Wiley Finance Series

The Advanced Fixed Income and Derivatives Management Guide

SAIED SIMOZAR

WILEY

Although TSIR (term structure of interest rates) influences not only the valuation of securities, but also most investment decisions and even monetary policy, what many investors still lack is a thorough and consistent method for systematically managing all their global fixed income assets based on TSIR. For one thing, many theoretical models of TSIR are not accurate enough. *The Advanced Fixed Income and Derivatives Management Guide* offers a new solution.

The author has created a completely novel framework for evaluating global fixed income investments based on a stable TSIR. In this packed guide, he provides over 700 equations and pages of explanation to give you the most detailed analysis of many fixed income instruments and sectors including inflation linked and corporate securities and bond option currently available.

The book is a practical reference guide and astonishingly detailed. The author implements his methodology to analyze valuation, risk measurement, performance attribution, security selection and portfolio construction across all sectors and their respective derivatives. You'll find numerous useful market based examples to help you find value and decompose the risks of bonds, mortgages, credits and currencies as well as their derivatives such as bond futures and options, callable bonds, credit default swaps, inflation swaps and swaptions.

There is coverage for the first ever estimation of recovery value from market prices and partial yields. Trading ideas, rich/cheap analysis and arbitrage opportunities are explored across all asset classes and their derivatives.

The book's comprehensive methodology can also be used for back office operations such as risk management and policy checking in a consistent way.

As an added bonus, many of the analytics are available on the book's dedicated and password protected website. You'll find worksheets and macros that you can download and use to measure risks and valuations based on the TSIR.

If you're an analyst, portfolio manager or trader seeking a more systematic and thorough way to manage even the most complex securities or investments, this book offers a practical new solution.

The Advanced Fixed Income and Derivatives Management Guide

SAIED SIMOZAR

WILEY

This edition first published 2015

Registered office
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

For details of our global editorial offices, for customer services and for information about how to apply for permission to reuse the copyright material in this book please see our website at www.wiley.com.

Library of Congress Cataloging-in-Publication Data is on file

ISBN 978-1-119-01414-0 (hardback) ISBN 978-1-119-01416-4 (ebk)

ISBN 978-1-119-01417-1 (ebk)

Cover Design: Wiley
Top Image: ©Shutterstock.com/bluebay
Bottom Image: ©Shutterstock.com/wongwean

Set in 10/12pt Times by SPi-Global, Chennai, India
Printed in Great Britain by TJ International Ltd, Padstow, Cornwall, UK

Contents

List of Tables

List of Figures

Abbreviations

CBF	Chebyshev basis function
CDS	Credit default swap
CFE	Cash flow engine
CSIA	Cumulative seasonal inflation adjustment
CTD	Cheapest to deliver
DUND	Drifted unit normal distribution
DV01	Dollar value of a basis point
EBF	Exponential basis function
EDFC	Eurodollar futures contract
EDTF	Exponentially decaying time function
IL	Inflation (indexed) linked
IRS	Interest rate swaps
ISDA	International Swaps and Derivatives Association
ISO	International Organization for Standardization
KBF	Key basis function
KRD	Key rate duration
KRS	Key rate security
LIBOR	London Inter-Bank Offered Rate
LP	Linear programming
MVBRR	Market value based recovery rate
OAS	Option adjusted spread
OBF	Orthogonal basis function
PBF	Polynomial basis function
PCA	Principal components analysis
PIK	Pay in kind
PSA	Prepayment speed assumption
RI	Refinancing incentive
STRIPS	Separate trading of registered interest and principal of securities
TIPS	Treasury inflation protected securities
TSD	Term structure duration
TSCR	Term structure of credit rates

TSCS	Term structure of credit spreads
TSDP	Term structure of default probability
TSIE	Term structure of inflation expectations
TSIR	Term structure of interest rates
TSKRD	Term structure based key rate duration
TSLR	Term structure of Libor rates
TSLV	Term structure of Libor volatility
TSRC	Term Structure of Real Credit
TSRR	Term structure of real rates
TSYV	Term structure of yield volatility
UND	Unit normal distribution
VBP	Value of a basis point
WAC	Weighted average coupon
WAM	Weighted average maturity

Notation

For notational convenience most variable names have been limited to a single character. Subscripts have been used to differentiate related variables. Subscripts i, j, and k have been used exclusively as running integers and are interchangeable. Other subscript letters are used to differentiate closely related names. For example, p_m and p_c are used for the market price and calculated price of a security, respectively. When these subscripts are mixed with running subscripts, a comma is inserted between them (e.g. $p_{m,i}$ or $p_{c,k}$).

SUBSCRIPTS

b	Bond specific – e.g., y_b is the yield of a bond
c	Constant – e.g., a constant or a fixed coupon rate Credit – e.g., $y_{t,c}$ is the credit yield calculated from the term structure of credit rates
e	Effective – e.g., y_e is the effective yield
f	Forward – e.g., y_f is the forward yield Floating – e.g., c_f is the floating coupon
g	Government or risk-free rate or simply interest rate
i	Usually, index of cash flows, e.g., t_i is the time to the ith cash flow of a bond
j	Usually, index number of a bond, e.g., $p_{t,j}$ is the term calculated price of bond j
k	kth component of the term structure or risk, e.g., ψ_k
l	Libor
m	Market – e.g. p_m is the market price
n	Inflation $a_{n,i}$ is the ith component of the term structure of inflation rates t_{in} time to the inflation reference point of cash flow i. $y_{r,in}$ real yield of cash flow i at its inflation reference point.
p	Principal – e.g., c_p is the principal cash flow of a bond
r	Real – e.g., y_r is the real yield of a bond; $y_{t,r}$ is the term structure real yield
s	Spot – e.g., y_s is the spot yield
t	Term structure – e.g., y_t is the term structure yield
v	Volatility related – $\psi_{v,k}$ is the kth component of volatility risk

VARIABLE NAMES

a Term structure component
- a_i ith component of the term structure of interest rates
- $a_{c,i}$ ith component of the term structure of credit rates
- $a_{g,i}$ ith component of the term structure of interest rates or government rates
- $a_{l,i}$ ith component of the term structure of Libor rates
- $a_{n,i}$ ith component of the term structure of inflation expectations
- $a_{r,i}$ ith component of the term structure of real rates

b_i ith component of the term structure of interest rates using key rate basis functions or the ith component of the term structure of yield volatility

c Cash flow or coupon
- $c_{c,i}$ ith fixed or constant cash flow of a bond
- $c_{e,i}$ ith effective cash flow of a bond
- $c_{f,i}$ ith forward or floating cash flow of a bond
- $c_{f,c,i}$ ith forward or floating cash flow of a credit security
- $c_{g,i}$ guaranteed cash flow of a bond
- c_i ith cash flow of a bond
- $c_{i,j}$ ith cash flow of bond j in a portfolio or index
- $c_{p,i}$ principal cash flow component of the ith cash flow of a bond
- $c_{r,i}$ recovery cash flow component of the ith cash flow of a bond

c_{ij} $|c_{ij}|$ conversion matrix elements for changing basis functions

d Discount function
- $d_{c,i}$ discount function for the ith cash flow of a credit bond
- d_i discount function for the ith cash flow of a bond

D Duration, distance
- D_c credit duration of a bond
- $D_{i,j}$ ith duration component of bond j in a portfolio or index
- D_k kth duration component of the term structure of interest rates
- D_m Macaulay duration of a bond
- D_v duration of volatility

Δy_k Change in yield due to the change in the kth component of the TSIR

e_{ij} Conversion matrix elements to convert from polynomial to key rate basis functions

$f(t)$ Instantaneous forward rate as a function of time

f_c Calculated forward rate as a function of time

f_s Market expected forward rate as a function of time

g_k Parameter representing the components of the term structure of interest rates or term structure of volatility

g_i ith component of cash flow guarantee

K_i Contribution to duration of the ith term structure in key rate basis functions

L Number of basis functions for the term structure of volatility

M Market value

n	Count or number of cash flows
N	Number of observations
N_B	Number of business days in a year
p	Price
	p_c calculated or model price based on the term structure
	$p_{c,i}$ calculated price of security i
	p_j price of security j
	Δp_{ij} change in price due to the change in the (i, j)th convexity
	Δp_k change in price due to the change in the kth component
	p_m market price plus accrued interest
	$p_{m,i}$ market price plus accrued interest for security i
	p_r price of a risky bond
	p_t term structure price
q_a	Contribution to performance due to factor a
Q	Recovery ratio of a defaulted bond as a fraction of its market price
r_c	Constant recovery rate of a defaulted bond as a fraction of its principal
r_i	Recovery rate for cash flow i
$r(t)$	Default rate per unit time at t
s	Spread
	s spread over the term structure of interest rate for a security
	$s(t)$ spot or credit spread as a function of time
	s_b spread of a bond or a security over its curve
	s_c calculated or implied spread or spot default probability
	$s_{d,i}$ spread of a credit (default-possible) security at ith cash flow
	$s_{l,i}$ Libor spread of at ith cash flow.
	s_s spot or market observed spread, adjusted for convexity
t	Time
	t_i time to ith cash flow
	t_{ij} time to ith cash flow of bond j in a portfolio or index
	t_m time to maturity
	t_{in} time to inflation reference point for the cash flow at time t_i
u_i	Face value weight of ith security in optimization for calculating the components of the TSIR
V	Velocity or speed; cash flow per unit of time
v	Volatility
	v_y relative yield volatility
	v_p price volatility
w	Absolute yield volatility; equal to relative yield volatility times yield
w_i	Weight of ith security
X	Overall convexity
X_{kl}	Cross-convexity of the kth and lth components of the term structure of rates

y Yield
- y_c credit yield
- $y_{c,i}$ credit yield at time t_i
- y_f forward; modifies all other yields to forwards
- $y_{f,c,i}$ forward credit at time t_i
- y_i yield at time t_i
- $y_{l,i}$ Libor yield at time t_i
- $y_{l,in}$ Libor yield at inflation reference point for cash flow at time t_i
- $y_{n,i}$ inflation yield at time t_i
- $y_{r,i}$ real yield at time t_i
- $y_{s,i}$ spot yield adjusted for convexity at time t_i
- $y_{s,c,i}$ spot credit yield at time t_i
- $y_{s,l,i}$ spot Libor yield at time t_i
- $y_{s,r,i}$ spot real yield at time t_i
- $y_{t,i}$ term structure (calculated) yield at time t_i
- $y_{t,c,i}$ term structure credit yield at time t_i
- y_x yield due to convexity
- $y(0)$ short term yield
- $y(\infty)$ long term yield

Z Optimization function
Z_i Derivative of the optimization function relative to the ith variable
Z_λ Derivative of the optimization function relative to λ
Z_{ij} Second derivative of the optimization function relative to the (i, j)th variables
$Z_{i\lambda}$ Second derivative of the optimization function relative to the ith variable and λ

α Decay coefficient
α_{cf} Decay coefficient estimated from cash flow
α_{dw} Decay coefficient estimated from duration weighting
α_{pv} Decay coefficient estimated from present value
β Market decay coefficient
Δ_i Optimization weight for calculating components of the TSIR
ε_v Absolute inflation volatility
z_{i} ith basis function for the term structure of volatility
η_i ith orthogonal basis function
λ Lagrange multiplier
μ Fraction of a floating rate payment for a floating rate coupon bond
ϖ Vega, price sensitivity relative to yield volatility
ϖ_s Vega, price sensitivity relative to spread volatility
$\rho(t)$ Survival probability of a risky bond by time t
ρ_{uv} Correlation coefficient between real rates and inflation expectations
σ_s Relative spread volatility
σ_u Relative real yield volatility
σ_v Relative inflation volatility

σ_y	Relative yield volatility
τ	Time unit in the EDTF
τ_m	Time to maturity in EDTF
ϕ_i	ith forward rate basis function
χ_i	ith KRD basis function
χ_{ik}	ith KRD basis function evaluated at the maturity of the kth key rate
ψ_i	ith basis function for the TSIR

This page intentionally left blank

Copyrighted Material

Preface

Fixed income management has become significantly more quantitative and competitive over the last 20 years or so, and the days where fund managers could make very large duration bets are mostly over. Most clients prefer portfolios with diversified sources of alpha and duration targets that are comparable to the risk profiles of their liabilities or their intended risk/return expectations. Developments of strategies that are quantifiable and repeatable are essential for the success of fixed income business.

Understanding the factors that contribute to risk and return are essential, in order to structure a sound portfolio. Risk management and return attribution require the quantification of sources of risk and return and thus are math intensive. A portfolio manager who is familiar with linear programming can structure an optimum portfolio based on analysts' recommendations, portfolios policies and guidelines as well as his own views of the markets that is likely to have a superior return than another portfolio of similar weights and risk profiles.

This book provides a comprehensive framework for the management of fixed income, both horizontally and vertically. It covers in detail all sectors of fixed income, including treasuries, mortgages, international bonds, swaps, inflation linked securities, credits and currencies and their respective derivatives. We develop a methodology for decomposing valuation metrics and risks into common components that can easily be understood and managed. Valuation, risk measurement and management, performance attribution, hedging and cheap/rich analysis are the natural byproducts of the framework.

Nearly all the concepts in the book were developed out of necessity over more than 20 years as a fund manager at DuPont Capital Management, Putnam Investments, Banc of America Capital Management and Nuveen Investments. Even though the book is rich in theory and mathematical derivations, the primary focus is alpha generation, understanding valuations and exploiting market opportunities.

The intended audience of the book includes the following:

- *Portfolio managers* – Throughout the book there are numerous strategies and valuation formulas to help portfolio managers structure optimal portfolios and identify value opportunities without changing their intended risk profile.
- *Analysts* – Estimation of default probability and recovery value from market prices of securities as well as recovery adjusted yield and duration can help analysts compare securities on a level playing field.

- *Traders* – Throughout the book there are numerous examples of cheap/rich analysis of securities to help traders identify trading opportunities. Synthetic securities can be constructed when a security that provides the necessary exposure does not exist or is not available for trading.
- *Hedge funds* – There is coverage for nearly all liquid fixed income derivatives together with methods for the identification of value and hedging the risks of derivatives. Several backtests demonstrate the efficacy of value identification and provide systematic approaches to long/short and leveraged strategies.
- *Proprietary trading desks* – There is broad coverage of risk decomposition and hedging for all securities and their derivatives, including credit securities and credit default swaps.
- *Risk measurement/management* – The risks of all securities are decomposed into components that can be separately measured or hedged by both the back office and portfolio managers.
- *Performance attribution* – Performance attribution and contribution at the security and portfolio levels for all asset classes and derivatives is performed using the same methodology. The performance of a treasury portfolio can be measured to within 1 basis point on an annual basis, with similar accuracy for other sectors.
- *Central bankers* – The analysis of default probability and recovery for sovereign countries based on the traded price of their securities and precise calculations of the term structure of inflation expectations provide methods for the measurements of systemic risk in global markets.
- *Academics* – There are a few concepts covered in the book that have not been published elsewhere, including:
  - proof that long term yields cannot change;
  - structural problems of swaps and why they are subject to arbitrage;
  - why corporate bonds violate the efficient market hypothesis;
  - real rates cannot have log-normal distribution.
- *Finance and financial engineering textbook* – This book can serve as an advanced book for graduate students in finance or financial engineering.

Many of the mathematical derivations are followed by practical examples or backtests to show how the analysis can be used to uncover value or measure risks in fixed income portfolios.

This book assumes that the reader is familiar with basic fixed income securities and their analysis. Knowledge of calculus, linear algebra and matrix operations is necessary to follow many of the quantitative aspects of the book. Some of the math concepts that are not covered in calculus can be easily found in online sources such as Wikipedia, including Chebyshev polynomials, the gamma function, principal components analysis, and eigenvalues and eigenvectors.

Most of the derivations in the book are original and therefore only a few external references have been mentioned. For some areas that have been extensively studied in the market, we provide comprehensive coverage within our framework, including:

- *Mortgage valuations* – We provide very detailed measurements of sensitivity to the term structure of volatility and rates by matching volatility across its surface precisely and using a method similar to a closed form solution. We show that hedging the volatility of mortgages requires multiple swaptions.

- *Corporate bonds* – We estimate the recovery value from the market price of securities and calculate the recovery adjusted spread and credit and interest rate durations. We show that option adjusted spread is not the best measure of value for corporate bonds.
- *Bond futures* – A self-consistent probability weighted method for the valuation and risk measurement is developed. The valuation result is used in backtests for long/short strategies that produce very respectable information ratios.
- *Inflation linked* – The decomposition of risks of inflation linked bonds and inflation swaps into the respective components of real and nominal along with seasonal adjustments provides very accurate hedging and valuations.
- *Bond options* – It is argued that Black-76 model is not arbitrage-free for bond options and we develop a model for pricing American bond options with the accuracy of a closed form solution, if one existed. In the options chapter we show that the most widely used platform to value American bond options is sometimes off by a factor of more than 2 at the time of this analysis.

The backbone of our framework is the term structure of rates, including interest rates, real rates, swap rates (Libor), credit rates and volatility. Through principal components analysis we show that the market's own modes of fluctuations of interest rates are nearly identical to the components of our term structure of interest rates. Essentially, our term structure model speaks the language of the markets. Thus, the model requires the minimum number of components to explain all changes in interest rates. Five components can price all zero coupon treasuries within 2 basis points (bps) of market rates. More importantly, a different number of components can be used for risk management than for valuation without loss of generality. Exact pricing of all interest rate swaps that is provided by our methodology can be used for valuation of swap transactions.

The components of the term structure model represent weakly correlated sectors of the yield curve and can be used for structuring and risk measurement of portfolios. The first component, level, is associated with the duration of the portfolio. The second component, slope, is associated with the flattening/steepening structure and can be used to structure a barbell trade. The third component, bend, represents the exposure of a portfolio at the long and short ends relative to the middle of the curve and is used to structure a butterfly trade.

Valuation metrics along with the term structure durations for the identification of sources of alpha and risk are provided for all asset classes. We introduce the concept of partial yields as a way to decompose the contribution of different sectors to the yield of a portfolio. It is not reasonable to aggregate the yield of a security that has a high probability of default in a portfolio, since the resulting portfolio yield is not likely to be realized. Partial yield addresses this issue, by calculating the default probability and decomposing the yield into components that can be used to aggregate a portfolio's yield.

The valuation metrics and term structure durations along with linear programming provide tools for portfolio construction at the security level. This is also known as the bottom-up approach to portfolio construction and is useful for daily maintenance of a portfolio. Sector allocations and analysis of the portfolio's mix of assets and durations and correlation among different asset classes are the subject of the top-down method of portfolio construction in fixed income. The two methods are complementary to each other; however, top-down is usually analyzed on a monthly or quarterly basis.

There is a step-by-step outline of building a spreadsheet based tool for designing new products or maintaining an existing portfolio. This tool provides the tracking error, marginal contribution to risk, and can be used for what-if analysis or to see how the portfolio would have performed during prior financial crises or how additions of new asset classes or sectors alter the risk profile of the portfolio. There is also a method to identify the structure of the competitive universe and design a product that could compete in that space.

We have provided detailed steps and formulation for the implementation of the framework that is outlined in the book. Many of the components can be built in spreadsheets; however, reliable and efficient analytics require the development of the necessary tools as separate programs. The benefits of such a framework and the potential performance improvements significantly outweigh its development costs.

Acknowledgement

You might think that following some of the seven hundred or so formulas in the book is not a trivial task, let alone deriving them. Kris Kowal, Managing Director and Chief Investment Officer of DuPont Capital Management, Fixed Income Division, offered to review the manuscript and re-derive nearly all the formulas in the book. Kris provided numerous helpful suggestions and comments that were instrumental in reshaping the book into its present form. In many cases, following Kris's recommendations additional steps were added to the derivations to make it easier for the reader to follow. Thanks Kris.

