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Field Theory.
Complex Variables

Workbook for the students of the second year of education

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Field Theory. Complex Variables: Workbook for the students of the second year of education.

In the workbook briefly developed main information and development of the theoretical and practical classes from the module #5 "Field Theory. Complex Variables." Solutions of the typical problems, examples and personal problems are presented from each paragraph.

Tutorial examined and approved by the methodological council of the TCS Faculty.

Protocol #9, dated by April 22, 2008.

Tutorial examined and approved by the Department of Mathematics.

Protocol #8, dated by March 6, 2008.

CONTENTS

Class Hours for Module:
„Field Theory. Complex Variables"

Structure of the Module

Contents of the Module	Lectures (hrs.)	Classes		Personal Work
		Practical	Labs	
1 Scalar and Vector Fields	18	18	-	40
2 Complex Variables	14	14	-	31
Total	32	32	-	71

Contents of the Final Module

1 Scalar and Vector Fields.
- 1.1 Scalar Field. Directional Derivative. Gradient.
- 1.2 Curvilinear Integral of the II-nd type. Green Formula.
- 1.3 Surface Integral of the II-nd type. Gauss – Ostrogradskiy Formula. Stokes Formula.
- 1.4 Vector Fields. Vector Lines. Vector Field Flow. Divergence of the Vector Field. Circulation. Rotor of the field. Hamiltonian.
- 1.5 Special Vector Fields. Helmholtz Theorem.

2 Complex Variables.
- 2.1 Limit and Continuity of the Function with the Complex Variables
- 2.2 Derivative of the Function with the Complex Variable. Cauchy-Riemann Condition. Analytical Functions.
- 2.3 Integral of the Function with the Complex Variables, its connection with the curvilinear integral of the II-nd type. Cauchy theorem. Cauchy Integral Formula. Cauchy Integral.
- 2.4 Series in the Complex Area. Taylor Series.
- 2.5 Laurent Series. Classification of the Isolated Critical Points.
- 2.6 Residues. The Main Theorem about Residues.

Syllabus.

№	Number, Topic of the Lecture	Час.
	1 Scalar and Vector Fields	
1	L.1 Scalar Field. Directional Derivative. Gradient of Function. Hamiltonian.	2
2	L.2 Vector Field. Curvilinear Integral of the second type and its calculation. T1 Usage of the Curvilinear Integral of the second type.	2
3	L.3 Green Formula and its usage.	2

4	L.4 Surface Integral of the second type. T2 Calculation of the Surface Integral of the second type.	2
5	L.5 Divergence of the Vector Field. Gauss – Ostrogradskiy Formula.	2
6	L.6 Rotor of the Vector Field. Stokes Formula. T3 Calculation of the Circulation of the Vector Field.	2
7	L.7 Properties of the Vector Fields.	2
8	L.8 Work in the Potential Field. Helmholtz Theorem. T4 Finding of the Potential of the Potential Field.	2
	2 Complex Variables	
9	L.9 Complex numbers, operations with them. Euler Formula.	2
10	L.10 Complex Variables. Main Definitions. T5 Usage of the Euler formula.	2
11	L.11 Derivative of the Function with the Complex Variable. Cauchy-Riemann condition. T6 Determining of the Analyticity of the function with the Complex Variable.	2
12	L.12 Integral of the Function with the Complex Variable and its Properties.	2
13	L.13 Cauchy Integral theorem for Simply Connected Domain and Multiply Connected Region.	2
14	L.14 Cauchy Integral Formula. Derivatives of the Analytical Function with the Complex Variable of the higher order.	2
15	L.15 Laurent and Taylor Series.	2
16	L.16 Classification of the Isolated Critical Points. Deductions and its usage. T7 Calculation of the Improper Integrals of the Function with the Real Variable using Deductions.	2

Syllabus of the Practical Classes

№	Number, Topic of the Lecture	Час.
	1 Scalar and Vector Fields	
1	Scalar Fields.	2
2	Characteristics of the Scalar Fields. Directional Derivatives. Gradient.	2
3	Characteristics of the Vector Fields.	2
4	Calculation of the Curvilinear Integrals of the II-nd type.	2
5	Calculation of the surface integrals of the II-nd type.	2
6	Vector Field Flow.	2
7	Divergence.	2
8	Circulation. Rotor of the field.	2
9	Special Vector Fields.	2
	2 Complex Variables	
10	Complex Variables. Basic Functions with the Complex Variables.	2
11	Derivative of the Function with the Complex Variable. Cauchy-	2

	Riemann Condition.	
12	Integral of the Function with the Complex Variable. Cauchy Theorem.	2
13	Cauchy Integral Formula. Cauchy Integral.	2
14	Taylor Series.	2
15	Laurent Series.	2
16	Deductions. Usage of the Deductions for Calculation of the Integral.	2

The List of Knowledge and skills

Student must know:

1 Conception of the Scalar and Vector Fields, its characteristics.

2 Properties of the Vector Fieds (solenoidality, potentiality, irrotationality).

3 Conception of the Function with the Complex Variable. Conception of the Limit, Derivative, Integral of the Function with the complex Variable.

4 Definition of the Derivative of the Function with the Complex Variable and Condition of the Defferentiability.

5 Conception of the Analytical Function with the Complex Variable. Cauchy Theorem. Cauchy Integral Formula.

6 Decomposition of the Functions with the Complex Variables in Laurent and Taylor Series.

7 Deductions. The main theorem about Deductions.

Student must have skills:

1 In finding main characteristics of the Scalar Field (surface lines of the level, directional derivatives, gradient).

2 In finding main characteristics of the Vector Field (equations of the vector lines, flow, divergence, circulation, rotor, potential)

3 In making operations with the complex numbers.

4 In finding Real and Imaginary Part of the Function with the Complex Variable.

5 In checking Cauchy-Riemann condition and in finding derivative of the Function with the Complex Variable.

6 In usage Cauchy Integral Formula for calculation of the Integrals and for calculation of the Curvilinear Integral of the Analytical Function with the Complex Variable.

7 In distributing in Laurent and Riemann Series, in using them.

8 In finding deductions and in calculation of the integrals using deductions.

1 FIELD THEORY

1.1 Scalar Field

Let G be any domain on the surface or in the space.

Definition. If there is a constant u connected to every point M of the domain G, we say, that the scalar field of that point is defined.

Example. Temperature field, pressure field.

Scalar field is defined by the scalar function $u = f(M)$.

Definition. A set of points, in which scalar function takes the same value $f(M) = ñ, (c = const)$, calls level line (level surface).

$$c = f(M),\ c = f(x, y) \text{ или } c = f(x, y, z).$$

Example 1. Find level lines of the function $u = \frac{y}{x^2}$.

Solution.

$c = \frac{y}{x^2}$, $y = cx^2$.

Level lines in that case is a set of parabolas (figure 1.1).

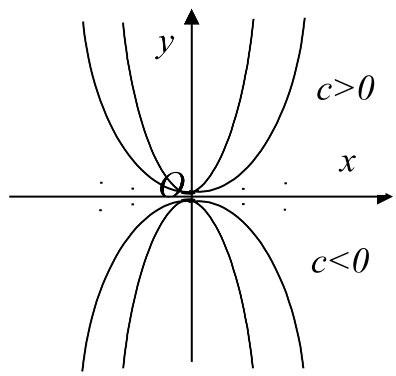


Figure 1.1

Example 2. Find level surfaces of the function $u = \sqrt{x^2 + y^2 + z^2}$.

Solution.

$c = \sqrt{x^2 + y^2 + z^2}$, $x^2 + y^2 + z^2 = c^2$, level surfaces is a set of the concentric spheres (figure 1.2).

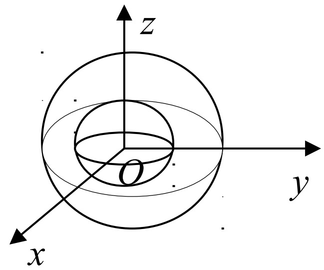


Figure 1.2

1.1.1 Directional Derivative

Let $u=f(x,y)$ be a function of the scalar field, $M\in G$.

Definition. The ratio $\frac{\Delta u}{\Delta l}$ is called the derivative of the function $u=f(x,y)$ at the point M in direction of the vector $\bar{l}$, when $\Delta l\to 0$ (if exist), labels as $\frac{\partial u}{\partial l}$. According to definition

$$\frac{\partial u}{\partial l}=\lim_{\Delta l\to 0}\frac{\Delta_l u}{\Delta l}. \qquad (1)$$

Consider $\frac{\partial u}{\partial l}=\lim_{\Delta l\to 0}\frac{\Delta_l u}{\Delta l}=\lim_{\Delta l\to 0}\frac{f(N)-f(M)}{\Delta l}$, $\overline{MN}\parallel\bar{l}$ (figure 1.3).

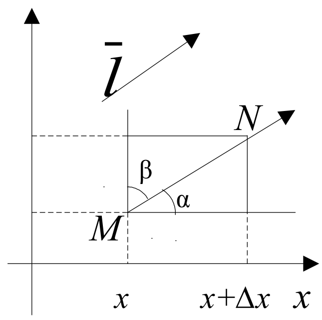


Figure 1.3

Let $MN=\Delta l$, then $\Delta x=\Delta l\cos\alpha$, $\Delta y=\Delta l\cos\beta$.

$$\bar{l}^{\circ}=\frac{\bar{l}}{|\bar{l}|}=\left(\frac{l_x}{\sqrt{l_x^2+l_y^2}},\ \frac{l_y}{\sqrt{l_x^2+l_y^2}}\right)=(\cos\alpha,\ \cos\beta).$$

Then, $\cos\alpha=\frac{l_x}{\sqrt{l_x^2+l_y^2}}$, $\cos\beta=\frac{l_y}{\sqrt{l_x^2+l_y^2}}$ and

$$\cos^2\alpha+\cos^2\beta=1.$$

Change: $\Delta_l u=f(N)-f(M)=f(x+\Delta x,y+\Delta y)-f(x,y)$. Rewrite that equality as follows $\Delta_l u=f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)+f(x,y+\Delta y)-f(x,y)$, then $\Delta_l u=\Delta_x u+\Delta_y u$. Substituting the last equality into the formula (1), we have:

$$\frac{\partial u}{\partial l}=\lim_{\Delta l\to 0}\left(\frac{\Delta_x u}{\Delta l}+\frac{\Delta_y u}{\Delta l}\right)=\lim_{\Delta l\to 0}\left(\frac{\Delta_x u}{\Delta x}\cos\alpha+\frac{\Delta_y u}{\Delta y}\cos\beta\right).$$

So, the derivative of the function of the scalar field *u=f(x, y)* in direction of the vector $\bar{l}$ equals to:

$$\frac{\partial u}{\partial l}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta.$$

Similar we can prove, that for the function with three variables $u=f(x,y,z)$ the derivative in direction can be calculated using the following formula:

$$\frac{\partial u}{\partial l}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma,$$

where $\bar{l}^{\circ}=(\cos\alpha,\cos\beta,\cos\gamma)$, $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$, and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called directional cosines of the vector $\bar{l}$.

Note. Conception of the directional derivative is generalization of the partial derivatives $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$, that can be considered as derivatives of the function $u=f(x,y,z)$ in the direction of the axes Ox, Oy, Oz.

Example. Find the derivative of the function $u = x^2 - 4yz$ at the point $M(-2, 1, 0)$ in the direction from that point to the point $M_1(2, 1, 3)$.

Solution. Find coordinates of the vector $\overline{MM_1} = \bar{l}$ and its directional cosines: $\overline{MM_1}(4, 0, 3)$, $\cos\alpha = \dfrac{4}{\sqrt{16+9}} = \dfrac{4}{5}$, $\cos\beta = 0$, $\cos\gamma = \dfrac{3}{\sqrt{16+9}} = \dfrac{3}{5}$.

Partial derivatives of the function at the point M: $\left.\dfrac{\partial u}{\partial x}\right|_M = 2x|_M = -4$, $\left.\dfrac{\partial u}{\partial y}\right|_M = -4z|_M = 0$, $\left.\dfrac{\partial u}{\partial z}\right|_M = -4y|_M = -4$.

So, $\dfrac{\partial u}{\partial l} = -4\cdot\dfrac{4}{5} + 0 - 4\cdot\dfrac{3}{5} = -\dfrac{28}{5}$.

Because $\dfrac{\partial u}{\partial l} < 0$, function is decreasing in the given direction.

Answer: –4.

1.1.2 Gradient of the function

Definition. Vector $\operatorname{grad} u = \dfrac{\partial u}{\partial x}\bar{i} + \dfrac{\partial u}{\partial y}\bar{j} + \dfrac{\partial u}{\partial z}\bar{k}$ or $\operatorname{grad} u = \left(\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial z}\right)$ is called gradient of the function $u = f(x, y, z)$.

Obvious, that $\dfrac{\partial u}{\partial l} = \operatorname{grad} u \cdot \bar{l}^{\circ}$.

Gradient of the function indicates direction in which change of the function is the largest.

Properties:

1) $\operatorname{grad}(u + \upsilon) = \operatorname{grad} u + \operatorname{grad} \upsilon$;
2) $\operatorname{grad}(c\cdot u) = c\cdot\operatorname{grad} u$;
3) $\operatorname{grad}(u\cdot\upsilon) = \upsilon\cdot\operatorname{grad} u + u\cdot\operatorname{grad}\upsilon$.

1.2 Hamiltonian

The main operations with the scalar field u and with the vector field $\bar{F}$ are: $\operatorname{grad} u$, $\operatorname{div}\vec{F}$, $\operatorname{rot}\bar{F}$ ($\operatorname{div}\vec{F}$, $\operatorname{rot}\bar{F}$ take a look later). Operations of the finding gradient, divergence and rotor are called vector operations of the first order (derivatives of the first order are involved). These operations can be easily written down using Hamiltonian (symbolic vector “nabla”):

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

It takes the meaning only in combination of the scalar or vector functions. Symbolic multiplication of the vector $\vec{\nabla}$ by the scalar u or vector $\overline{F}$ can be done according to the rules of the vector algebra, and "multiplication" of the symbols $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ by the values u, P, Q, R is understood as finding of the partial derivative of those values.

For example, $\text{grad}\, u = \vec{\nabla} \times u$, where $\vec{\nabla} = \frac{\partial}{\partial x}\bar{i} + \frac{\partial}{\partial y}\bar{j} + \frac{\partial}{\partial z}\bar{k}$.

1.3 Vector fields

Definition. If there is a vector quantity $\overline{F}$ connected to every point M of the domain G, we say that the vector field of that quantity is defined.

Example. Speed field, voltage field.

If there is a rectangular coordinate system in space, then vector field is defined by the vector function

$$\overline{F}(M) = P(M)\bar{i} + Q(M)\bar{j} + R(M)\bar{k},$$

where $P(M) = P(x,y,z), Q(M) = Q(x,y,z), R(M) = R(x,y,z)$ – scalar functions

$\overline{F}(x,y,z) = P(x,y,z)\bar{i} + Q(x,y,z)\bar{j} + R(x,y,z)\bar{k}$ – spatial vector field.

At the plane, vector field is defined as $\overline{F}(x,y) = P(x,y)\bar{i} + Q(x,y)\bar{j}$ (plane vector field).

Definition. Vector line of the field $\overline{F}(M)$ is called a curve, for which every tangent line in each point of the curve consists with the direction of the vector $\overline{F}(M)$.

Example. In the speed field of the spreading fluid, vector lines are the lines, along which parts of the fluid move (current streamlets); for magnetic field of the Earth, vector (force) lines are the lines, that come out of the North Pole and end up at the South Pole.

Total of all the vector lines of the field, which go through some closed curve, is called vector tube.

Let vector field be defined by the vector $\overline{F} = P\bar{i} + Q\bar{j} + R\bar{k}$, where $P = P(x,y,z), Q = Q(x,y,z), R = R(x,y,z)$ are continuous functions of the variables x, y, z, that have continuous partial derivatives of the first order.

Vector lines of the field $\overline{F}(x,y,z)$ can be found from the system of the differential equations:

$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$ – for spatial vector field,

$\frac{dx}{P(x,y)} = \frac{dy}{Q(x,y)}$ – for plane field, that can be written as:

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \to \; y' = f(x,y).$$

Above equations follow from the condition of collinearity of the vectors $\overline{F}(P,Q,R)$ and $\overline{dr}(dx,dy,dz)$ (figure 1.4).

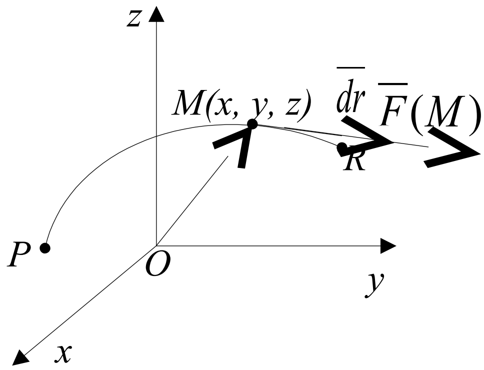


Figure 1.4

1.3.1 Integral by region (II-nd type)

We consider only two domains in vector field: lines and surfaces.

Definition. Line is called oriented, if direction is defined in every point of that line, concurring with the direction of the tangent line (figure 1.5).

Figure 1.5

Definition. Surface is called oriented, if direction is defined in every point of that surface, concurring with the direction of the normal line at that point. (figure 1.6).

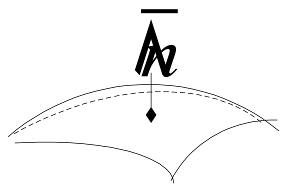


Figure 1.6

We denote oriented domain as G, and we set unit vector as oriented vector, that defines direction at random point M, and denote it as $\bar{l}(M)$.

Let G be any oriented domain, $\bar{l}(M)$ be oriented vector, and $\overline{F}(M)$ be a vector function at every point of that domain. Let's split randomly domain G by n parts: $G_1,..., G_n$ ($G_1 \cup G_2 \cup ... \cup G_n = G$). Let $\Delta\mu_i$ be measures of the domains G_i and let $\lambda = \max_i |\Delta\mu_i|$. Randomly, in every domain G_i, we choose a point M_i and calculate the value of the vector function $\overline{F}(M_i)$.

Let's trace the vector $\overline{\Delta\mu_i}$ at the point M_i, whose direction concurs with the direction of the oriented vector at the point M_i, and the length equals to the measure of the domain G_i, i.e. $\overline{\Delta\mu_i} = \Delta\mu_i \times \bar{l}(M_i)$. Calculate scalar product: $\overline{F}(M_i) \times \overline{\Delta\mu_i}$.

Consider sum

$$\sum_{i=1}^{n} \overline{F}(M_i)\overline{\Delta\mu_i} = \sum_{i=1}^{n} \left(\overline{F}(M_i) \times \bar{l}(M_i)\right)\Delta\mu_i .$$

That sum is called integral sum for the vector function $\overline{F}(M_i)$ over the oriented domain G.

Definition. If integral sum $\sum_{i=1}^{n} \overline{F}(M_i)\overline{\Delta\mu_i} = \sum_{i=1}^{n} \left(\overline{F}(M_i) \times \bar{l}(M_i)\right)\Delta\mu_i$ has limit when $\lambda = \max_i |\Delta\mu_i| \to 0$, that does not depend on the way domain G was splitted into parts, does not depend on the way intermediate points M_i were chosen, then we call it the integral from the vector function $\overline{F}(M_i)$ over the oriented domain G (second type integral) and denote as

$$\int_G \overline{F}(M)\overline{d\mu} = \lim_{\lambda \to 0} \sum_{i=1}^{n} \overline{F}(M_i)\overline{\Delta\mu_i} .$$

1.3.2 Curvilinear integral of the second type.

Definition. If domain G is the line L, and its oriented vector at the random point $M - \bar{l}(M)$, then the integral

$$\int_L \bar{F} \times \overline{dl} = \int_L P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$$

is called curvilinear integral of the second type over the space line L.

