac{r^n}{2\pi} \frac{2\pi\rho M}{\rho^n (\rho - r)} = K\left(\frac{r}{\rho}\right)^n,$$

where K is constant independent of n.

Since $r < \rho$ we see that $|R_n| \to 0$, as $n \to \infty$.

Theorem. Let C_1 and C_2 be two circles of centre z_0 with radii R_1 and R_2 ($R_1 < R_2$); then, if $f(z)$ be analytic on the circles and within the annulus between C_1 and C_2,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n, \qquad (2.8.2)$$

z being any point of the annulus. The coefficients c_n are given by

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(t)dt}{(t-z_0)^{n+1}}, \; n = 0, \pm 1, \pm 2, \ldots \qquad (2.8.3)$$

where C is any closed contour, lying within the annulus between C_1 and C_2.

The proof of this theorem is analogous to that of the previous theorem. The series (2.8.2) is called Laurent's series.

Now we determine the relation between Laurent and Fourier series. Let the function $f(z)$ be analytic within the annulus $1-\varepsilon<|z|<1+\varepsilon$, then this function can be represented by Laurent's series

$$f(z)=\sum_{n=-\infty}^{+\infty} c_n z^n,$$

where

$$c_n=\frac{1}{2\pi i}\int_{|\tau|=1}\frac{f(\tau)d\tau}{\tau^{n+1}}=\frac{1}{2\pi}\int_0^{2\pi} f\left(e^{i\theta}\right)e^{-in\theta}d\theta \ \ (n\in Z)$$

In particular, for all points $z=e^{it}$ lying on the unit circle we have

$$F(t)=f\left(e^{it}\right)=\sum_{n=-\infty}^{+\infty} c_n e^{\text{int}}, \tag{2.8.4}$$

where

$$c_n=\frac{1}{2\pi}\int_0^{2\pi} F(\theta)e^{-in\theta}d\theta \ \ (n\in Z) \tag{2.8.5}$$

The series (2.8.4) with coefficients (2.8.5) is Fourier series of the function $F(t)$ in a complex form since it can be rewritten in the form

$$F(t)=c_0+\sum_{n=1}^{\infty}\left(c_n e^{int}+c_{-n}e^{int}\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nt+b_n\sin nt\right),$$

where $a_0=2c_0$, $a_n=c_n+c_{-n}$ and $b_n=i\left(c_n-c_{-n}\right)$.

On the bases of $a_0=\frac{1}{\pi}\int_0^{\pi}F(\theta)d\theta$, $a_n=\frac{1}{\pi}\int_0^{\pi}F(\theta)\cos n\theta\, d\theta$ and $b_n=\frac{1}{\pi}\int_0^{\pi}F(\theta)\sin n\theta\, d\theta$ $(n\in N)$.

If we consider Laurent's series on a unit circle of a function of a real argument *t*, then this series is Fourier series of the function $F(t) = f(e^{it})$.

Example. Expand the function $f(z) = \dfrac{1}{(z^2-1)^2}$ into Laurent's series within the annulus $0 < | z+1 | < 2$.

Solution. The given function is analytic within mentioned annulus therefore we can find c_n by formulas (2.8.3)

$$c_n = \frac{1}{2\pi i}\int_{\Gamma} \frac{\frac{1}{(z^2-1)^2}}{(z+1)^{n+1}}dz = \frac{1}{2\pi i}\int_{\Gamma} \frac{dz}{(z-1)^2(z+1)^{n+3}} \quad (n \in Z),$$

where Γ is an arbitrary circle center $z = -1$ within the annulus $0 < | z+1 | < 2$.

If $n+3 \le 0$ then the integrand $\dfrac{1}{(z-1)^2(z+3)^{n+3}}$ is analytic at all points within a circle Γ including the point $z_0 = -1$, therefore by Cauchy's theorem we get $c_n = 0 \ (n = -3, -4, \ldots)$.