This page intentionally left blank

Foreword

In 1998, shortly after arriving at Putnam Investments, Saied Simozar began work on a model for the term structure of interest rates that was to become a cornerstone of an entire complex of portfolio management tools and infrastructure. It was fortuitous timing because that rate model had the dual benefits of being derived through current market pricing structure (rather than historical regressions) and the flexibility to quickly incorporate new security types.

The late 1990s marked something of a sea change in the fixed income markets. The years leading up to that period had been defined by big global themes and trends like receding global inflation rates and the development of out of benchmark sectors like high yield corporate bonds and emerging market debt, as well as global interest rate convergence under the nascent stages of European Monetary Union. Under these broad trends, return opportunities, portfolio positioning, and risk could easily be characterized in terms of duration and sector allocation percentages.

Much of that changed in 1998 when the combination of increasingly complex security types, rapid globalization of financial markets, and large mobile pools of capital set the stage for a series of rolling financial crises that rocked global financial markets and eventually led to the collapse of one of the most sophisticated hedge funds of that era – Long Term Capital Management. In the aftermath, it became clear that traditional methods of monitoring portfolio positioning and risk were insufficient to manage all the moving parts in modern fixed income portfolios.

Fortuitously, that term model (and the portfolio management tools built around it) allowed Putnam to effectively navigate through that financial storm. Perhaps more importantly, it provided the basis for an infrastructure that could easily adapt and change with the ever evolving fixed income landscape. Today, while many of the original components of that infrastructure have been augmented and updated, the basic tenants of the philosophical approach remains in place.

In his book, Saied lays out a blueprint for a set of integrated tools that can be used in all aspects of fixed income portfolio management from term structure positioning, analysis of spread product, security valuation, risk measurement, and performance attribution. While the work is firmly grounded in mathematical theory, it is conceptually intuitive and imminently practical to implement. Whether you are currently involved in the management of fixed income portfolios or are looking to get a better understanding of all the inherent complexities, you won't find a more comprehensive and flexible approach.

D. William Kohli
Co-Head of Fixed Income
Putnam Investments

This page intentionally left blank

About the Author

Saied Simozar, PhD has spent almost 30 years in fixed income portfolio management, fixed income analytics, scientific software development and consulting. He is a principal at Fipmar, Inc., an investment management consulting firm in Beverly Hills, CA. Prior to that, Saied was a Managing Director at Nuveen Investments, with responsibilities for all global fixed income investments. He has also been a Managing Director at Bank of America Capital Management responsible for all global and emerging markets portfolios of the fixed income division. Prior to that, he was a senior portfolio manager at Putnam Investments and DuPont Pension Fund Investments.

This page intentionally left blank

Introduction

One of the keys to managing investment portfolios is identification and measurement of sources of risk and return. In fixed income, the most important source is the movement of interest rates. Even though changes in interest rates at different maturities are not perfectly correlated, diversifying a portfolio across the maturity spectrum will not lead to interest rate risk reduction. In general, a portfolio of one security that matches the duration of a benchmark tends to have a lower tracking error with the benchmark than a well-diversified portfolio that ignores duration.

Historically, portfolio managers have used Macaulay or modified duration to measure the sensitivity of a portfolio to changes in interest rates. With the increased efficiency of the markets and clients' demands for better risk measurement and management, several approaches for modeling the movements of the term structure of interest rates (TSIR) have been introduced.

A few TSIR models are based on theoretical considerations and have focused on the time evolution or stochastic nature of interest rates. These models have traditionally been used for building interest rate trees and for pricing contingent claims. For a review of these models, see Boero and Torricelli [1].

Another class of TSIR models is based on parametric variables, which may or may not have a theoretical basis, and their primary emphasis is to explain the shape of the TSIR. An analytical solution of the theoretical models would also lead to a parametric solution of the TSIR; see Ferguson and Raymar for a review [2]. Parametric models can be easily used for risk management and they almost always lead to an improvement over the traditional duration measurement. Willner [3] has applied the term structure model proposed by Nelson and Siegel [4] to measure level, slope and curvature durations of securities.

Key rate duration (KRD) proposed by Ho [5] is another attempt to account for non-parallel movements of the TSIR. A major shortcoming of KRD is that the optimum number and maturity of key rates are not known, and often on-the-run treasuries are used for this purpose. Additionally, key rates tend to have very high correlations with one another, especially at long maturities, and it is difficult to attach much significance to individual KRDs. The most important feature of KRD is that the duration contribution of a key rate represents the correct hedge for that part of the curve.

Another approach that has recently received some attention for risk management is the principal components analysis (PCA) developed by Litterman and Scheinkman [6]. In PCA, the most significant components of the yield curve movements are calculated through the statistical analysis of historical yields at various maturities. A very attractive feature of principal components, as far as risk management is concerned, is that they are orthogonal to each other (on the basis of historical data). The first three components of PCA usually account for more than 98% of the movements of the yield curve.

Another class of yield curve models is based on splines. Cubic splines are widely used for fitting the yield curve and are useful for valuation purposes, to the extent that the yield curve is smooth. Cubic splines can be unstable, especially if the number of bonds is relatively low. For a review of different yield curve models, see *Advanced Fixed Income Analysis* by Moorad Choudhry [7].

All of the above models are useful either for risk management or pricing, but not for both. For portfolio management applications, it is quite difficult to translate either KRDs or PCA durations into positions in a portfolio. Likewise, it is not straightforward to convert valuations from a cubic spline curve into risk metrics. For global portfolios, it would be impossible to compare the relative value of securities or the cheapness/richness of the areas of global yield curves using KRDs, PCA or cubic splines. Each currency requires a separate PCA, which in turn requires the availability of historical data.

In this book we will develop a market driven framework for fixed income management that addresses all aspects of fixed income portfolio management, including risk measurement, performance attribution, security selection, trading, hedging and analysis of spread products. For risk management, the model is as accurate as PCA and its first three components are very similar to those of PCA. For trading and hedging, the model can be easily transformed into KRDs. This framework has been successfully applied to the management of global portfolios, risk measurement and management, credit and emerging markets securities, derivatives, mortgage bonds and prepayment models, and for the construction of replicating portfolios.

The movements of interest rates are decomposed into components that are weakly correlated with each other and can be viewed as independent and diversifying components of a fixed income portfolio strategy. These interest rate components can be viewed as different sectors of the treasury curve. However, TSIR components tend to be more weakly correlated with one another in the medium term horizon than typical sectors of the equity market and therefore can offer better diversification potential.

First, we develop a parametric term structure model that can price the treasury curve very accurately. The model is highly flexible and stable and its movements are very intuitive. The components of the model represent the modes of fluctuations of the yield curve, namely, level, slope, bend etc. and in well behaved markets all bonds can be priced with an average error of less than 2 bps. The components of the yield curve or the basis functions, as we call them, can be converted to other basis functions such as Key Rate components. We will also compare the components of our model to PCA and to an economic indicator.

The model is then applied to risk measurement and management for treasuries. The components of the term structure directly translate into trades that fixed income practitioners are accustomed to such as bullets, barbells and butterfly trades of the yield curve. The level duration of a portfolio measures the net duration or bullet duration, while the slope duration measures the barbell strategies and bend duration measures the butterfly strategies. We also compare historical data using different basis functions.

In the performance attribution section, we show that the performance of a treasury portfolio can be measured with an accuracy of less than 1 basis point per year, by decomposing performance yield, duration and convexity and security selection components. We will further delineate the difference between various representations of the yield curve and provide some evidence associated with the weaknesses of Key Rate basis functions.

A few characteristics of the TSIR model are as follows:

- It is driven by current market prices and accurately prices treasuries using only five parameters.
- Risk measurement and portfolio replication do not require a historical correlation matrix for a country where the information is not available.
- Risk management, valuation, performance attribution and portfolio management can be integrated.
- It can be easily expanded if a higher number of components are desired without changing the value of primary components significantly.
- It is intuitive, is easy to use, implement and manipulate. Its components are readily identified with portfolio positions of duration, flattening/steepening, butterfly, etc.
- It is flexible and can be easily applied to mortgage prepayment models, emerging markets, multi-currency portfolios, inflation linked bonds, derivatives analysis, etc.
- It can be used as an indicator of relative value or relative curve positions in a consistent way across currencies and credits.
- The model is easily applied to all global rates, term structure of Libor, term structure of real rates and term structure of credit rates.
- The model is very stable and, unlike cubic splines, can be easily differentiated multiple times if necessary.

Throughout this book we have provided detailed examples of the applications of our model to risk measurement, performance attribution and portfolio management. We first introduce the concept of linear and non-linear time space and then construct the components of our term structure model and forward rates. Next, we derive duration and convexity components and calculate performance attribution from duration components.

In Chapter 6 Libor and interest rate swaps are covered and the model is applied to the term structure of Libor rates. It is shown that interest rate swaps have a structural problem that makes them subject to arbitrage. In Chapters 7 and 8 trading and portfolio optimization and security selection are examined. In Chapter 9 a model for the term structure of volatility surface is developed, and in Chapter 10 the effects of convexity and volatility on the shape of the TSIR are analyzed and the convexity adjusted TSIR model is developed. The convexity adjustment to eurodollar futures is also covered and potential arbitrage opportunities are pointed out. In Chapter 11 there is a very detailed and precise coverage of inflation linked bonds along with the application of the term structure of real rates to global inflation linked bonds as well as inflation swaps.

In Chapter 12 credit securities are analyzed and the term structure of credit rates (TSCR) with its application to performance attribution and risk measurement is analyzed. In Chapter 13 default and recovery or cash flow guarantees of credit securities are analyzed and for the first time the TSCR is used to estimate the market implied recovery rate. The application of the TSCR to credit default swaps and construction of performance attribution for complex portfolios are also analyzed in this chapter.

Analysis of global bond futures and their hedging, replication, arbitrage and performance attribution are covered in Chapter 14. Bond options and callable bonds are covered in Chapter 15 along with a very detailed analysis of American bond options with accuracy approaching closed form solutions. The weaknesses of the Black-76 model are pointed out and the model is applied to corporate bond options and exotic securities. It is shown that credit bond prices cannot follow the efficient market hypothesis and there are long term opportunities in the credit markets for fund managers.

In Chapter 16 currencies as an asset class along with their options and futures are covered and models to take advantage of currencies in a portfolio are explored. Chapters 17 and 18 cover the application of the TSIR to prepayments and development of mortgage analysis. In Chapter 19 product design and portfolio construction are covered and a method is developed to analyze the competitive universe of a bond fund. Chapter 20 covers detailed mathematical derivations of the parameters of the TSIR and TSCR and estimation of recovery value, and Chapter 21 covers implementation notes and short-cuts.

CHAPTER 1

Review of Market Analytics

This chapter reviews some of the basic analytics for fixed income securities and provides evidence for the inadequacies of the existing models. The simplest and most straightforward fixed income instrument is a bond. A bond is a security that pays interest at prescribed intervals, called coupon dates, and pays back the principal and final coupon on the maturity date.

Consider a company or a government that borrows \$100 million for a period of 5 years at a rate of 7% per year payable at semi-annual intervals. The borrower, also known as the bond issuer, will have to make coupon payments equal to 3.5% of the borrowed amount or \$3.5 million every 6 months to lenders, also known as bond holders or investors. At the end of 5 years, the borrower pays \$3.5 million of interest plus the \$100 million principal back to the lenders.

The above example is a typical bond, where the borrower, unlike mortgage borrowers, cannot pay back the principal earlier than scheduled. The bond holder can usually sell the bond in the secondary market and receive a fair price for it.

The primary risk of a bond holder, other than default, is a rise in interest rates. If inflation expectations increase, bond investors demand higher interest rates to compensate them for anticipated inflation that will lower their future buying power. Likewise, if inflationary expectations fall, interest rates are likely to fall as well. During rapid economic growth, demand for money rises, which can lead to higher interest rates. During recessions or low economic activity, demand for money falls, usually resulting in lower interest rates.

1.1 BOND VALUATION

If interest rates fall, the value of an existing bond increases since investors will pay a premium price for a bond that has a higher coupon payment than a newly issued bond with a lower coupon. This brings us to the simplest and most fundamental of all pricing formulas in the fixed income market, namely the present value of a bond, defined with a principal amount of 100 as

$$p = \sum_{i=1}^{n} \frac{c}{\left(1 + y_m/m\right)^i} + \frac{100}{\left(1 + y_m/m\right)^n} \tag{1.1}$$

where p is the present value of the bond, y_m is the market yield or effective interest rate of the bond, m is the frequency of coupon payment (if the bond pays semi-annual interest, then $m = 2$, if it pays quarterly, then $m = 4$), c is the periodic coupon payment, and n is the number of interest payments. It can be easily shown that if the present value of the bond on issue date is equal to 100, then the following relationship holds:

$$c = 100\,\frac{y_m}{m} \tag{1.2}$$

In our prior example, the semi-annual coupon payment per 100 of principal would be 3.5. Inserting this value for c, and using $m = 2$, leads to a yield of 0.07 or 7%. Thus, on issue date, the yield of a bond priced at 100 (par) is equal to the annual coupon payment of the bond per 100 principal amount divided by 100. At all other times, the price/yield function of a bond is a little more complicated.

Nearly all bonds in the market are traded on the basis of what is known as the clean price. The clean price does not include the amount of interest that has been accrued but not paid to the bond holder. Accrued interest is the pro rata share of the next coupon payment that is due the seller at the time of the trade settlement. In our previous example, if after 3 months the bond holder sells his bonds, then the buyer has to pay half of the next coupon payment to the seller for holding the bonds for half the period of coupon payment.

Different bond markets have different conventions on how the accrued interest is calculated. Accrued interest for US treasuries is calculated on an actual/actual basis with semi-annual payments. For example, a bond that pays coupon on February 15 and August 15, if it is purchased on March 15 of a non-leap year, the amount of accrued interest would be calculated by the ratio 28/181 multiplied by the amount of semi-annual coupon payment. This ratio is the number of days between February 15 and March 15 (28) divided by the number of days in the period between February 15 and August 15 (181).

Corporate or agency bond markets use the 30/360 convention, implying that a month is 30 days and a year is 360. In the above example, the number of days between February 15 and March 15 would be 30 and the number of days from February 15 to August 15 would be 180.

If we denote the fractional accrual period by x, our present value formula will change to

$$p + xc = \sum_{i=1}^{n} \frac{c}{(1 + y_m/m)^{i-x}} + \frac{100}{(1 + y_m/m)^{n-x}} \tag{1.3}$$

In this equation, $1 - x$ is the fractional period to the next cash flow or coupon payment. We can convert it to the fraction of a year by multiplying it by m. Thus,

$$p + xc = \sum_{i=1}^{n} \frac{c}{(1 + y_m/m)^{mt_i}} + \frac{100}{(1 + y_m/m)^{mt_n}} \tag{1.4}$$

If we denote the cash flow at time t_i by c_i and the invoice price by p_m, we can simplify the above equation to

$$p_m = \sum_{i=1}^{n} c_i \left(1 + \frac{y_m}{m}\right)^{-mt_i} \tag{1.5}$$

Equation (1.5) is a generalization of (1.4) and allows for cash flows to be different. It can be used for bonds with step coupons or sinking or capitalizing principals. As can be seen, the market yield of a security depends on the accrual frequency. For example, German government bonds (Bunds) accrue on an annual basis while US treasuries pay coupon semi-annually. If you buy 100 units of a Bund at a yield of 6%, after 1 year the value of principal and interest will be 106. For US treasuries with the same yield, there is a semi-annual interest payment of 3%, which if reinvested at the same rate will result in 106.09. The effective yield of the US treasury is 6.09%. We therefore need to analyze all bonds on the same footing to be able to make fair comparisons.

1.2 SIMPLE BOND ANALYTICS

A problem that bond managers are faced with on a regular basis is the impact of changes in interest rates on the price of a bond. For a small change in interest rates, we can expand the pricing function using Taylor series as follows:

$$p_m(\Delta y) = p_m(0) + \frac{\partial p_m}{\partial y_m}\Delta y_m + \frac{1}{2}\frac{\partial^2 p_m}{\partial y_m^2}(\Delta y_m)^2 \ldots \tag{1.6}$$

After some simplification, the first term in the expansion is

$$\frac{\partial p_m}{\partial y_m} = -\frac{1}{1 + y_m/m}\sum_{i=1}^{n} c_i t_i \left(1 + \frac{y_m}{m}\right)^{-mt_i} \tag{1.7}$$

The expression within the summation is the weighted average time to future cash flows multiplied by the price and is called the Macaulay duration. The negative sign implies that the price of bonds falls if interest rates rise. The modified duration of a bond is defined as

$$\begin{aligned} D_m &= -\frac{1}{p_m}\frac{\partial p_m}{\partial y_m} = \frac{D}{1 + y_m/m} \\ \Delta p &\approx D_m p_m \Delta y_m \end{aligned} \tag{1.8}$$

where D is the Macaulay duration of the bond. Modified duration measures the price sensitivity of a bond to changes in interest rates. For example, if the modified duration of a bond that is priced at 104 is 11 years, for a change of 10 bps in interest rates ($10/10{,}000 = 0.001 = 0.1\%$), the change in the price of the bond is expected to be $0.001 \times 11 \times 104 = 1.144$.

The second order term in (1.6) can be simplified to

$$\frac{\partial^2 p_m}{\partial y_m^2} = \sum_{i=1}^{n} c_i t_i \, \frac{1 + m t_i}{m} \left(1 + \frac{y_m}{m}\right)^{-m t_i - 2} \tag{1.9}$$

This expression, denoting convexity multiplied by price, is always positive for bonds with fixed coupon payments. Market yield, modified duration, and convexity of bonds depend on coupon frequency and therefore cannot be used to compare bonds with different coupon frequencies. For example, the duration of a corporate bond that pays quarterly coupons cannot be combined with the duration of a treasury bond that pays semi-annually in a portfolio. We need to do all the calculations using the same accrual convention. Our solution is to use a continuously compounded framework.

Consider a bond with principal continuously growing at a rate of y per year. The change in the principal after a short time dt is

$$dp = p y dt \tag{1.10}$$

Integrating the above equation leads to

$$p = p_0 e^{yt} \tag{1.11}$$

where p is the future value of an initial investment of p_0. Likewise, the present value of a future cash flow p will be

$$p_0 = p e^{-yt} \tag{1.12}$$

The present value of a number of cash flows discounted by the same yield will be

$$p = \sum_{i=1}^{n} c_i e^{-y t_i} \tag{1.13}$$

Comparing (1.13) with (1.5), we find that they are identical if we make the following substitutions:

$$\begin{aligned} e^y &= \left(1 + \frac{y_m}{m}\right)^m \\ y &= m \ln\left(1 + \frac{y_m}{m}\right) \\ y_m &= m\left(e^{y/m} - 1\right) \end{aligned} \tag{1.14}$$

We can derive the continuously compounded yield and durations in the limit as $m \to \infty$:

$$\lim_{m\to\infty} \left(1 + \frac{y_m}{m}\right)^m = \lim_{m\to\infty} e^{y_m} = e^y \tag{1.15}$$

In the continuously compounded framework, duration (D) and convexity (X) become much simpler to handle, and modified duration and Macaulay duration converge to the same value:

$$D = \frac{1}{p}\sum_{i=1}^{n} c_i t_i e^{-yt_i} \tag{1.16}$$

$$X = \frac{1}{p}\sum_{i=1}^{n} c_i t_i^2 e^{-yt_i} \tag{1.17}$$

The change in the price of a security due to a small change in its yield in the continuously compounded framework is

$$\Delta p = -pD\,\Delta y + \frac{1}{2} pX \left(\Delta y\right)^2 + \ldots \tag{1.18}$$

1.3 PORTFOLIO ANALYTICS

A bond portfolio can be composed of many bonds along the maturity, credit quality, and currency spectrums. For regulatory, policy, or strategy purposes, the portfolio manager needs to know the duration of the portfolio. Since different market sectors may have different coupon frequencies, it is important that all calculations for the duration be done on a consistent basis.