Problem (computation of work of the vector field). Let variable force $\bar{F}(M) = P(M)\bar{i} + Q(M)\bar{j}$ move material point along the curve L from the point M to the point N (figure 1.7). We need to calculate work of that force at the given path. For that we split the curve L up by n parts using points $M = M_0, M_1, M_2, ..., M_n = N$, then

$$A_n = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \bar{F}(M_i)\Delta \bar{l}_i,$$

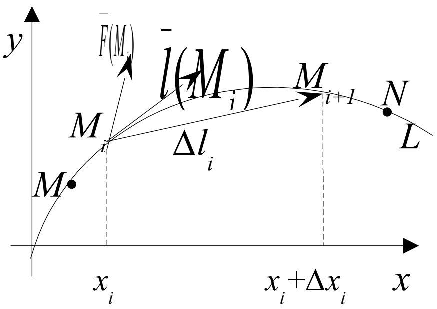


Figure 1.7

where $\Delta \bar{l}_i = \Delta x_i \bar{i} + \Delta y_i \bar{j}$.

Work equals to the limit of the sequence of integral sums when $\lambda = \max_i |\Delta l_i| \to 0$ and denotes as $A = \int_L \bar{F} \times \overline{dl}$, $d\bar{l} = dx\bar{i} + dy\bar{j}$, but it is curvilinear integral of the second type by definition.

Because $A_n = \sum_{i=1}^{n} P(M_i)\Delta x_i + Q(M_i)\Delta y_i$, then $A = \int_L P(x,y)dx + Q(x,y)dy$ is work of the vector field $\bar{F}(M) = P(M)\bar{i} + Q(M)\bar{j}$ along the arch $\breve{MN}$.

If field is spatial $\bar{F}(M) = P(M)\bar{i} + Q(M)\bar{j} + R(M)\bar{k}$, then work can be calculated using the formula:

$$A = \int_L P(M)dx + Q(M)dy + R(M)dz.$$

Note 1. Self-contained curve L is called a closed loop. The integral is called circulation of the vector $\bar{F}$ over the self-contained loop L and denotes $C = \oint_L \bar{F}(M)d\bar{l}$.

Note 2. Curvilinear integral of the second type depends on the direction of the bypass of the curve, $\int_{\breve{AB}} \bar{F}d\bar{l} = -\int_{\breve{BA}} \bar{F}d\bar{l}$.

For the self-contained loop L, counterclockwise bypass is a positive direction.

1.3.3 Calculation of the curvilinear integral of the second type

Let curve L be parametrically defined $x = x(t)$, $y = y(t)$, $z = z(t)$, $t_1 \le t \le t_2$, then

$$\int\limits_{\breve{AB}} \overline{F} d\overline{l} = \int\limits_{\breve{AB}} P(M)dx + Q(M)dy + R(M)dz =$$

$$= \int\limits_{t_1}^{t_2} P\left[x(t), y(t), z(t)\right] x'(t)\,dt + Q\left[x(t), y(t), z(t)\right] y'(t)\,dt +$$

$$+ R\left[x(t), y(t), z(t)\right] z'(t)\,dt.$$

Example. Calculate $\oint\limits_L \frac{xdy - ydx}{x^2 + y^2}$, where L is a triangle ABC with the vertices $A(1, 0)$, $B(1, 1)$, $C(0, 1)$.

Solution.

$$\oint\limits_L \frac{xdy - ydx}{x^2 + y^2} = \int\limits_{\breve{AB}} \frac{xdy - ydx}{x^2 + y^2} + \int\limits_{\breve{BC}} \frac{xdy - ydx}{x^2 + y^2} + \int\limits_{\breve{CA}} \frac{xdy - ydx}{x^2 + y^2}.$$

Write down the equations of the lines *AB, BC, CA* (figure 1.8) parametrically:

$$AB: \begin{cases} x = 1, \\ y = t; \end{cases} \quad 0 \le t \le 1;$$

$$BC: \begin{cases} x = -t, \\ y = 1; \end{cases} \quad -1 \le t \le 0;$$

$$CA: \begin{cases} x = t, \\ y = 1 - t; \end{cases} \quad 0 \le t \le 1.$$

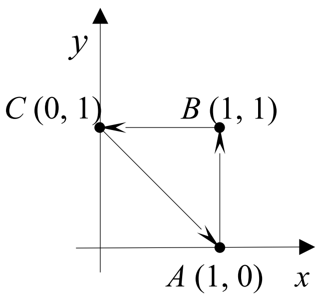


Figure 1.8

Then $$\int\limits_{AB} \frac{ydx - xdy}{x^2 + y^2} = \left[\begin{array}{l} AB: \begin{cases} x = 1, \\ y = t, \end{cases} \\ 0 \le t \le 1 \end{array} \right] = \int\limits_0^1 \frac{t \cdot 0 \cdot \mathrm{d}t - 1 \cdot \mathrm{d}t}{1 + t^2} =$$

$$= -\int\limits_0^1 \frac{\mathrm{d}t}{1 + t^2} = -\operatorname{arctg} t\Big|_0^1 = -\frac{\pi}{4};$$

$$\int\limits_{BC} \frac{ydx - xdy}{x^2 + y^2} = \left[\begin{array}{l} BC: \begin{cases} x = -t, \\ y = 1, \end{cases} \\ -1 \le t \le 0 \end{array} \right] = \int\limits_{-1}^0 \frac{1 \cdot (-\mathrm{d}t) + t \cdot 0 \cdot \mathrm{d}t}{t^2 + 1} = -\int\limits_{-1}^0 \frac{\mathrm{d}t}{t^2 + 1} = -\operatorname{arctg} t\Big|_0^1 = -\frac{\pi}{4};$$

$$\int_{CA}\frac{ydx-xdy}{x^2+y^2}=\left[CA:\begin{cases}x=t,\\ y=1-t,\end{cases}\ 0\le t\le 1\right]=\int_0^1\frac{(1-t)\cdot dt-t\cdot(-dt)}{t^2+1-2t+t^2}=\int_0^1\frac{dt}{2t^2-2t+1}=\frac{1}{2}\int_0^1\frac{dt}{t^2-t+\frac{1}{2}}=$$

$$=\frac{1}{2}\int_0^1\frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}=\operatorname{arctg}2\left(t-\frac{1}{2}\right)\Big|_0^1=\frac{\pi}{2}.$$

So, $\oint_L\frac{xdy-ydx}{x^2+y^2}=-\frac{\pi}{4}-\frac{\pi}{4}+\frac{\pi}{2}=0$.

Answer: 0.

1.3.4 Calculation of the area of the plane figures using the curvilinear integral of the second type

Let D be the correct domain, so that $D:\begin{cases}a\le x\le b,\\ y_1(x)\le y\le y_2(x).\end{cases}$ (figure 1.9*a*) or $D:\begin{cases}c\le y\le d,\\ x_1(y)\le x\le x_2(y).\end{cases}$ (figure 1.9*b*).

a)

Figure 1.9

Then

$$S_D=\int_a^b\left(y_2(x)-y_1(x)\right)dx=\int_a^b y_2(x)dx-\int_a^b y_1(x)dx=\int_{AB_2C}ydx-\int_{AB_1C}ydx=$$

$$=-\int_{AB_1C}ydx-\int_{CB_2A}ydx=-\oint_L ydx.$$

So,

$$S_D=-\oint_L ydx. \qquad (2)$$

Similarly (figure 1.9b),

$$S_D=\int_c^d\left(x_2(y)-x_1(y)\right)dy=\int_c^d x_2(y)dy-\int_c^d x_1(y)dy=\int_{B_1CB_2}xdy+\int_{B_2AB_1}xdy=\oint_L xdy=\oint_L xdy.$$

$$S_D = \oint_L x dy. \qquad (3)$$

Sum up (2) and (3), we get: $S_D = \frac{1}{2}\oint_L x dy - y dx$. (4)

Example. Calculate the area of ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution.

Write down the equation of ellipse parametrically:

$$\begin{cases} x = a\cos t, \\ y = b\sin t, \end{cases} \quad 0 \le t \le 2\pi .$$

Using formula (4), we can calculate the area of ellipse:

$$S = \frac{1}{2}\oint_L x dy - y dx = \frac{1}{2}\int_0^{2\pi} a\cos t\, b\cos t\, dt + b\sin t\, a\sin t\, dt = \frac{ab}{2}\int_0^{2\pi} dt = \pi ab .$$

Answer: πab.

1.3.5 Connection of the curvilinear integral of the second type with the double integral (Green formula)

If $\overline{F}$ is a plane vector field and functions $P(x, y), Q(x, y), P_y'(x, y), Q_x'(x, y)$ are continuous in the domain D and at the boundary L, then Green formula takes place:

$$\oint_L P(x, y)dx + Q(x, y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy ,$$

Where bypath of the loop L is in the positive direction.

Proof.

$$\iint_D \frac{\partial P}{\partial y} dxdy = \int_a^b dx \int_{y_1(x)}^{y_2(x)} \frac{\partial P}{\partial y} dy = \int_a^b P(x, y)\Big|_{y_1(x)}^{y_2(x)} dx = \int_a^b P\left(x, y_2(x)\right) dx - \int_a^b P\left(x, y_1(x)\right) dx =$$

$$= \int_{AB_2C} P(x, y)dx - \int_{AB_1C} P(x, y)dx = -\int_{CB_2A} P(x, y)dx - \int_{AB_1C} P(x, y)dx = -\oint_L P(x, y)dx . \qquad (5)$$

Similarly, changing the order of integration, we get:

$$\iint_D \frac{\partial Q}{\partial x} dxdy = \int_c^d dy \int_{x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x} dx = \int_c^d Q(x, y)\Big|_{x_1(y)}^{x_2(y)} dy = \int_c^d Q\left(x_2(y), y\right) dy - \int_c^d Q\left(x_1(y), y\right) dy =$$

$$= \int_{B_1CB_2} Q(x, y)dy - \int_{B_1AB_2} Q(x, y)dy = \int_{B_1CB_2} Q(x, y)dy + \int_{B_2AB_1} Q(x, y)dy = \oint_L Q(x, y)dy . \qquad (6)$$

Sum up (5) and (6), we get Green formula.

1.3.6 Surface integral of the second type. Flow of the vector field.

Take a look at the integral over the oriented domain $\int\limits_G \overline{F}\times\overline{d\mu}$ in case, when the domain G is the smooth double-faced surface σ (figure 1.10).

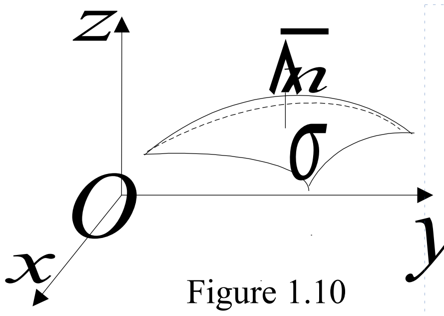


Figure 1.10

Definition. Surface σ is called smooth, if we can transact tangent plane in every point.

Definition. Surface σ is called double-faced, if normal line to the surface comes back to the reference position when bypasses over arbitrary closed loop.

Assume, that in every point of the surface σ , vector field $\overline{F}(M)$ is defined.

Let σ , double-faced surface, be the domain of integration of the vector function $\overline{F}(M)$, oriented vector is the unit vector $\overline{n}(M)$ to the surface σ at the point M, whose direction defines one of the sides of the surface.

Definition. Integral over such oriented surface is called surface integral of the second type and defines

$$\int\limits_\sigma \overline{F}(M)\overline{d\sigma},$$

where $\overline{d\sigma} = \overline{n}(M)\,d\sigma$ is a vector, which length equals to the area $d\sigma$ of an element of the surface σ , and direction concurs with the direction of the normal line to that surface at the point M.

So,

$$\int\limits_\sigma \overline{F}(M)\times\overline{d\sigma} = \int\limits_\sigma \overline{F}(M)\times\overline{n}(M)\,d\sigma .$$

Definition. Quantity of fluid, passing through the surface σ in time unit, is called the vector flow.

Problem (about vector flow through the surface).

Consider speed field $\overline{\upsilon}(M)$ of fluid in space (fluid does not compress).

Calculate flow of the vector $\overline{\upsilon}(M)$. We split the surface σ by n parts σ_i, $\Delta\sigma_i$ are the areas of those parts. In each part we choose random point $M_i(x_i, y_i, z_i)$. Vector $\overline{n_i}$ is the unit vector of the normal line to the sector σ_i.

Quantity of fluid, that passes through σ_i in time unit, approximately equals to the volume of cylinder (cylindrical pole), whose height equals

$$h_i = np_{\overline{n}_i}\overline{\upsilon}_i = \left|\overline{\upsilon}_i\right|\cos(\overline{n}_i,\overline{\upsilon}_i) = \overline{\upsilon}_i\overline{n}_i .$$

Flow of the vector $\overline{\upsilon}_i$ through σ_i equals: $\Pi_i \approx h_i\Delta\sigma_i = \overline{\upsilon}_i\overline{n}_i\Delta\sigma_i = \overline{\upsilon}_i\overline{\Delta\sigma_i}$.

Length of the vector $\overline{\Delta\sigma_i}$ equals to the area of the surface part.

That is why flow of the vector $\overline{\upsilon}$ through the surface σ approximately equals to:

$$\Pi = \sum_{i=1}^{n}\Pi_i \approx \sum_{i=1}^{n}\overline{\upsilon}_i\overline{\Delta\sigma_i} = \sum_{i=1}^{n}\overline{\upsilon}_i\overline{n}_i\Delta\sigma_i .$$

Definition. The limit of that sum when $n\to\infty$, if $\lambda\to 0$ (λ is the maximum diameter of the sections), is called the flow of the vector field $\overline{\upsilon}(M)$ through the surface σ and defines as:

$$\Pi = \iint_{\sigma}\overline{\upsilon}(M)\overline{n}d\sigma = \iint_{\sigma}\overline{\upsilon}(M)\overline{d\sigma} .$$

Integral on the right hand is called the surface integral of the second type.

Thereby, physical meaning of the surface integral over the surface σ is the flow of the vector field $\overline{F}(M)$ through the given surface.

Surface integral has the following properties:

$$\Pi = \iint_{\sigma^+}\overline{F}(M)\overline{n}(M)d\sigma = -\iint_{\sigma^-}\overline{F}(M)\overline{n}(M)d\sigma ,$$

where σ^+,σ^- are different sides of the surface σ .

We reduce the calculation of the surface integral of the second type to the calculation of the regular integral. For that, we need to present the unit vector $\overline{n}$ of the normal line to the surface by the instrumentality of the direction cosines:

$$\overline{n} = \cos(\widehat{\overline{n},\overline{i}})\overline{i} + \cos(\widehat{\overline{n},\overline{j}})\overline{j} + \cos(\widehat{\overline{n},\overline{k}})\overline{k} .$$

Define $\alpha = (\widehat{\overline{n},\overline{i}}), \beta = (\widehat{\overline{n},\overline{j}}), \gamma = (\widehat{\overline{n},\overline{k}})$, then unit vector can be written as:

$$\overline{n} = \cos\alpha\,\overline{i} + \cos\beta\,\overline{j} + \cos\gamma\,\overline{k} = \left\{\cos\alpha ,\cos\beta ,\cos\gamma\right\} .$$

If the equation of the surface σ is presented by the formula $z = g(x,y)$, then $u = g(x,y) - z = 0$ is the surface of the level when $c = 0$. Vector $\operatorname{grad}u$ is going to be perpendicular to that surface. Because $\operatorname{grad}u = \left\{g'_x, g'_y, -1\right\}$, then $\overline{N} = \left\{g'_x, g'_y, -1\right\}$, $\overline{n} = \dfrac{\overline{N}}{\left|\overline{N}\right|}$, $\left|\overline{N}\right| = \sqrt{1 + g'^2_x + g'^2_y}$,

$$\overline{n} = \left\{\frac{g'_x}{\left|\overline{N}\right|},\frac{g'_y}{\left|\overline{N}\right|},\frac{-1}{\left|\overline{N}\right|}\right\} \text{ or } \overline{n} = \left\{\frac{-g'_x}{\left|\overline{N}\right|},\frac{-g'_y}{\left|\overline{N}\right|},\frac{1}{\left|\overline{N}\right|}\right\} .$$

Positive direction of the vector $\overline{n}$ is the direction when the angel between vectors $\overline{n},\overline{k}$ is acute, e.g. $\cos\gamma > 0$.

Define $D_{x,y}$ as a projection of σ over xOy, similarly $D_{x,z}$, $D_{y,z}$ are projections of σ over xOz and yOz, and their measures are respectively: $\Delta S_{x,y}, \Delta S_{x,z}, \Delta S_{y,z}$.

Then,

$$\Delta S_{x,y} = \Delta\sigma\,|\cos\gamma| = \Delta\sigma\,\left|\cos(\vec{n},z)\right|,$$

$$\Delta S_{x,z} = \Delta\sigma\,|\cos\beta| = \Delta\sigma\,\left|\cos(\vec{n},y)\right|,$$

$$\Delta S_{y,z} = \Delta\sigma\,|\cos\alpha| = \Delta\sigma\,\left|\cos(\vec{n},x)\right|.$$

Because, $\Delta S_{x,y} \Rightarrow dx\,dy$, when $\Delta\sigma \to d\sigma$, then

$$d\sigma = \frac{dx\,dy}{|\cos\gamma|},$$

similarly

$$d\sigma = \frac{dx\,dz}{|\cos\beta|} \text{ и } d\sigma = \frac{dy\,dz}{|\cos\alpha|}.$$

Hence, $\Pi = \iint\limits_{\sigma} \overline{F}\times\overline{n}\,d\sigma = \iint\limits_{\sigma}\left[P(M)dydz + Q(M)dxdz + R(M)dxdy\right] =$

$$= \iint\limits_{D_{yz}} P\left(x(y,z),y,z\right)\text{sign}\left(\cos\alpha\right)dydz + \iint\limits_{D_{xz}} Q\left(x,y\left(x,z\right),z\right)\text{sign}\left(\cos\beta\right)dxdz +$$
$$+ \iint\limits_{D_{xy}} R\left(x,y,z\left(x,y\right)\right)\text{sign}\left(\cos\gamma\right)dxdy.$$

Definition. The flow of the vector through the closed surface is the difference between outgoing and incoming flows (for example, fluids).

E.g., $\Pi = \Pi_{âèõ} - \Pi_{âõ}$, $\Pi = \oint\limits_{\sigma} \overline{F}(M)\overline{n}(M)d\sigma$.

If $\Pi = 0$ through the closed surface, then there are no sources inside the surface, limited by σ.

If $\Pi > 0$ through the closed surface, then there is a positive source of the vector field.

If $\Pi < 0$, there are outflows inside, e.g. negative sources.