If $n+3 > 0$ by Cauchy's integral formula we have

$$c_n = \frac{n+3}{2^{n+4}} \quad (n = -2, -1, 0, 1, \ldots).$$

Hence

$$\frac{1}{(z^2-1)^2} = \sum_{n=-2}^{+\infty} \frac{n+3}{2^{n+4}}(z+1)^n.$$

2.9. The Residue Theorem

If $f(z)$ is analytic within a given domain D, we have seen that it can be expanded in a Taylor series about any point $z = z_0$ of D and

$$f(z)=\sum_{n=0}^{\infty} c_n (z-z_0)^n .$$

If $c_0 = c_1 = c_2 = \ldots = c_{m-1} = 0$, $c_m \neq 0$, the first term in the Taylor expansion is $c_m (z-z_0)^m$. In this case $f(z)$ is said to have a zero of order *m* at $z = z_0$.

A singularity of a function $f(z)$ is a point at which the function ceases to be analytic.

If $f(z)$ is analytic within a domain *D*, except at the point $z = z_0$, which is an isolated singularity of $f(z)$, then we can draw two concentric circles of center z_0, both lying within *D*. The radius of the smaller circle R_1 may be as small as we please, and the radius R_2 of the larger circle of any length, subject to restriction that the circle lies wholly within *D*. In the annulus between two circles, $f(z)$ has a Laurent expansion of the form

$$f(z)= \sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} c_n (z-z_0)^n + \sum_{n=0}^{\infty} c_{-n} (z-z_0)^n .$$

The second term on the right is called the **principal part** of $f(z)$ at $z = z_0$. It may happen that $c_{-m} \neq 0$, while $c_{-m-1} = c_{-m-2} = \ldots = 0$. In this case the principal part consists of the finite number of terms

$$\frac{c_{-1}}{z-z_0} + \frac{c_{-2}}{(z-z_0)^2} + \ldots + \frac{c_{-m}}{(z-z_0)^m},$$

and the singularity at $z = z_0$ is called a **pole of order *m*** of $f(z)$ and the coefficient c_{-1}, which may in certain cases be zero, is called the **residue** of $f(z)$ at the pole $z = z_0$.

If the pole be of order **one**, $c_{-1} = \lim_{z \to z_0} (z-z_0) f(z)$.

If the principal part is an infinite series, the singularity is an **isolated essential singularity**.

Now we consider poles of $f(z)$ at infinity. In complex variable theory we have seen that it is convenient to regard infinity as a single point. The behavior of $f(z)$ “at infinity’ is considered by making the substitution $z = \dfrac{1}{\zeta}$ and examining

$f\left(\frac{1}{\zeta}\right)$ at $\zeta = 0$. We say that $f(z)$ is analytic, or has a simple pole, or has an essential singularity at infinity according as $f\left(\frac{1}{\zeta}\right)$ has the corresponding property at $\zeta = 0$. We know that $f\left(\frac{1}{\zeta}\right)$ has a pole of order m at $\zeta = 0$, near $\zeta = 0$ we have

$$f\left(\frac{1}{\zeta}\right) = \sum_{n=0}^{\infty} c_n \zeta^n + \frac{c_{-1}}{\zeta} + \frac{c_{-2}}{\zeta^2} + \ldots + \frac{c_{-m}}{\zeta^m},$$

and so, near $z = \infty$

$$f(z) = \sum_{n=0}^{\infty} c_n z^{-n} + c_{-1} z + c_{-2} z^2 + \ldots + c_{-m} z^m.$$

Thus, when $f(z)$ has a pole of order m at infinity, the principal part of $f(z)$ at infinity is finite series in ascending powers of z.

Since

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots,$$

the function $\sin z$ has an isolated essential singularity, at infinity the principal part being an infinite series.

The residue can also be defined as follows. If the point $z = z_0$ is the only singularity of $f(z)$ inside a closed contour, and if $\frac{1}{2\pi i}\int_C f(z)dz$ has a value, that value is the residue of $f(z)$ at $z = z_0$, and denoted by

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \operatorname{Re} s\,(f(z); z_0) \qquad (2.9.1)$$

The residue of $f(z)$ at infinity may also be defined.