Most bond portfolios are managed against a benchmark. The benchmark can be an index or it can be the peer group. In the cases of indices, such as the Barclays Aggregate Bond Index, the composition of the index is known on or before the last business day of a month for the following month. Portfolio managers can adjust the duration of the portfolio in relation to the changes in the duration of the index. A benchmark can be a peer group where the duration of the benchmark cannot be measured, but can be estimated through market movements. We will cover this issue in more detail in Chapter 19.

Before we calculate the duration of a portfolio, we introduce the concept of the value of a basis point (VBP) or dollar value of a basis point (DV01), which is the change in the market value of a portfolio resulting from the change of 1 basis point in the level of interest rates:

$$VBP_j = DV01_j = \frac{M_j D_j}{10{,}000} \tag{1.19}$$

where M_j and D_j are the market value and duration of a bond j in the portfolio. Consider a portfolio of N securities, each with multiple cash flows. The total market value of the portfolio can be written as

$$M = \sum_{j=1}^{N} w_j p_j = \sum_{j=1}^{N} w_j \sum_{i} c_{ij} e^{-y_j t_{ij}} \tag{1.20}$$

where p_j is the price of security j, w_j is the weight or face amount of security j, c_{ij} is the ith cash flow of security j and t_{ij} is the time to that cash flow. For a small uniform change in the yield of all bonds in the portfolio, it can easily be shown that the change in the market value will be

$$\Delta M = -\sum_{j=1}^{N} w_j p_j D_j \Delta y = -\sum_{j=1}^{N} M_j D_j \Delta y \tag{1.21}$$

Alternatively,

$$\Delta M = -10{,}000 \Delta y \sum_{j=1}^{N} VBP_j = -10{,}000 \Delta y VBP \tag{1.22}$$

If D is the duration of the portfolio, the change in market value for a change of Δy in yield will be

$$\Delta M = -DM\Delta y \tag{1.23}$$

The duration of the portfolio is

$$D = \frac{1}{M}\sum_{j=1}^{n} w_j p_j D_j = \frac{1}{M}\sum_{j=1}^{n} M_j D_j = 10{,}000\,\frac{VBP}{M} \tag{1.24}$$

Thus, the duration of a portfolio is the market value weighted sum of the duration of all bonds in the portfolio. Alternatively, the duration of a portfolio is the sum of all the VBPs divided by the market value and multiplied by 10,000.

To estimate the yield of the portfolio, we note that the market value of the portfolio can be calculated by discounting all bonds in the portfolio by their respective yields or by discounting all the cash flows in the portfolio by the portfolio yield:

$$M = \sum_j w_j \sum_i c_{ij} e^{-y t_{ij}} = \sum_j w_j \sum_i c_{ij} e^{-y_j t_{ij}} \tag{1.25}$$

where y is the overall yield of the portfolio and y_j is the yield of bond j. Subtracting the summations and expanding the resulting difference by Taylor series and retaining only the first two components leads to

$$0 = \sum_j w_j \sum_i c_{ij}\; e^{-y_j t_{ij}} \left(1 - e^{-\left(y - y_j\right) t_{ij}}\right) \approx \sum_j w_j \sum_i c_{ij}\; e^{-y_j t_{ij}} \left(y - y_j\right) t_{ij} \tag{1.26}$$

The portfolio yield can now be calculated:

$$y \approx \frac{\sum_j w_j y_j \sum_j c_{ij} t_{ij} e^{-y_j t_{ij}}}{\sum_j w_j \sum_j c_{ij} t_{ij} e^{-y_j t_{ij}}} = \frac{\sum_j w_j p_j D_j y_j}{\sum_j w_j p_j D_j} \tag{1.27}$$

The yield of a portfolio as calculated by (1.27) can be significantly different from the market value weighted yield in a non-flat yield curve environment.

The conventional duration of a portfolio also requires some adjustments in a non-flat yield curve environment. If M and D are the market value and duration of a portfolio respectively, and D_j and X_i are the duration and convexity of a security, then

$$\begin{aligned} MD &= \sum_j M_j D_j = \sum_j w_j \sum_j c_{ij} t_{ij} e^{-y t_{ij}} \\ &= \sum_j w_j \sum_i c_{ij} t_{ij} e^{-y_j t_{ij} - (y - y_j) t_{ij}} \end{aligned} \tag{1.28}$$

Expanding the summation using Taylor series and keeping only the first two components leads to

$$MD \approx \sum_j w_j p_j D_j + \sum_j w_j (y_j - y) p_j X_j \tag{1.29}$$

For a portfolio of two zero coupon bonds, (1.29) simplifies to

$$MD = M_1 D_1 + M_2 D_2 + \frac{M_1 D_1 M_2 D_2 (y_2 - y_1)(D_2 - D_1)}{2(M_1 D_1 + M_2 D_2)} \tag{1.30}$$

TABLE 1.1 Yield and duration of a portfolio

Line	Instrument	Duration	Yield	Face	Price	Market Value
1	A	2.00	1.000%	10,202,013	98.020	10,000,000
2	B	18.00	4.000%	20,544,332	48.675	10,000,000
3	Average	10.00	2.500%			20,000,000
4	Duration Weighted	10.43	3.700%			20,000,000
5	A + B One Security	10.42	3.714%			20,000,000
6	A + B Two Securities	10.00	3.700%			20,000,000

Table 1.1 shows an example of the yield and duration of a portfolio of two zero coupon bonds. The market value weighted yield of the portfolio is 2.5% (line 3) which is significantly below the actual yield (3.714%). The actual yield is calculated by iteratively finding the yield that correctly reproduces the market value of the portfolio. The duration weighted yield on line 4 (3.7%) is very close to the actual yield of the portfolio in line 5.

The correct method for calculating the duration of A + B as one security is by discounting each cash flow by its respective discount yield and then summing the contributions from all cash flows, resulting in a duration of 10 years. Calculating the duration of the combined securities by discounting all the cash flows by a yield of 3.714% on line 5 will result in a duration measurement that is off by about 0.42 years from the correct duration.

The weighted duration calculated from equation (1.30), shown on line 4, is very close to the market convention duration of the combined securities.

When we combine two securities into one, in a steep yield curve environment, the yield of the combined security will be lower than the yield of the longer duration security. Therefore, the market value of the longer duration security will be higher than if it was discounted by its correct yield. On the other hand, the market value of the shorter duration security will be lower than its actual market value, resulting in the market value of the combined security being equal to the sum of individual securities. In the combined security, the calculated duration is overestimated in a steep yield curve environment and underestimated in an inverted yield curve environment.

This example suggests that the duration of a 20-year coupon bond, in a steep yield curve environment, will be higher than the duration of a portfolio of zero coupon bonds with exactly the same cash flows. The correct method for calculating the duration of a portfolio is to discount all cash flows by their actual discount yield. In our example, the correct duration for the portfolio is the market value weighted average, which is equal to 10 years.

Since in a steep yield curve environment the durations of all bonds are overestimated, a portfolio manager who structures a portfolio using zero coupon bonds is likely to construct a portfolio that is longer than he desires for the respective index. Portfolios composed of coupon bonds will also overestimate the duration and the two overestimates offset each other. In 2012, the duration of the Barclays Treasury Bond Index was overestimated by about 0.09 years. This is a much larger uncertainty than most portfolio managers are willing to accept and is the extra duration that a portfolio manager who uses zero coupon bonds will have in his portfolio. For example, if the duration of the benchmark is 5.6 years and the portfolio manager desires to be long 0.20 years, he will target a duration of 5.8 years and rebalance at the end of the month when the index changes or when a transaction takes place to maintain 0.20 years' relative duration.

The mismatch in duration measurement can be significantly larger for credit securities, in particular high yield and emerging markets and at times when the spreads widen. We will see in Section 12.4 that the duration mismatch for credit securities can be longer than 1 year and hedging with coupon securities will not offset the duration mismatch. Using modified duration, the duration mismatch will be even larger, since the duration is scaled by the yield, which is larger for zero coupon bonds in a steep yield curve environment than the combined durations of the security.

1.4 KEY RATE DURATIONS

Key rate duration, proposed by Ho [5], is an attempt to measure the risks of a portfolio across the yield curve. The key rate is usually referred to a very liquid security such as an on-the-run treasury. Usually the yield curve is broken into about 10 duration buckets, each representing one key rate. The most widely used key rates are the 6 month, 1 year, 2, 3, 4, 5, 7, 10, 20, and 30 year points on the curve. To calculate the exposure of a security to a key rate, the yield of the security is shifted at the respective key rate by a small amount, while maintaining all other key rates constant, and the impact of the changes in the price of the security is calculated. The yield shift is linearly interpolated between the key rate and the preceding or following key rates. For example, to calculate the 5-year key rate duration of a security, if we shift the 5-year yield by 10 bps, the shift for a cash flow that is at 5.50 years will be

$$\frac{7-5.5}{7-5}10 = 7.5 \text{ bps}$$

While key rates address the exposure of a portfolio to different parts of the curve, they have many shortcomings for complex securities and some derivatives. Additionally, they do not address the incorrect duration calculation that was mentioned in the previous section. Table 1.2 lists the KRDs of the combined security that was shown in Table 1.1 along with the correct KRDs based on individual securities.

TABLE 1.2 Key rate duration of a portfolio

Key Rate	Combined KRD	Individual KRD
2.00	0.947	1.000
10.00	1.895	1.800
20.00	7.580	7.200
	10.423	10.000

Key rate duration is a good measure of the risk of a security or a portfolio to interest rates. However, the curve exposure of credit and inflation linked securities and many derivatives is either not measured by key rates or the exposure cannot be used effectively. For example, knowing ten key rate credit durations of a security is not useful for hedging its risks.

One of the biggest shortcomings of the KRD is its relation to performance attribution and return calculation. To calculate the performance of a security from its KRDs, we need to multiply the changes in the key rates by the respective durations. This implies that we must have an unbiased measurement of the change in the yields of the key rates. On-the-run securities, due to their liquidity, are often very rich relative to the rest of the treasury market and cannot be used as unbiased indicators of the level of interest rates at a given maturity. As on-the-run securities season, they underperform the rest of the market. For example, if the yield of a newly issued 10-year bond with 8 years of duration is 5 bps below the previously issued 10-year bond, over time, as its yield normalizes, it will have a negative cumulative performance of $0.0005 \times 8 = 0.4\%$. Thus, a security whose yield does not change will benefit from an apparent gain in performance as a result of the underperformance of the key rate security. Therefore, the use of KRDs requires calculation of a smoothed curve for the treasury market. The smoothed curve has to be derived from a pool of treasuries that do not have a liquidity premium and thus cannot include on-the-run securities.

Positive relative KRDs can also be an issue for a portfolio that cannot use derivatives. If the relative exposure of a portfolio at the 10-year part of the curve is –0.2 years, it can be hedged by buying 0.2 years of the 10-year zero coupon bond. However, if the portfolio is long duration, then selling 0.2 years of the 10-year zero coupon bond may not be practical if it is not already part of the portfolio, without changing the structure of the portfolio.

Since KRDs are based on localized changes in the yield curve, it is very difficult to compare competing trades that have similar goals. For example, there is no way to compare a barbell trade that is overweight 10-year, underweight 2-year treasuries with a similar trade that is overweight 20-year, underweight 2-year treasuries. Similarly, there is no way to compare a 2–5–10-year butterfly trade with a 2–10–30-year butterfly. Hedging the risks of credit securities where only a few bonds are available is not practical by using key rate credit exposures. Some derivative securities can have interest rate exposures that require treasuries that are longer than 30 years for their effective hedging. Likewise, long dated inflation linked bonds have a small exposure to nominal rates due to the inflation lag which may require longer than 30-year treasuries for their hedging.

Linear time interpolation of key rates can be a source of overestimation or underestimation of duration at some key rates. The correlation between 2-year and 4-year rates (2, 4) is significantly lower than the correlation between (18, 20). For a constant difference in the maturity of two key rates, the longer their maturity, the higher their correlation is. Historical correlation of (18, 20) rates is very similar to correlations of (2, 2.25) year maturities. Similarly, the correlation between (18, 20) is higher than the correlations between (10, 12), (12, 14), (14, 16), and (16, 18) maturities. We now return to the above example and calculate the 20-year KRD of the combined security in Table 1.2. We assumed that for a change of 5 bps in the yield of the 20-year key rate, the yield of the 18-year cash flow changed proportionally by $\frac{18-10}{20-10} \times 5 = 4$ bps. Likewise, for a change of 5 bps in the yield of the 10-year key rate, the yield of the same cash flow changed by $\frac{20-18}{20-10} \times 5 = 1$ basis point. Based on our argument, for a change of 5 bps in the yield of the 20-year key rate, the change in the yield of the 18-year cash flow should be slightly higher than 4 bps due to its higher correlation. Assume that for a change of 5 bps in the yields of 20-year and 10-year key rates, the yield of the cash flow at 18 years changes by respectively 4.2 and 0.8 bps (their sum has to add up to 5 bps). Based on this change the 10- and 20-year KRDs in Table 1.2 would change to 1.44 and 7.56 respectively, resulting in a lower exposure to the 10-year rate, as we suspected. We will review this issue in more detail in Section 4.5 and show that the KRD based on the term structure of rates provides a better hedge than ordinary KRDs. In Section 8.3 we will see that only five components of duration provide the same tracking error as ten KRDs. Additionally, KRD can have unintended performance biases which could adversely affect a portfolio. Fewer duration components provides flexibility to enhance performance by hedging in both long or short portfolios.

CHAPTER 2

Term Structure of Rates

In the previous chapter, we briefly discussed some of the shortcomings of the traditional measurements of risk and return in the treasury markets. Analysis of more complex fixed income instruments such as options and futures, credit products and mortgages requires more elaborate mathematical analysis and cannot be handled using the simple price/yield formulas. As we discussed previously, the result of yield or duration calculation of a portfolio was path dependent, that is, the calculated yield and duration were different if we treated all cash flows as one security or calculated the yield and duration for each cash flow separately and then combined the results. The primary reason for this path dependency was the use of different discount yields in one path versus another. Our primary objective in this chapter is to develop a term structure of interest rates (TSIR) model that provides a basis for discounting all cash flows at the correct discount yield. We will then provide examples of market derived yield curves based on our methodology.

2.1 LINEAR AND NON-LINEAR SPACE

Perhaps the most important issue in developing a TSIR model is the choice of reference frame. To motivate the development of a logical reference frame, we will compare an investment instrument to a pedestrian.

Consider a fixed income instrument that pays or receives a constant cash flow of c at regular intervals such as a fixed rate bond or a home mortgage loan. Likewise, consider a pedestrian who walks with a uniform step size of c. After n steps, the sum of cash flows or the traveled distance for the pedestrian is

$$D = cn \tag{2.1}$$

The Advanced Fixed Income and Derivatives Management Guide. Saied Simozar

Since long term cash flows are worth less than near term ones, the value of cash flows is different from the sum of cash flows. Using a continuous compounding method, we can calculate the present value of cash flows discounted by a yield of y as

$$p = \sum_{i=1}^{n} ce^{-yt_i} \tag{2.2}$$

Equations (2.1) and (2.2) are examples of discrete processes. We can also write equivalent equations in a continuous form. If the pedestrian walks at a speed of v per unit of time or a fixed income instrument has a cash flow of v per unit of time, we can write (2.1) as either

$$D = vt \tag{2.3}$$

or

$$D = \int_0^t v\, dt \tag{2.4}$$

Likewise, (2.2) can be written as

$$p = \int_0^t ve^{-yt}\, dt \tag{2.5}$$

Equation (2.4) is the integral form of (2.3); for our example (2.3) and (2.4) are identical. The most important difference between (2.4) and (2.5) is that (2.4) is linear and (2.5) is non-linear (exponentially decaying). The marginal value of a step in the linear case is the same whether it is at the beginning of the walk process or at the end of it. On the other hand, the marginal value of a cash flow in the future is less than the present cash flow.

Let us define the function u as

$$u = \frac{-1}{y} e^{-yt} \tag{2.6}$$

so that

$$du = e^{-yt}\, dt \tag{2.7}$$

By substituting the integrand in (2.5) using (2.7), we have

$$p = \int_0^t v\, du \tag{2.8}$$

The similarity between (2.8) and (2.4) is striking. Equation (2.4) is linear in the normal time space t, while (2.8) is linear in the exponentially decaying time space u. The marginal value of a cash flow, such as an interest payment, is an exponentially decaying function of time; however, it is linear in the exponentially decaying time frame. On this basis, we expect measuring the yield level would also have a closer linear relationship in an exponentially decaying time frame (EDTF) than in a linear time frame.

If we use a normal time frame to model the TSIR, all areas of the yield curve will be given similar significance, while in practice the front end of the yield curve tends to have much more structure and importance than the long end. The concentration of cash flows, macro-economic fundamentals and inflation expectations all have more influence at the front end of the yield curve. Therefore, any logical yield curve model has to give greater weight to the short end of the yield curve than the long end. An easy and efficient way to achieve this objective is through the use of the EDTF. It is possible to use an EDTF u in such a way that the yield curve has a comparable structure in all subintervals of u.

Thus, we will use an EDTF as the reference frame to model the TSIR. The choice of the decay coefficient and the functional form of the yield curve will be discussed in the following sections.

2.2 BASIS FUNCTIONS

In the EDTF, the importance of cash flows and the changes in the yield of 25–30-year maturities are significantly less than those for 0–5-year maturities. In such a frame, we will represent the yield curve in a polynomial form as follows:

$$y = a_0 b_{00} + a_1(b_{10} + b_{11}x) + a_2(b_{20} + b_{21}x + b_{22}x^2) + \cdots \tag{2.9}$$

Where, x is an exponentially decaying function of time, similar to u, b_{ij} is the jth coefficient of the ith component of the yield curve a_i and is the strength of that component. The decay coefficient for x is selected in such a way that is consistent with historical market behavior. We will refer to each one of the polynomials in parenthesis as a Basis Function (BF). We simply write x as

$$x = e^{-\alpha t} \tag{2.10}$$

In equation (2.9), each basis function represents one of the movements or modes of fluctuations of the TSIR. The first component of the yield curve represents the level of interest rates or a parallel shift of one unit for the entire curve, thus $b_{00} = 1$. The second component represents the slope of the yield curve, which we will show by short rates falling by one unit and long spot rates rising by one unit in EDTF, that is,

$$b_{10} + b_{11}x = \begin{cases} -1, & t = 0, x = 1 \\ +1, & t = \infty, x = 0 \end{cases} \tag{2.11}$$

This leads to

$$\begin{aligned} b_{10} &= 1 \\ b_{11} &= -2 \end{aligned} \tag{2.12}$$

Equation (2.9) can now be written as

$$y = a_0 + a_1\,(1 - 2x) + a_2(b_{20} + b_{21}x + b_{22}x^2) + a_3\,(b_{30} + \cdots) + \cdots \tag{2.13}$$

Likewise, we can represent the third component as the hump, bend or the butterfly of the yield curve or the simultaneous rise of one unit for the long and short end of the curve and a fall of one unit for the middle part of the curve. Specifically, we require that the minimum value of the hump in the range of $(x = 0, x = 1)$ be equal to -1:

$$\begin{aligned} b_{20} + b_{21}x + b_{22}x^2 &= \begin{cases} +1, & t = 0, x = 1 \\ +1, & t = \infty, x = 0 \end{cases} \\ \min(b_{20} + b_{21}\, x + b_{22}\, x^2) &= -1 \end{aligned} \tag{2.14}$$

This leads to

$$\begin{aligned} b_{20} &= +1 \\ b_{21} &= -8 \\ b_{22} &= +8 \end{aligned} \tag{2.15}$$

We can construct other components of the yield curve, piece by piece, in a similar fashion, by requiring that the minimum and maximum amplitudes for each basis function be equal to ± 1.