Methods of calculation of the flow of the vector field

1 Method of the projection over the one of the coordinate planes. Not closed surface σ projects over the plane xOy in the domain D_{xy}. In that case plane σ is defined by the equation $z = g(x,y)$, and because the element of the area $d\sigma$ of that surface equals

$$d\sigma = \frac{dx\,dy}{|\cos\gamma|},$$

then calculation of the flow of the vector field $\overline{F}$ reduces to the calculation of the double integral by the formula

$$\Pi = \iint_{\sigma} \overline{F}\times\overline{n}\,d\sigma = \iint_{D_{xy}} \left.\frac{\overline{F}\times\overline{n}}{|\cos\gamma|}\right|_{z=g(x,y)} dxdy. \qquad (7)$$

Unit vector of the normal $\overline{n}$ to the chosen side of the surface σ can be found by the formula

$$\overline{n} = \pm\frac{-g'_x\overline{i} - g'_y\overline{j} + \overline{k}}{\sqrt{g'^2_x + g'^2_y + 1}}, \qquad (8)$$

a $\cos\gamma$ equals to the coefficient of the unit vector $\overline{k}$, formula (8):

$$\cos\gamma = \pm\frac{1}{\sqrt{g'^2_x + g'^2_y + 1}}. \qquad (9)$$

If angel γ between axes Oz and normal vector $\overline{n}$ is acute, then in formula (8) and in formula (9) we need to take „+” sign, if angel is obtuse, then „–” sign.

If it is easier to project surface σ at the coordinate plane yOz or xOz, then formulas (7), (8), (9) are:

$$\Pi = \iint_{\sigma} \overline{F}\times\overline{n}\,d\sigma = \iint_{D_{yz}} \left.\frac{\overline{F}\times\overline{n}}{|\cos\alpha|}\right|_{x=\varphi(y,z)} dydz \text{ or } \Pi = \iint_{\sigma} \overline{F}\times\overline{n}\,d\sigma = \iint_{D_{xz}} \left.\frac{\overline{F}\times\overline{n}}{|\cos\beta|}\right|_{y=\psi(x,z)} dxdz,$$

where vectors of the normal lines are, respectively:

$$\overline{n} = \pm\frac{\overline{i} - \varphi'_y\overline{j} - \varphi'_z\overline{k}}{\sqrt{1 + \varphi'^2_y + \varphi'^2_z}} \text{ or } \overline{n} = \pm\frac{-\psi'_x\overline{i} + \overline{j} - \psi'_z\overline{k}}{\sqrt{\psi'^2_x + 1 + \psi'^2_z}},$$

and directional cosines:

$$\cos\alpha = \pm\frac{1}{\sqrt{1 + \varphi'^2_y + \varphi'^2_z}}, \quad \cos\beta = \pm\frac{1}{\sqrt{\psi'^2_x + 1 + \psi'^2_z}}.$$

Note. In case, when the surface σ is defined implicitly by the equation $\Phi(x,y,z) = 0$, unit vector of the normal line can be found by the formula:

$$\overline{n} = \pm\frac{\Phi'_x\overline{i} + \Phi'_y\overline{j} + \Phi'_z\overline{k}}{\sqrt{\Phi'^2_x + \Phi'^2_y + \Phi'^2_z}}.$$

Example 1. Find the flow of the vector field $\overline{F} = (x - 3z)\overline{i} + (x + 3y + z)\overline{j} + (5x + y)\overline{k}$ through the upper side of the triangle ABC with the vertices $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 2)$.

Solution. The equation of the plane of the triangle ABC is: $2x + 2y + z = 2$, we get $z = 2 - 2x - 2y$. Triangle ABC is projecting one-to-one at the plane xOy in the domain D_{xy}, which is triangle OAB (figure 1.11).

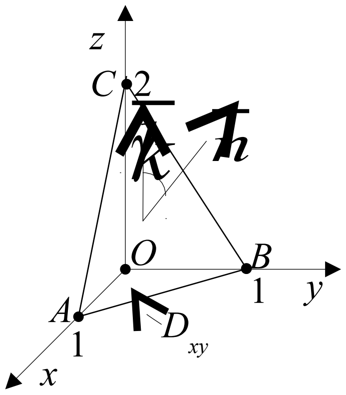


Figure 1.11

Angle γ is acute, that is why in formulas (8) and (9) we take „+” sign, we get:

$$\overline{n} = \frac{2\overline{i} + 2\overline{j} + \overline{k}}{\sqrt{4+4+1}} = \frac{2}{3}\overline{i} + \frac{2}{3}\overline{j} + \frac{1}{3}\overline{k} \text{ and } \cos\gamma = \frac{1}{3}.$$

Scalar product is

$$\overline{F} \times \overline{n} = (x - 3z)\frac{2}{3} + (x + 3y + z)\frac{2}{3} + (5x + y)\frac{1}{3} = \\ = \frac{9x + 7y - 4z}{3}.$$

Substitute results that we got into the formula (7) and calculate flow:

$$\Pi = \iint_{D_{xy}} \frac{\overline{F}\times\overline{n}}{|\cos\gamma|}\bigg|_{z=g(x,y)} dxdy = \iint_{D_{xy}} \left(9x + 7y - 4z\right)\Big|_{z=2-2x-2y} dxdy = \int_0^1 dx \int_0^{1-x} \left(17x + 15y - 8\right) dy =$$

$$= \int_0^1 \left(17x(1-x) + \frac{15}{2}(1-x)^2 - 8(1-x)\right) dx = -\frac{19}{6} + 5 - \frac{1}{2} = 1\frac{1}{3}.$$

Answer: $1\frac{1}{3}$.

2 Method of the projection ower three coordinate planes. Let σ be the surface that projects one-to-one at all three coordinate planes. Denote D_{xy}, D_{xz}, D_{yz} as projections of σ at the planes xOy, xOz, yOz respectively.

Suppose that the equation $\Phi(x,y,z) = 0$ of the surface σ can be one-to-one solved with respect to each variable $x = x(y,z),\ y = y(x,z),\ z = z(x,y)$. Then the flow of the vector field $\overline{F}$ can be calculated by the formula

$$\Pi = \pm\iint_{D_{yz}} P\left(x(y,z),y,z\right) dydz \pm \iint_{D_{xz}} Q\left(x,y\left(x,z\right),z\right) dxdz \pm \iint_{D_{xy}} R\left(x,y,z\left(x,y\right)\right) dxdy. \quad (10)$$

Sign in front of every integral depends on the sign of the appropriate directional cosine of the normal vector (value of the cosine of the acute angle is positive, value of the obtuse angle is negative).

Example 2. Find the flow of the vector field $\overline{F} = \{xy,\ yz,\ xz\}$ through the face of the sphere $x^2 + y^2 + z^2 = 1$, located in the first octant.

Solution. Because the part of the surface is located in the first octant (angles between normal vector and coordinate axes is acute), then in formula (10) in front of every integral we need to take ”+” sign. Taking into consideration, that $P = xy,\ Q = yz,\ R = xz$, and from the equation of the sphere

$$z = \sqrt{1 - x^2 - y^2},\ y = \sqrt{1 - x^2 - z^2},\ x = \sqrt{1 - z^2 - y^2},$$

we get

$$\Pi = \iint_{D_{yz}} xy\,dydz + \iint_{D_{xz}} yz\,dxdz + \iint_{D_{xy}} xz\,dxdy =$$

$$= \iint_{D_{yz}} \sqrt{1-z^2-y^2}\,y\,dydz + \iint_{D_{xz}} \sqrt{1-z^2-x^2}\,z\,dxdz + \iint_{D_{xy}} x\sqrt{1-x^2-y^2}\,dxdy\,.$$

Pass to the polar coordinates and calculate the third integral, which is in the right side of the last equality (the first and the second integrals can be calculated analogically).

$$\iint_{D_{xy}} x\sqrt{1-x^2-y^2}\,dxdy = \left[\begin{cases} x = \rho\cos\varphi\,,\ 0 \le \varphi \le \dfrac{\pi}{2}, \\ y = \rho\sin\varphi\,,\ \ 0 \le \rho \le 1. \end{cases}\right] = \iint_{D_{xy}} \rho^2\sqrt{1-\rho^2}\cos\varphi\, d\varphi\, d\rho =$$

$$= \int_0^{\frac{\pi}{2}} \cos\varphi\, d\varphi \int_0^1 \rho^2\sqrt{1-\rho^2}d\rho = \int_0^1 \rho^2\sqrt{1-\rho^2}d\rho = \left[\begin{matrix} \rho = \sin t, \\ d\rho = \cos t\,dt \end{matrix}\right] = \int_0^{\frac{\pi}{2}} \sin^2 t\cos^2 t\,dt =$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2t dt = \frac{1}{8}\int_0^{\frac{\pi}{2}} (1-\cos 4t)dt = \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16}\,.$$

The flow that we were looking for equals to $\Pi = \frac{\pi}{16} + \frac{\pi}{16} + \frac{\pi}{16} = \frac{3\pi}{16}$.

Answer: $\frac{3\pi}{16}$.

1.3.7 Divergence of the vector field

Definition. Divergence of the vector field $\overline{F}$ at the point M denotes by the symbol $\operatorname{div}\overline{F}(M)$ and can be found as:

$$\operatorname{div}\overline{F}(M) = \lim_{\substack{\lambda \to 0 \\ (V \to M)}} \frac{\oiint_{\sigma} \overline{F}(M)\overline{n}d\sigma}{V},$$

e.g., divergence of the vector field is the limit of the ratio of the flow of the vector field through the closed surface σ to the volume V, limited by that surface, given that domain goes to the point M (λ − its diameter).

Divergence characterizes the capacity density of the source of the vector field. That scalar quantity can be calculated by the formula

$$\operatorname{div}\overline{F}(M) = \overline{\nabla} \times \overline{F},$$

or

$$\operatorname{div}\overline{F}(M) = \frac{\partial P(M)}{\partial x} + \frac{\partial Q(M)}{\partial y} + \frac{\partial R(M)}{\partial z}.$$

Properties:

1) $\operatorname{div}\overline{c} = 0$, $\overline{c}$ − constant vector;

2) $\operatorname{div}(\alpha \overline{F}_1 + \beta \overline{F}_2) = \alpha \operatorname{div}\overline{F}_1 + \beta \operatorname{div}\overline{F}_2$, α, β - $const$;

3) $\operatorname{div}\left(\varphi(M)\times\overline{F}(M)\right) = \operatorname{grad}\varphi \times\overline{F} + \varphi \times\operatorname{div}\overline{F}$, because

$(\varphi \times P)'_x + (\varphi \times Q)'_y + (\varphi \times R)'_z = \varphi'_x \times P + \varphi \times P'_x + \varphi'_y \times Q + \varphi \times Q'_y + \varphi'_z \times R + \varphi \times R'_z$;

4) $\operatorname{div}(\varphi \times\overline{c}) = \operatorname{grad}\varphi \times\overline{c}$;

5) $\operatorname{div}\operatorname{grad}u = \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$, where Δ − Laplace operator:

$\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ ($\Delta u = 0$ − Laplace equation).

1.3.8 Gauss – Ostrogradskiy Formula

Gauss – Ostrogradskiy theorem. Let vector field $\overline{F}(M)$ have continuous partial derivatives P'_x, Q'_y, R'_z in any domain V and at its boundary, let closed surface σ limits any domain V. Then

$$\oiint_{\sigma} \overline{F}\times\overline{n}d\sigma = \iiint_{V} \operatorname{div}\overline{F}(M)dv,$$

e.g., the flow of the vector $\overline{F}$ through the closed surface σ equals to the triple integral over the domain V from the divergence of that vector.

In coordinates:

$$\iiint_{V}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)dv = \oiint_{\sigma}\left(Pdydz + Qdxdz + Rdxdy\right). \qquad (11)$$

Proof. Let domain D be the projection of the surface σ at the plane xOy, $z = z_1(x, y)$ and $z = z_2(x, y)$ are the equations of the parts of the surface σ (bottom part σ_1 and upper part σ_2), z_1, z_2 are continuous in D.

Denote $P = F_x, Q = F_y, R = F_z$.

Consider

$$I = \iiint_{V}\frac{\partial F_z}{\partial z}dxdydz = \iint_{D} dxdy \int_{z_1(x, y)}^{z_2(x, y)} \frac{\partial F_z}{\partial z} dz.$$

Calculate inside integral using the Newton-Leibniz formula:

$$I = \iint_{D}\left[F_z\left(x, y, z_2(x, y)\right) - F_z\left(x, y, z_1(x, y)\right)\right]dxdy.$$

Express double integral through the surface integral of the second order (when to the random points $M(x, y)$ of the domain D respond points (x, y, z), that circumscribe the surface σ as points of entry and as output points through the surface)

$$I = \iint_{\sigma_2^+} F_z(x, y, z)\,dxdy - \iint_{\sigma_1^-} F_z(x, y, z)\,dxdy.$$

Substituting in the second integral interior by the outer face, we get:

$$I = \iint_{\sigma_2^+} F_z(x, y, z)\,dxdy + \iint_{\sigma_1^+} F_z(x, y, z)\,dxdy = \oiint_{\sigma^+} F_z(x, y, z)\,dxdy .$$

So, $I = \iiint_V \frac{\partial F_z}{\partial z} dxdydz = \oiint_{\sigma^+} F_z(x, y, z)\,dxdy$.

Similarly we prove, that

$$\iiint_V \frac{\partial F_x}{\partial x} dxdydz = \oiint_{\sigma^+} F_x(x, y, z)\,dydz, \quad \iiint_V \frac{\partial F_y}{\partial y} dxdydz = \oiint_{\sigma^+} F_y(x, y, z)\,dxdz .$$

Adding these equalities by parts, we get

$$\iiint_V \operatorname{div} \overline{F}(M) dv = \oiint_{\sigma^+} F_x dydz + \oiint_{\sigma^+} F_y dxdz + \oiint_{\sigma^+} F_z dxdy = \oiint_{\sigma} \overline{F} \times \overline{d\sigma}$$

Example. Find the flow of the vector field $\overline{F} = \{x - 3z, x + 3y + z, 5x + y\}$ through the outer face of the pyramid *CAOB* with the vertices *O*(0, 0, 0), *A*(1, 0, 0), *B*(0, 1, 0), *C*(0, 0, 2).

Solution. Because $P = x - 3z, Q = x + 3y + z, R = 5x + y$, then, using the formula (11), we get

$$\Pi = \iiint_V (1 + 3 + 0) dv = 4\iiint_V dv = 4V_{CAOB} = 1\frac{1}{3} .$$

Answer: $1\frac{1}{3}$.

1.3.9 Rotor of the vector field

Definition. Vector $\operatorname{rot} \overline{F}(M)$ is called rotor of the vector field, can be found as:

$$\operatorname{rot} \overline{F}(M) = \overline{\nabla} \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} ,$$

e.g.

$$\operatorname{rot} \overline{F}(M) = \{R'_y - Q'_z; P'_z - R'_x; Q'_x - P'_y\} .$$

Direction of the rotor is a direction, around which circulation has the maximum value in comparison with the circulation around any direction that does not concur with the normal line to the plane domain, restrained by the closed loop.

Properties:

1) $\operatorname{rot} \overline{c} = 0$;

2) $\operatorname{rot}(\alpha \overline{F}_1 + \beta \overline{F}_2) = \alpha \operatorname{rot} \overline{F}_1 + \beta \operatorname{rot} \overline{F}_2$;

3) $\operatorname{rot}\left(\varphi(M)\cdot\overline{F}(M)\right) = \operatorname{grad}\varphi \times \overline{F} + \varphi\cdot\operatorname{rot}\overline{F}$;

4) $\operatorname{rot}\left(\varphi(M)\cdot\overline{c}\right) = \operatorname{grad}\varphi \times \overline{c}$;

5) $\operatorname{rot}(\operatorname{grad}\varphi) = 0$;

6) $\operatorname{div}(\operatorname{rot}\overline{F}) = 0$.

1.3.10 Stokes formula

Stokes theorem. Circulation of the vector field $\overline{F}(M)$ over the closed loop L equals to the flow of the curl of that vector through the surface σ, stretched at the loop L, e.g.

$$C = \oint_L \overline{F}(M)\overline{dl} = \iint_\sigma \operatorname{rot}\overline{F}(M)\cdot\overline{n}(M)d\sigma = \iint_\sigma \operatorname{rot}\overline{F}\cdot\overline{d\sigma}.$$

In coordinate form

$$\oint_L Pdx + Qdy + Rdz = \iint_\sigma (R'_y - Q'_z)dydz + (P'_z - R'_x)dxdz + \left(Q'_x - P'_y\right)dxdy.$$

Pass over the loop L can be chosen so, that if you look from the end of the normal vector at the motion over the loop L, the direction of the motion has to be counterclockwise.

In particular case of the Stokes formula, when the vector field is flat, then $R \equiv 0$, $z = 0$, $\overline{F}(M) = P(M)\overline{i} + Q(M)\overline{j}$, and Stokes formula is the following:

$$\oint_L Pdx + Qdy = \iint_D (Q'_x - P'_y)dxdy,$$

got Green formula.

Physical meaning of the Stokes formula is the statement that the flow of the vector field $\overline{F}$ equals to the quantity of the liquid weaving through the surface σ in time unit

$$\oint_L \overline{F}\overline{dl} = \iint_\sigma \operatorname{rot}\overline{F}\overline{d\sigma}.$$

1.3.11 Properties of the vector fields

1 Solenoid vector field.

Vector field $\overline{F} = P\overline{i} + Q\overline{j} + R\overline{k}$ is called solenoid in the domain V, if $\operatorname{div}\overline{F}(M) \equiv 0$ in each point M of the domain V.

Then from the Gauss-Ostrogradskiy formula: $\Pi = \iiint_{V'} \operatorname{div}\overline{F}dv \equiv 0$, e.g. the flow of the solenoid vector field through any closed surface σ', that restricts the domain $V' \subset V$, equals to zero.

2 Irrotational vector field.

Vector field $\overline{F}(M)$ is called irrotational in any domain V, if in every point of that domain $\operatorname{rot}\overline{F}(M)=0$. Then from Stokes formula, $C=0$ over any closed loop L, belonged to the domain V.

3 Potential vector field.

Vector field is called potential in any domain V, if there exists such scalar function $u(M)$ or $u(x,y,z)$, that $\overline{F}(M)=\operatorname{grad}u(M)=\overline{\nabla}\times u$, $M\in V$.

Function $u(M)$ is called potential of the vector field.

Theorem 1. For vector field to be potential in any domain V, it is necessary and sufficient to have $\operatorname{rot}\overline{F}(M)=0$, $M\in V$.

Proof.

Necessity. Let $\overline{F}(M)$ be potential field in the domain V. Then

$$\overline{F}(M)=\operatorname{grad}u(M)=\overline{\nabla}\times u,\ \operatorname{rot}\overline{F}(M)=\overline{\nabla}\times\overline{F}.$$

So, $\operatorname{rot}\overline{F}(M)=\overline{\nabla}\times(\overline{\nabla}\times u)=0$ (because $\overline{\nabla}$ è $\overline{\nabla}\times u$ are collinear vectors).

Sufficiency. Given that $rot\overline{F}(M)=0$ in the domain V, e.g. $\overline{F}(M)$ – irrotational field.

$\operatorname{rot}\overline{F}(M)=\left\{R'_y-Q'_z;\ P'_z-R'_x;\ Q'_x-P'_y\right\}=0$, if $R'_y=Q'_z$; $P'_z=R'_x$; $Q'_x=P'_y$.

Look at the following function:

$$u(x,y,z)=\int_{x_0}^{x}P(x,y,z)dx+\int_{y_0}^{y}Q(x_0,y,z)dy+\int_{z_0}^{z}R(x_0,y_0,z)dz.$$

Calculating, we get:

$u'_x=P(x,y,z)$;

$u'_y=\int_{x_0}^{x}P'_ydx+Q(x_0,y,z)+0=\int_{x_0}^{x}Q'_xdx+Q=Q$, так как $P'_y=Q'_x$;

$$u'_z=\int_{x_0}^{x}P'_zdx+\int_{y_0}^{y}Q'_zdy+R(x_0,y_0,z)=\int_{x_0}^{x}R'_xdx+\int_{y_0}^{y}R'_y(x_0,y,z)dy+R(x_0,y_0,z)=$$

$$=R(x,y,z)-R(x_0,y,z)+R(x_0,y,z)-R(x_0,y_0,z)+R(x_0,y_0,z)=R,$$

because $P'_z=R'_x$ and $Q'_z=R'_y$.