If $f(z)$ has an isolated singularity at infinity, or is analytic there, and if C is a large circle which encloses all the finite singularities of $f(z)$, then the residue at $z = \infty$ is defined to be taken round C in the negative sense (negative with respect to the origin), provided that this integral has a definite value. If we apply the transformation $z = \frac{1}{\zeta}$ to the integral it becomes

$$\frac{1}{2\pi i}\int_C - f\left(\frac{1}{\zeta}\right)\frac{d\zeta}{\zeta^2}$$

taken positively round a small circle, center the origin.

It follows that if

$$\lim_{\zeta\to 0}\left(-\frac{f\left(\frac{1}{\zeta}\right)}{\zeta}\right) = \lim_{z\to\infty}\left(-zf(z)\right)$$

has a definite value, that value is the residue of $f(z)$ at infinity.

Note that a function may be analytic at $z = \infty$ but yet have a residue there. The function $f(z) = \frac{A}{z}$ has a residue A at $z = 0$ and a residue $-A$ at $z = \infty$, although $f(z)$ is analytic at $z = \infty$.

Theorem. (Cauchy's Residue Theorem)

Let $f(z)$ be continuous within and on a closed contour C and analytic, save for a finite number of poles, within C. Then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z); z_k), \tag{2.9.2}$$

where $\sum_{k=1}^{n} \operatorname{Res}(f(z); z_k)$ is the sum of residues of $f(z)$ at its poles within C.

Let $z_1, z_2, ..., z_n$ be the n poles within C. Draw a set of circles γ_k of radius δ and center z_k, which do not intersect and which all lie inside C. Then $f(z)$ is certainly analytic in the region between C and these small circles γ_k. We can therefore deform C until it consists of the small circles γ_k and a polygon P which joins together the small circles. Then

$$\int_C f(z)\,dz = \int_P f(z)\,dz + \sum_{k=1}^{n}\int_{\gamma_k} f(z)dz = \sum_{k=1}^{n}\int_{\gamma_k} f(z)dz,$$

for the integral round the polygon P vanishes because $f(z)$ is analytic within and on P. Assume, that a point $z = z_0$ is a pole of order one and function $f(z)$ has a form

$$f(z) = \frac{f_1(z)}{f_2(z)}, \tag{2.9.3}$$

where $f_1(z)$ and $f_2(z)$ are analytic at a point z_0 and also z_0 is a zero of order one of the function $f_2(z)$, and $f_1(z_0) \neq 0$. According to

$$\operatorname{Re} s\,(f(z); z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

we have

$$\operatorname{Re} s\,(f(z); z_0) = \lim_{z \to z_0} (z - z_0) \frac{f_1(z)}{f_2(z)} = \lim_{z \to z_0} \frac{f_1(z)}{\frac{f_2(z)}{z - z_0}} = \frac{f_1(z)}{\lim\limits_{z \to z_0} \frac{f_2(z)}{z - z_0}} =$$

$$= \frac{f_1(z)}{\lim\limits_{z \to z_0} \frac{f_2(z) - f_2(z_0)}{z - z_0}} = \frac{f_1(z)}{f_2'(z_0)}. \tag{2.9.4}$$

Example 2.9.1. Compute $\operatorname{Re} s\,(\cot z; 0)$.

We get $\cot z = \dfrac{\cos z}{\sin z}$. The point $z = 0$ is a zero of order one of the function $\sin z$, $\cos 0 \neq 0$ and by the formula (2.9.4) we obtain

$$\operatorname{Re} s\,(\cot z; 0) = \frac{\cos 0}{\cos 0} = 1.$$

Example 2.9.2. Calculate $\displaystyle\int_C \frac{(z+1)\,dz}{z^2 + 4}$, where C is a circle $|\, z \,| = 3$.

The function $f(z) = \dfrac{z+1}{z^2 + 4}$ has singular points $z = \pm 2i$. These points are poles of order one since $z^2 + 4 = (z - 2i)(z + 2i)$.