Let us define a new variable τ as

$$\tau = 1 - 2x = 1 - 2e^{-\alpha t} \tag{2.16}$$

For t in $[0,\infty)$ the corresponding range for variable τ is $[-1,+1)$ and the midpoint corresponds to $\tau = 0$. It can be shown by iteration that our process of constructing the components of the yield curve leads to Chebyshev polynomials in τ space. Expanding (2.13) in τ space in Chebyshev basis functions results in

$$y = a_0 + a_1\,\tau + a_2\,(2\tau^2 - 1) + a_3\,(4\tau^3 - 3\tau) + a_4\,(8\tau^4 - 8\tau^2 + 1) + \cdots \tag{2.17}$$

We will refer to $a_0, a_1, \ldots$ as the components of the TSIR, and 1, τ and the functions in parentheses as basis functions. We denote the basis functions by ψ and write the term structure function as

$$y(t) = \sum_{i=0}^{n-1} a_i \psi_i(t) \tag{2.18}$$

where

$$\begin{aligned} \psi_0 &= 1 \\ \psi_1 &= \tau \\ \psi_2 &= 2\tau^2 - 1 \\ \psi_3 &= 4\tau^3 - 3\tau \\ \psi_4 &= 8\tau^4 - 8\tau^2 + 1 \end{aligned} \tag{2.19}$$

In general, Chebyshev polynomials satisfy the recursion formula

$$\psi_i = 2\tau\psi_{i-1} - \psi_{i-2} \tag{2.20}$$

Chebyshev polynomials can be represented in trigonometric form as

$$\psi_n = \cos(n \arccos(\tau)) \tag{2.21}$$

It is easy to see from this formulation that the short and long rates are calculated respectively as follows:

$$y(0) = a_0 - a_1 + a_2 - a_3 + \cdots \tag{2.22}$$

$$y(\infty) = a_0 + a_1 + a_2 + a_3 + \cdots \tag{2.23}$$

Equation (2.18) represents our standard TSIR model, and we will use it extensively throughout the remainder of the book for its applications and the derivation of our generalized TSIR model. Only the first three components are necessary for most applications.

For the purpose of calculations in this book, we will use $\alpha = 0.13$ in (2.16). See Section 2.9 for a semi-empirical derivation of the time decay coefficient α. In Chapter 5 we will show that a relatively wide range of decay coefficients provide reasonable accuracy as long as it is applied consistently.

Figures 2.1 and 2.2 show the shapes of basis functions in τ space and in time space, respectively. We have named the basis functions by the order of their polynomials as level (0th), slope (twist) (1st), bend (butterfly, quadratic) (2nd), cubic (3rd), quartic (4th), quintic (5th), etc. For notational as well as computational convenience, we use continuously compounded yield in this book (1.14).

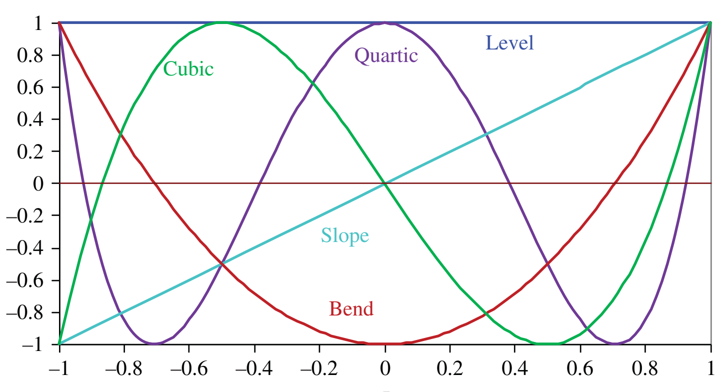


FIGURE 2.1 Chebyshev term structure components in τ space

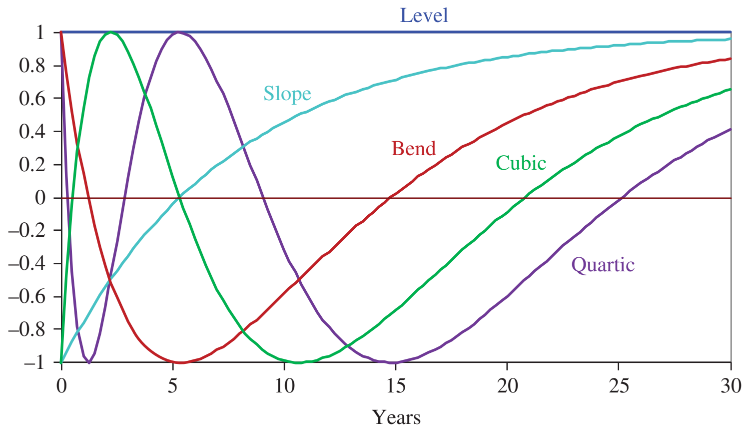


FIGURE 2.2 Chebyshev term structure components in time space

Chebyshev polynomials are orthogonal to each other relative to the weighting function $1/\sqrt{1-\tau^2}$:

$$\int_{-1}^{1} \frac{\psi_i \psi_j}{\sqrt{1-\tau^2}} d\tau = \begin{cases} 0 & i \neq j \\ \pi & i = j = 0 \\ \dfrac{\pi}{2} & i = j \neq 0 \end{cases} \tag{2.24}$$

It is also possible to represent the TSIR in basis functions that are orthogonal to each other without a weighting function; we will cover different basis functions in Chapter 3.

Mathematically speaking, Chebyshev and orthogonal polynomials are identical to any other polynomial representation of basis functions in τ space. The advantages of Chebyshev and orthogonal polynomials are that they provide an easy framework for portfolio structuring, risk management, performance attribution, communication, and near orthogonality of the basis functions. In Chapters 3–5 we will show why the choice of Chebyshev or orthogonal polynomials provides the most compact form of basis functions for risk measurement and performance attribution.

2.3 DECAY COEFFICIENT

The decay coefficient is a measure of the pivot point or the symmetry point of the yield curve. The decay coefficient defines the point in time where the slope is zero. Setting $\tau = 0$ in (2.16), we have

$$t = \frac{\ln(2)}{\alpha} \tag{2.25}$$

Therefore α is related to the half-life of the distribution. For $\alpha = 0.13$, the pivot point of the distribution will be about 5.3 years, which is close to the duration of most bond indexes. Higher values of the decay coefficient correspond to shorter time horizons and higher emphasis on shorter cash flows, while lower values of the decay coefficient emphasize longer cash flows.

2.4 FORWARD RATES

If we define $y_f(t)$ to be the instantaneous forward rate, the relationship between the spot and forward rate is

$$y(t)t = \int_0^t y_f(t)dt \tag{2.26}$$

The forward yield curve can be calculated by differentiating the above equation:

$$y_f(t) = y(t) + t\frac{\partial y(t)}{\partial t} \tag{2.27}$$

Substituting (2.18) in (2.27), the components of the term structure of forward rates can be calculated as

$$y_f = \sum_{i=0}^{n-1} a_i \varphi_i(t) \tag{2.28}$$

where

$$\begin{aligned}
\varphi_0 &= \psi_0 + t\frac{\partial \psi_0}{\partial t} = 1 \\
\varphi_1 &= \psi_1 + t\frac{\partial \psi_1}{\partial t} = \tau + \alpha t(1-\tau) \\
\varphi_2 &= \psi_2 + t\frac{\partial \psi_2}{\partial t} = 2\tau^2 - 1 + 4\alpha t\tau(1-\tau) \\
\varphi_3 &= \psi_3 + t\frac{\partial \psi_3}{\partial t} = 4\tau^3 - 3\tau + \alpha t(1-\tau)(12\tau^2 - 3) \\
\varphi_4 &= \psi_4 + t\frac{\partial \psi_4}{\partial t} = 8\tau^4 - 8\tau^2 + 1 + \alpha t\tau(1-\tau)(32\tau^2 - 16)
\end{aligned} \tag{2.29}$$

Figures 2.3 and 2.4 show the shapes of forward rate basis functions in τ space and in time space.

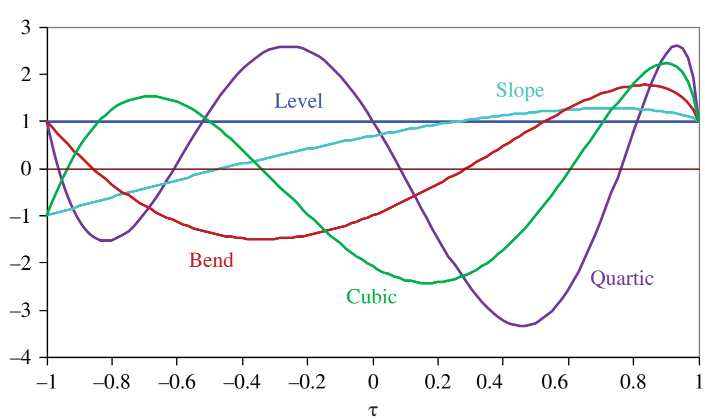


FIGURE 2.3 Forward rate components in τ space

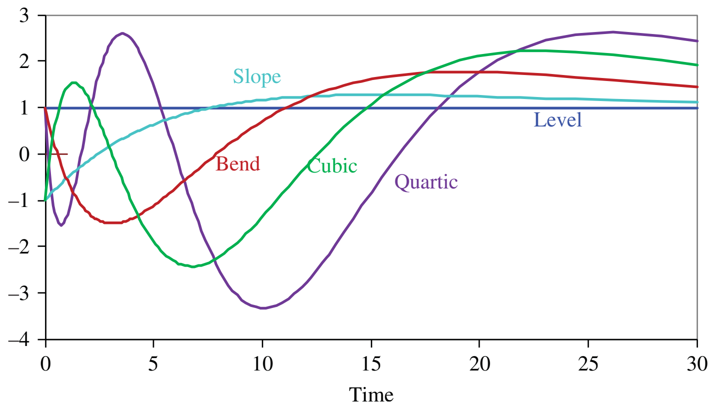


FIGURE 2.4 Forward rate components in time space

2.5 PAR CURVE

If we define the par curve to be the current coupon curve, we can calculate the constant continuously compounded coupon of a bond with maturity t_m as

$$100 + ct_0 = \sum_{i=0}^{N} ce^{-y(t_i)t_i} + 100\, e^{-y(t_m)t_m} \tag{2.30}$$

where c is the coupon rate and t_0 is the interval from the dated date to the present time. We can solve the above equation for the coupon rate.

2.6 APPLICATION TO THE US YIELD CURVE

The yield curve of the US and other government bonds can be fitted to our TSIR model very accurately. Figure 2.5 shows the calculated US spot curve as well as coupon and principal Strips (separate trading of registered interest and principal of securities) yields calculated from market prices using the first five components of the TSIR. The calculated curve is very close to the coupon Strips curve. The average yield error between the calculated yield curve and the coupon Strips curve is about 2 bps.

The term structure is very useful for identifying pricing errors and/or cheap/rich analysis of securities. Securities that are below the calculated curve are rich and those above it are cheap. The spread of a security to the curve is a very good quantitative measure of the cheapness/richness of a security.

Figure 2.6 shows the contribution of level, slope, bend, cubic and quartic components to the calculated TSIR. The value of the level component for this calculation is 1.83%. The slope of the yield curve is 2.09% and the maximum contribution from other components is less than 0.4%. Figures 2.7–2.9 show the change in the shape of the yield curve both in time and in τ space for the level, slope and bend components when they are shifted by 50 bps.

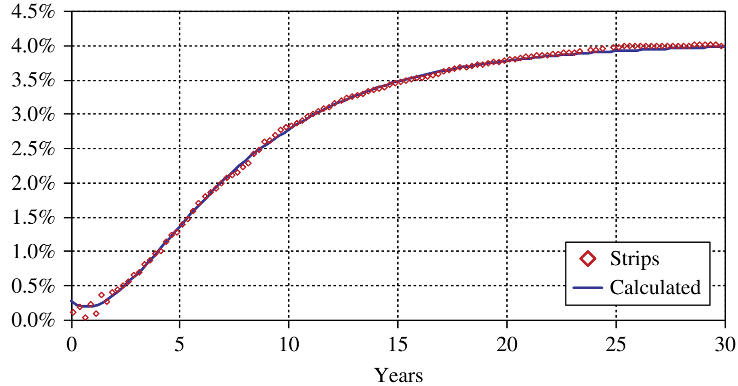


FIGURE 2.5 US term structure of interest rates for September 30, 2010

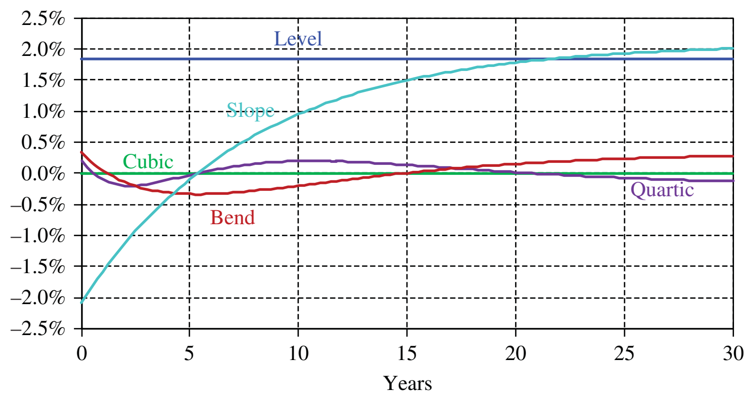


FIGURE 2.6 Components of US yield curve for September 30, 2010

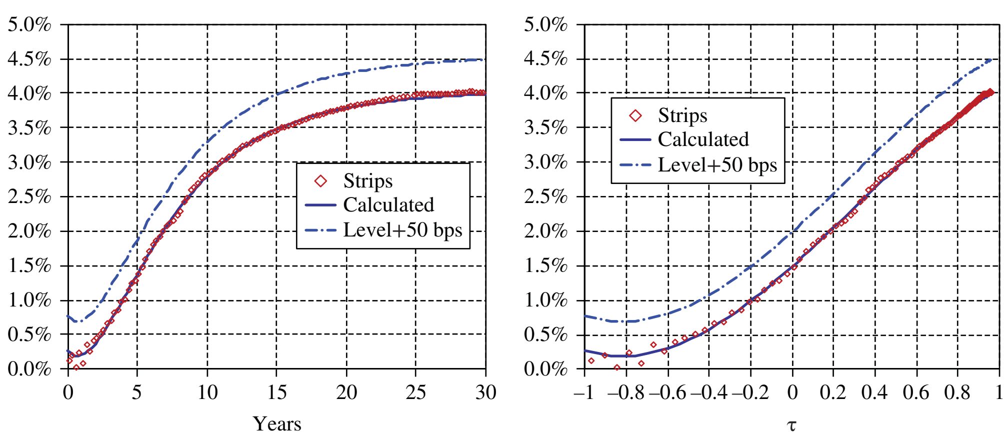


FIGURE 2.7 Level of yield curve shifted by 50 bps.

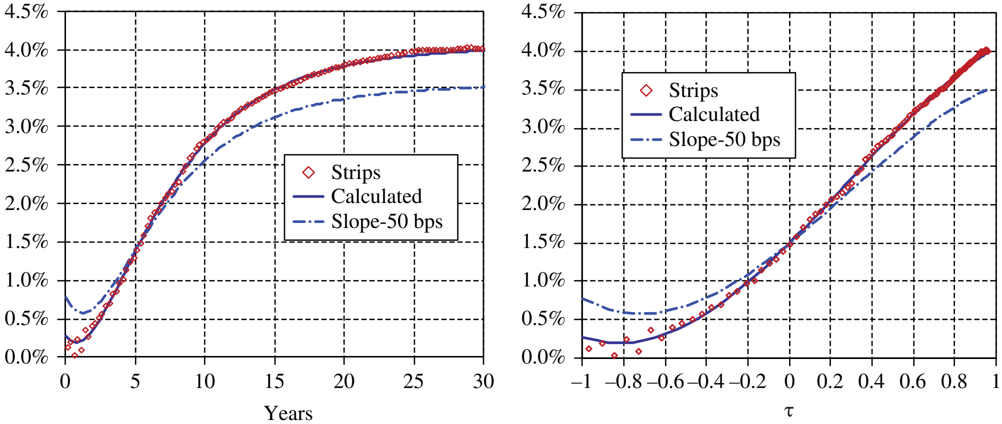


FIGURE 2.8 Slope of yield curve shifted by 50 bps.

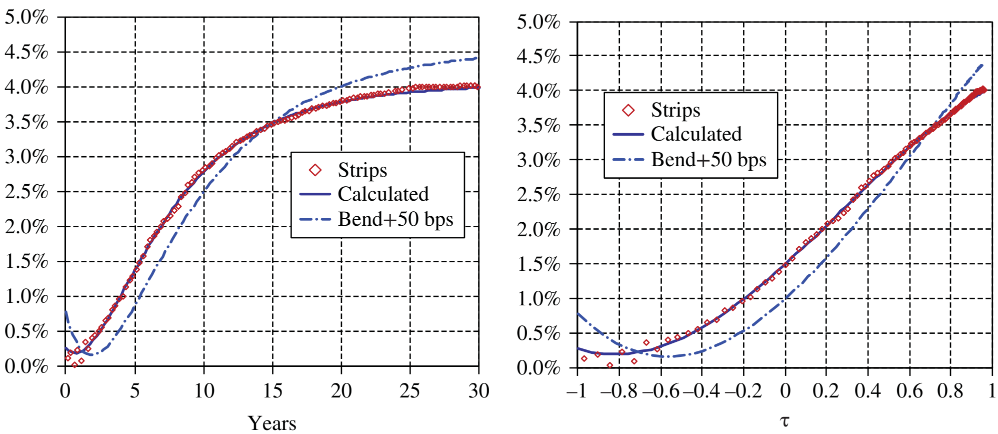


FIGURE 2.9 Bend of yield curve shifted by 50 bps.

2.7 HISTORICAL YIELD CURVE COMPONENTS

Table 2.1 shows the annual components of the TSIR from 1992 to 2011. The components are calculated by price-yield optimization (see Chapter 20) of US government bonds. The annualized volatility of the components of the TSIR, measured in terms of standard deviation of changes, is the highest for the level of interest rates and falls rapidly for each successive component.