So,

$$\operatorname{grad}u(M)=\overline{F}(M).$$

Theorem 2. If field $\overline{F}(M)$ is irrotational in the domain V, then curvelinear integral $\int_{AB}\overline{F}\times\overline{dl}$ (2[nd] type) does not depend on the way of integration in the domain V.

Proof.

$\overline{F}(M)$ is irrotational, that is why $\operatorname{rot}\overline{F}(M) = 0, M \in V$. Then, by the Stokes formula: $\oint\limits_L \overline{F} \cdot \overline{dl} = 0$.

$$\oint\limits_L \overline{F} \cdot \overline{dl} = \int\limits_{AMB} \overline{F} \cdot \overline{dl} + \int\limits_{BNA} \overline{F} \cdot \overline{dl} = \int\limits_{AMB} \overline{F} \cdot \overline{dl} - \int\limits_{ANB} \overline{F} \cdot \overline{dl} = 0,$$

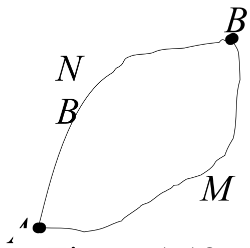


Figure 1.12

e.g., $\int\limits_{AMB} \overline{F} \cdot \overline{dl} = \int\limits_{ANB} \overline{F} \cdot \overline{dl}$.

It follows from the theorem, that for curvelinear integral of the second type not to depend on the way of integration it is necessary and sufficient, for the field to be potential.

If the vector field $\overline{F}(M)$ is potential in the domain V, then $\oint\limits_L \overline{F} \cdot \overline{dl} = \iint\limits_\sigma \operatorname{rot} \overline{F} \cdot \overline{d\sigma} = \begin{bmatrix} \operatorname{rot}\overline{F} \equiv 0 \text{ in the domain } V \\ \text{for the potential field} \end{bmatrix} = 0$.

1.3.12 Work in the potential field

Physical meaning of the curvilinear integral of the second type is work, that is why if vector field $\overline{F}(M)$ is potential in the domain V, then

$$A = \int\limits_L \overline{F} \cdot \overline{dl} = \int\limits_L Pdx + Qdy + Rdz = \begin{bmatrix} \text{the potential field} \\ P = u'_x,\ Q = u'_y,\ R = u'_z \end{bmatrix} =$$

$$= \int\limits_L u'_x dx + u'_y dy + u'_z dz = \int\limits_L du = u(B) - u(A),$$ where $AB = L$ (open curve).

So, work of the vector $\overline{F}(M)$ in the potential field equals to the difference of the potentials at the start and at the final points.

1.3.13 Helmholtz theorem

Theorem. Any vector field $\overline{F}(M)$ can be presented as a sum of two vector fields, one of them is potential, another one is solenoid.

Proof. Let $\overline{F}(M)$ be any vector field. Then $\operatorname{div}\overline{F}(M) = f(M)$ is a scalar function. Find potential field $\overline{F}_1(M) = \operatorname{grad} u(M)$, potential u is a solution of the Laplace heterogeneous equation: $\Delta u = f(M)$.

E.g., $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = f(M)$ or $\Delta u = \overline{\nabla} \cdot \overline{\nabla} u$.

Let's show, that vector field $\overline{F_2} = \overline{F} - \overline{F}_1$ is solenoid. To do that we need to find divergence:

$$\operatorname{div}\overline{F}_2 = \operatorname{div}\overline{F} - \operatorname{div}\overline{F}_1 = f(M) - \operatorname{div}(\overline{\nabla} u) = f(M) - \overline{\nabla} \cdot \overline{\nabla} u =$$

$= f(M) - \Delta u = f(M) - f(M) \equiv 0$, that is why $\overline{F_2}$ is a solenoid field.

Consequently, $\overline{F}(M) = \underbrace{\overline{F}_1(M)}_{\text{potential}} + \underbrace{\overline{F}_2(M)}_{\text{solenoid}}$.

1.4 Tasks for personal work

1.4.1 Defined scalar field $u(x,y,z)$, vector $\bar{l}$ and point M. Find:

1) Derivative of the field $u(x,y,z)$ at the point M in the direction of the vector $\bar{l}$;
2) Gradient of the field $u(x,y,z)$ at the point M;
3) The maximum velocity of increasing of the field $u(x,y,z)$ at the point M.

1.01.	$u = x^2 - y^2 + z^3$,	$\bar{l} = 2\bar{i} - \bar{j} + 2\bar{k}$,	$M(1;2;-1)$.
1.02.	$u = xy + yz - xz$,	$\bar{l} = \bar{i} - 2\bar{j} - 2\bar{k}$,	$M(0;1;-1)$.
1.03.	$u = x^2y + y^2z + xz^2$,	$\bar{l} = 3\bar{i} - 4\bar{k}$,	$M(1;0;2)$.
1.04.	$u = xz + y^2x - yz^2$,	$\bar{l} = 4\bar{i} + 3\bar{j}$,	$M(2;1;0)$.
1.05.	$u = xyz^2 + zy$,	$\bar{l} = -2\bar{i} + 2\bar{j} + \bar{k}$,	$M(-1;0;2)$.
1.06.	$u = 3x^2 + 2y^2 - 4zx$,	$\bar{l} = \bar{i} - \bar{j} + \sqrt{2}\bar{k}$,	$M(2;1;-1)$.
1.07.	$u = 2x^3 + xy^2 + 3z^2$,	$\bar{l} = 2\bar{i} + \bar{j} - \bar{k}$,	$M(2;2;0)$.
1.08.	$u = x^2y^2z^2$,	$\bar{l} = 4\bar{i} - 2\bar{j} + 4\bar{k}$,	$M(1;2;3)$.
1.09.	$u = xyz - y^2z^3$,	$\bar{l} = 4\bar{i} + 4\bar{j} - 2\bar{k}$,	$M(2;0;1)$.
1.10.	$u = x^2yz + y^2 + xyz^2$,	$\bar{l} = 2\bar{i} + 2\bar{j} + \bar{k}$,	$M(1;-1;0)$.
1.11.	$u = xy^2z^3$,	$\bar{l} = \bar{i} - 2\bar{j} + \bar{k}$,	$M(0;1;2)$.
1.12.	$u = 2xy + 3yz^3$,	$\bar{l} = 2\bar{i} - 2\bar{j} + \sqrt{8}\bar{k}$,	$M(-1;2;1)$.
1.13.	$u = 3xy + 2yz^3$,	$\bar{l} = \bar{i} + \bar{j} + 2\bar{k}$,	$M(2;0;2)$.
1.14.	$u = 3x^2 + 2y^2 - 4z^2$,	$\bar{l} = 4\bar{j} - 3\bar{k}$,	$M(1;1;0)$.
1.15.	$u = 2xz + 3xy + 4yz$,	$\bar{l} = \bar{i} - \bar{j} + \sqrt{2}\bar{k}$,	$M(1;2;-2)$.
1.16.	$u = 3x^2 + yz - xz^2$,	$\bar{l} = 2\bar{i} - 2\bar{j} + \bar{k}$,	$M(0;-1;2)$.
1.17.	$u = 3xy^2 + 2yz^3$,	$\bar{l} = 4\bar{i} - 3\bar{k}$,	$M(2;0;1)$.
1.18.	$u = 2xy - 3yz^3$,	$\bar{l} = 2\bar{i} - \bar{j} + 2\bar{k}$,	$M(1;1;0)$.
1.19.	$u = x^2y + xy^2 - yz^2$,	$\bar{l} = \bar{i} - \bar{j} + \sqrt{7}\bar{k}$,	$M(1;-1;1)$.
1.20.	$u = 2xyz^2 - 3xzy^2$,	$\bar{l} = \bar{i} + 2\bar{j} + 2\bar{k}$,	$M(2;1;-1)$.
1.21.	$u = xy^2z + x^2z^2$,	$\bar{l} = 2\bar{i} + 2\bar{j} + \bar{k}$,	$M(0;-1;3)$.
1.22.	$u = 4xz + 3yz^2$,	$\bar{l} = \bar{i} - \bar{j} + \sqrt{7}\bar{k}$,	$M(1;0;-3)$.
1.23.	$u = x^3y^2 + yz^2$,	$\bar{l} = -\bar{i} + \bar{j} - 7\bar{k}$,	$M(2;1;-3)$.
1.24.	$u = 4xz + 3yz^2$,	$\bar{l} = 2\bar{i} + \bar{j} - \bar{k}$,	$M(3;0;1)$.
1.25.	$u = 2xy - 3yz + 4xz$,	$\bar{l} = \bar{i} + 2\bar{j} - \bar{k}$,	$M(1;3;-1)$.

1.26. $u = xy^2z + 2x^2z^2$, $\bar{l} = 2\bar{i} - \bar{j} + 2\bar{k}$, $M(1;3;-2)$.
1.27. $u = 2xyz + 3x^2z^2$, $\bar{l} = 4\bar{i} + 3\bar{k}$, $M(2;-3;-1)$.
1.28. $u = 3xy^2 + 4yz^2$, $\bar{l} = -3\bar{i} - 4\bar{k}$, $M(-2;0;3)$.
1.29. $u = 2xy - 3yz^3$, $\bar{l} = 2\bar{i} - 2\bar{j} + \bar{k}$, $M(-1;0;1)$.
1.30. $u = 3xy + yz^3$, $\bar{l} = \bar{i} + \bar{j} + 2\bar{k}$, $M(2;0;2)$.

1.4.2 Defined vector field $\overline{F}(x,y,z)$ and closed surface $\sigma = \sigma_1 \cup \sigma_2$. Find:

1) The flow of the vector field $\overline{F}$ through the surface σ using definition and using the Gauss-Ostrogradskiy formula;

2) Circulation of the field $\overline{F}$ over the closed loop $L = \sigma_1 \cap \sigma_2$ using definition and using Stokes theorem.

2.01. $\overline{F} = (x-y)\bar{i} + (y+z)\bar{j} + (y-z)\bar{k}$; $\sigma_1: x^2 - y^2 - z^2 = 1$; $\sigma_2: x^2 + y^2 + z^2 = 3$ $(z > 0)$.
2.02. $\overline{F} = (x-y)\bar{i} + (y+z)\bar{j} - 2x\bar{k}$; $\sigma_1: x^2 + y^2 + z^2 = 9$; $\sigma_2: z = 0$ $(z \geq 0)$.
2.03. $\overline{F} = yz\bar{i} + x\bar{j} - y\bar{k}$; $\sigma_1: x^2 + y^2 = z^2$; $\sigma_2: z = 1$ $(0 \leq z \leq 1)$.
2.04. $\overline{F} = (x-y)\bar{i} + (y-z)\bar{k}$; $\sigma_1: x^2 - y^2 - z^2 = 1$; $\sigma_2: x = \sqrt{5}$.
2.05. $\overline{F} = (2x-y)\bar{i} + (3x+y)\bar{j} + z\bar{k}$; $\sigma_1: y = x^2 + z^2$; $\sigma_2: y = 1$.
2.06. $\overline{F} = 2x\bar{i} - y\bar{j} + z\bar{k}$; $\sigma_1: x^2 + y^2 + z^2 = 4$; $\sigma_2: 3z = x^2 + y^2$.
2.07. $\overline{F} = -y\bar{i} + x\bar{j} + z\bar{k}$; $\sigma_1: x^2 + y^2 = z^2$; $\sigma_2: x^2 + y^2 + z^2 = 2$ $(z > 0)$.
2.08. $\overline{F} = x\bar{i} - y\bar{j} + z\bar{k}$; $\sigma_1: x^2 + y^2 = z$; $\sigma_2: z = 3 + 2y$ $(z \geq 0)$.
2.09. $\overline{F} = (x+z)\bar{i} + y\bar{j}$; $\sigma_1: z = 8 - x^2 - y^2$; $\sigma_2: z = x^2 + y^2$.
2.10. $\overline{F} = (x-y)\bar{i} + (x+z)\bar{j} + 2y\bar{k}$; $\sigma_1: x = y^2 + z^2$; $\sigma_2: x^2 + y^2 + z^2 = 2$.
2.11. $\overline{F} = x\bar{i} + (x+z)\bar{j} - 2y\bar{k}$; $\sigma_1: x = y^2 + z^2$; $\sigma_2: x^2 + y^2 + z^2 = 2$.
2.12. $\overline{F} = -3y\bar{i} + 2x\bar{j} + (x+z)\bar{k}$; $\sigma_1: x^2 - y^2 - z^2 = 1$; $\sigma_2: x^2 + y^2 + z^2 = 3$ $(z > 0)$.
2.13. $\overline{F} = -2y\bar{i} + 3z\bar{k}$; $\sigma_1: x^2 = y^2 + z^2$; $\sigma_2: x^2 + y^2 + z^2 = 8$ $(x \geq 0)$.
2.14. $\overline{F} = (x-3y)\bar{i} + (y+5z)\bar{j} + 2x\bar{k}$; $\sigma_1: x^2 + y^2 = z^2$; $\sigma_2: z = 3$.
2.15. $\overline{F} = 3x\bar{i} - z\bar{j}$; $\sigma_1: z = 6 - x^2 - y^2$; $\sigma_2: x^2 + y^2 = z^2$ $(z \geq 0)$.
2.16. $\overline{F} = x\bar{i} - (x+2y)\bar{j} + y\bar{k}$; $\sigma_1: z = x^2 + y^2$; $\sigma_2: z = 1$.
2.17. $\overline{F} = x\bar{i} + z\bar{j} - y\bar{k}$; $\sigma_1: z = 4 - 2(x^2 + y^2)$; $\sigma_2: z = 2(x^2 + y^2)$.
2.18. $\overline{F} = x\bar{i} - z\bar{j} + y\bar{k}$; $\sigma_1: x^2 + y^2 + z^2 = 1$; $\sigma_2: x^2 + y^2 = z^2$ $(z \geq 0)$.
2.19. $\overline{F} = (x+z)\bar{i} + y\bar{j}$; $\sigma_1: z = 8 - x^2 - y^2$; $\sigma_2: z = x^2 + y^2$.
2.20. $\overline{F} = (x+y)\bar{i} - 4y\bar{j} + 2x\bar{k}$; $\sigma_1: z = x^2 + y^2$; $\sigma_2: z = 1$.
2.21. $\overline{F} = x\bar{i} - 2z\bar{j} + 3y\bar{k}$; $\sigma_1: x^2 + y^2 - z = 0$; $\sigma_2: z - 2x = 0$.
2.22. $\overline{F} = x\bar{i} + \bar{j} + y\bar{k}$; $\sigma_1: z = 4 - x^2 - y^2$; $\sigma_2: z = 0$ $(z > 0)$.
2.23. $\overline{F} = 3x\bar{i} - 2y\bar{k}$; $\sigma_1: x^2 - y^2 - z^2 = 1$; $\sigma_2: x = 2$.

2.24. $\overline{F}=(x-2y)\overline{i}+(z-x)\overline{j}+x\overline{k}$; $\sigma_1: x^2-y^2-z^2=1$; $\sigma_2: y=2$.
2.25. $\overline{F}=y\overline{i}+2x\overline{j}+z\overline{k}$; $\sigma_1: x^2+y^2+z^2=1$; $\sigma_2: z=0\ (z>0)$.
2.26. $\overline{F}=(x+z)\overline{i}+(y+z)\overline{k}$; $\sigma_1: z=x^2+y^2$; $\sigma_2: z=1$.
2.27. $\overline{F}=(x-z)\overline{i}+y\overline{j}$; $\sigma_1: z=1-x^2-y^2$; $\sigma_2: z=x^2+y^2$.
2.28. $\overline{F}=(x+y)\overline{i}+2x\overline{k}$; $\sigma_1: z=x^2+y^2$; $\sigma_2: z=4$.
2.29. $\overline{F}=x\overline{i}+z\overline{j}+3y\overline{k}$; $\sigma_1: x^2+y^2-z=0$; $\sigma_2: z-2y=0$.
2.30. $\overline{F}=x\overline{i}+\overline{j}+y\overline{k}$; $\sigma_1: z=9-x^2-y^2$; $\sigma_2: z=0\ (z>0)$.

1.4.3 Defined the vector field $\overline{F}(x,y,z)$. Prove, that the vector field $\overline{F}(x,y,z)$ is potential. Find the potential of the field $\overline{F}(x,y,z)$.

3.01. $\overline{F}=3x^2\overline{i}+2y\overline{j}-4\overline{k}$.
3.02. $\overline{F}=(2x+1)\overline{i}-4y^2\overline{j}-2z\overline{k}$.
3.03. $\overline{F}=(x^2+x)\overline{i}+4\overline{j}-2z\overline{k}$.
3.04. $\overline{F}=4\overline{i}-(y^2+1)\overline{j}+3z\overline{k}$.
3.05. $\overline{F}=5x\overline{i}-(y+2)\overline{j}+(z^2+1)\overline{k}$.
3.06. $\overline{F}=(2x^2+x)\overline{i}+y^2\overline{j}-(3z+2)\overline{k}$.
3.07. $\overline{F}=5\overline{i}-(y+3)\overline{j}-(z^2-1)\overline{k}$.
3.08. $\overline{F}=(1-3x)\overline{i}+5\overline{j}+(3z^2+2)\overline{k}$.
3.09. $\overline{F}=(4x^2+1)\overline{i}+(3y-2)\overline{j}+z\overline{k}$.
3.10. $\overline{F}=(3x+4)\overline{i}+(y^2-2y)\overline{j}+4\overline{k}$.
3.11. $\overline{F}=(2-3x)\overline{i}+(4y^2+3y)\overline{j}+(2z+1)\overline{k}$.
3.12. $\overline{F}=(1-2x^2)\overline{i}+(3-y)\overline{j}+(4z+1)\overline{k}$.
3.13. $\overline{F}=2x^2\overline{i}-y^2\overline{j}+\overline{k}$.
3.14. $\overline{F}=(5x+2)\overline{i}-(3y+1)\overline{j}+(2z-1)\overline{k}$.
3.15. $\overline{F}=(2x^2+x)\overline{i}+y^2\overline{j}-(3z+2)\overline{k}$.
3.16. $\overline{F}=(1+3x)\overline{i}-2y\overline{j}+(4-3z)\overline{k}$.
3.17. $\overline{F}=(3x-x^2)\overline{i}+(2-4y)\overline{j}+3\overline{k}$.
3.18. $\overline{F}=(2x^2+4)\overline{i}+(3y-1)\overline{j}+(z+2)\overline{k}$.
3.19. $\overline{F}=3x\overline{i}+(2-4y^2)\overline{j}+(3z-z^2)\overline{k}$.
3.20. $\overline{F}=(x^2-2)\overline{i}+(y-2)\overline{j}+(z^2+2z)\overline{k}$.
3.21. $\overline{F}=(1-2x+x^2)\overline{i}+(4y-1)\overline{j}-2z\overline{k}$.
3.22. $\overline{F}=2\overline{i}-(3y^2+y)\overline{j}+(4-3z)\overline{k}$.
3.23. $\overline{F}=(2x-x^2)\overline{i}+(1-4y)\overline{j}+(z^2+1)\overline{k}$.