On using the formula (2.9.4) we get

$$\operatorname{Re} s\,(f(z); -2i) = \left.\frac{z+1}{(z^2 + 4)'}\right|_{z=-2i} = \left.\frac{z+1}{2z}\right|_{z=-2i} = -\frac{1 - 2i}{4i};$$

$$\operatorname{Re}s\,(f(z);2i)=\left.\frac{z+1}{2z}\right|_{z=2i}=\frac{1+2i}{4i}.$$

On the bases of Cauchy's residue theorem we have

$$\int_C \frac{(z+1)\,dz}{z^2+4}=2\pi i\left(\frac{1+2i}{4i}-\frac{1-2i}{4i}\right)=2\pi i.$$

Example 2.9.3. Calculate $\displaystyle\int_C \frac{z\,dz}{1-2\sin^2 z}$, where C is a circle of radius 2 center at the origin.

Since $1-2\sin^2 z=2\left(\frac{\sqrt{2}}{2}-\sin z\right)\left(\frac{\sqrt{2}}{2}+\sin z\right)$ the integrand has two simple poles at the points $z_1=\frac{\pi}{4}$ and $z_2=-\frac{\pi}{4}$ $\left(\frac{3\pi}{4}>2\right)$. With accordance with Cauchy's residue theorem we have

$$\int_C \frac{z\,dz}{1-2\sin^2 z}=2\pi i\left(\operatorname{Re}s\left(\frac{z}{1-2\sin^2 z};\frac{\pi}{4}\right)+\operatorname{Re}s\left(\frac{z}{1-2\sin^2 z};-\frac{\pi}{4}\right)\right).$$

Using the formula (2.9.4) and taking into account that

$$\left(1-2\sin^2 z\right)'=-2\sin 2z$$

we find

$$\operatorname{Re}s\left(\frac{z}{1-2\sin^2 z};\frac{\pi}{4}\right)=\frac{\frac{\pi}{4}}{-2\sin\frac{\pi}{2}}=-\frac{\pi}{8},$$

$$\operatorname{Re}s\left(\frac{z}{1-2\sin^2 z};-\frac{\pi}{4}\right)=\frac{-\frac{\pi}{4}}{-2\sin\left(-\frac{\pi}{2}\right)}=-\frac{\pi}{8}.$$

Hence

$$\int_C \frac{z\,dz}{1-2\sin^2 z}=2\pi i\left(-\frac{\pi}{8}-\frac{\pi}{8}\right)=-\frac{\pi^2 i}{2}.$$

Suppose that $z = z_0$ is a pole of order *m*, then in the neighborhood of this point

$$f(z) = \varphi(z) + \frac{c_{-1}}{z - z_0} + \frac{c_{-2}}{(z - z_0)^2} + \ldots + \frac{c_{-m}}{(z - z_0)^m},$$

where $\varphi(z)$ is analytic at $z = z_0$. In this case it is possible to prove that

$$c_{-1} = \operatorname{Re} s(f(z); z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z). \qquad (2.9.5)$$

Example 2.9.4. Find the residue of the function $\frac{1}{(z^2+1)^3}$ at the point $z = i$.

The point $z = i$ is a pole of order three since

$$\frac{1}{(z^2+1)^3} = \frac{1}{(z-i)^3 (z+i)^3}.$$

By the formula (2.9.5) we get

$$\operatorname{Re} s\left(\frac{1}{(z^2+1)^3}; i\right) = \frac{1}{2!} \lim_{z \to i} \frac{d^2}{dz^2} (z-i)^3 \frac{1}{(z-i)^3 (z+i)^3} = \frac{1}{2} \lim_{z \to i} \frac{d^2}{dz^2} (z+i)^{-3} =$$
$$\frac{1}{2} \lim_{z \to i} (-3)(-4)(z+i)^{-5} = \frac{6}{(2i)^5} = -\frac{3}{16} i.$$

2.10. Integration Round the Unit Circle

We consider the evaluation by contour integration of integrals of the type

$$\int_0^{2\pi} \varphi(\cos\theta, \sin\theta) d\theta,$$

where $\varphi(\cos\theta, \sin\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$. If we write $z = e^{i\theta}$, then $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$, $\frac{dz}{iz} = d\theta$; and so

$$\int_0^{2\pi} \varphi(\cos\theta, \sin\theta)d\theta = \int_C \psi(z)dz,$$

where $\psi(z)$ is rational function of z, and C is the unit circle $|z| = 1$. Hence

$$\int_C \psi(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname{Re} s(\psi(z); z_k),$$

where z_k are poles inside C.