The error column is the standard deviation of the calculated yield from the term structure versus the market yield for each respective year. The error component is a stronger function of the stability of the market than the goodness of the fit. During the tumultuous year of 2008, with the bankruptcy of Lehman Brothers, where the yield difference between on-the-run bonds and older bonds was about 40 bps, the error term was very large. Likewise, during 1998, with the collapse of LTCM, and

TABLE 2.1 US historical term structure components

	Level	Slope	Bend	Cubic	Quartic	Error
12/31/91	5.94%	2.16%	−0.18%	−0.03%	0.06%	0.048%
12/31/92	5.70%	2.59%	−0.47%	0.13%	0.05%	0.047%
12/31/93	5.14%	1.94%	−0.13%	0.05%	0.00%	0.037%
12/30/94	7.44%	0.65%	−0.39%	0.23%	−0.15%	0.034%
12/29/95	5.51%	0.61%	0.12%	0.02%	−0.03%	0.037%
12/31/96	6.05%	0.89%	−0.20%	0.12%	−0.09%	0.030%
12/31/97	5.69%	0.34%	−0.03%	0.05%	−0.05%	0.045%
12/31/98	4.70%	0.81%	−0.13%	0.19%	−0.24%	0.073%
12/31/99	6.21%	0.63%	−0.35%	0.07%	−0.20%	0.047%
12/29/00	5.32%	0.16%	0.21%	−0.04%	−0.10%	0.051%
12/31/01	3.71%	3.01%	−1.17%	0.38%	−0.35%	0.057%
12/31/02	2.85%	2.77%	−0.25%	0.08%	−0.22%	0.054%
12/31/03	3.05%	2.94%	−0.41%	0.13%	−0.16%	0.041%
12/31/04	3.67%	1.61%	−0.13%	0.12%	−0.16%	0.028%
12/30/05	4.35%	0.24%	−0.04%	0.07%	−0.11%	0.027%
12/29/06	4.77%	0.00%	−0.10%	−0.02%	−0.09%	0.028%
12/31/07	3.69%	0.91%	−0.10%	−0.20%	−0.12%	0.082%
12/31/08	1.37%	2.02%	−0.60%	−0.10%	−0.44%	0.081%
12/31/09	2.62%	2.66%	−0.26%	−0.19%	−0.07%	0.031%
12/31/10	2.29%	2.46%	−0.18%	−0.26%	0.03%	0.021%
12/31/11	1.36%	1.57%	−0.38%	−0.13%	−0.04%	0.015%
AVG	4.34%	1.49%	−0.13%	0.03%	−0.11%	
StdeV	0.82%	0.55%	0.42%	0.22%	0.16%	

the widening of spreads, the error term was relatively large. The dispersions were the results of premiums that traders were willing to pay for liquidity, creating bonds that were in very high demand for borrowing. Such bonds are said to be on-special.

A bond that is on-special can be borrowed at much lower interest rates than Libor or prevailing short rates. For example, if the overnight deposit rate is 5%, a bond that is on-special can be borrowed at 2%. The holder of the bond lends it at a rate of 2% for 1 week and receives cash equivalent to its market value minus a small variance, which can be invested at 5%, earning the holder of such a bond an additional return. When a bond goes on-special, its yield falls to compensate for the financing disparity. Even though in the long run the financing incentive does not compensate the bond holder for its lower yield, traders prefer such bonds since they have much lower transaction costs to trade. This is especially true in times of distress.

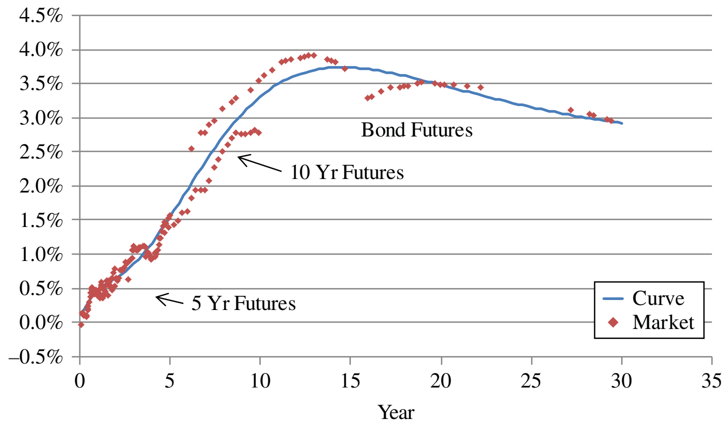


FIGURE 2.10 Yield curve on December 11, 2008

Figure 2.10 shows the yield curve for December 11, 2008 along with constituent coupon bonds. After the collapse of Lehman Brothers and the associated liquidity crisis, bond futures traded at a premium due to liquidity. Likewise, bonds that qualified as deliverable into the bond futures contracts traded at a premium relative to other bonds. For example, the yield of the treasury 9%, 11/15/2018 traded at a yield premium of about 0.75% compared to the yield of the treasury 3.75% of 11/15/2018 which was deliverable into the futures contract. The marked areas on the graph show treasury bonds that are deliverable into one of the futures contracts. Needless to say, it is not possible to calculate a discount function that would price both bonds correctly or to argue which one has the correct price.

The calculated curve attempts to price the universe of bonds as efficiently as possible. It is very easy to screen the bonds that are on-special or have bad pricing, by excluding bonds that have an error of more than three standard deviations from the curve to obtain a more efficient curve. Such a procedure lowers the error term significantly and is very useful for calculating an efficient curve most of the time.

Table 2.1 shows that during calm periods, the yield of all bonds can be calculated within 2 bps. We will see when we discuss performance attribution in Chapter 5 that, even at times of crisis, we can calculate the performance of nearly all portfolios with an error of less than 1 basis point per year.

Table 2.2 shows the historical volatility of the components of the TSIR for US from 1992 through 2012 for varying values of the decay coefficient. The mean error is the average of the daily yield error of calculated versus market yield of bonds which, as Figure 2.10 shows, included market inefficiencies during times of crises.

The Stdev Error column in Table 2.2 is the standard deviation of the daily changes in the error term. The fact that this error is smaller by a factor of 20 than the mean error points to the persistence and large serial correlation of daily error values. In other words, the daily error values were not random deviations of the calculated versus market prices, but rather were results of persistent cheap or rich pricing of certain bonds, as is evidenced in Figure 2.10.

TABLE 2.2 US historical volatility of term structure components

Decay	Level	Slope	Bend	Cubic	Quartic	Mean Error	Stdev Error	Ratio
0.09	0.802%	0.513%	0.407%	0.213%	0.177%	0.037%	0.0020%	4.536
0.1	0.806%	0.516%	0.401%	0.209%	0.167%	0.039%	0.0020%	4.838
0.11	0.809%	0.526%	0.403%	0.210%	0.161%	0.040%	0.0020%	5.029
0.12	0.812%	0.539%	0.410%	0.215%	0.158%	0.042%	0.0020%	5.134
0.13	0.815%	0.554%	0.420%	0.221%	0.158%	0.044%	0.0020%	5.174
0.14	0.819%	0.572%	0.433%	0.231%	0.158%	0.045%	0.0020%	5.168
0.15	0.822%	0.592%	0.449%	0.242%	0.160%	0.046%	0.0021%	5.125
0.16	0.825%	0.616%	0.468%	0.255%	0.163%	0.047%	0.0021%	5.051
0.17	0.829%	0.643%	0.492%	0.270%	0.167%	0.048%	0.0021%	4.948
0.18	0.832%	0.676%	0.520%	0.287%	0.173%	0.048%	0.0022%	4.818
0.19	0.836%	0.716%	0.554%	0.306%	0.179%	0.048%	0.0022%	4.666
0.20	0.840%	0.765%	0.594%	0.329%	0.187%	0.048%	0.0022%	4.497
0.21	0.846%	0.823%	0.642%	0.354%	0.196%	0.048%	0.0023%	4.317

A relatively large range of decay coefficients results in a similar goodness of fit for the TSIR with consistent error values. The volatility of components shows a steadily decreasing pattern for all decay coefficients which, as we will see in later chapters, is very beneficial for risk management and performance attribution. The Ratio column in Table 2.2 is the ratio of the volatility of the level to quartic component and is a measure of the steepness of the decline in the volatility of components. This ratio is highest for a decay coefficient of 0.13.

For the year 2012, the average yield error was about 1 basis point for most decay coefficients in the middle of the above range. The difference between calculated and market prices is very small for all decay coefficients from 0.11 to 0.16 corresponding to 6.3 to 4.3 years at which the slope is zero. In this book we use a decay coefficient of 0.13 for all calculations.

2.8 SIGNIFICANCE OF THE TERM STRUCTURE COMPONENTS

The level of interest rates is associated with the general level of inflation and growth expectations. During periods of rising or high (falling or low) inflation, the level of interest rates rises (falls).

The slope of interest rates is a measure of the future path of interest rates. At times, when the Federal Reserve or central bank lowers interest rates, the slope increases, and during a period of rate hikes, the slope decreases. Oftentimes, a steepening or flattening of the yield curve takes place in anticipation of action by the Fed. During recessions, the slope of the yield curve rises, and during recovery, when inflation falls, the level of rates falls.

The bend of the yield curve is a more subtle property and has to do with the performance of the 5-year part of the curve relative to the long and short rates. When market participants expect lower rates in the future, but the central bank has not started cutting rates, the middle part of the curve outperforms the market. This behavior will be reflected in the yield curve as a rise in the curvature or bend of the yield curve. Likewise, when the market participants expect higher rates in the medium to long future, the 5-year rates rise more than the average of short and long rates. This will result in a fall in the bend of the yield curve.

The Institute of Supply Management (ISM) manufacturing index is based on a monthly survey of purchasing managers who are asked to describe their business activity as better, worse or about the same. The responses are weighted according to the size of the manufacturing companies and combined and seasonally adjusted to calculate the monthly index values. The ISM is a diffusion index with a range of 0–100, but it is mostly in the range of 40–60. It is closely followed by market participants as a leading indicator of manufacturing activity.

Figure 2.11 is a chart of the monthly ISM survey on the left axis as well as the monthly average of the bend component of the Libor (LBR) curve on the right axis. The correlation between the two series is about 40%. The correlation has been weaker in the last few years, partly due to high unemployment rates and market participants' assumption that the central bank is not likely to raise rates any time soon and therefore the 5-year part of the curve has been rich. It should be noted that the ISM index is by definition highly mean reverting. If the index is at 45 and in the following month's surveys the average response is "no change", the index jumps to 50 even though economic activity is at weak levels. Likewise, if the index is at 60 and the following month's survey response is "no change" in activity, the index drops to 50. However, the similarity between the seemingly unrelated data is very interesting and points to the importance of the components of the term structure of rates.

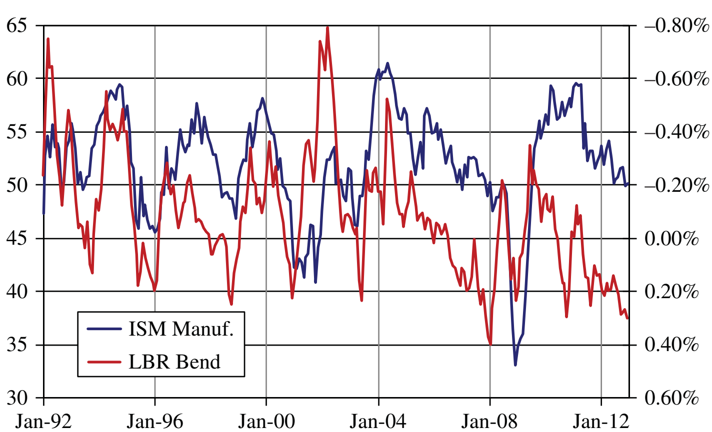


FIGURE 2.11 Comparison of ISM manufacturing index and bend of the TSIR

2.9 ESTIMATING THE VALUE OF THE DECAY COEFFICIENT

The objective of the decay coefficient, as explained in Section 2.1, is to create a reference frame in which all the subintervals have similar significance. In order to preserve simplicity and practicality, we limit the choice of reference frames to exponentially decaying time functions.

We first try to model the cash flow structure of the market, which is front loaded due to the maturity of older issues and concentration of issuance in the 2–5-year maturities. We now assume that the cash flow structure in the market can be approximated by an exponentially decaying function of time,

$$c(t) = c_0\, e^{-\beta t} \tag{2.31}$$

The Barclays Aggregate Bond Index provides a relatively good measure of the cash flow structure of the market. One measure of this cash flow is the duration to worst value, which is not very sensitive to daily interest rate movements and provides a reasonable estimate of the duration of callable bonds and mortgages.

Assuming that the term structure of interest rates is flat, we can calculate the duration of the market as

$$D = \frac{\displaystyle\int_0^\infty c_0 t e^{-(\beta+y)t}\,dt}{\displaystyle\int_0^\infty c_0 e^{-(\beta+y)t}\,dt} = \frac{1}{\beta + y} \tag{2.32}$$

Knowing the duration and yield of Barclays Aggregate, we can estimate the value of cash flow decay coefficient β. It can be easily shown that the present value of all cash flows in a subinterval $d\tau$, defined as

$$d\tau = e^{-(\beta+y)t}\,dt = e^{-\alpha t}\,dts \tag{2.33}$$

are equal. If we assume that in a reference frame where the present value of cash flows are uniformly distributed the yield curve would have close to linear behavior, the decay coefficient α in (2.16) based on present value arguments will be equal to

$$\alpha_{pv} = \beta + y \tag{2.34}$$

It can also be argued that, for equal present values, the yield curve would be more sensitive to a future cash flow than to a present one. If we assume that the yield curve sensitivity is related to the duration impact of a cash flow, then the yield curve reference frame would have the form

$$d\tau = te^{-(\beta+y)t}\,dt \tag{2.35}$$

Equation (2.35) is not a true exponentially decaying reference frame; it also implies that the yield curve sensitivity to short term cash flows is almost zero. To get an idea of the average decay coefficient that (2.35) implies, we will calculate the duration of the impact of cash flows and equate it to the duration of an exponentially decaying reference frame, that is,

$$\frac{\int t d\tau}{\int d\tau} = \frac{\int e^{-\alpha t} t dt}{\int e^{-\alpha t} dt} \tag{2.36}$$

$$\alpha_{dw} = \frac{\beta + y}{2} \tag{2.37}$$

The estimated duration weighted decay coefficient in (2.37) is half that of (2.34) and, as expected, gives a higher weight to longer term cash flows. Equations (2.34) and (2.37) should serve as upper and lower ends of the range for the decay coefficient.

One can also argue that the yield sensitivity should be related to the available cash flows. For this scenario, the decay coefficient would be

$$\alpha_{cf} = \beta \tag{2.38}$$

Figure 2.12 shows the calculated implied historical decay coefficients based on present value (2.34), duration weighted (2.37), and cash flow (2.38) for the Barclays Bond index.

We can use a similar argument using the treasury market. Figure 2.13 shows the implied decay coefficient using treasury market data and issuance. The average calculated cash flow decay coefficient from 12/1991 through 12/2012 is 0.134, with a standard deviation of 0.017. In the following chapters we will see that any α in the range of 0.09–0.21 can be used for risk measurement and performance attribution of

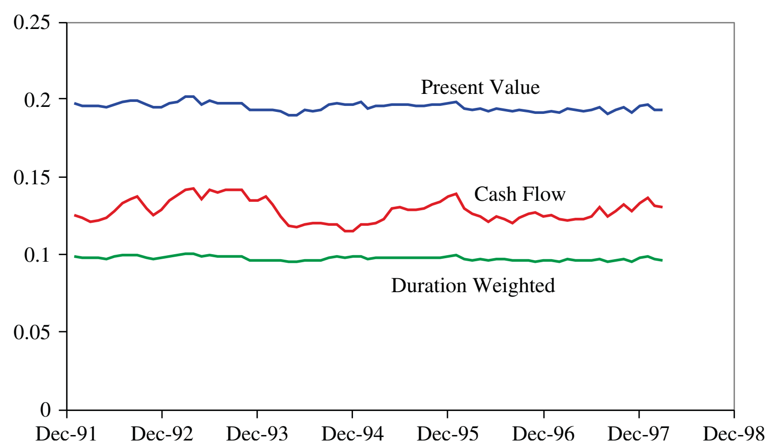


FIGURE 2.12 Implied historical decay coefficient

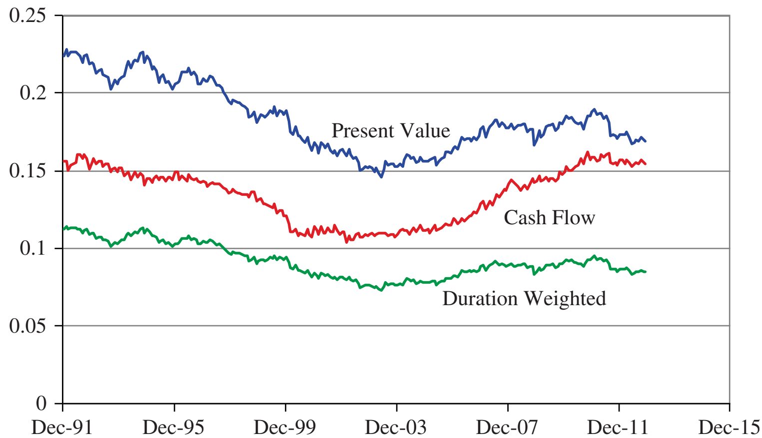


FIGURE 2.13 Implied historical decay coefficient from treasury market

a portfolio with comparable accuracy. We have used $\alpha = 0.13$ throughout this book, which implies a middle point for the term structure of rates of about 5.3 years.

For practical applications, α should be a constant. Changing the value of α results in shifting the relative risks, especially slope, quadratic and higher components of risk, which becomes impractical for portfolio management applications. Given that a range of decay coefficients can be used with similar risk management accuracy, it is best to fix the value of decay coefficient for all currencies and asset classes.

CHAPTER 3

Comparison of Basis Functions

As suggested in the previous chapter, Chebyshev polynomials are just one set of basis functions that we could use to represent the term structure components. In practice, there are infinite number of functions that can be used with the same overall mathematical accuracy. The choice of basis functions is more a matter of practical application than mathematical accuracy. Different basis functions have different applications when it comes to portfolio management and understanding how market forces affect different components. In this chapter we will show the implications of different basis functions and how they can be used for trading or portfolio management and how to transform from one set of basis functions to another.

3.1 POLYNOMIAL BASIS FUNCTIONS

Let us first examine how we can transform Chebyshev basis functions (CBFs) to polynomial basis functions (PBFs). Consider a five-parameter term structure model

$$y = a_0 + a_1\tau + a_2(2\tau^2 - 1) + a_3(4\tau^3 - 3\tau) + a_4(8\tau^4 - 8\tau^2 + 1) \tag{3.1}$$

We can write equation (3.1) in PBF form as

$$y = b_0 + b_1\tau + b_2\tau^2 + b_3\tau^3 + b_4\tau^4 \tag{3.2}$$

with

$$\begin{aligned}
b_0 &= a_0 - a_2 + a_4 \\
b_1 &= a_1 - 3a_3 \\
b_2 &= 2a_2 - 8a_4 \\
b_3 &= 4a_3 \\
b_4 &= 8a_4
\end{aligned}$$

Equations (3.1) and (3.2) are two mathematical representations of the same underlying yield curve and are mathematically identical.

3.2 EXPONENTIAL BASIS FUNCTIONS

The next set of basis functions that we use are exponential basis functions (EBFs). From the definition of τ in (2.16), we can write

$$e^{-\alpha t} = \frac{1-\tau}{2} \tag{3.3}$$

The ith component of the EBF is defined as

$$\chi_i(\tau) = \left(\frac{1-\tau}{2}\right)^i = e^{-\alpha i t} \tag{3.4}$$

3.3 ORTHOGONAL BASIS FUNCTIONS

Another set of basis functions can be constructed in such a way that they are orthogonal to each other in the interval ($-\tau$, τ). As mentioned in Section 2.2, Chebyshev polynomials are orthogonal relative to a weighting function. The orthogonal basis functions (OBFs) can be constructed by an iterative process, similar to the derivation of Chebyshev polynomials. This is accomplished by requiring that every basis function is orthogonal to all the lower order basis functions. Thus, if η_n is the nth orthogonal basis function, we can write

$$\int_{-1}^{+1} \eta_n \eta_i \, d\tau = 0, \quad i = 0, 1, \ldots, n-1 \tag{3.5}$$

Additionally, we require that the maximum amplitude of each basis function be equal to 1. The first two components of orthogonal polynomials are identical to Chebyshev basis functions. The remaining components are somewhat different. The orthogonal basis functions can be written as

$$\begin{aligned}
\eta_0 &= 1 \\
\eta_1 &= \tau \\
\eta_2 &= \tfrac{3}{2}\tau^2 - \tfrac{1}{2} \\
\eta_3 &= \tfrac{5}{2}\tau^3 - \tfrac{3}{2}\tau \\
\eta_4 &= \tfrac{35}{8}\tau^4 - \tfrac{15}{4}\tau^2 + \tfrac{3}{8}
\end{aligned} \tag{3.6}$$

Figure 3.1 shows the components of the orthogonal basis functions.