3.24. $\overline{F} = (4x^2 + 1)\overline{i} - (3y + 2)\overline{j} + (1 - z^2)\overline{k}$.

3.25. $\overline{F} = (1 + 2x^2)\overline{i} + (3y - 1)\overline{j} + 4z\overline{k}$.

3.26. $\overline{F} = 3\overline{i} + (4y^2 + 3y)\overline{j} + (2z + 3)\overline{k}$.

3.27. $\overline{F} = (4 - 3x^2)\overline{i} + (2y^2 - y)\overline{j} + (1 - 2z)\overline{k}$.

3.28. $\overline{F} = (x^2 + 1)\overline{i} + (3y - 2)\overline{j} + 4z^2\overline{k}$.

3.29. $\overline{F} = 2x^2\overline{i} - (3y^2 + 2y)\overline{j} + (z^2 - 3z)\overline{k}$.

3.30. $\overline{F} = (3x - 2)\overline{i} + (2y + 1)\overline{j} + (z - 1)\overline{k}$.

COMPLEX VARIABLES

2.1 Complex numbers and operations with them

Definition. Complex numbers are the numbers

$$z = x + iy, \quad (1)$$

where x and y are real numbers, and i is imaginary unit ($i^2 = -1$). Number x is a real part of the complex number and denotes $x = \mathrm{Re}\,z$, y is imaginary part: $y = \mathrm{Im}\,z$.

Two complex numbers are equal if and only if equal their real and imaginary parts.

Number $\bar{z} = x - iy$ is called adjoined to the number $z = x + iy$.

(1) is an algebraic form of the complex number.

Complex number $z = x + iy$ corresponds at the plane to the point with the coordinates (x, y). Coordinate plane in that case is called complex plane.

Complex number is presented as a radius-vector. The length of the radius-vector is:

$$r = |z| = \sqrt{x^2 + y^2}$$

is called the module of the complex number ($r \geq 0$). Angel φ, generated by the radius-vector oz and the positive direction of the axes Ox is called the argument of the complex number and denotes $\varphi = \arg z = \mathrm{arctg}\dfrac{y}{x}$ $(x \neq 0)$.

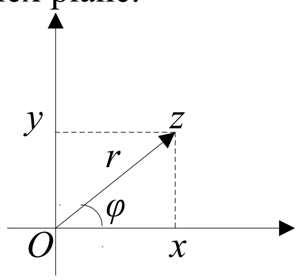


Figure 2.1

The argument of the complex number is a multiple-meaning value: $\mathrm{Arg}\,z = \arg z + 2\pi k$ $(k = 0, -1, 1, -2, 2, ...)$, where arg*z* is a main meaning of the argument, that lies in the interval $(-\pi; \pi]$.

From the figure 2.1 we see that:

$$x = r\cos\varphi,\ y = r\sin\varphi,\ r = |z| = \sqrt{x^2 + y^2},\ \mathrm{tg}\varphi = \frac{y}{x}.$$

If in algebraic form substitute x and y with the $x = r\cos\varphi$ and $y = r\sin\varphi$, then complex number $z = x + iy$ can be written as:

$$z = r\left(\cos\varphi + i\sin\varphi\right). \quad (2)$$

It is trigonometric form of the complex number.

The equality $\cos\varphi + i\sin\varphi = e^{i\varphi}$ was proved by Euler, using it, we can write

$$z = re^{i\varphi}. \quad (3)$$

It is exponential form of the complex number.

Let $z_1 = x_1 + iy_1$ и $z_2 = x_2 + iy_2$ be two complex numbers in algebraic form. Then:

1) $z_1 \pm z_2 = \left(x_1 \pm x_2\right) + i\left(y_1 \pm y_2\right)$;

2) $z_1 \times z_2 = \left(x_1 + iy_1\right) \times \left(x_2 + iy_2\right) = \left(x_1x_2 - y_1y_2\right) + i\left(x_1y_2 + x_2y_1\right)$;

3) $\dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} \cdot \dfrac{x_2 - iy_2}{x_2 - iy_2} = \dfrac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} = \dfrac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\dfrac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}$.

If complex numbers are in trigonometric form, e.g. $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$, then

1) $z_1 \cdot z_2 = r_1(\cos\varphi_1 + i\sin\varphi_1) \cdot r_2(\cos\varphi_2 + i\sin\varphi_2) = r_1 \cdot r_2 \cdot (\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$;

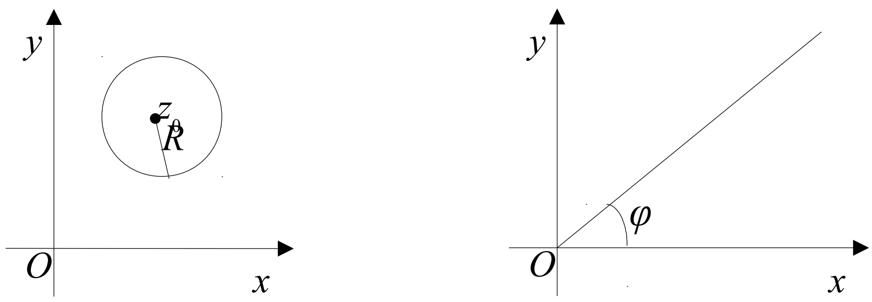
2) $\dfrac{z_1}{z_2} = \dfrac{r_1(\cos\varphi_1 + i\sin\varphi_1)}{r_2(\cos\varphi_2 + i\sin\varphi_2)} = \dfrac{r_1}{r_2}[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)]$;

3) $z^n = r^n(\cos\varphi_1 + i\sin\varphi_1)^n = r^n(\cos n\varphi + i\sin n\varphi)$, $n \in \mathbb{N}$ – Moivre formula;

4) $\sqrt[n]{z} = \sqrt[n]{r(\cos\varphi + i\sin\varphi)} = \sqrt[n]{r}\left(\cos\dfrac{\varphi + 2k\pi}{n} + i\sin\dfrac{\varphi + 2k\pi}{n}\right)$, $k = 0, 1, 2, ..., n-1$.

2.2 Concept of the function with the complex variable

Further we consider different sets of complex numbers. They are defined as equalities or inequalities. For example, condition $|z_0| = R, R = const$ defines a circle with the radius R and with the center at the point z_0 (figure 2.2*a*); condition $\arg z = const$ is a ray, that comes out from the origin angularly $\varphi = \arg z$ (figure 2.2*b*).



a) *b*)

Figure 2.2

Definition 1. If to every complex number z, that belongs to the set D, put adequacy any single complex number or any collection of the complex numbers ω, we say, that ω is a function with the variable z, defined in the set D and denotes

$$\omega = f(z). \tag{4}$$

If take into account, that $z = x + iy$ and put $\omega = u + iv$, then for extension of a definition of the function ω it is enough to define two functions with real variables $u = u(x, y)$ и $v = v(x, y)$.

Consequently,

$$\omega = u(x, y) + iv(x, y). \tag{5}$$

From (4) we can come to (5). Such transition is called elimination of the real and imaginary parts.

Example. Eliminate real and imaginary part of the function: $\omega = z^2$.

Solution. Because $z = x + iy$, then $\omega = (x+iy)^2 = x^2 - y^2 + 2xyi$. So, $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$.

Definition 2. Complex number $\omega_0 = u_0 + iv_0$ is called the limit of the function $\omega = u(x,y) + iv(x,y) = f(z)$ of the complex variable $z = x + iy$ when $z \to z_0(x_0, y_0)$, if for every, as small as possible, positive number ε we can indicate such positive number σ, that from the inequality $|z - z_0| < \sigma$ follows inequality $|f(z) - \omega_0| < \varepsilon$.

Given definition can be written as: $\lim\limits_{z \to z_0} f(z) = \omega_0$.

From the definition it follows, that if ω_0 is a limit of the function $\omega = f(z)$ when $z \to z_0$, then the value ω_0 does not depend on the way along which point z comes to the point z_0. From the definition it also follows that if the limit of the function exists, then the following limits exist:

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u(x,y) = u_0 \text{ and } \lim_{\substack{x \to x_0 \\ y \to y_0}} v(x,y) = v_0.$$

Definition 3. $f(z)$ is called continuous function at the point z_0, if it defined in any neighborhood of that point and $\lim\limits_{z \to z_0} f(z) = f(z_0)$. Function $f(z)$ is continuous in the domain D, if it continuous in every point of that domain.

Continuous functions with the complex variable have same properties as continuous functions with the real variables. Particularly, if function $\omega = f(z)$ is continuous in the closed domain D, then that function:

1) is absolutely limited in that domain, e.g. $|f(z)| < M$;
2) reaches its maximum and minimum value in the closed domain D.

2.3 The main basic functions with the complex variable

Define the main basic functions with the complex variable $z = x + iy$.

Exponential function. Exponential function $w = e^z$ is defined by the formula

$$w = e^z = e^x(\cos y + i \sin y).$$

Let $y = 0$ in that equation, determine, that for the real values $z = x$ exponential function e^z concurs with the exponential function with the real variable: $e^z = e^x$.

Logarithmic function. Logarithmic function defines as a function, inverse to the exponential function: number w is called logarithm of the number $z \neq 0$, if $e^w = z$ and denotes $w = \operatorname{Ln} z$. Because the value of the exponential function $e^w = z$ is always different from zero, then logarithmic function $w = \operatorname{Ln} z$ is defined everywhere at the surface z, except the point $z = 0$.

Put $z = re^{i\varphi}$, $w = u + iv$, we get, by the definition of the logarithmic function,

$$e^{u+iv} = r \times e^{i\varphi} \text{ або } e^u \times e^{iv} = r \times e^{i\varphi}.$$

From here we have:

$$e^u = r,\ v = \varphi + 2k\pi \text{ , e.g. } u = \ln r,\ v = \varphi + 2k\pi \quad (k = 0, \pm 1, \pm 2, ...).$$

Therefore, $w = \mathrm{Ln}\, z = u + iv = \ln r + i(\varphi + 2k\pi) = \ln|z| + i(\arg z + 2k\pi)$, e.g. $\mathrm{Ln}\, z = \ln|z| + i(\arg z + 2k\pi)$ or $\mathrm{Ln}\, z = \ln|z| + i\mathrm{Arg}\, z$, where $\mathrm{Arg}\, z = \arg z + 2k\pi$.

The last formula shows, that logarithmic function with the complex variable has infinitely many values, e.g. $w = \mathrm{Ln}\, z$ – multiple-valued function

One-valued path of that function can be extracted by the substitution in the last formula specified value k. Let $k = 0$, we get one-valued function , that called the main value of the logarithm $\mathrm{Ln}\, z$ and denotes as $\ln z$:

$$\ln z = \ln|z| + i\arg z \text{, where } -\pi < \arg z \le \pi \ .$$

If z – is a real positive number, then $\arg z = 0$ and $\ln z = \ln|z|$, e.g. the main value of the logarithm with the real positive number concurs with the regular natural logarithm of the number.

Logarithmic function with the complex variable can be presented as:

$$\mathrm{Ln}\, z = \ln z + 2k\pi i \, .$$

Power function. If n – is a natural number, then the power function is defined by the equation $w = z^n = r^n(\cos n\varphi + i\sin n\varphi)$. Function $w = z^n$ – is an one-valued function. If $n = \frac{1}{q}$ $(q \in ¥)$, then in that case

$$w = z^{\frac{1}{q}} = \sqrt[q]{z} = \sqrt[q]{|z|}\left(\cos\frac{\arg z + 2k\pi}{q} + i\sin\frac{\arg z + 2k\pi}{q}\right), \text{ where } k = 0, 1, 2, ..., q - 1.$$

Here function $w = z^{\frac{1}{q}}$ is a multiple-valued function (q-valued). It is possible to get one-valued path of that function, attaching to k definite value, for example $k = 0$.

If $n = \frac{p}{q}$ $(p, q \in ¥)$, then the power function is defined by the equality

$$w = z^{\frac{p}{q}} = \left(\sqrt[q]{z}\right)^p = \sqrt[q]{|z|^p}\left(\cos\frac{p(\arg z + 2k\pi)}{q} + i\sin\frac{p(\arg z + 2k\pi)}{q}\right).$$

Function $w = z^{\frac{p}{q}}$ – is a multiple-valued function.

Trigonometric function. Trigonometric function with the complex variable $z = x + iy$ is defined by the equalities:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \mathrm{tg}\, z = \frac{\sin z}{\cos z}, \mathrm{ctg}\, z = \frac{\cos z}{\sin z}.$$

Trigonometric functions with the complex variable have the same properties as the functions with the real variables.

Hyperbolic functions. Hyperbolic functions are defined by the equalities:

$$\operatorname{sh} z = \frac{e^z - e^{-z}}{2},\ \operatorname{ch} z = \frac{e^z + e^{-z}}{2},\ \operatorname{th} z = \frac{\operatorname{sh} z}{\operatorname{ch} z},\ \operatorname{cth} z = \frac{\operatorname{ch} z}{\operatorname{sh} z}.$$

The connection between hyperbolic and trigonometric functions is:

$$\operatorname{sh} iz = i\sin z,\ \sin z = -i\operatorname{sh} iz,\ \operatorname{ch} iz = \cos z.$$

From the definition of the hyperbolic functions follows, that functions $\operatorname{sh} z, \operatorname{ch} z$ are periodic with the period $2\pi i$; functions $\operatorname{th} z, \operatorname{cth} z$ have the period πi.

2.4 Differentiability and analyticity of the function with the complex variable

Definition. Let $f(z)$ be a function defined in any neighborhood of the point z. Then, if the limit

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{df(z)}{dz} = f'(z),$$

exists, then it calls the derivative of the function $f(z)$ with the complex variable z, and the function $f(z)$ is called differentiable at the point z.

Recall, that the function with two real variables $u(x,y)$ is called differentiable, or the one that has total differential at that point (x,y), if the leading linear part and the infinitesimal part with the higher order of infinitesimally with respect to $\Delta x, \Delta y$ can be extracted in total increment of that function at that point, e.g.

$$u(x+\Delta x,\ y+\Delta y) - u(x,y) = A\Delta x + B\Delta y + \alpha\Delta x + \beta\Delta y, \qquad (6)$$

$\alpha, \beta \to 0$ when $\Delta x, \Delta y \to 0$, also $A = \frac{\partial u}{\partial x},\ B = \frac{\partial u}{\partial y}$.

Theorem. If the function $f(z) = u(x,y) + iv(x,y)$ is defined in any neighborhood of the point $z = x + iy$, then for function $f(z)$ to have derivative at that point, it is necessary and sufficient for the functions u and v to be differentiable in the point $z(x,y)$ with respect to x and with respect to y and have place the following condition:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ (Cauchy-Riemann condition).}$$

Proof.

Necessity. Let the function $f(z)$ be differentiable at the point $z = x + iy$. It is necessary to prove, that functions $u(x,y)$ and $v(x,y)$ are differentiable at that point (x,y) and Cauchy-Riemann condition is satisfied.

Because function $f(z)$ is differentiable at the point z, then its increment can be written as:

$$f(z+\Delta z) - f(z) = c\Delta z + \gamma\Delta z,$$

and $\gamma \to 0$ when $\Delta z \to 0$, $c = f'(z)$, e.g.

$$f(z+\Delta z) - f(z) = f'(z)\Delta z + \gamma\Delta z.$$

By the definition: $z = x + iy$, $\Delta z = \Delta x + i\Delta y$;

$$f(z)=u(x,y)+iv(x,y);$$
$$f(z+\Delta z)=u(x+\Delta x,\ y+\Delta y)+iv(x+\Delta x,\ y+\Delta y).$$
$$f'(z)=A+iB,\ \gamma=\alpha+i\beta\ .\ \alpha,\beta\to 0 \text{ при } \Delta x\to 0,\ \Delta y\to 0.$$
Then
$$f(z+\Delta z)-f(z)=[u(x+\Delta x,\ y+\Delta y)-u(x,y)]+i[v(x+\Delta x,\ y+\Delta y)-v(x,y)]=$$
$$=(A+iB)(\Delta x+i\Delta y)+(\alpha+i\beta)(\Delta x+i\Delta y).$$
Hence,
$$u(x+\Delta x,\ y+\Delta y)-u(x,y)=A\Delta x-B\Delta y+\alpha\Delta x-\beta\Delta x, \quad (7)$$
$$v(x+\Delta x,\ y+\Delta y)-v(x,y)=B\Delta x+A\Delta y+\alpha\Delta y+\beta\Delta x. \quad (8)$$
Expressions (7) and (8) look like the expression (6). Comparing them, we note
$$A=\frac{\partial u}{\partial x} \text{ and } B=-\frac{\partial u}{\partial y};\ A=\frac{\partial v}{\partial y} \text{ and } B=\frac{\partial v}{\partial x},$$
because $f'(z)=A+iB=u'_x+iv'_x$.
Hence,
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}. \quad (9)$$
So, it is proved that if the function $f(z)$ is differentiable at the point $z=x+iy$, then its real and imaginary parts $u(x,y), v(x,y)$ are differentiable at the point (x,y) and satisfy conditions (9).

Taking into account Cauchy-Riemann condition, the derivative of the function $f(z)=u(x,y)+iv(x,y)$ can be found by the formulas
$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x};\ f'(z)=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y};\ f'(z)=\frac{\partial u}{\partial x}-i\frac{\partial u}{\partial y};\ f'(z)=\frac{\partial v}{\partial y}+i\frac{\partial v}{\partial x}.$$

Example. Check Cauchy-Riemann condition and find derivatives of the functions: 1) $w=f(z)=z^2$; 2) $w=z\,\mathrm{Re}\,z$.

Solution.

1) $z=x+iy$, then $\omega=(x+iy)^2=x^2-y^2+2xyi$, that is why $u(x,y)=x^2-y^2$, $v(x,y)=2xy$.

Because $\frac{\partial u}{\partial x}=2x,\ \frac{\partial v}{\partial y}=2x;\ \frac{\partial u}{\partial y}=-2y,\ \frac{\partial v}{\partial x}=2y$, then Cauchy-Riemann conditions $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ are satisfied.

Hence, $\frac{\partial w}{\partial z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=2x+i2y=2(x+iy)=2z$.

2) $\omega=z\,\mathrm{Re}\,z=(x+iy)x=x^2+xyi$, then $u(x,y)=x^2$, $v(x,y)=xy$. Find $\frac{\partial u}{\partial x}=2x,\ \frac{\partial v}{\partial y}=x;\ \frac{\partial u}{\partial y}=0,\ \frac{\partial v}{\partial x}=y.$

Because $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} \neq \frac{\partial v}{\partial x}$, then Cauchy-Riemann conditions are not satisfied and the function $\omega = z \operatorname{Re} z$ does not have the derivative.

Definition. If the function $f(z)$ is differentiable not only at the given point z_0, and also in any neighborhood of that point, then it calls analytical at that point z_0.

Definition. Function $f(z)$, analytical in every point of the domain D, is called analytical in the domain D.

2.5 Integral of the function with the complex variable. Cauchy theorem

Let $f(z)$ be continuous function with the complex variable, defined in every point of the arch *AB*.

Split up the arch *AB* by *n* parts, randc choosing points $A = z_0, z_1, z_2, \dots, z_{n-1}, z_n =$ (figure 2.3).