Example Prove that, if $a > b > 0$,

$$I = \int_0^{2\pi} \frac{\sin^2\theta\, d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$$

Now of making the above change of variable, if C is the unit circle $|z| = 1$,

$$I = \frac{i}{2b}\int_C \frac{(z-1)^2\, dz}{z^2\left(z^2 + \frac{2az}{b} + 1\right)} = \frac{i}{2b}\int_C \frac{(z-1)^2\, dz}{z^2(z-\alpha)(z+\beta)} = \frac{i}{2b}\int_C f(z)dz,$$

where

$$\alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}, \quad \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

are the roots of the quadratic $z^2 + \dfrac{2a}{b}z + 1 = 0.$

Since the product of the roots α, β is unity, we have $|\alpha||\beta| = 1$ where $|\alpha| < |\beta|$, and so $z = \alpha$ is the only simple pole inside C. The origin is the pole of order two. We calculate the residues at $z = \alpha$ and $z = 0$.

$$\operatorname{Re} s(f(z); \alpha) = \lim_{z\to\alpha}(z-\alpha)f(z) = \lim_{z\to\alpha}\frac{\left(z^2 - 1\right)^2}{z^2(z-\beta)} = \frac{\left(\alpha - \frac{1}{\alpha}\right)^2}{\alpha - \beta}$$

$$= \frac{(\alpha - \beta)^2}{\alpha - \beta} = \alpha - \beta = \frac{2\sqrt{a^2 - b^2}}{b}.$$

Residue at the point $z=0$ is the coefficient of $\frac{1}{z}$ in $\dfrac{\left(z^2-1\right)^2}{z^2\left(z^2+\dfrac{2a\,z}{b}+1\right)}$, where z is small. Now

$$\frac{\left(z^2-1\right)^2}{z^2\left(z^2+\dfrac{2a\,z}{b}+1\right)}=\frac{1-2z^2+\ldots}{z^2\left(z^2+\dfrac{2a\,z}{b}+1\right)}$$ and coefficient of $\frac{1}{z}$ is plainly $-\frac{2a}{b}$.

Hence

$$I=\frac{i}{2b}2\pi i(\operatorname{Re}s(f(z);\alpha)+\operatorname{Re}s(f(z);0))=-\frac{\pi}{b}\left(-\frac{2a}{b}+\frac{2\sqrt{a^2-b^2}}{b}\right),$$

which proves the result.

2.11. Evaluation of Integral of Meromorphic Function

Definition. A function $f(z)$, whose only singularities in the finite of the plane are poles, is called a **meromorphic function**.

We now prove a very useful theorem.

If $f(z)$ is meromorphic inside a closed contour C and is not zero at any point on the contour, then

$$\frac{1}{2\pi i}\int_C\frac{f'(z)}{f(z)}dz=N-P, \tag{2.11.1}$$

where N is the number of zeros and P the number of poles inside C. (A pole or zero of order m must be counted m times.)

Suppose that $z=z_0$ is a zero of order m, then, in the neighbourhood of this point

$$f(z)=(z-z_0)^m\varphi(z),$$

where $\varphi(z)$ is analytic and not zero. Hence

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{\varphi'(z)}{\varphi(z)}.$$

Since the last term is analytic at $z = z_0$, we see that $\frac{f'(z)}{f(z)}$ has a simple pole at $z = z_0$ with residue *m*. Similarly, if $z = \xi_0$ is a pole of order *k*, we see that $\frac{f'(z)}{f(z)}$ has a simple pole at $z = \xi_0$ with residue – *k*. It follows, by the formula (2.6.1), that the left-hand side of (2.11.1) is equal to $\sum m - \sum k = N - P$.