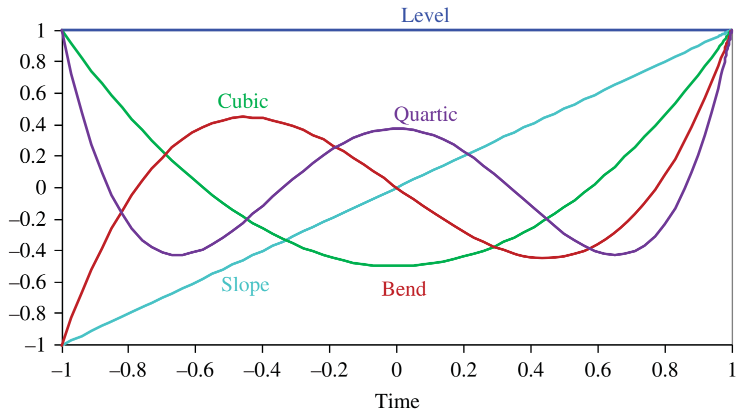


FIGURE 3.1 Orthogonal term structure components in τ space

The minimum value of the bend component for orthogonal and Chebyshev basis functions is respectively -0.5 and -1.0, while the maximum value for both functions is 1.0. The orthogonal model thus defines a simultaneous rise of one unit for long and short rates and a fall of half a unit in medium rates as a unit of bend of the yield curve. In Chebyshev polynomials, the positive and negative amplitudes of each component are equal. On the other hand, in the orthogonal model, the positive and negative areas of each component are equal.

3.4 KEY BASIS FUNCTIONS

Our final set of basis functions is the key basis functions (KBFs). These are based on points on the yield curve called key rates, such that at any key rate only one of the basis functions contributes to the yield and the value of all other key rates is zero. To illustrate how we can change the basis functions, we start with a simple example. Consider a two-parameter term structure model, that is, level and slope components of the TSIR. From (2.18) we can write

$$\begin{aligned} y(t) &= a_0\psi_0 + a_1\psi_1 \\ &= (a_0 - \tfrac{1}{2}a_1)(\tfrac{1}{2}\psi_0 - \psi_1) + (a_0 + \tfrac{1}{2}a_1)(\tfrac{1}{2}\psi_0 + \psi_1) \end{aligned} \tag{3.7}$$

or

$$y(t) = b_0(\tfrac{1}{2}\psi_0 - \psi_1) + b_1(\tfrac{1}{2}\psi_0 + \psi_1) = b_0\chi_0 + b_1\chi_1 \tag{3.8}$$

where

$$\begin{aligned} \chi_0 &= \tfrac{1}{2}\psi_0 - \psi_1 \\ \chi_1 &= \tfrac{1}{2}\psi_0 + \psi_1 \\ b_0 &= a_0 - \tfrac{1}{2}a_1 \\ b_1 &= a_0 + \tfrac{1}{2}a_1 \end{aligned} \tag{3.9}$$

From (2.19) we know that $\psi_0 = 1$. For $\psi_1 = -\frac{1}{2}$ in (3.8) the yield is equal to $y = b_0$ and for $\psi_1 = \frac{1}{2}$ the yield is $y = b_1$. The yield curve is thus explained by the new basis functions χ_0 and χ_1 and the yields of two points b_0 and b_1, instead of the usual level and slope components. The key rates using a decay coefficient of 0.13 will be at

$$\begin{aligned} \psi_0 &= \tau_0 = -\tfrac{1}{2} = 1 - 2e^{-\alpha t_0} \rightarrow t_0 = 2.21 \\ \psi_1 &= \tau_1 = +\tfrac{1}{2} = 1 - 2e^{-\alpha t_1} \rightarrow t_1 = 10.66 \end{aligned} \tag{3.10}$$

In general the following basis functions define one set of key rates:

$$\chi_i(\tau) = \frac{\prod_{j \neq i}^{n-1} (\tau_i - \tau_j)}{\prod_{j \neq i}^{n-1} (\tau - \tau_j)} \tag{3.11}$$

For example, the third key rate basis function ($i = 2$) will be

$$\chi_2(\tau) = \frac{(\tau - \tau_0)(\tau - \tau_1)(\tau - \tau_3)(\tau - \tau_4)}{(\tau_2 - \tau_0)(\tau_2 - \tau_1)(\tau_2 - \tau_3)(\tau_2 - \tau_4)} \tag{3.12}$$

We can also see that for two components, (3.11) results in

$$\begin{aligned} \chi_0 &= \tfrac{1}{2} - \tau \\ \chi_1 &= \tfrac{1}{2} + \tau \end{aligned} \tag{3.13}$$

By simple examination it is clear that at every key rate $\tau_k \neq \tau_i$, one of the factors in the numerator of (3.11) is zero, except at $\tau_k = \tau_i$, where $\chi_i = 1$. Therefore, the coefficient of the basis function at every key rate is simply equal to the yield of the key rate.

The natural set of key rates that we choose are the points where a Chebyshev polynomial of order equal to the number of key rates is zero. For example, for a five-parameter yield curve, we choose a Chebyshev polynomial of degree 5, find the zeros and use those values for the maturity of key rates. From equation (2.21) we can write

$$\tau_j = -\cos\left(\frac{\pi}{2n}(1 + 2j)\right) \tag{3.14}$$

For a five-parameter curve, the optimal key rate points will be at τ = –0.951, –0.588, 0, 0.588, and 0.951.

Having established the five different basis functions, we will analyze the properties of historical yield curves based on each basis function in Section 3.8.

3.5 TRANSFORMATION OF BASIS FUNCTIONS

In the previous section we showed how we could derive the PBF from the CBF. We will now derive the formal transformation process from polynomial to Chebyshev, orthogonal and exponential basis functions.

In general, we can define a new set of basis functions as a linear combination of an existing set of basis functions:

$$y(\tau) = \sum_i a_i \psi_i(\tau) = \sum_k b_k \chi_k(\tau) \tag{3.15}$$

and

$$\chi_j = \sum_i m_{ji} \psi_i \tag{3.16}$$

Consider the case of key rate basis functions. We will expand the yield curve y as a linear combination of functions χ such that at key rate maturity τ_k (expressed in τ space), χ_i is defined as

$$\begin{aligned} \chi_i(\tau_k) &= \chi_{ik} = \delta_{ik} \\ \delta_{ik} &= \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \end{aligned} \tag{3.17}$$

Substituting for χ from (3.17) into (3.15) and evaluating the latter at key rate maturity τ_k leads to

$$b_k = y(\tau_k) = \sum_i a_i \psi_i(\tau_k) = \sum_i a_i \psi_{ik} \tag{3.18}$$

Substituting for χ from (3.16) into (3.15) leads to

$$y = \sum_j a_j \psi_j = \sum_k b_k \sum_j m_{kj} \psi_j \tag{3.19}$$

Equation (3.19) implies that

$$a_j = \sum_k b_k \, m_{kj} \tag{3.20}$$

Substituting for b_k in (3.20) from (3.18), we arrive at

$$a_j = \sum_i \sum_k a_i \psi_{ik} m_{kj} = \sum_k \left(\sum_i a_i \psi_{ik} \right) m_{kj} \tag{3.21}$$

Thus,

$$[\mathbf{M}][\boldsymbol{\Psi}] = [\mathbf{I}] \tag{3.22}$$

where the (j, k)th element Ψ_{jk} of matrix $\boldsymbol{\Psi}$ is defined in (3.18) and the (j, k)th element m_{jk} of matrix $\mathbf{M}$ is defined in (3.16). By inverting $\boldsymbol{\Psi}$ matrix, the coefficients of the transformation matrix $\mathbf{M}$ can be calculated. We can then use (3.16) to calculate key rate basis functions.

For a three-parameter term structure model, at key rate points $\tau = 0$ and $\tau = \pm\,\tau_0$, the transformation matrix **M** is

$$\mathbf{M} = \frac{1}{4\tau_0^2}\begin{bmatrix} 1 & 2(2\tau_0^2 - 1) & 1 \\ -2\tau_0 & 0 & 2\tau_0 \\ 1 & -2 & 1 \end{bmatrix} \tag{3.23}$$

In general, transforming from PBFs to KBFs is more straightforward and the matrix elements can be calculated analytically. We will show that the trial yield curve $y_t(\tau)$ defined as

$$y_t(\tau) = \sum_i \left(y(\tau_i)\frac{\prod_{j\neq i}^{n}(\tau - \tau_j)}{\prod_{j\neq i}^{n}(\tau_i - \tau_j)} \right) \tag{3.24}$$

is equal to the calculated yield curve $y(t)$. We can see by inspection that the right hand side of (3.24) is a polynomial whose value is equal to $y(\tau_i)$ for every key rate maturity τ_i. For example, if there are three key rates at maturities corresponding to τ_0, τ_1, and τ_2, $y_t(\tau)$ will be equal to

$$y_t(\tau) = y(\tau_0)\frac{(\tau - \tau_1)(\tau - \tau_2)}{(\tau_0 - \tau_1)(\tau_0 - \tau_2)} + y(\tau_1)\frac{(\tau - \tau_0)(\tau - \tau_2)}{(\tau_1 - \tau_0)(\tau_1 - \tau_2)} + y(\tau_2)\frac{(\tau - \tau_0)(\tau - \tau_1)}{(\tau_2 - \tau_0)(\tau_2 - \tau_1)}$$

It is clear that at $\tau = \tau_1$, $y_t(\tau) = y(\tau_1)$. Thus $y_t(\tau)$ intersects $y(\tau)$ at all key rate maturities. Since there is one and only one polynomial of order $n - 1$ that would go through n different points, $y_t(\tau) = y(\tau)$. From (3.24) we can write the basis functions in KBFs as

$$y(\tau) = \sum_{i=0}^{n-1} y(\tau_i)\chi_i(\tau)$$

$$\chi_i(\tau) = \frac{\prod_{j\neq i}^{n-1}(\tau - \tau_j)}{\prod_{j\neq i}^{n-1}(\tau_i - \tau_j)} \tag{3.25}$$

Equation (3.25) satisfies (3.17) at all key rate maturities. Expanding the numerator of the right hand side of (3.25) leads to

$$\prod_{j\neq i}^{n-1}(\tau - \tau_j) = \tau^{n-1} - \tau^{n-2}\sum_{j\neq i}^{n-1}\tau_j + \tau^{n-3}\sum_{(j>k)\neq i}^{n-1}\tau_j\tau_k + \cdots + \prod_{j\neq i}^{n-1}\tau_j = \sum_j^{n-1} e_{ij}\tau^j \tag{3.26}$$

where

$$
\begin{aligned}
e_{i,n-1} &= (-1)^0\, 1 \\
e_{i,n-2} &= (-1)^1 \sum_{j\neq i} \tau_j \\
e_{i,n-3} &= (-1)^2 \sum_{(j>k)\neq i} \tau_j \tau_k \\
&\dots \\
e_{i,0} &= (-1)^{n-1} \prod_{\substack{j=0 \\ j\neq i}}^{n-1} \tau_j
\end{aligned}
\tag{3.27}
$$

Combining (3.25) and (3.26) leads to

$$
\chi_i(\tau) = \frac{\sum_j e_{ij}\tau^j}{\prod_{j\neq i}(\tau_i - \tau_j)} \tag{3.28}
$$

In the polynomial basis notation, if we replace Ψ_j by τ^j in (3.16) and equate the resulting equation to (3.28), we find that

$$
c_{ij} = \frac{e_{ij}}{\prod_{j\neq i}(\tau_i - \tau_j)} \tag{3.29}
$$

Equation (3.29) is the analytical solution for the transformation matrix to convert from PBFs to KBFs.

To transform from CBFs or OBFs to PBFs and vice versa, we can write

$$
y = \sum_j a_j \psi_j = \sum_j b_j \tau^j \tag{3.30}
$$

where

$$
\psi_j = \sum_{i=0}^{j} c_{ji}\tau^i \tag{3.31}
$$

Since the matrix **C** for the CBF or OBF coefficients is triangular, the transformation coefficients can be easily calculated using the formula

$$
b_i = \sum_{j=i}^{n-1} a_j c_{ji} \tag{3.32}
$$

Likewise, to transform from PBFs to CBFs or OBFs, we can use the recursive formula

$$a_i = \frac{b_i - \sum_{j=i+1}^{n-1} a_j c_{ji}}{c_{ii}} \tag{3.33}$$

To transform from PBFs to EBFs we note the relationship

$$\tau^i = \left(1 - 2e^{-\alpha t}\right)^i = (1 - 2x)^i = \sum_{j=0}^{i} \binom{i}{j} (-1)^j 2^j x^j \tag{3.34}$$

Thus, PBFs can be considered as a special set of basis functions of the exponential representation of the TSIR. To transform from PBFs to EBFs we use the same equation that we used for transforming from CBFs or OBFs to PBFs and vice versa.

The matrices [**CP**], [**OP**], [**CE**], and [**PE**] in equations (3.35)–(3.38) are used for CBF–PBF, OBF–PBF, CBF–EBF, and PBF–EBF conversions, respectively:

$$[\mathbf{CP}] = \left[C_{ji}\right] = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ -1 & 0 & 2 & & & & \\ 0 & -3 & 0 & 4 & & & \\ 1 & 0 & -8 & 0 & 8 & & \\ 0 & 5 & 0 & -20 & 0 & 16 & \\ -1 & 0 & 18 & 0 & -48 & 0 & 32 \end{bmatrix} \tag{3.35}$$

$$[\mathbf{OP}] = \left[C_{ji}\right] = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ -\frac{1}{2} & 0 & \frac{3}{2} & & & & \\ 0 & -\frac{3}{2} & 0 & \frac{5}{2} & & & \\ \frac{3}{8} & 0 & -\frac{15}{4} & 0 & \frac{35}{8} & & \\ 0 & \frac{15}{8} & 0 & -\frac{35}{4} & 0 & \frac{63}{8} & \\ -\frac{5}{16} & 0 & \frac{105}{16} & 0 & -\frac{315}{16} & 0 & \frac{231}{16} \end{bmatrix} \tag{3.36}$$

$$[\mathbf{CE}]=\left[C_{ji}\right]=\begin{bmatrix}1 & & & & & & \\ 1 & -2 & & & & & \\ 1 & -8 & 8 & & & & \\ 1 & -18 & 48 & -32 & & & \\ 1 & -32 & 160 & -256 & 128 & & \\ 1 & -50 & 400 & -1120 & 1280 & -512 & \\ 1 & -72 & 840 & -3584 & 6912 & -6144 & 2048\end{bmatrix} \tag{3.37}$$

$$[\mathbf{PE}]=\left[C_{ji}\right]=\begin{bmatrix}1 & & & & & & \\ 1 & -2 & & & & & \\ 1 & -4 & 4 & & & & \\ 1 & -6 & 12 & -8 & & & \\ 1 & -8 & 24 & -32 & 16 & & \\ 1 & -10 & 40 & -80 & 80 & -32 & \\ 1 & -12 & 60 & -160 & 240 & -192 & 64\end{bmatrix} \tag{3.38}$$

For example, to transform from CBFs to PBFs, we use the following matrix operation for five components:

$$\begin{aligned}\left[b_0 \quad b_1 \quad b_2 \quad b_3 \quad b_4\right] &= \left[a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4\right]\times[\mathbf{CP}] \\ &= \left[a_0 - a_2 + a_4, \quad a_1 - 3a_3, \quad 2a_2 - 8a_4, \quad 4a_3, \quad 8a_4\right]\end{aligned} \tag{3.39}$$

To transform from PBFs to CBFs, we can either use the inverse of (3.35) or solve it recursively from highest order to lowest order components respectively. The inverse of (3.35) is

$$[\mathrm{PC}]=\left[C_{ji}\right]=\begin{bmatrix}1 & & & & & & \\ 0 & 1 & & & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & & & \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} & & & \\ \frac{3}{8} & 0 & \frac{1}{2} & 0 & \frac{1}{8} & & \\ 0 & \frac{5}{8} & 0 & \frac{5}{16} & 0 & \frac{1}{16} & \\ \frac{1}{4} & 0 & \frac{15}{32} & 0 & \frac{3}{16} & 0 & \frac{1}{32}\end{bmatrix} \tag{3.40}$$

Given that the matrices (3.35)–(3.40) are triangular, the transformation coefficients can be calculated recursively instead of using the inverse matrix as in (3.40). For example, to calculate the CBF coefficients (a_i) from PBF coefficients (b_i) using matrix C_{ji} in (3.35), we write

$$\begin{aligned} a_i &= \frac{b_i - d_i}{C_{ii}} \\ d_i &= \sum_{j=i+1}^{n-1} C_{ji} a_j \end{aligned} \tag{3.41}$$

For example, d_2 is calculated as

$$d_2 = C_{32}a_3 + C_{42}a_4 = -8a_4$$

Thus,

$$a_2 = \frac{b_2 + 8a_4}{2}$$

The transformation to and from KBFs is not as straightforward as for other cases. The key maturities can be arbitrarily selected and in practice they are selected in such a way that they coincide with on-the-run treasuries. We define the natural key rates as the points where the roots of Chebyshev polynomials are zero. The respective τs are calculated from (2.21) as

$$\begin{aligned} \psi_n &= \cos(n \arccos(\tau)) = 0 \\ \tau_i &= -\cos\left(\frac{\pi(2i+1)}{2n}\right), \quad i = 0, \ldots, n-1 \end{aligned} \tag{3.42}$$

The negative sign is required to provide a sequence of increasing τ from the most negative to the most positive. The natural key maturities will be defined from (2.16):

$$\begin{aligned} \tau_i &= 1 - 2e^{-\alpha t_i} \\ t_i &= -\frac{1}{\alpha} \ln\left(\frac{1-\tau_i}{2}\right) \end{aligned} \tag{3.43}$$

To convert from PBFs to KBFs, for $n = 5$, $\tau_i = 0, \pm 0.95106, \pm 0.58779$, we create the following matrix based on (3.18):

$$\psi_{ji} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -0.95106 & -0.58779 & 0 & 0.58779 & 0.95106 \\ 0.90451 & 0.34549 & 0 & 0.34549 & 0.90451 \\ -0.86024 & -0.20307 & 0 & 0.20307 & 0.86024 \\ 0.81814 & 0.11936 & 0 & 0.11936 & 0.81814 \end{bmatrix} \tag{3.44}$$

If the PBF coefficients of the TSIR are

$$a_j = \begin{bmatrix} 0.0174 & 0.021 & 0.0028 & -0.004 & 0.0032 \end{bmatrix} \tag{3.45}$$

the KBF coefficients from (3.18) will be

$$b_k = \sum_j a_j \psi_{jk} = \begin{bmatrix} 0.00602 & 0.00722 & 0.0174 & 0.03028 & 0.03908 \end{bmatrix} \tag{3.46}$$

The coefficients b_k are the value of spot yields at the respective key rate maturities. Inverting the above matrix, we will find that the KBF coefficients are

$$m_{ki} = \begin{bmatrix} 0 & 0.32492 & -0.34164 & -0.94046 & 0.98885 \\ 0 & -1.37638 & 2.34164 & 1.52169 & 2.58885 \\ 1 & 0 & -4 & 0 & 3.2 \\ 0 & 1.37638 & 2.34164 & -1.52169 & 2.58885 \\ 0 & -0.32492 & -0.34164 & 0.94046 & 0.98885 \end{bmatrix}. \tag{3.47}$$

3.6 COMPARISON WITH THE PRINCIPAL COMPONENTS ANALYSIS

Principal components analysis (PCA) is a statistical/linear algebra method of analyzing a large number of correlated variables and identifying the most significant components. For example, there are many economic indicators that are used to gauge the state of the economy such as unemployment rate, durable goods orders, inflation rate, S&P earnings, and manufacturing indexes. In many econometric models an average of a sample of important economic factors is calculated as a gauge of economic activity, and its relation with economic downturns and upturns is analyzed to predict future recessions or recoveries. PCA can be used to identify the best unbiased sets of variables based on historical data that describe economic factors.