At each part, randomly choos int ξ_i and get sum: $\sum_{k=1}^{n} f(\xi_k)\Delta z_k$, where $\Delta z_k = z_{k+1} - z_k$

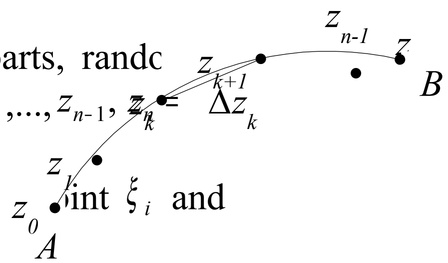


Figure 2.3

The limit of that sum (when $\max|\Delta z_k|$ goes to zero) is called the integral of the function $f(z)$ along the edge *AB* and denotes $\int_{\overset{\frown}{AB}} f(z)dz$.

E.g. $\int_{\overset{\frown}{AB}} f(z)dz = \lim_{\max|\Delta z_k| \to 0} \sum_{k=1}^{n} f(\xi_k)\Delta z_k$, where $|\Delta z_k|$ is the length of the chord of the elementary arch $\overset{\frown}{z_k, z_{k+1}}$.

Properties of the integral.

The following properties come from the definition:

1. $\int_{\overset{\frown}{AB}} [f_1(z) \pm f_2(z)]dz = \int_{\overset{\frown}{AB}} f_1(z)dz \pm \int_{\overset{\frown}{AB}} f_2(z)dz$.
2. $\int_{\overset{\frown}{AB}} c f(z)dz = c\int_{\overset{\frown}{AB}} f(z)dz$, $c - const$.

3. $\int\limits_{\overset{\frown}{AB}} f(z)dz = -\int\limits_{\overset{\frown}{BA}} f(z)dz$.

4. If the edge *AB* is splitted by point C, then

$$\int\limits_{\overset{\frown}{AB}} f(z)dz = \int\limits_{\overset{\frown}{AC}} f(z)dz + \int\limits_{\overset{\frown}{CB}} f(z)dz.$$

5. Estimation of the absolute value of the integral

If for curve $\overset{\frown}{AB}$, that has length L: $|f(z)| \le M$, then $\left|\int\limits_{\overset{\frown}{AB}} f(z)dz\right| \le ML$.

Proof.

$\left|\sum\limits_{k=1}^{n} f(\xi_k)\Delta z_k\right| \le \sum\limits_{k=1}^{n} |f(\xi_k)||\Delta z_k| \le M\sum\limits_{k=1}^{n} |\Delta z_k|$, $|\Delta z_k|$ is the distance between points z_k and z_{k-1}, and $\sum\limits_{k=1}^{n} |\Delta z_k|$ is the length of the kinked curve, inscribed in the arch $\overset{\frown}{AB}$, that is why $\sum\limits_{k=1}^{n} |\Delta z_k| \le L$.

Therefore, $\left|\sum\limits_{k=1}^{n} f(\xi_k)\Delta z_k\right| \le ML$.

Coming to the limit, we get $\left|\int\limits_{\overset{\frown}{AB}} f(z)dz\right| \le ML$

6. Expression of the integral of the function with the complex variable in terms of the curvilinear integral of the second order.

Because $f(z) = u(x,y) + iv(x,y)$ and $z_k = x_k + iy_k$, then

$$\sum_{k=1}^{n} f(z_k)\Delta z_k = \sum_{k=1}^{n} [u(x_k,y_k) + iv(x_k,y_k)](\Delta x_k + i\Delta y_k) =$$

$$= \sum_{k=1}^{n} [u(x_k,y_k)\Delta x_k - v(x_k,y_k)\Delta y_k] + i\sum_{k=1}^{n} [v(x_k,y_k)\Delta x_k + u(x_k,y_k)\Delta y_k].$$

These sums are the integral sums of the curvilinear integrals of the second type. That is why the passage to the limit with the condition $\max\limits_{k}|\Delta z_k| \to 0$, e.g., $\max\limits_{k}|\Delta y_k| \to 0$ gives the possibility to write down the formula

$$\int\limits_{\overset{\frown}{AB}} f(z)dz = \int\limits_{\overset{\frown}{AB}} u(x,y)dx - v(x,y)dy + i\int\limits_{\overset{\frown}{AB}} v(x,y)dx + u(x,y)dy.$$

In that formula real and imaginary parts can be segregated.

7. Transformation of the integral with the complex variable into the regular integral from the complex function with the real variable.

If the arch *AB* is defined by the parametric equation $\begin{cases} x = x(t), \\ y = y(t), \end{cases} \quad t_1 \le t \le t_2$, then $z = z(t) = x(t) + iy(t)$ is the complex parametric equation of the arch *AB*.

Then

$$\int_{\overset{\frown}{AB}} f(z)dz = \int_{t_1}^{t_2} f[x(t)+iy(t)]d[x(t)+iy(t)] =$$

$$= \int_{t_1}^{t_2} f[x(t)+iy(t)]\frac{d}{dt}[x(t)+iy(t)]dt = \int_{t_1}^{t_2} f[z(t)]z'(t)dt.$$

Given formula brings calculation of the integral with the complex variable to the calculation of the definite integral with the real variable.

Example 1. Find $\oint_\gamma \frac{dz}{z-z_0}$, where γ is a circle with the radius R and with the center at the point $z_0 = \alpha + i\beta$.

Solution.

The equation of the circle is:

$(x-\alpha)^2 + (y-\beta)^2 = R^2$ (figure 2.4).

Let $x-\alpha = R\cos t$, $y-\beta = R\sin t$.

The equation $\begin{cases} x = \alpha + R\cos t, \\ y = \beta + R\sin t, \end{cases}$ $0 \le t \le 2\pi$ is a parametric equation of the circle, then its complex parametric equation is:

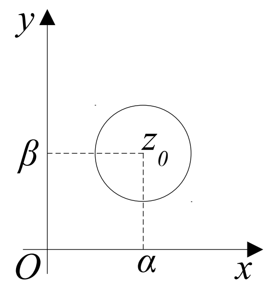


Figure 2.4

$z = x+iy = \alpha + i\beta + R(\cos t + i\sin t) = z_0 + Re^{it}$, and $dz = d(z_0 + Re^{it}) = iRe^{it}dt$.

That is why $\oint_\gamma \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{iRe^{it}dt}{Re^{it}} = 2\pi i$.

Answer: $2\pi i$.

The main Cauchy theorem. If the function $f(z)$ is analytical in simply connected domain D, limited by the closed loop Γ, and also at the points of that loop, the integral of that function over the loop Γ equals to zero:

$$\oint_\Gamma f(z)dz = 0.$$

Proof.

Let the function $w = f(z) = u(x,y) + iv(x,y)$ be analytical in simply connected domain D. It follows existence of the continuous partial derivatives of the functions $u(x,y)$, $v(x,y)$ in the domain D. Let Γ be a closed loop, limiting domain D (figure 2.5).

Then $\oint_\Gamma f(z)dz = \oint_\Gamma u(x,y)dx - v(x,y)dy + i\oint_\Gamma v(x,y)dx + u(x,y)dy$.

Using Green formula

$$\oint_\Gamma P(x,y)dx + Q(x,y)dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy,$$

we get

$$\oint_\Gamma f(z)dz = \iint_D \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy.$$

Because function $f(z)$ is analytical, then Cauchy-Riemann conditions take place:

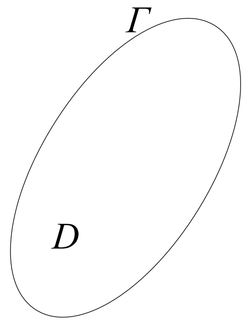


Figure 2.5

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Therefore, $\oint_{\Gamma} f(z)dz=0$.

Example 2. Calculate the integrals:

1) $\oint_{\Gamma}(z^2-3z)dz$;

2) $I=\oint_{\Gamma}\frac{dz}{z-z_0}$, if *a*) $\Gamma:|z-z_0|=R$;

в) Γ is a loop that does not contain point z_0.

Solution.

1) Because function $f(z)=z^2-3z$ is analytical at all plane, then $\oint_{\Gamma}(z^2-3z)dz=0$ over any closed loop.

2) $I=\oint_{\Gamma}\frac{dz}{z-z_0}$, if *a*) $\Gamma:|z-z_0|=R$.

From proved above (Example 1) $I=\oint_{\gamma}\frac{dz}{z-z_0}=2\pi i$. Note, that the point z_0 is inside the circle $|z-z_0|=R$ and function $f(z)$ is not analytical at that point.

б) If Γ is the loop, that does not contain the point z_0, then $I=\oint_{\Gamma}\frac{dz}{z-z_0}=0$.

Answer: 0; $2\pi i$, 0.

Theorem. If the function $f(z)$ is analytical in the multiply-connected region D, limited by the external loop Γ and inside loops $\gamma_1, \gamma_2, \dots, \gamma_n$, and also by the loops $\Gamma, \gamma_1, \gamma_2, \dots, \gamma_n$, the following formula takes place

$$\oint_{\Gamma} f(z)dz=\sum_{k=1}^{n}\oint_{\gamma_k} f(z)dz.$$

Proof.

Let in the domain, limited by the loop Γ, exist two loops γ_1 and γ_2 and the function $f(z)$, analytical in the domain, between loops Γ and γ_1, γ_2, and also at the above mentioned loops.

We prove theorem for n=2. Trace arches: lk, mn, pg, connecting loops Γ with γ_1, γ_1 c γ_2, γ_2 with Γ and denote as C_1 closed loop $lkzmntpgfl$, and as C_2 – $klrgpunmsk$ (figure 2.6). Then domains inside the loops C_1 and C_2 are simply connected and the main Cauchy theorem takes place:

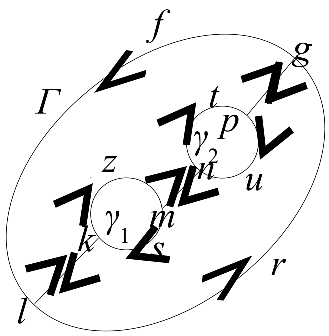


Figure 2.6

$$\oint_{C_1} f(z)dz = 0, \quad \oint_{C_2} f(z)dz = 0.$$

Adding these equalities and using the third property for integrals, we can note, that integration over every arch lk, mn, pq performed twice in different directions, and bypasses of the loops γ_1, γ_2 clockwise (e.g., in opposite direction). So,

$$\oint_{\Gamma} f(z)dz - \oint_{\gamma_1} f(z)dz - \oint_{\gamma_2} f(z)dz = 0,$$

the necessity follows.

2.6 Cauchy integral. Cauchy integral formula

Theorem. Let $f(z)$ be analytical function in the simply connected domain D, limited by and on the closed loop Γ. Then the value of the function in any point $z_0 \in D$ can be found by the formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - z_0} dz.$$

Proof.

Let $f(z)$ be the function in the domain limited by and on the closed loop Γ. Fix the point z_0 inside the loop Γ and consider the function

$$\varphi(z) = \frac{f(z) - f(z_0)}{z - z_0}.$$

That function is analytic in every point of the loop and on the loop Γ, except the point z_0. E.g., when $z = z_0$ we get uncertainty $\left[\frac{0}{0}\right]$, that expands, because when $z \to z_0$ $\varphi(z) \to f'(z_0)$. If we redefine function $\varphi(z)$ at the point z_0 by the condition $\varphi(z) = f'(z_0)$, then $\varphi(z)$ be the continuous function in all domain limited by the closed loop.

Therefore, the function itself is going to be limited and $|\varphi(z)| < M$, where M is any positive number.

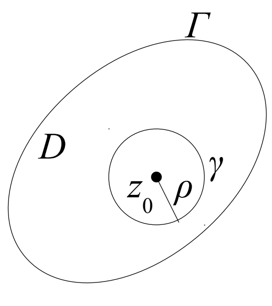


Figure 2.7

Let γ be the circle with the radius ρ, with the center at the point z_0, that is inside Γ. In the domain limited by the loop Γ and γ, function $\varphi(z)$ is analytical, because point $z = z_0$ (at that point analyticity is broken) is taken away from the domain.

Using Cauchy theorem for multiply connected region, we get

$$\oint_{\Gamma} \varphi(z)dz = \oint_{\gamma} \varphi(z)dz.$$

According to the rule of estimation of the absolute value of the integral, we get:

$$\left|\oint_{\gamma} \varphi(z)dz\right| \le ML_{\Gamma} = 2\pi\rho M.$$

Passing to the limit when $\rho \to 0$ in the last equality, we get

$$\oint_{\Gamma} \varphi(z)dz = 0,$$

or

$$\oint_{\Gamma} \frac{f(z) - f(z_0)}{z - z_0} dz = 0 \Rightarrow \oint_{\Gamma} \frac{f(z)dz}{z - z_0} - f(z_0)\oint_{\Gamma} \frac{dz}{z - z_0} = 0. \qquad (10)$$

According to the Cauchy theorem for multiply connected region $\oint_{\Gamma} \frac{dz}{z - z_0} = \oint_{\gamma} \frac{dz}{z - z_0} = 2\pi i$. That is why equality (10) takes on form: $\oint_{\Gamma} \frac{f(z)dz}{z - z_0} - 2\pi i f(z_0) = 0$.

From here,

$$f(z_0) = \frac{1}{2\pi i}\oint_{\Gamma} \frac{f(z)}{z - z_0} dz.$$

That formula is called Cauchy integral formula, and the integral on the right is the Cauchy integral. From that formula we see that the value of the analytical function inside the loop Γ defines by the value of that function on the boundary of the domain C.

Example. Calculate $I = \frac{1}{2\pi i}\oint_{\Gamma} \frac{z^2 dz}{z + 1}$, where Γ: 1) $|z| = 2$; 2) $|z| = \frac{1}{2}$.

Solution.

1) By the condition $I = \frac{1}{2\pi i}\oint_{\Gamma} \frac{z^2 dz}{z + 1}$ function $f(z) = z^2$ is analytical everywhere. Then $z_0 = -1$ lies inside the circle $|z| < 2$, that is why by the Cauchy formula $I = f(-1) = 1$.

2) the point $z_0 = -1$ lies outside the circle $|z| \le \frac{1}{2}$, that is why by the Cauchy theorem $I = 0$.

Answer: 1; 0.

2.7 Higher order derivatives of the analytic function

Theorem. Analytical function is infinitely many times differentiable, and the following formula have place

$$f^{(n)}(z)=\frac{n!}{2\pi i}\oint_{\Gamma}\frac{f(\xi)d\xi}{(\xi-z)^{n+1}}.$$

Proof.

We showed that if the function $f(z)$ is analytical in the domain D, limited by and on the loop Γ, then the value of the function at any point z, that belongs to D, can be found by the Cauchy formula

$$f(z)=\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(\xi)}{\xi-z}d\xi\ .$$

No matter what point z is, it is always possible to choose such value Δz that new point $z+\Delta z$ is going to lie inside the domain D. Let, for example, $|\Delta z|$ be the shortest distance from the point z to the boundary of the domain. Then, for the new point $z+\Delta z$, by the Cauchy formula

$$f(z+\Delta z)=\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(\xi)}{\xi-z-\Delta z}d\xi\ .$$

Consider

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}=\frac{1}{2\pi i\Delta z}\oint_{\Gamma}\left[\frac{f(\xi)}{\xi-z-\Delta z}-\frac{f(\xi)}{\xi-z}\right]d\xi\ .$$

After transformation

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}=\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(\xi)}{(\xi-z-\Delta z)(\xi-z)}d\xi\ .$$

When $\Delta z\to 0$, we get

$$f'(z)=\frac{1}{2\pi i}\oint_{\Gamma}\frac{f(\xi)}{(\xi-z)^2}d\xi\ . \qquad (11)$$

We got formula for $n=1$. Substituting in the formula (11) z by $z+\Delta z$ and forming new ratio: $\frac{f'(z+\Delta z)-f'(z)}{\Delta z}$, we get formula for the second derivative:

$$f''(z)=\frac{2}{2\pi i}\oint_{\Gamma}\frac{f(\xi)}{(\xi-z)^3}d\xi \quad \text{and so on.}$$

So, $f^{(n)}(z)=\frac{n!}{2\pi i}\oint_{\Gamma}\frac{f(\xi)}{(\xi-z)^{n+1}}d\xi$.

From the analyticity of the function in any point, follows existence, in neighborhood of that point, of the derivatives of any order, and therefore their analyticity.

2.8 Series of the analytic functions

Definition. Power series are the series $\sum_{n=0}^{\infty} C_n(z-z_0)^n$, where C_n are constant complex numbers (coefficients of the series). For power series the following properties have place:

1. For the power series $\sum_{n=0}^{\infty} C_n(z-z_0)^n$, that have both points of convergence and points of divergence, there always exists such a real number $R \geq 0$, that inside the circle $|z-z_0|< R$ the given series converge, and outside diverge. The domain $|z-z_0|< R$ is called the domain of convergence, and the number R is the radius of convergence of the power series.

2. The sum of the power series is the analytical function in the domain of convergence.

3. Power series in the circle with the radius $\rho < R$ converge evenly. It can be termwise differentiated and termwise integrated over any arch, that lies in the domain of convergence. The radius of convergence of every newly built series equals to the radius of convergence of original series, and the same actions are performed with the series sum.

Example. Find the domain of convergence $\sum_{n=0}^{\infty} \frac{(z-1+i)^n}{2^n}$.

Solution. Use Dalamber property

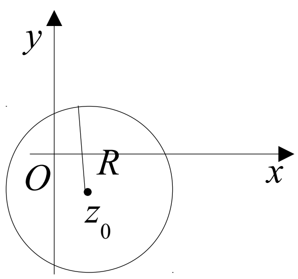


Figure 2.8

$$\lim_{n\to\infty}\left|\frac{\left(z-1+i\right)^{n+1}2^n}{2^{n+1}\left(z-1+i\right)^n}\right| = \frac{1}{2}|z-1+i|< 1, \text{ then } |z-1+i|< 2,$$

wher $z_0 = 1-i$ and $R = 2$.

Because

$$|z-1+i| = |x+iy-1+i| < 2 \Rightarrow \sqrt{\left(x-1\right)^2+\left(y+1\right)^2} < 2,$$

and the last inequality defines domain, limited by the circle with the center at the point $z_0(1,-1)$ and by the radius $R = 2$.

2.8.1 Taylor series

Power series inside its domain of convergence define analytical function − the sum of the series. The converse preposition is also true.

Theorem. Any function $f(z)$, analytical inside the circle with the center at the point z_0, develops, inside that circle, in the power series:

$$f(z) = \sum_{n=0}^{\infty} C_n(z-z_0)^n,$$

Coefficients can be calculated by the formula:

$$C_n = \frac{f^{(n)}(z_0)}{n!},\ n = 0, 1, 2, ...$$

Proof.

Consider function $f(z)$, analytical inside the circle k with the center at the point z_0.

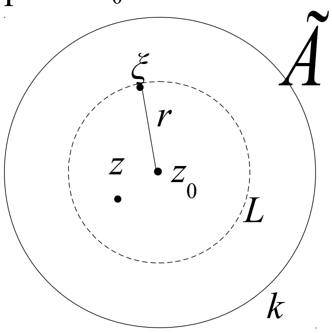


Figure 2.9

Let z be any point of the circle. Then by the Cauchy integral formula $f(z)=\dfrac{1}{2\pi i}\oint_{\Gamma}\dfrac{f(\xi)}{\xi-z}d\xi$.

Draw inside the circle k the circle L with the center at the point z_0 with the radius r so, that the point z is inside of that circle (figure 2.9).

Then the equation of the circle L: $|\xi-z_0|=r$. The distance between points z and z_0 is less than r.

Consider fraction

$$\frac{1}{\xi-z}=\frac{1}{\xi-z_0-\left(z-z_0\right)}=\frac{1}{\xi-z_0}\cdot\frac{1}{1-\dfrac{z-z_0}{\xi-z_0}}, \tag{12}$$

where $\dfrac{z-z_0}{\xi-z_0}=q\Rightarrow\ |q|<1$.