If $f(z)$ is analytic in *C*, then *P* = 0, and the integral on the left of (2.11.1) is equal to *N*.

Since $\frac{d}{dz}\ln f(z) = \frac{f'(z)}{f(z)}$, we may write the result in another form,

$$\int_C \frac{f'(z)}{f(z)}dz = \Delta_C \ln f(z),$$

where Δ_C denotes the variation of $\ln f(z)$round the contour *C*. The value of the logarithm with which we start is immaterial, and since

$$\ln f(z) = \ln|f(z)| + i \arg f(z)$$

and $\ln|f(z)|$ is one-valued, the formula may be written

$$N = \frac{1}{2\pi}\Delta_C \arg f(z).$$

This result is known as **the principle of the argument**.

2.12. Evaluation of a Type of Infinite Integral

Let $f(z)$be a function of z satisfying the conditions:

1) $f(z)$ is meromorphic in the upper half-plane;

2) $f(z)$ has no poles on the real axis;

3) $zf(z) \to 0$ uniformly, as $|z| \to \infty$ for $0 \le \arg z \le \pi$;

4) $\int\limits_{0}^{+\infty} f(x)dx$ and $\int\limits_{-\infty}^{0} f(x)dx$ both converge, then

$$\int\limits_{-\infty}^{=\infty} f(x)dx = 2\pi i\sum_{k=1}^{n} \operatorname{Re} s(f(z); z_k), \qquad (2.12.1)$$

where z_k are poles of $f(z)$ in the upper half-plane.

Choose as contour a semicircle, center the origin and radius *R*, in the upper half-plane. Let the semicircle be denoted by *Γ*, and choose R large enough for the semicircle to include all the poles of $f(z)$. Then, by the reside theorem,

$$\int\limits_{-R}^{R} f(x)dx + \int\limits_{\Gamma} f(z)dz = 2\pi i\sum_{k=1}^{n} \operatorname{Re} s(f(z); z_k).$$

From (3)), if *R* be large enough, $|zf(z)| < \varepsilon$ for all points on *Γ*, and so

$$\left|\int\limits_{\Gamma} f(z)dz\right| = \left|\int\limits_{0}^{\pi} f\left(\operatorname{Re}^{i\theta}\right)\operatorname{Re}^{i\theta} id\theta\right| < \varepsilon\int\limits_{0}^{\pi} d\theta = \rho\varepsilon.$$

Hence, as $R \to \infty,$ the integral round *Γ* tends to zero. If (4)) is satisfied, it follows that

$$\int\limits_{-\infty}^{+\infty} f(x)dx = 2\pi i\sum_{k=1}^{n} \operatorname{Re} s(f(z); z_k).$$

Example. Prove that, if $a > 0$

$$\int\limits_{0}^{+\infty} \frac{dx}{x^4 + a^4} = \frac{\pi}{2\sqrt{2}a^3}.$$

If $z^4 + a^4 = 0,$ we have $z^4 = a^4 e^{i\pi}$, and the simple poles of the integrand are at $ae^{\frac{\pi}{4}i}$, $ae^{\frac{3\pi}{4}i}$, $ae^{\frac{5\pi}{4}i}$, $ae^{\frac{7\pi}{4}i}$, Of these, only the first two are in the upper-half plane. The conditions of the theorem are plainly satisfied, and so

$$\int\limits_{-\infty}^{+\infty} \frac{dx}{x^4 + a^4} = 2\pi i\sum\left(\text{Resedues at } z = ae^{\frac{\pi}{4}i}, ae^{\frac{3\pi i}{4}}\right).$$

Let k denote any one of these, then $k^4=-a^4$ and the residue at the simple pole $z=k$ is $\lim\limits_{z\to k}(z-k)\left(z^4-k^4\right)^{-1}$. This may be evaluated by Cauchy's formula, as applied to the evaluation of limits of expressions of the indeterminate form $\left[\frac{0}{0}\right]$, and so

$$\lim_{z\to k}\frac{(z-k)}{z^4-k^4}=\lim_{z\to k}\frac{1}{4z^3}=\frac{1}{4k^3}=-\frac{k}{4a^4}.$$