The process of performing PCA is to first calculate the correlation matrix of the changes of all relevant variables. Suppose we take monthly time series of unemployment rates and durable goods orders. We then calculate the monthly changes of each time series and calculate the correlation of the changes. Thus, if we have 50 economic time series, we will construct a 50 × 50 correlation matrix. The principal components are equal to the eigenvectors of the correlation matrix, and the weights of the eigenvectors are equal to the eigenvalues of the matrix. An eigenvector of a matrix **A** is defined as a vector that when multiplied by the matrix, will result in itself multiplied by a constant:

$$\mathbf{A}\vec{v} = \lambda \vec{v} \tag{3.48}$$

In this equation λ is called the eigenvalue and $\vec{v}$ is called the eigenvector. The eigenvalues can be calculated by introducing the identity matrix **I** into the right hand side of the above equation,

$$\begin{aligned} \mathbf{A}\vec{v} &= \lambda \mathbf{I}\vec{v} \\ (\mathbf{A} - \lambda \mathbf{I})\vec{v} &= 0 \\ \left[a_{ij} - \lambda \delta_{ij}\right] &= 0 \\ \delta_{ij} &= \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \end{aligned} \tag{3.49}$$

For example, suppose that we are analyzing the ISM manufacturing index and housing starts and do not know which one to use if we have room for just one more indicator. Assume that the two series have a correlation of 0.5. We will then construct the correlation matrix as

$$\mathbf{A} = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$

Eigenvalues can be calculated from

$$\det \begin{bmatrix} 1-\lambda & 0.5 \\ 0.5 & 1-\lambda \end{bmatrix} = 0$$

giving $\lambda_1 = 1.5$ and $\lambda_1 = 0.5$. Once the eigenvalues are calculated, we can solve for eigenvectors using the set of linear equations in (3.49). The corresponding eigenvectors are

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda_1 = 1.5$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \lambda_2 = 0.5$$

The value of each element of an eigenvector corresponds to the weight of the corresponding variable. In our example, the first eigenvector has equal weights of the original variables, namely, ISM manufacturing index and housing starts. The absolute value of an eigenvalue is an indication of the importance of the corresponding eigenvector. The largest eigenvalue corresponds to the first principal component, the second largest eigenvalue to the second principal component, etc. Thus, the first eigenvector which corresponds to using equal weights for ISM manufacturing index and housing starts is a superior indicator compared to using either one of them or any other combination of the two variables.

Eigenvectors of a matrix are uncorrelated and their multiplication leads to zero. In the above example, $\vec{v}_1^T \times \vec{v}_2 = 0$. It is standard practice to normalize an eigenvector by scaling it to a unit vector. When the correlation matrix is constructed, each variable is scaled by its standard deviation. Similarly, the weights of variables in the eigenvectors have to be scaled by their respective standard deviations before analysis. In the above example, if housing starts is four times as volatile as the ISM manufacturing index, it has to be scaled back so that both have similar contributions, on average, to the principal components.

PCA can be used for the analysis of components of the term structure of rates. The variables used for such an analysis are the spot yields of different points on the treasury curve. Since all yields are highly correlated, PCA is a very useful method to estimate the principal modes of fluctuations of the term structure of rates. However, the choice of maturities affects the resulting eigenvectors, and thus there is no unique set of principal components. For example, if we assume that the first principal component is equal to the level of rates, the average value of the elements of the second component has to be zero in order for it to be orthogonal to the first component. If we use many points at the long end of the curve, then the second principal component will have a different shape than if we use fewer points at long maturities.

The maturities of the spot curve used for the calculation of the principal components act like a weighting function that will affect the shape of the principal components to ensure their orthogonality. Orthogonal basis functions are orthogonal to each other without a weighting function (or a weighting function of unity) and Chebyshev basis functions are orthogonal to each other with a weighting function of $1/\sqrt{1-\tau^2}$, as was mentioned in Section 2.2. The implied weighting function for PCA is a discrete function of the selected maturities of the spot yields.

Considering that the average maturity of most market benchmarks is around 5 years, we constructed a set of maturities that represented the market structure by having an equal number of points with maturities below and above the 5-year part of the curve. One way of selecting the maturities is to use equally spaced maturities of the spot yields in the τ space. If we used equally spaced maturities in time, the long end of the curve would be overemphasized. For example, using quarterly maturities, there will be 100 maturities between 5 years and 30 years, and only 20 between the zero and 5-year part of the curve. Figure 3.2 shows the calculated principal components as well as the

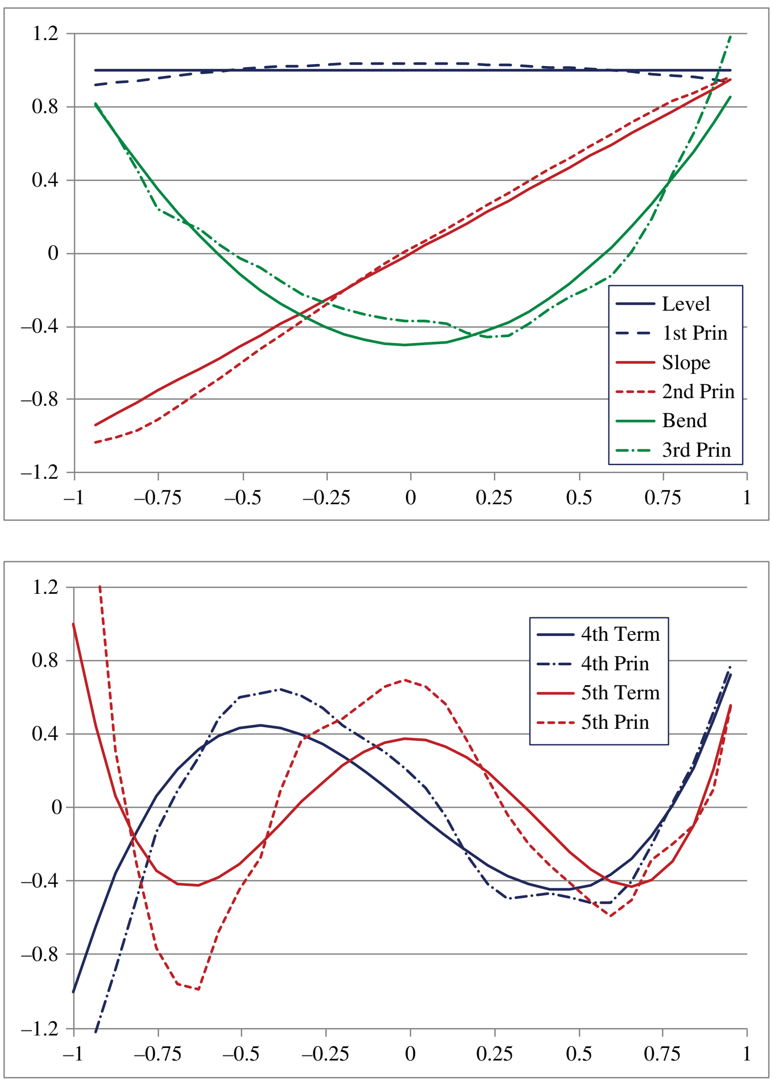


FIGURE 3.2 Orthogonal term structure and principal components in τ space, 1992–2012

orthogonal basis functions of the term structure of interest rates in the US. The PCA was performed for the period 1992–2012. The first three components are almost identical and the last two components have very similar shapes and peaks and valleys. The fifth principal component is almost identical to the fifth component of the term structure in Chebyshev basis functions and thus can be obtained by linear transformation.

Since each vector of the principal components had a different scale from our term structure components, we simply divided even numbered components, level, bend, and quartic, by a constant number equal to the average value of all the vector elements to be equal to one. For odd numbered components (slope and cubic) we divided all elements by half the range of values.

Figure 3.3 shows the orthogonal basis functions and the volatility adjusted principal components in tau space. These graphs show that the modes of fluctuations

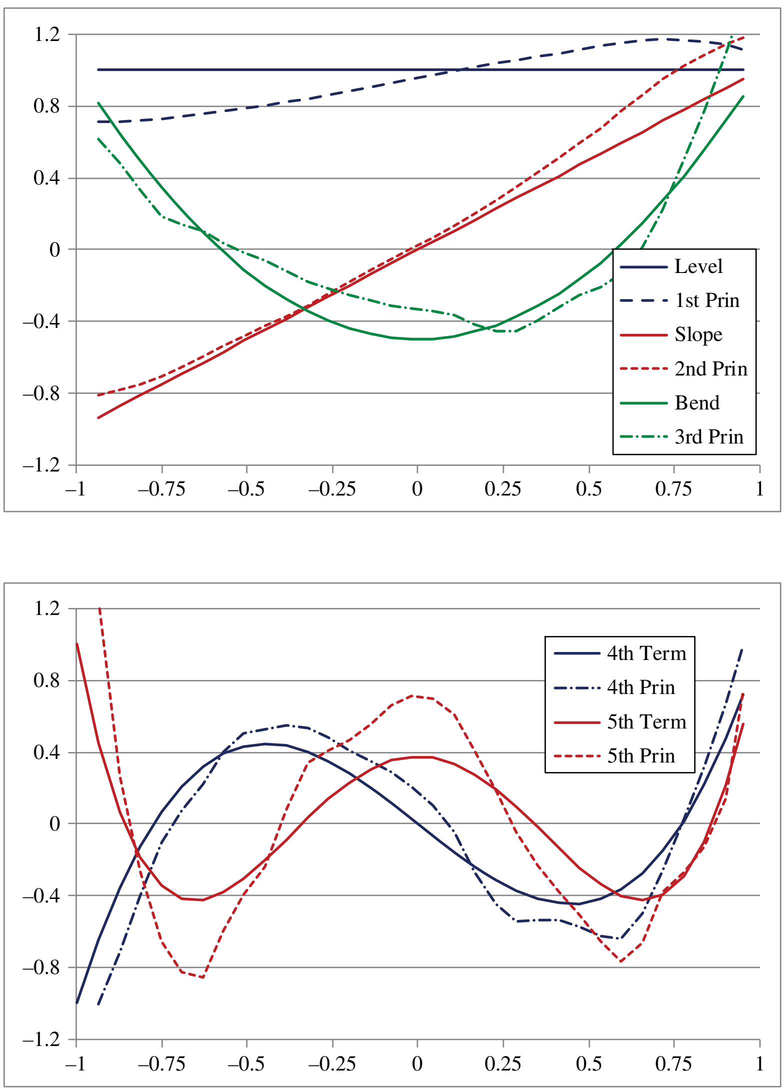


FIGURE 3.3 Term structure and volatility adjusted principal components in τ space, 1992–2012

of the term structure of interest rates can be best approximated by our term structure model in the τ space. Not only are the shapes very similar to the Chebyshev or orthogonal basis functions, but also the peaks and valleys are nearly at the same points. We have already learned in this chapter that the basis functions can be transformed to each other without loss of generality. Likewise, we can construct linear transformations that would construct the fourth and fifth principal components almost exactly. In essence, our term structure of interest rates speaks the language of the markets.

For the calculation of the principal components we used maturities in such a way that they were linear in τ space. This implied that the resulting principal components would be closer to orthogonal basis functions than to Chebyshev. Recall that in the orthogonal basis functions all components are orthogonal to each other, while Chebyshev requires a weighting function for orthogonality. For example, the bend component is orthogonal to the level which is equal to one, and thus

$$\int_{-1}^{1} (\tfrac{3}{2}\tau^2 - \tfrac{1}{2})d\tau = 0$$

The third principal component is similarly orthogonal to the first one, which is almost a constant. Thus, the shape of the third principal component must be closer to the bend component of the orthogonal basis function than to Chebyshev. The first three components account for more than 99% of the variations in the shape of the yield curve. Table 3.1 shows the weight of principal components that is proportional to the value of the eigenvalues of the principal components.

The orthogonality of basis functions is a very useful feature to construct a portfolio of uncorrelated strategies. In practice, this does not work because correlations are very unstable and volatility of the market changes. At times of stress, when the volatility is highest and a portfolio could benefit most from diversification, the correlations approach 1 or –1. At other times, depending on the level of inflation and the actions of the central bank, the correlations can be positive or negative. For example, during tightening of monetary policy, the curve tends to flatten while at the same time the level of rates rise (slope falling, level rising), leading to a negative correlation. During recovery periods, when inflation is falling, both the level and slope of the curve are likely to fall, leading to positive correlation. In the long run, these positive and negative correlations can cancel each other out, but during a typical investment horizon of 3 months to 1 year, correlations tend to be persistent.

TABLE 3.1 Weights of principal components, 1992–2012

	Weight	Total
1st	92.32	92.32
2nd	7.19	99.51
3rd	0.32	99.83
4th	0.10	99.93
5th	0.03	99.96

3.7 MEAN REVERSION

Mean reversion refers to the property of a distribution that is more likely to have changes in the direction of its long term mean than away from it. A random walk does not have a statistically significant mean reversion tendency. In our term structure model, the bend parameter tends to have a mean reversion tendency. If the bend is large and positive, the 5-year part of the curve becomes very rich. Fund managers structure barbell portfolios to take advantage of the superior yields below and above 5-year treasuries, leading to the underperformance of the 5-year area of the curve. Likewise, when the bend is large and negative, the 5-year part of the curve will be cheap relative to the short and long ends and the opposite dynamic works to revert it to mean.

A simple way of modeling a mean reverting process is to note that changes in the dependent variable u are proportional to the distance of the variable from its mean, thus

$$du = -m(u - \bar{u})dt \tag{3.50}$$

where m is the proportionality constant and is a positive number. This equation is similar to a spring function, where the tendency to revert to the mean position is stronger, the farther the spring is from its mean. Solving for this equation, we obtain

$$\ln\left(\frac{u - \bar{u}}{u_0 - \bar{u}}\right) = -mt \tag{3.51}$$

We define the half-life as the time that it takes for a distribution to retrace half of its distance from its long term mean by setting the value of the argument in ln to 0.5:

$$t_{1/2} = \frac{\ln(2)}{m} \tag{3.52}$$

Table 3.2 shows the half-life of the term structure components in different basis functions. Table 3.3 shows the t-test for the validity of calculating the half-life for each of the basis functions. A t-test value greater than 2 is statistically significant.

For the CBF, the first component (level) has a half-life of 14.1 years and a t-statistic of 0.47. Obviously, this is not statistically significant. Also considering that we used 21 years of data, a half-life of 10 years or more would not provide enough instances of

TABLE 3.2 Historical half-life (mean reversion) of US treasury term structure components

	First	Second	Third	Fourth	Fifth
CBF	14.10	3.51	0.68	0.48	0.67
OBF	12.27	3.63	0.67	0.48	0.67
PBF	7.79	2.06	0.66	0.48	0.67
EBF	4.13	0.93	0.84	0.75	0.67
KBF	7.89	15.60	7.79	4.98	4.36

TABLE 3.3 *t*-test of half-life of US treasury term structure components

	First	Second	Third	Fourth	Fifth
CBF	0.47	1.49	3.18	4.02	3.49
OBF	0.50	1.47	3.18	4.02	3.49
PBF	0.70	1.92	3.53	4.02	3.49
EBF	1.28	2.83	3.10	3.32	3.49
KBF	0.83	0.45	0.70	0.94	1.18

mean reversion for the data to be significant. Looking at the third component (bend), we can see a half-life of 0.68 years and a *t*-statistic of 3.18. Given that the mean reversion is very significant, we can use a mean reversion trade, for example, when the bend component is more than one standard deviation from its mean.

The exponential basis function provides a mean reversion trade on the slope of the yield curve with a reasonable half-life of 0.93 years and a *t*-statistic of 2.83.

For a mean reversion process to be statistically significant, there have to be many oscillations in the data and the mean has to be crossed multiple times. Figure 3.4 is an example of a bend component of the US treasury curve in CBF which is mean reverting. The mean of the distribution is −0.14% and there are multiple oscillations around the mean.

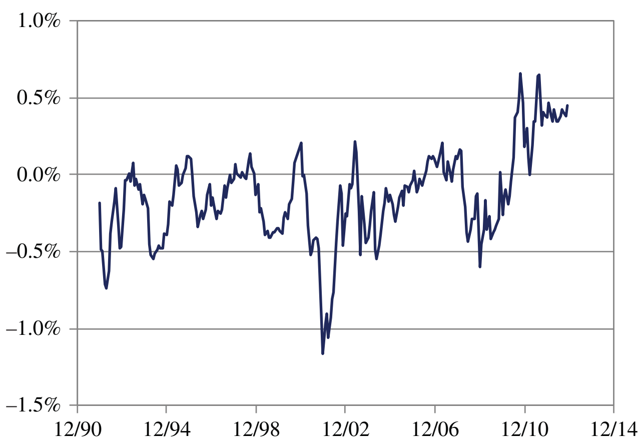


FIGURE 3.4 Historical bend of the Chebyshev basis function

3.8 HISTORICAL TABLES OF BASIS FUNCTIONS

Table 3.4 lists the average of the components of the US treasury term structure data in the period 1991–2012 for different basis functions. Notice how the absolute value of each component becomes smaller for CBF than the previous one (except for the fifth). This is one of the very attractive properties of CBF and we will cover this in more detail in Chapters 4 and 5.

TABLE 3.4 Average value of US treasury term structure components

	First	Second	Third	Fourth	Fifth
Cheby	4.34%	1.49%	−0.13%	0.03%	−0.11%
Ortho	4.39%	1.47%	−0.10%	0.04%	−0.20%
Poly	4.37%	1.41%	0.60%	0.10%	−0.87%
Exp	5.61%	1.12%	−17.22%	26.97%	−13.89%
Key	2.77%	3.62%	4.37%	5.32%	5.63%

TABLE 3.5 Annualized absolute volatility of US treasury term structure components

	First	Second	Third	Fourth	Fifth
Cheby	1.03%	0.75%	0.17%	0.14%	0.03%
Ortho	0.98%	0.23%	0.13%	0.07%	0.03%
Poly	1.31%	2.11%	1.06%	1.54%	0.05%
Exp	9.95%	22.47%	23.10%	8.61%	0.13%
Key	0.04%	0.83%	1.71%	2.15%	0.90%

Table 3.5 provides the annualized volatility of components of the US treasury term structure for different basis functions. Note that the CBF volatility falls steadily for each successive component. This is another very attractive property of CBF for risk measurement and risk management. Additionally, for most portfolios, the duration exposure of the term structure of rates falls for each successive component, leading to a rapid decline in contribution to risk for each successive component of CBF. None of the other basis functions has such a strong declining contribution to risk property.

CHAPTER 4

Risk Measurement

There are many sources of risk in a fixed income portfolio, including interest rate risk, credit risk, liquidity risk, currency risk, prepayment risk, and market risk. In this chapter we cover the most straightforward risk to measure and manage (hedge), namely interest rate risks. Other risks will be covered in later chapters.