Consider expression (12) as a sum of decreasing geometric progression with the first member $\dfrac{1}{\xi-z_0}$ and with the denominator $\dfrac{z-z_0}{\xi-z_0}$. Then according to the sum of the geometric progression $\dfrac{a}{1-q}=a+aq+aq^2+\ldots$, we get:

$$\frac{1}{\xi-z}=\frac{1}{\xi-z_0}+\frac{z-z_0}{(\xi-z_0)^2}+\frac{(z-z_0)^2}{(\xi-z_0)^3}+\ldots=\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\xi-z_0)^{n+1}}. \tag{13}$$

Multiplying series (13) by $f(\xi)$, we get:

$$\frac{f(\xi)}{\xi-z}=\frac{f(\xi)}{\xi-z_0}+\frac{\left(z-z_0\right)f(\xi)}{(\xi-z_0)^2}+\frac{(z-z_0)^2 f(\xi)}{(\xi-z_0)^3}+\ldots=\sum_{n=0}^{\infty}\frac{(z-z_0)^n f(\xi)}{(\xi-z_0)^{n+1}}. \tag{14}$$

Because $|\xi-z_0|=r$, taking onto account analyticity of the function $|f(\xi)|<M$ when $\xi\in L$, then

$$\left|\frac{(z-z_0)^n f(\xi)}{(\xi-z_0)^{n+1}}\right|=\left|\frac{z-z_0}{\xi-z_0}\right|^n\times\frac{|f(\xi)|}{|\xi-z_0|}\le|q|^n\times\frac{M}{r}.$$

Series $\sum\limits_{n=0}^{\infty}|q|^n\times\dfrac{M}{r}=\dfrac{M}{r}\sum\limits_{n=0}^{\infty}|q|^n$ converge and are majorant to (14), that is why series (14) converge and can be termwise integrated

$$f(z)=\oint_L\left[\frac{f(\xi)}{\xi-z_0}+\frac{\left(z-z_0\right)f(\xi)}{\left(\xi-z_0\right)^2}+\frac{\left(z-z_0\right)^2 f(\xi)}{\left(\xi-z_0\right)^3}+\ldots\right]d\xi=$$

$$= \frac{1}{2\pi i}\oint_L \frac{f(\xi)}{\xi - z_0}d\xi + \frac{(z-z_0)}{2\pi i}\oint_L \frac{f(\xi)}{(\xi - z_0)^2}d\xi + \frac{(z-z_0)^2}{2\pi i}\oint_L \frac{f(\xi)}{(\xi - z_0)^3}d\xi + \ldots =$$

$$= f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \ldots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \ldots . \quad (15)$$

Therefore,

$$f(z) = \sum_{n=0}^{\infty} C_n (z-z_0)^n \text{, where } C_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\oint_L \frac{f(\xi)}{(\xi - z_0)^{n+1}}d\xi .$$

Power series (15) are called Taylor series for the function $f(z)$.

2.8.2 Laurent series

It was earlier shown, that the domain of convergence of the power series

$$C_0 + C_1(z-z_0) + C_2(z-z_0)^2 + \ldots + C_n(z-z_0)^n + \ldots = \sum_{n=0}^{\infty} C_n(z-z_0)^n$$

is a circle: $|z - z_0| < R$.

Consider series

$$\frac{C_1}{z-z_0} + \frac{C_2}{(z-z_0)^2} + \ldots + \frac{C_n}{(z-z_0)^n} + \ldots = \sum_{n=1}^{\infty} C_n (z-z_0)^{-n} . \quad (16)$$

Make substitution $\frac{1}{z-z_0} = \zeta$ and get series:

$$C_1\zeta + C_2\zeta^2 + \ldots + C_n\zeta^n + \ldots = \sum_{n=1}^{\infty} C_n\zeta^n .$$

That power series converge in the circle $|\zeta| < \rho$. Coming from ζ to the variable z, we get: $\frac{1}{|z-z_0|} < \rho$, $|z-z_0| > \frac{1}{\rho}$, e.g. the domain of convergence of the series (16) is exterior of the circle with the center at the point z_0 and with the radius $z = \frac{1}{\rho}$.

Consider series, infinite in both sides

$$\ldots + \frac{C_{-n}}{(z-z_0)^n} + \ldots + \frac{C_{-1}}{(z-z_0)^1} + C_0 + C_1(z-z_0) + \ldots + C_n(z-z_0)^n + \ldots = \sum_{n=-\infty}^{\infty} C_n(z-z_0)^n . \quad (17)$$

That series are convergent, if simultaneously converge series:

$$\sum_{n=0}^{\infty} C_n(z-z_0)^n \text{ and } \sum_{n=-\infty}^{-1} C_n(z-z_0)^n .$$

The domain of convergence of the first series is a circle with the radius R, with the center at the point z_0. The domain of convergence of the second series is exterior of any circle with the radius r, with the center at the point z_0. If $0 < r < R$ (figure 2.10), then their shared part, the ring, is the domain of convergence of the series (17).

A set of the points z, that are in the ring, satisfies condition

$$r < |z - z_0| < R.$$

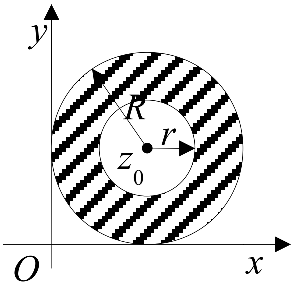


Figure 2.10

If $r > R$, then series (17) do not have points of convergence. If $R = r$, then series (17) can have points of convergence only on the circle $r = R$.

Sum of the series (17) is analytic function in the ring of convergence.

Converce proposition is true, it is called Laurent theorem.

Laurent theorem. Any function, analytical inside of the ring $0 < r < |z - z_0| < R$ with the center at the point z_0, can be decomposed in series inside of that ring

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n,$$

coefficients can be found from the formula $C_n = \frac{1}{2\pi i} \oint_L \frac{f(z)}{(z - z_0)^{n+1}} dz$, $n = 0; \pm 1; \pm 2; \ldots$, and L is any circle with the center at the point z_0, that is inside of the given ring (without proof).

That series are called Laurent series for the function $f(z)$ in the ring under consideration.

2.9 Isolated special points

Points of the plane, in which function $f(z)$ is analytical, are called true points of that function, and points, in which function $f(z)$ is not analytical, particularly, points, in which function $f(z)$ is not defined, are called special points. Special point is called isolated, if there are no more special points in any neighborhood of that point.

If z_0 is isolated special point of the function $f(z)$, then in small enough circle with the pricked center z_0, that is the ring with the internal radius, equal to zero, a function $f(z)$ is going to be analytical and can be decomposed in Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n = \sum_{n=0}^{\infty} C_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{C_{-n}}{(z - z_0)^n}. \quad (18)$$

Series $\sum_{n=0}^{\infty} C_n (z - z_0)^n$ are called true part, and series $\sum_{n=1}^{\infty} \frac{C_{-n}}{(z - z_0)^n}$ are called the main part of decomposition of the function $f(z)$ (18).

Three cases are possible:

1. The main part is missed in the decomposition (18).

In that case point z_0 is called removable special point (RSP).

$$f(z)=C_0+C_1(z-z_0)+\ldots+C_n(z-z_0)^n+\ldots,$$
$$\lim_{z\to z_0} f(z)=C_0.$$

So, if redefine function $f(z)$ in the point z_0, setting $f(z_0)=C_0$, then point z_0 becomes true point.

Example 1. Find special points of the function $f(z)=\frac{\sin z}{z}$ and define their type.

Solution. $z=0$ is a special point. Decomposition of the function $f(z)=\sin z$ in series: $\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ldots$, then $\frac{\sin z}{z}=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots$ and $\lim_{z\to 0}\frac{\sin z}{z}=1$.

Setting $f(0)=1$, uncertainty $\frac{0}{0}$ can be removed. Therefore, $z=0$ is RSP.

Answer: $z=0$ is RSP.

If z_0 is removable special point (RSP) of the function $f(z)$, then the finite limit of the function $\lim_{z\to z_0} f(z)$ exists.

2. The main part contains the finite number of terms.

The point z_0 is called the pole of the *k*th order (P*к*O), if

$$f(z)=\sum_{n=0}^{\infty} C_n(z-z_0)^n+\frac{C_{-1}}{(z-z_0)}+\ldots+\frac{C_{-k}}{(z-z_0)^k},$$ where k is the pole order.

Example 2. Find special points of the function $f(z)=\frac{\cos z}{z}$ and define their type.

Solution. $z=0$ is a special point. Decomposition of the function $f(z)=\cos z$ in series: $\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4}-\ldots$, then $\frac{\cos z}{z}=\frac{1}{z}-\frac{z}{2!}+\frac{z^3}{4!}-\ldots$

Therefore, $z=0$ is a pole of the first order (P1O).

Answer: $z=0$ is the P1O.

If z_0 is a pole (P), then the limit $\lim_{z\to z_0} f(z)=\infty$. The pole of the first order is called a simple pole. A point z_0 is the pole of the *к*th order of the function $f(z)$, if $\lim_{z\to z_0} f(z)(z-z_0)^k=C\neq 0$.

3. The main part contains infinite number of terms.

A point z_0 is called considerablyg special point (CSP).

$$f(z)=\sum_{n=-\infty}^{\infty} C_n(z-z_0)^n.$$

Example 3. Find special points of the function $f(z)=e^{\frac{1}{z}}$ and define their type.

Solution. $z=0$ is a special point. Decomposition of the function in series: $e^{\frac{1}{z}}=1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\ldots$

Therefore, $z=0$ is a considerably special point (CSP).

Answer: $z=0$ is a CSP.

If z_0 is a considerably special point (CSP), then the limit $\lim\limits_{z\to z_0} f(z)$ does not exist.

2.10 Residues. The main theorem about residues

Let z_0 be isolated special point of the function $f(z)$. In the neighborhood of that point, the function $f(z)$ can be decomposed in Laurent series:

$$f(z)=\sum_{n=0}^{\infty} C_n (z-z_0)^n+\frac{C_{-1}}{z-z_0}+\frac{C_{-2}}{(z-z_0)^2}+\ldots,$$

coefficients can be found from the formula:

$$C_n=\frac{1}{2\pi i}\oint_{\tilde{A}}\frac{f(z)}{(z-z_0)^{n+1}}dz,\ n=0;\pm 1;\pm 2;\ldots$$

Definition. Coefficients by the $(z-z_0)^{-1}$ in Laurent decomposition, e.g. the number C_{-1} is called the residue of the function $f(z)$ with respect to the special point z_0 and denotes as $C_{-1}=\operatorname*{Res}\limits_{z=z_0} f(z)$ or $C_{-1}=\operatorname*{Âû÷}\limits_{z_0} f(z)$.

From the formula for the coefficients of Laurent series follows, that

$$C_{-1}=\operatorname*{Res}_{z=z_0} f(z)=\frac{1}{2\pi i}\oint_{\gamma} f(z)dz.$$

The main theorem about residues. If the function $f(z)$ is analytical inside and on the closed loop $\tilde{A}$, except the finite number of the points $z_1, z_2,\ldots,z_n$ inside $\tilde{A}$, that are poles, then $\oint_{\tilde{A}} f(z)dz$ equals to the product of $2\pi i$ by the sum of the residues with respect to the special points of the function $f(z)$, that are inside of the domain, limited by the loop $\tilde{A}$.

$$\oint_{\tilde{A}} f(z)dz=2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z). \qquad (19)$$

Proof.

Special points, that are inside the domain, limited by the loop $\tilde{A}$, separate by the neighborhoods $\gamma_1, \gamma_2,\ldots, \gamma_n$ of so small radius, that they lie inside the domain, limited by the loop $\tilde{A}$ and do not cross (figure 2.11).

From Cauchy theorem for multiply connected regions, we have:

$$\oint_{\tilde{A}} f(z)dz=\sum_{k=1}^{n}\oint_{\gamma_k} f(z)dz. \qquad (20)$$

By the definition of residues:

$$\oint_{\gamma_k} f(z)dz=2\pi i\operatorname*{Res}_{z=z_k} f(z). \qquad (21).$$

Substituting (21) in (20), we get formula (19).

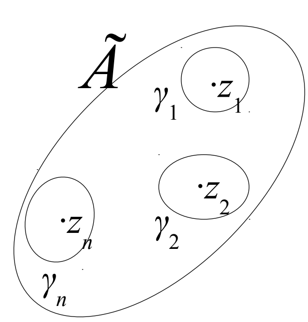


Figure 2.11

Therefore, to calculate integral, it is necessary to know the formula for calculating the residues.

Calculation of residues

1. If the function $f(z)$ is analytical at the point z_0 or z_0, a removable special point (RSP) of the function $f(z)$, then $\underset{z=z_0}{\text{Res}} f(z) = 0$.

2. Let z_0 be a simple pole of the function $f(z)$, then

$$f(z) = \sum_{n=0}^{\infty} C_n (z - z_0)^n + \frac{C_{-1}}{z - z_0}.$$

Multiplying term by term from the right and from the left by the $(z - z_0)$, we get:

$$f(z)(z - z_0) = \sum_{n=0}^{\infty} C_n (z - z_0)^{n+1} + C_{-1}.$$

Coming to the limit when $z \to z_0$, then $C_{-1} = \lim_{z \to z_0} f(z)(z - z_0)$.

So, if z_0 is a simple pole of the function $f(z)$, then

$$\underset{z=z_0}{\text{Res}} f(z) = \lim_{z \to z_0} \left[f(z)(z - z_0) \right].$$

If the function $f(z)$ can be presented as $f(z) = \dfrac{\varphi(z)}{\psi(z)}$, where the functions $\varphi(z)$, $\psi(z)$ are analytic at the point z_0, $\psi(z_0) = 0$, $\psi'(z_0) \neq 0$, then

$$\underset{z=z_0}{\text{Res}} f(z) = \frac{\varphi(z_0)}{\psi'(z_0)}.$$

If the point z_0 is the pole of the *k*th order of the function $f(z)$, then

$$\underset{z=z_0}{\text{Res}} f(z) = \frac{1}{(k-1)!} \lim_{z \to z_0} \left[f(z)(z - z_0)^k \right]^{(k-1)}$$

or

$$\underset{z=z_0}{\text{Res}} f(z) = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}\left[f(z)(z - z_0)^k \right]}{dz^{k-1}}.$$

3. If z_0 is a considerably special point (CSP) of the function $f(z)$, then its residue at the given point can be found from the decomposition of the function $f(z)$ in Laurent series $\operatorname*{Res}_{z=z_0} f(z) = C_{-1}$.

Example. Calculate the integral $\oint\limits_{\Gamma} \frac{dz}{(z-1)^2(z^2+1)}$, where $\Gamma : |z-1-i| = 2$.

Solution.

Special points of the integral function are: $z_{1,2} = 1$, $z_3 = -i$, $z_4 = i$.

Points $z_{1,2} = 1$, $z_4 = i$ can be found in the domain, limited by the loop $\Gamma : |z-1-i| = 2$ (figure 2.12).

Define the type of the special points:

1) $z_{1,2} = 1$. $\lim\limits_{z\to 1} \frac{1}{(z-1)^2(z^2+1)} = \infty$.

So, $z_{1,2} = 1$ is a pole. Find the order of the pole:

$$\lim_{z\to 1} \frac{(z-1)^2}{(z-1)^2(z^2+1)} = \lim_{z\to 1} \frac{1}{z^2+1} = \frac{1}{2} \neq 0.$$

Then $z_{1,2}=1$ is the pole of the second order (P2P).

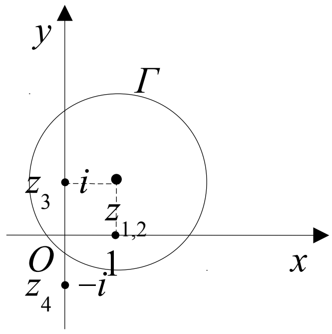


Figure 2.12

2) $z_4 = i$. $\lim\limits_{z\to i} \frac{1}{(z-1)^2(z^2+1)} = \lim\limits_{z\to i} \frac{1}{(z-1)^2(z+i)(z-i)} = \infty$. $z_4 = i$ is a pole. The order of the pole is: $\lim\limits_{z\to i} \frac{(z-i)}{(z-1)^2(z+i)(z-i)} = \lim\limits_{z\to i} \frac{1}{(z-1)^2(z+i)} = \frac{1}{4}$. So, $z_4 = i$ is the simple pole (SP).

Knowing the types of the special points, find the residues of the function at the given points:

1) because $z_{1,2} = 1$ is P2P, then $\operatorname*{Res}\limits_{z=1} f(z) = \lim\limits_{z\to 1} \left[\frac{(z-1)^2}{(z-1)^2(z^2+1)} \right]' = \lim\limits_{z\to 1} \frac{-2z}{(z^2+1)^2} = -\frac{1}{2}$;

2) $z_4 = i$ is SP, then $\operatorname*{Res}\limits_{z=i} f(z) = \lim\limits_{z\to i} \frac{(z-i)}{(z-1)^2(z+i)(z-i)} = \lim\limits_{z\to i} \frac{1}{(z-1)^2(z+i)} = \frac{1}{4}$.

Then $\oint\limits_{|z-1-i|=2} \frac{dz}{(z-1)^2(z^2+1)} = 2\pi i \left(-\frac{1}{2} + \frac{1}{4} \right) = -\frac{\pi i}{2}$.

For $z_4 = i$ (SP), the function $\frac{1}{(z-1)^2(z^2+1)}$ can be presented as $\frac{\frac{1}{(z-1)^2(z+i)}}{z-i}$, then $\operatorname*{Res}\limits_{z=i} f(z) = \lim\limits_{z\to i} \frac{\frac{1}{(z-1)^2(z+i)}}{(z-i)'} = \lim\limits_{z\to i} \frac{1}{(z-1)^2(z+i)} = \frac{1}{4}$.

Answer: $-\frac{\pi i}{2}$.

2.11 Calculation of the improper integrals

Improper integrals with the real variable can be found using residues:

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z), \tag{22}$$

where $z = z_k$ are isolated special points of the function $f(z)$, that are above the axes x, and the function $f(z)$ satisfies to the condition: $|z| \times f(z) \to 0$ when $|z| \to \infty$.

If $z = z_i$ are isolated special points of the function $f(z)$, that are on the axes x and $|z| \times f(z) \to 0$ when $|z| \to \infty$, then

$$\int_{-\infty}^{\infty} f(x)dx = \pi i \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z). \tag{23}$$

Example 1. Calculate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$.

Solution. Write the function $f(z) = \frac{1}{1+z^2}$. Because $\frac{z}{1+z^2} \to 0$ when $|z| \to \infty$, then we can use above mentioned formulas.

$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \oint_C \frac{dz}{1+z^2} = \oint_C \frac{dz}{(z-i)(z+i)}$, C is a boundary of the semicircle with the enough big radius that contains all of the special points of the half plane

Special points of the function: $z_1 = i, z_2 = -i$.

There is a point $z_1 = i$ above the axes x, that is the simple pole of the function $f(z) = \frac{1}{(z-i)(z+i)}$, then

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z \to i} \frac{(z-i)}{(z-i)(z+i)} = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}.$$

Because special point is above axes x, then, using formula (22), we get $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = 2\pi i \frac{1}{2i} = \pi$.

Answer: π.

Example 2. Calculate $\int_{-\infty}^{\infty} \frac{dx}{\left(1+x^2\right)^2}$.

Solution. Work in the function $f(z) = \frac{1}{\left(1+z^2\right)^2}$. Because $\frac{z}{\left(1+z^2\right)^2} \to 0$ when $|z| \to \infty$, we can use above mentioned formulas.