Hence

$$\int_{-\infty}^{+\infty}\frac{dx}{x^4+a^4}=-2\pi i\frac{1}{4a^4}\left(ae^{\frac{\pi}{4}i}+ae^{\frac{3\pi}{4}i}\right)=-\frac{\pi i}{2a^3}\left(e^{\frac{\pi}{4}i}-ae^{-\frac{\pi}{4}i}\right)=$$

$$=-\frac{\pi i}{2a^3}2i\sin\frac{\pi}{4}=\frac{\pi}{\sqrt{2}a^3}.$$

Hence $\displaystyle\int_{0}^{+\infty}\frac{dx}{x^4+a^4}=\frac{\pi}{2\sqrt{2}a^3}$.

III. Miscellaneous Problems

1. Show that the functions $w=\sin z$, $w=\cos z$, and $w=z^n$ (n is an integer) satisfy the Cauchy – Riemann equations.
2. Find the analytic function $f(z)$, the real part of which equals

   a) x^3-3xy^2 b) x^2-y^2+2x c) $\dfrac{x}{x^2+y^2}$

   d) $\dfrac{x}{x^2+y^2}-2y$ e) $2e^x\sin y$

3. Find the analytic function $f(z)$, the imaginary part of which equals

   a) $-\dfrac{y}{(x+1)^2+y^2}$ b) $2xy+3x$ c) $\tan^{-1}\dfrac{y}{x}$, $x>0$

   d) $e^x(y\cos y+x\sin y)+x+y$
4. Show that the function $f(z)=\sqrt{|xy|}$ is not analytic at the origin although the Caushy – Riemann equations are satisfied at this point.
5. Prove that the functions

a) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

b) $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

both satisfy Laplace's equation, and determine the corresponding analytic function $u + iv$ in each case.

6. Evaluate $\int_C \operatorname{Im} z dz$ if a contour C is the straight line segment joining points $z = 0$ and $z = 2 + i$.

7. Compute the integral $\int_C \frac{z^2 dz}{z - 2i}$ if C is the circle of radius 3 with center at 0.

8. Evaluate $\int_C \frac{\sin z dz}{z + i}$ where C is the circle with center at $z = -i$.

9. Compute the integral $\int_C \frac{dz}{z^2 + 9}$, where C is the circle of radius 2 with center at $z = 2 + i$.

10. Evaluate the integral $\int_C \frac{dz}{(z-1)^3 (z+1)^3}$, where C is the circle of radius R<2 with center at $z = 1$.

11. When C is the circle $|z| = 2$, use Cauchy's integral theorem or Caushy's integral formula to prove that

a) $\int_C \frac{zdz}{z^2 + 9} = 0$ b) $\int_C \sec \frac{z}{2} dz = 0$

c) $\int_C \frac{z^2 + 4}{z - 1} dz = 10\pi i$ d) $\int_C \frac{\sinh z}{2z + \pi i} dz = \pi$

12. Establish these expansions in the region $|z| < 1$

a) $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$ b) $\frac{1}{1 - z^2} = \sum_{n=0}^{\infty} z^{2n}$

13. Find the zeros of the functions and determine their order:

a) $(z^2 + 9)(z^2 + 4)^5$ b) $(1 - e^z)(z^2 - 4)$

c) $z \sin z$ d) $\frac{\sin^3 z}{z}$

14. Expand the function $f(z)=z^2e^{1/z}$ in a Laurent's series about $z=0$.

15. Expand the function $f(z)=\cos\frac{z^2-4z}{(z-2)^2}$ in a Laurent's series about $z=2$.

16. Examine a behavior of the functions

a) $\frac{z^2}{3+z^2}$ b) $\frac{z}{5-z^4}$ c) e^{-z}

d) $\sin z$ e) $e^{1/z}$

17. Classify the points $z=0$, $z=1$, and the point at infinity in relation to the function $f(z)=\frac{z-2}{z^2}\sin\frac{1}{z}$, and find the residues of $f(z)$ at $z=0$, and $z=1$.

18. Find the residues of these functions at their singular points:

a) $\frac{z+1}{z-1}$ b) $\frac{1}{\sin z}$ c) $\frac{\cos z}{(z-z_0)^2}$

d) $z^3\cosh\frac{1}{z^2}$ e) $\frac{\sin z^2}{z^5}$

Answers. a) 2; b) ± 1; c) $-\sin z_0$; d) $\frac{1}{2}$; e) 0.

19. Show that the singular point $z=0$ of the function $f(z)=\frac{1}{\sin\frac{\pi}{z}}$ is not isolated.

20. With the aid of the Cauchy-Riemann conditions prove that the components u and v of an analytic function $u+iv$ are harmonic functions, that is, they are continuous with continuous partial derivatives up to the second order, and satisfy Laplace's equation

$$u_{xx}(x,y)+u_{yy}(x,y)=0.$$

21. Find the residues of the following functions at their poles:

a) $\frac{z^2+1}{z-2}$ b) $\frac{e^{\pi z}}{z-i}$ c) $\frac{1}{(z^2+1)^3}$

d) $\frac{z^{2n}}{(z-1)^n}\ (n>0)$ e) $\frac{z^2}{(z^2+1)^2}$ f) $\frac{1}{z^3-z^5}$

g) $\frac{\sin 2z}{(z+1)^4}$ h) $\tan z$ i) $\cot z$

22. Find the residues of the functions

a) $e^{\frac{1}{z}}$ b) $\cos\frac{1}{z}$ c) $\sin\frac{1}{z}$

23. Compute the integral $\int\limits_C \frac{dz}{z^4+1}$, where C is the positively oriented circle $x^2+y^2=2x$.

24. Evaluate the integral $\int\limits_C \frac{dz}{(z-1)^2(z^2+1)}$, where C is the positively oriented circle $x^2+y^2=2x+2y$.

25. Calculate with the aid of the Cauchy's residue theorem the integrals

a) $\int\limits_{-\infty}^{+\infty} \frac{x^2+1}{x^4+1}dx$ b) $\int\limits_{-\infty}^{+\infty} \frac{dx}{(x^2+1)^n}, n>0$

c) $\int\limits_{-\infty}^{+\infty} \frac{x^2}{x^2+a^2}dx, a>0$ d) $\int\limits_{-\infty}^{+\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$, $a>0, \mathrm{b}>0$

e) $\int\limits_{-\infty}^{+\infty} \frac{\cos x}{x^2+9}dx$ f) $\int\limits_{-\infty}^{+\infty} \frac{x\sin x}{x^2+4x+20}dx$

IY. APPENDIX

Fundamental Elementary Functions of Complex Variables

- $e^{iz}=\cos z+i\sin z$
- $e^{-iz}=\cos z-i\sin z$
- $\sin z=\frac{e^{iz}-e^{-iz}}{2i}$
- $\cos z=\frac{e^{iz}+e^{-iz}}{2}$
- $\sinh z=\frac{e^{z}-e^{-z}}{2}$
- $\cosh z=\frac{e^{z}+e^{-z}}{2}$
- $\ln z=\ln|z|+i\arg z \quad (\arg z\in(-\pi;\pi))$
- $\mathrm{Ln}\, z=\ln z+i2\pi k,\ k\in Z$

$z=x+iy$ $x=\frac{z+\bar{z}}{2}$ $y=\frac{z-\bar{z}}{2i}$

- $\sin z=\sin x\cosh y+i\cos x\sinh y$
- $\cos z=\cos x\cosh y-i\sin x\sinh y$
- $\sinh z=\sinh x\cos y+i\cosh x\sin y$
- $\cosh z=\cosh x\cos y-i\sinh x\sin y$
- $\ln z=\frac{1}{2}\ln(x^2+y^2)+i\arctan\frac{y}{x}$