$\int_{-\infty}^{\infty} \frac{dx}{\left(1+x^2\right)^2} = \oint_C \frac{dz}{\left(1+z^2\right)^2} = \oint_C \frac{dz}{(z-i)^2(z+i)^2}$, C is a boundary of the semicircle with the enough big radius, that contains all of the special points of the half plane.

Special points of the function are: $z_1 = i, z_2 = -i$.

Above the axes x, there is the point $z_1 = i$, that is the pole of the second order of the function $f(z) = \frac{1}{(z-i)^2(z+i)^2}$. Then,

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i}\left[\frac{(z-i)^2}{(z-i)^2(z+i)^2}\right]' = \lim_{z\to i}\left[\frac{1}{(z+i)^2}\right]' = \lim_{z\to i}\left[\frac{-2}{(z+i)^3}\right] = -\frac{2}{8i^3} = -\frac{i}{4}.$$

According to the formula (22), we have $\int_{-\infty}^{+\infty}\frac{dx}{\left(1+x^2\right)^2} = 2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2}$.

Answer: $\frac{\pi}{2}$.

Example 3. Calculate $\int_{-\infty}^{\infty}\frac{dx}{\left(x^2+1\right)\left(x-1\right)}$.

Solution. Work in the function $f(z) = \frac{1}{\left(z^2+1\right)\left(z-1\right)}$. $\frac{z}{\left(z^2+1\right)\left(z-1\right)} \to 0$ when $|z| \to \infty$, we can use formulas (22) and (23).

$\int_{-\infty}^{\infty}\frac{dx}{\left(x^2+1\right)\left(x-1\right)} = \oint_C \frac{dz}{\left(z^2+1\right)\left(z-1\right)} = \oint_C \frac{dz}{(z-i)(z+i)\left(z-1\right)}$, C is a boundary of the semicircle with the enough big radius, that contains all of the special points of the half plane.

Special points of the function are: $z_1 = i,\ z_2 = -i,\ z_3 = 1$.

Above the axes x, there is the point $z_1 = i$ (P1O), on the axes x, there is a point $z_3 = 1$ (P1O) of the function $f(z) = \frac{1}{(z-i)(z+i)\left(z-1\right)}$. Then,

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i}\frac{(z-i)}{(z-i)(z+i)\left(z-1\right)} = \frac{1}{-2-2i} = \frac{-1+i}{4};$$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}\frac{(z-1)}{(z^2+1)\left(z-1\right)} = \frac{1}{2}.$$

Using formulas (22) and (23), we get

$$\int_{-\infty}^{+\infty}\frac{dx}{\left(x^2+1\right)\left(x-1\right)} = 2\pi i\left(\frac{-1+i}{4}\right) + \pi i\left(\frac{1}{2}\right) = -\frac{\pi}{2}.$$

Answer: $-\frac{\pi}{2}$.

2.12 Tasks for personal work

2.12.1 Upbuild the domain G in the complex plane (geometric sense).

1.01. $G: |z+1-zi| = 3$.

1.02. $G: |z-1-i| = 2$.

1.03. $G: |z+1-i| = 1$.

1.10. $G: |z-1+i| < 2$.

1.11. $G: \operatorname{Re} z > 2$.

1.12. $G: \operatorname{Im} z \le -1$.

1.04. $G: \operatorname{Re} z - \operatorname{Im} z = 2$.

1.05. $G: 0 \le \arg z \le \pi/4$.

1.06. $G: |z-1| + |z+1| = 3$.

1.07. $G: |z+2| - |z-1| = 3$.

1.08. $G: \operatorname{Re}\frac{1}{z} = 2$.

1.09. $G: \left|\frac{z-1}{z+1}\right| = 1$.

1.13. $G: 0 < \operatorname{Re} z < 2$.

1.14. $G: 0 \le Imz \le 3$.

1.15. $G: |z| + \operatorname{Re} z \le 1$.

1.16. $G: 0 < \arg z < \pi/4$.

1.17. $G: -\frac{\pi}{4} \le \arg z \le 0$.

1.18. $G: |z-2| + |z+2| = 5$.

1.19. $G: 0 < \operatorname{Re}(iz) < 1$.

1.20. $G: |z| = 1 + \operatorname{Re} z$.

1.21. $G: \operatorname{Re} z + \operatorname{Im} z < 1$.

1.22. $G: |2z| > |1+z|$.

1.23. $G: |z-2| - |z+2| = 3$.

1.24. $G: |z-1-2i| = 2$.

1.25. $G: |2z+1-2i| = 2$.

1.26. $G: |2z-1-i| = 1$.

1.27. $G: |2z+1-i| = 2$.

1.28. $G: \operatorname{Re}(2z) - \operatorname{Im}(2z) = 1$.

1.29. $G: \arg z = \pi/4, \operatorname{Re} z \le 2$.

1.30. $G: \operatorname{Re}(z) - \operatorname{Im}(z) = 1$.

2.12.2 Find out, if the function is analytic: if yes, find its derivative at the given point.

2.01. $\omega = (\operatorname{Re} z)^2 \cdot e^z$, $z_0 = -1$.

2.02. $\omega = \operatorname{Re} z \cdot \operatorname{Im} z \cdot \sin z$, $z_0 = 0$.

2.03. $\omega = \operatorname{Im} z \cdot \cos z$, $z_0 = \pi/3$.

2.04. $\omega = \operatorname{Re} z \cdot e^z$, $z_0 = 0$.

2.05. $\omega = z - \ln z$, $z_0 = 1$.

2.06. $\omega = 9z - ie^z$, $z_0 = i$.

2.07. $\omega = 8z^2 - \sin z$, $z_0 = 1 - i$.

2.08. $\omega = 2z^2 - i(z+1)$, $z_0 = -i$.

2.09. $\omega = (iz)^4 - i$, $z_0 = i$.

2.10. $\omega = ze^z + ie^z$, $z_0 = i$.

2.11. $\omega = (z - \bar{z})\cos z$, $z_0 = 0$.

2.12. $\omega = z \cdot \bar{z} \cdot e^z$, $z_0 = 1$.

2.13. $\omega = z \cdot \cos \bar{z}$, $z_0 = \pi/3$.

2.14. $\omega = \sin(i+z) + z^2 + 1$, $z_0 = -1$.

2.15. $\omega = i(1-z) + e^z$, $z_0 = -i$

2.16. $\omega = (z-1)^2 + z + i + 1$, $z_0 = 0$.

2.17. $\omega = (z-1) \cdot e^{z+i}$, $z_0 = 1 - i$.

2.18. $\omega = (1-z)\sin z$, $z_0 = \pi/4$.

2.19. $\omega = e^{-iz^2}$, $z_0 = i$.

2.20. $\omega = i + e^{-iz}$, $z_0 = 1 - i$.

2.21. $\omega = z - e^z$, $z_0 = -1 - i$.

2.22. $\omega = z + \sin z$,. $z_0 = 2i - 1$

2.23. $\omega = (i+1) + e^z$, $z_0 = 2i$.

2.24. $\omega = i\sin(z+1)$, $z_0 = i$.

2.25. $\omega = \frac{1}{i}\cos(z-i)$, $z_0 = i + 1$.

2.26. $\omega = \left(\operatorname{Re}\frac{z}{3}\right)^2 \cdot e^{\frac{z}{3}}$, $z_0 = 2i$.

2.27. $\omega = \operatorname{Re}\left(\frac{z}{2}\right)\operatorname{Im}\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right)$, $z_0 = \pi$.

2.28. $\omega = \operatorname{Im}\left(\frac{z}{2}\right) \cdot \sin\left(\frac{z}{2}\right)$, $z_0 = \pi/6$.

2.29. $\omega = \operatorname{Re}\left(\frac{z}{2}\right) \cdot e^{\frac{z}{2}}$, $z_0 = 1$.

2.30. $\omega = 2z - \ln(2z)$, $z_0 = 2$.

2.12.3 Calculate the integral over the given curve.

3.01. $\int_{AB} \bar{z}dz$, AB: is a line segment, connecting points $A = 0$ and $B = 3 + 2i$.

3.02. $\int_{AB} \bar{z}dz$, AB: is a semicircle $|z| = 1$, $0 \le \varphi \le \pi$.

3.03. $\int_{AB} \operatorname{Im}\bar{z}dz$, AB: is a radius vector of the point $z = 1 + i$.

3.04. $\int_{AB} \frac{z}{\bar{z}}dz$, AB: is a line segment, connecting points $A = 0$ and $B = 1 + i$.

3.05. $\int_{AB} \frac{z}{\bar{z}}dz$, AB: is an arch of the parabola $y = x^2$, connecting points $A = 0$ and $B = 1 + i$.

3.06. $\int_{AB} \frac{z}{\bar{z}}dz$, AB: is an arch $y = \sqrt{x}$, connecting points $A = 1 + i$ and $B = 0$.

3.07. $\int_{AB} z^3dz$, AB: is a line segment, connecting points $A = 2 + 4i$ and $B = 0$.

3.08. $\int_{AB} z^3dz$, AB: is an arch $y = x^2$, connecting points $A = 0$ and $B = 2 + 4i$.

3.09. $\int_{AB} z^3dz$, AB: is an arch $y = 2\sqrt{2x}$, connecting points $A = 0$ and $B = 2 + 4i$.

3.10. $\int_{AB} z^3dz$, AB: is a line segment, connecting points $A = i$ and $B = 1$.

3.11. $\oint_{\Gamma} z^3dz$, Γ: is a circle $|z| = 2$.

3.12. $\int_{AB} \operatorname{Re} zdz$, AB: is a radius vector of the point $z = 2 + i$.

3.13. $\int_{AB} \operatorname{Im} zdz$, AB: is a radius vector of the point $z = 2 - i$.

3.14. $\int_{AB} xdz$, AB: is a semicircle $|z| = 1$, $0 \le \varphi \le \pi$.

3.15. $\int_{AB} ydz$, AB: is a part of the circle $|z| = 2$, $0 \le \varphi \le \pi/2$.

3.16. $\int_{AB} |z|dz$, AB: is a radius vector of the point $(-1 - i)$.

3.17. $\int_{AB} |z|dz$, AB: is a semicircle $|z| = 2$, $0 \le \varphi \le \pi$.

3.18. $\int_{AB} |z|\bar{z}dz$, AB: is a part of the circle $|z| = 2$, $\pi/2 \le \varphi \le \pi$.

3.19. $\int_{AB} |z|zdz$, AB: is a semicircle $|z| = 3$, $0 \le \varphi \le \pi$.

3.20. $\int\limits_{AB} \frac{dz}{\sqrt{z}}$, AB: is a part of the circle $|z| = 1, \quad 0 \le \varphi \le \pi/2$.

3.21. Ñ$_{Ã}(z^2 - 5)dz$, $Ã$: is a circle $|z - 5| = 1$.

3.22. Ñ$_{Ã}(\bar{z} + 1)dz$, $Ã$: is a circle $|z| = 1$.

3.23. Ñ$_{Ã}e^{z}dz$, $Ã$: is a circle $|z| = 1$.

3.24. $\int\limits_{AB} (\bar{z} + 1)dz$, AB: is a line segment, connecting points $A = 1 + i$ and $B = 2 - i$.

3.25. $\int\limits_{AB} (\bar{z} + 1)dz$, AB: is a semicircle $|z| = 2$, $0 \le \varphi \le \pi$.

3.26. Ñ$_{Ã}(5i + 4)(1 + \sin z)dz$, $Ã$: is a circle $|z - 1| = 1$.

3.27. Ñ$_{Ã}(5i - \cos z)dz$, $Ã$: is a circle $|z - i| = 2$.

3.28. $\int\limits_{AB} e^{z}(i + 1)dz$, AB: is a line segment, connecting points $A = -i$ and $B = -1$.

3.29. $\int\limits_{AB} \frac{(z+1)^2}{\bar{z}}dz$, AB: is a line segment, connecting points $A = -i$ and $B = 1 + i$.

3.30. $\int\limits_{AB} \frac{(z+1)^2}{\bar{z}}dz$, AB: is a line segment, connecting points $A = i$ and $B = 1 - i$.

2.12.4 Decompose function in Laurent series in the neighborhood of the given function (or in the ring).

4.01. $\varphi = \frac{1}{z-2}$, $z_0 = 0$.

4.02. $\varphi = \frac{1}{z+1}$, $z_0 = 1$.

4.03. $\varphi = \frac{1}{(z-1)(z-2)}$, $z_0 = 0$.

4.04. $\varphi = \frac{1}{(z-1)(z-3)}$, $z_0 = 1$.

4.05. $\varphi = \frac{1}{(z-2)(z-4)}$, $2 < |z| < 4$.

4.06. $\varphi = \frac{1}{(z-i)(z-1)}$, $z_0 = 1$.

4.07. $\varphi = \frac{1}{z^2+1}$, $z_0 = 0$.

4.12. $\varphi = \frac{z^2 - 2z + 5}{(z-2)(z^2+1)}$, $z_0 = 2$.

4.13. $\varphi = \frac{z^2 - 2z + 5}{(z-2)(z^2+1)}$, $1 < |z| < 2$.

4.14. $\varphi = \frac{z}{1+z^2}$, $z_0 = i$.

4.15. $\varphi = \frac{z}{1+z^2}$, $z_0 = -i$

4.16. $\varphi = \frac{z}{(z-1)(z-i)}$, $z_0 = 1$.

4.17. $\varphi = \frac{2z}{(z-1)(z+i)}$, $z_0 = -i$.

4.18. $\varphi = \frac{3z}{(z+1)(z-1)}$, $z_0 = 1$.

4.08. $\varphi = \dfrac{1}{z^2+1}$, $z_0 = i$

4.09. $\varphi = \dfrac{3}{z^2+1}$, $z_0 = i$.

4.10. $\varphi = \dfrac{4}{z^2+2}$, $z_0 = \sqrt{2}i$.

4.11. $\varphi = \dfrac{1}{(z-i)(z-2)}$, $1<|z|<2$.

4.19. $\varphi = z^3\exp\dfrac{1}{z}$, $z_0 = 0$.

4.20. $\varphi = \cos\dfrac{z}{z-2}$, $z_0 = 2$.

4.21. $\varphi = \sin\dfrac{z}{z-i}$, $z_0 = i$.

4.22. $\varphi = (z-1)\sin\dfrac{z}{1-z}$, $z_0 = 1$.

4.23. $\varphi = e^{\frac{1}{1-z}}$, $z_0 = 1$.

4.24. $\varphi = z^2\exp\dfrac{1}{z}$, $z_0 = 0$.

4.25. $\varphi = \dfrac{5}{(z-1)(z-5)}$, $1<|z|<5$.

4.26. $\varphi = \dfrac{1}{2z-1}$, $z_0 = 1$.

4.27. $\varphi = \dfrac{1}{2z+i}$, $z_0 = 0$.

4.28. $\varphi = \dfrac{1}{(2z-1)(z-1)}$, $z_0 = 1$.

4.29. $\varphi = \dfrac{1}{(3z-1)(z-1)}$, $z_0 = 1$.

4.30. $\varphi = \dfrac{1}{(z-1)(3z-2)}$, $z_0 = 1$.

2.12.5 Calculate the integral of the function with the complex variable, using the theorem about residues and using the Cauchy formula.

5.01. $\oint_\Gamma \dfrac{dz}{z^2+1}$, $\Gamma: |z-i| = 1$.

5.02. $\oint_\Gamma \dfrac{dz}{z^2-1}$, $\Gamma: |z| = 2$.

5.03. $\oint_\Gamma \dfrac{dz}{(z-1)^2(z+2)}$, $\Gamma: |z| = 3$.

5.04. $\oint_\Gamma \dfrac{dz}{z^4-1}$, $\Gamma: |z| = 2$.

5.05. $\oint_\Gamma \sin\dfrac{1}{z}dz$, $\Gamma: |z| = 1$.

5.06. $\oint_\Gamma \sin^2\dfrac{1}{z}dz$, $\Gamma: |z| = 1$.

5.07. $\oint_\Gamma e^{\frac{1}{z}}dz$, $\Gamma: |z| = 1$.

5.08. $\oint_\Gamma z^2 e^{\frac{3}{z}}dz$, $\Gamma: |z| = 1$.

5.12. $\oint_\Gamma \dfrac{dz}{z^4+1}$, $\Gamma: x^2+y^2 = 2x$.

5.13. $\oint_\Gamma \dfrac{dz}{z^4+1}$, $\Gamma: |z-2| = \dfrac{1}{2}$.

5.14. $\oint_\Gamma \dfrac{z^3dz}{2z^4+1}$, $\Gamma: |z| = 1$.

5.15. $\oint_\Gamma \dfrac{e^z}{z^2(z^2-9)}dz$, $\Gamma: |z| = 1$.

5.16. $\oint_\Gamma \dfrac{\sin\frac{1}{z}}{2\pi i}dz$, $\Gamma: |z| = 2$.

5.17. $\oint_\Gamma \dfrac{1}{2\pi i}\sin^2\dfrac{1}{z}dz$, $\Gamma: |z| = 3$.

5.18. $\oint_\Gamma \dfrac{dz}{(z-3)(z^2-1)}$, $\Gamma: |z| = 2$.

5.19. $\oint_\Gamma \dfrac{1}{2\pi i}\cdot z^2\cdot e^{\frac{2}{z}}dz$, $\Gamma: |z| = 2$.

5.09. $\oint\limits_{\Gamma} \cos\frac{1}{z}\exp\frac{2}{z}dz$, $\quad \Gamma: |z| = 1$.

5.10. $\oint\limits_{\Gamma} \frac{\sin\frac{2}{z}}{\exp\frac{1}{z}}dz$, $\quad \Gamma: |z| = 1$.

5.11. $\oint\limits_{\Gamma} \frac{(z-1)\sin\frac{2}{z}}{z^2+1}dz$, $\quad \Gamma: |z| = 2$.

5.20. $\oint\limits_{\Gamma} (1+z)e^{\frac{1}{z-1}}dz$, $\quad \Gamma: |z| = 3$.

5.21. $\oint\limits_{\Gamma} (1+z)^2 e^{\frac{1}{z}}dz$, $\quad \Gamma: |z| = 3$.

5.22. $\oint\limits_{\Gamma} (1+z+z^2)e^{\frac{1}{z-2}}dz$, $\quad \Gamma: |z| = 3$.

5.23. $\oint\limits_{\Gamma} \frac{zdz}{(z-2)(z+i)}$, $\quad \Gamma: |z-3| = 33$.

5.24. $\oint\limits_{\Gamma} \frac{z+1}{z^2+1}$, $\quad \Gamma: |z-4| = 44$.

5.25. $\oint\limits_{\Gamma} \frac{z+1}{z^2(z-1)}$, $\quad \Gamma: |z-0{,}1| = 0{,}2$.

5.26. $\oint\limits_{\Gamma} \frac{dz}{2z^2+3}$, $\quad \Gamma: |z+i| = 1$.

5.27. $\oint\limits_{\Gamma} \frac{dz}{2z^2-3}$, $\quad \Gamma: |z-1| = 3$.

5.28. $\oint\limits_{\Gamma} \frac{2zdz}{(z-2)^2(2z+i)}$, $\quad \Gamma: |z| = 4$.

5.29. $\oint\limits_{\Gamma} \frac{dz}{3z^4-1}$, $\quad \Gamma: |z-1| = 3$.

5.30. $\oint\limits_{\Gamma} \sin\frac{z-1}{z}dz$, $\quad \Gamma: |z-i| = 0{,}5$.

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