4.1 INTEREST RATE RISKS

The calculated price of a risk-free non-contingent bond can be written as

$$p_t = \sum_i c_i e^{-y(t_i)t_i} = \sum_i c_i e^{-t_i \sum_j a_j \psi_j} \tag{4.1}$$

Duration is defined as the present value weighted average time to cash flows of a bond:

$$D = \frac{1}{p_t} \sum_i c_i t_i \, e^{-y_i t_i} \tag{4.2}$$

Mathematically, duration can be defined as the derivative of the logarithm of price with respect to yield. Using the term structure model, we can calculate duration and convexity relative to changes in the level, slope, bend, etc. components of the yield curve, by calculating the first and second derivatives relative to a_0, a_1, etc. Thus,

$$\vec{D} = [D_k] = -\frac{1}{p_t}\frac{\partial p_t}{\partial a_k} = \frac{1}{p_t}\sum_i c_i t_i \psi_k \, e^{-t_i \sum\limits_j a_j \psi_j} = \left[\left\langle t\psi_k(t)\right\rangle\right]_{k=0,1,\ldots n-1} \tag{4.3}$$

$$\begin{aligned} \mathbf{X} = [X_{kl}] &= \frac{1}{p_t}\frac{\partial^2 p_t}{\partial a_k \partial a_l} = \frac{1}{p_t}\sum_i c_i t_i^2 \, \psi_k \psi_l \, e^{-t_i \sum\limits_j a_j \psi_j} \\ &= \left[\left\langle t^2 \psi_k(t)\psi_l(t)\right\rangle\right]_{k,l=0,1,\ldots n-1} \end{aligned} \tag{4.4}$$

The Advanced Fixed Income and Derivatives Management Guide. Saied Simozar

Equation (4.3) represents the vector of duration components of a bond relative to the TSIR, and (4.4) is the matrix of term structure cross-convexities. In the above equations, p_t is the calculated price of the security based on the TSIR. Using the calculated price is preferable to the market price to ensure that identical cash flows will have identical duration measurements, regardless of their market prices.

The first three duration components obtained by substituting k=0, 1, and 2 in (4.3) are the level, slope, and bend duration components, respectively.

Substituting $\psi_0 = 1$ from (2.19) into (4.3), it is evident that the level duration is close to the modified or Macaulay duration of a bond, which are identical to each other when using continuous compounding of the yield. Chapter 5 provides some evidence that the level duration of a typical portfolio is the largest source of return and risk. The importance of higher order components falls significantly to the point, where the contributions from the fourth or fifth component become practically zero for risk management.

Aside from the curve exposure, the duration components of the TSIR provide a consistent measure of duration at all aggregate levels. Since every cash flow is discounted by the market discount function, aggregation of cash flows into securities does not impact the duration contribution of individual cash flows. A fundamental weakness of the conventional duration measurement is that all cash flows in a security are discounted by the same average yield, which in a non-flat yield curve environment results in internal inconsistencies. In Chapter 1 we showed that the conventional measure of duration for a portfolio requires adjustments which are not easy to make.

Since the volatility of the level component of the TSIR is the largest, X_{00} is likely to be the most important contributor to the overall convexity. Unlike ordinary convexity, cross-convexity components are not positive all the time. If the level, slope, and other components of the TSIR are weakly correlated, the contribution from cross-convexity will be very small.

The following is a representation of the cross-convexity matrix:

$$\begin{bmatrix} X_{00} & X_{01} & X_{02} & \cdots \\ X_{10} & X_{11} & X_{12} & \cdots \\ X_{20} & X_{21} & X_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} \left\langle t^2 \right\rangle & \left\langle t^2\tau \right\rangle & \left\langle t^2(2\tau^2-1) \right\rangle & \cdots \\ \left\langle t^2\tau \right\rangle & \left\langle t^2\tau^2 \right\rangle & \left\langle t^2\tau(2\tau^2-1) \right\rangle & \cdots \\ \left\langle t^2(2\tau^2-1) \right\rangle & \left\langle t^2\tau(2\tau^2-1) \right\rangle & \left\langle t^2(2\tau^2-1)^2 \right\rangle & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{4.5}$$

The convexity contribution of diagonal components defined below will always be positive:

$$X_{kk} = \frac{1}{p}\frac{\partial^2 p}{\partial a_k^2} = \frac{1}{p}\sum_i c_i t_i^2 \psi_k^2 \, e^{-t_i \sum_i a_j \psi_j} = \left\langle t^2 \psi_k^2 \right\rangle \tag{4.6}$$

For portfolio structuring and risk management, the first component of the TSIR duration defines the sensitivity to the changes in the level of interest rates. For example, a portfolio that is overweight in level sensitivity outperforms the market when interest rates fall.

The second component, slope, measures the sensitivity to the slope of the yield curve. In portfolio management jargon, it is called a flattening or a steepening trade; an overweight in slope sensitivity results in outperformance relative to the market in a flattening environment and vice versa.

The third component, bend, measures the sensitivity to the curvature of the yield curve or a butterfly trade. A portfolio that is long the butterfly will outperform the market if the yields at the two ends of the yield curve fall compared to the middle part of the curve. In this case the curvature of the yield curve falls in value or becomes more negative.

Since successive components of Chebyshev basis functions have more oscillations, their volatility and overlap with the prior components tend to be rapidly declining. Likewise, in a typical portfolio, exposure to higher order duration components tends to be small due to the even distribution of cash flows and lower volatility of those interest rate components. This implies that higher order duration parameters would have lower and lower impact on the performance of a portfolio when using CBFs.

For a five-parameter term structure, there are 15 independent convexity components. In CBFs, the contribution of each successive component of cross-convexity declines significantly relative to the previous component. This is not the case for all other basis functions. For example, in key basis functions, all components have comparable volatility and average (see Tables 3.4 and 3.5). For most applications, only four components of convexity capture about 98% of convexity contribution to risk and return; these components are: X_{00}, X_{01}, X_{02} and X_{11}.

The decline in the volatility and contribution of risk of successive components of the CBF is very important to integrate risk measurement and valuation. For example, the value of individual securities can be measured by using the first five components of the TSIR, while the first three components may be sufficient for risk management.

Applying the term structure risk measurement across all fixed income asset classes can provide for a very powerful platform for portfolio applications and hedging. This is especially true of derivatives securities such as eurodollar futures contracts and floating rate notes that have a constant or zero sensitivity to the level of interest rates but have very strong exposure to the slope or bend components of the curve.

The application of this model to more complex securities such as spread products and derivatives is also quite straightforward. One can easily overlay a term structure of credit spreads on the risk-free term structure of interest rates and discount complex cash flows such as credit default swaps or Brady bonds using the respective treasury or credit discount functions.

The exponential nature of the basis functions leads to easy differentiation for calculating forwards or higher order derivatives of risk components.

To illustrate the use and interpretation of duration components we will provide a few examples.

4.2 ZERO COUPON BONDS EXAMPLES

The calculation of duration components for a zero coupon bond is very easy, since the summation in (4.3) is replaced with the single cash flow at maturity:

$$\begin{aligned} D_0 &= \frac{1}{p}\frac{\partial p}{\partial a_0} = t_m \\ D_1 &= \frac{1}{p}\frac{\partial p}{\partial a_1} = t_m \tau_m \\ D_2 &= \frac{1}{p}\frac{\partial p}{\partial a_2} = t_m (2\tau_m^2 - 1) \end{aligned} \tag{4.7}$$

TABLE 4.1 Duration components of zero coupon bonds

Maturity Years	Level Duration	Slope Duration	Bend Duration
1.22	1.22	−0.86	0.00
5.33	5.33	0.00	−5.33
14.78	14.78	10.45	0.00

where t_m and τ_m are time and τ to maturity for the zero coupon bond. Table 4.1 shows the duration components of zero coupon bonds maturing in 1.22, 5.33, and 14.78 years. The durations of these zero coupon bonds correspond to the approximate duration of 1-year, 5–7-year, and 30-year coupon treasuries corresponding to the short end, middle and long end of the yield curve, respectively.

Consider three separate portfolios, A, B and C, each with a level duration of 6 years, and each constructed by using zero coupon bonds maturing in 1.22, 5.33, or 14.78 years. The portfolios have the same market values; however, they can borrow or lend cash at overnight rates to meet their duration targets. Table 4.2 shows the structure of these portfolios. Let us examine the performance of these portfolios under different interest rate scenarios.

Since the *level* duration of all portfolios is the same (6.00 years), they all respond in the same way to changes in the level of interest rates or parallel shifts of the yield curve. For example, if the yield levels rise by 10 bps, each portfolio loses 60 bps of market value.

The interpretation of the *slope* change is similar to the level change. When the slope of the yield curve falls, a portfolio that is long the slope duration outperforms the market. If the slope of the yield curve falls by 10 bps, then portfolio A loses $10 \times 4.24 = 42.4$ bps, while portfolio C gains 42.4 bps and portfolio B does not change. Intuitively, this is what we expect. When the slope falls, the yield curve flattens and the long bond outperforms the 1-year treasury bill.

To quantify further the meaning of slope change, recall that the slope of the yield curve is represented by τ (equation (2.19)). The interpretation of a fall of 10 bps in the slope is a fall of long rates ($\tau = 1$, $t = \infty$) by 10 bps and a rise of short rates ($\tau = -1$, $t = 0$) by 10 bps. This would translate into a fall of 7.07 bps (10 bps $\times\ \tau$ at $t = 14.78$) in interest rates at a maturity of 14.78 years and a rise of 7.07 bps in rates at a maturity of 1.22 years. The performance of portfolios A and C, each with a duration of 6 years, is expected to be $6 \times (-7.07) = -42.4$ and 42.4 bps, respectively. By buying one unit of

TABLE 4.2 Curve exposure of portfolios of zero coupon bonds

Portfolio	Level Duration	Slope Duration	Bend Duration
A	6.00	−4.24	0.00
B	6.00	0.00	−6.00
C	6.00	4.24	0.00

portfolio A and selling one unit of portfolio C, we can create a portfolio that is exposed only to the slope of the yield curve.

The *bend* component is a representative of the curvature or hump of the yield curve. When the bend component of the yield curve falls by 10 bps, the yield of long ($\tau = 1$, $t = \infty$) and short ($\tau = -1$, $t = 0$) term securities falls by 10 bps and the yield of medium ($\tau = 0$, $t = 5.3$) term securities rises by 10 bps. Consider a portfolio D that is constructed by buying one unit of each A and C and selling two units of B. This portfolio is called a long butterfly or a long barbell–short bullet. Portfolio D will exhibit positive performance if short and long term yields fall relative to medium term yields. When the bend component of the yield curve falls a butterfly trade exhibits positive outperformance and vice versa. The bend duration provides a convenient way to compare the magnitude and effectiveness of various butterfly trades.

4.3 EURODOLLAR FUTURES CONTRACTS EXAMPLES

Eurodollar futures contracts (EDFCs) trade on the basis of expected futures deposit (certificate of deposit) rates; for example, EDH21 (March 2021) contract trades on the expected 3-month deposit rate in March 2021. The traded price of a eurodollar contract is 100 minus the implied future deposit rate, so that when rates fall, prices appreciate and vice versa.

Since the contracts are based on 3-month deposit rates, their assumed duration is 0.25 years. Obviously, such a simple classification of the duration for all EDFCs is inaccurate, since EDH21 (March 2021) depends on short term rates in 2021 while EDH15 depends on rates in 2015. We can use the TSIR model to calculate the sensitivity of these contracts.

The first task in calculating the duration components of EDFCs is to understand their cash flows. Consider a 3-month (0.25-year) certificate of deposit for \$1 million. It involves investing \$1 million for 3 months and receiving the principal plus interest after 3 months. For EDFCs, no up-front cash is required; cash investment at the inception of the certificate of deposit is implied. This means that each EDFC is a combination of two cash flows as follows:

$$p = -c_1(t_1) + c_2(t_2) \tag{4.8}$$

Cash flows c_1 and c_2 are selected in such a way that the present value of each is equal to \$1 million and $t_2 = t_1 + 0.25$ years. The net price of any EDFC at initiation is zero (i.e., only margin requirements must be met). We can write the previous equation as

$$p = -c_1 e^{-(y_1+s_1)t_1} + c_2 e^{-(y_2+s_2)t_2} \tag{4.9}$$

where s_1 and s_2 are the EDFC credit spread over treasuries at t_1 and t_2. Since the market value of an EDFC is zero, its duration is undefined. However, the interest rate risk of these contracts can be represented in terms of the value of a basis point (VBP). The VBP of an EDFC is \$25, that is, for a change of 1 basis point in yield the value of an EDFC changes by \$25, which is realized in the form of daily margin movement. The VBP components of the TSIR for EDFCs are simplified as follows:

$$\begin{aligned} VBP_0 &= 100(t_2 - t_1) \\ VBP_1 &= 100(t_2\tau_2 - t_1\tau_1) \\ VBP_2 &= 100\left[t_2(2\tau_2^2 - 1) - t_1(2\tau_1^2 - 1)\right] \end{aligned} \tag{4.10}$$

Figure 4.1 shows the VBP for different components of the term structure sensitivity of EDFC.

Table 4.3 shows the VBP sensitivity of three selected EDFCs to level, slope and bend components of the TSIR. These sensitivities can also be obtained from Figure 4.1. By comparing Tables 4.2 and 4.3, it is evident that the risk profiles of EDFCs with expiration dates of 0.45, 2.3, and 7.77 years are similar to those for zero coupon bonds with maturities of 1.2, 5.3, and 14.8 years, respectively.

The EDFC risk profile in Table 4.3 is a simplified version of the risk. More accurate risks are provided in Section 10.4 along with the convexity bias of eurodollar futures.

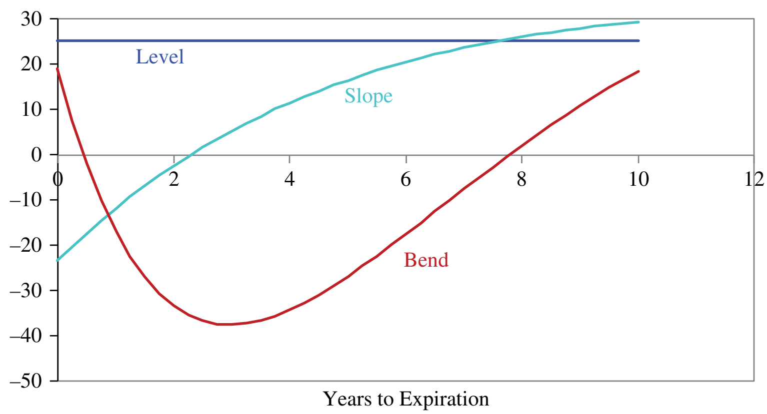


FIGURE 4.1 Eurodollar futures contracts VBP

TABLE 4.3 Curve exposure of eurodollar futures contracts

Contract Expiration In Years	Level VBP	Slope VBP	Bend VBP
0.45	25.0	–18.0	0.0
2.30	25.0	0.0	–35.5
7.77	25.0	25.7	0.0

4.4 CONVENTIONAL DURATION OF A PORTFOLIO

Consider a steep yield curve environment such as existed in early 1993 or 2011 in the US. We construct a portfolio A + B of two zero coupon bonds similar to the example in Section 1.4 as shown in Table 4.4.

TABLE 4.4 Conventional yield and duration of portfolios of securities

Portfolio	Face Amount	Maturity Years	Yield	Price	Implied Market Value	Duration
A	57,623,070	3	4.73%	86.7708	50,000,000	3
B	178,631,139	17	7.49%	27.9906	50,000,000	17
A+B MV Weighted		17	6.11%		111,192,409	10
A+B MVD Weighted		17	7.08%		100,247,769	10
A+B Actual		17	7.10%		100,000,000	10.48

The market value weighted average yield of this portfolio is 6.11%; however, discounting the cash flows by this yield results in a market value of $111,192,409! The correct way to estimate the yield of this portfolio is by market value duration weighting as explained in Chapter 1. This results in a yield and market value of respectively 7.08% and $100,247,769 (the accurate yield is 7.1%).

Likewise, the market value weighted duration of this portfolio is 10 years while the aggregate duration, using the same discount rate for both cash flows, is 10.48 years.

In conventional duration measurements, the calculated value of duration for a portfolio depends on how cash flows are aggregated. The inconsistency of these duration measurements is due to the non-uniqueness of the yield as it is applied to different cash flow aggregates.

Using the TSIR model, the level duration of a portfolio is equal to the calculated market value weighted duration of different securities. Since each cash flow is discounted by its specific yield, different aggregation methods would not impact the overall duration measurement and a unique duration value is calculated regardless of the aggregation method.

4.5 RISKS AND BASIS FUNCTIONS

The concepts of partial duration and key rate duration (see Ho [5]) are very closely related. These concepts have been introduced to analyze the sensitivity of portfolios to the shape of the TSIR.

The key rate duration measures the sensitivity to the curve by analyzing the impact of localized changes in the yield of a security on its price. In order to measure the sensitivity of a security to the 2-year part of the curve, the yield of the 2-year cash flows is moved up or down by 1 basis point. The resulting change in the yield is used to discount the cash flows and to calculate the changes in the security's market value. To confine the changes to the 2-year part of the curve, the impact of the plus or minus 1 basis point change is linearly interpolated to zero at the neighboring 1- and 3-year parts of the curve. For example, if we use 1-, 2-, and 3-year key rates to calculate the key rate duration of a cash flow at 1.5 years, for a change of 1 basis point in the yield of the 2-year curve, the change in yield of the cash flow will be 0.5 bps.

In practice, most applications that use or calculate KRD, use modified duration at selective maturities or the maturities of on-the-run bonds. Using continuously compounded yield as in our methodology, the modified and Macaulay durations are the same and are very close to the level of the TSIR. Typically key rates at 6 months, 1, 2, 3, 4, 5, 7, 10, 20, and 30 years are used for the calculation.

The relationship between modified duration and Macaulay duration for a bond with no contingent cash flow is

$$D_{\text{mod}} = \frac{D_{\text{mac}}}{1 + y/m} \tag{4.11}$$

where D_{mod} is the modified duration of the bond, D_{mac} is the Macaulay duration of the bond, y is market yield and m is the coupon frequency of the bond (for US treasuries $m = 2$, and for European bonds $m = 1$).

We can transform the basis functions of the vector of durations using the matrices that were developed in Section 3.5. The key rate basis functions provide a natural way to calculate yield curve sensitivity to various points on the yield curve. While the concepts of level, slope, bend, etc. are very useful for portfolio construction, they are somewhat abstract and hard to visualize. For hedging purposes, it may not be intuitively clear how best to hedge the bend duration component of a portfolio. Given the flexibility of the model, it is possible to define interest rate sensitivity on any arbitrary set of points on the curve and to transform one set of risk sensitivity into another by using linear transformations.

From (4.3) we note that a duration component is the weighted time average of the respective basis function:

$$D_k = \langle t\psi_k(t) \rangle \tag{4.12}$$

Defining a new basis function χ as in (3.16) and substituting it in the above equation, we can write the duration components in the new basis function as

$$D_k^{\chi} = \langle t\,\chi_k(t) \rangle = \left\langle t \sum_j m_{kj}\psi_j \right\rangle = \sum_j m_{kj} \langle t\psi_j \rangle = \sum_j m_{kj} D_j \tag{4.13}$$

Given the transformation matrix m_{ij} and the vector of durations D_j, we can calculate the durations in a different set of basis functions as

$$\left[\mathbf{D}_i^{\chi}\right] = \left[\mathbf{M}_{ij}\right] \times \left[\mathbf{D}_j\right] \tag{4.14}$$

Recall from (3.22) that the matrix $\mathbf{M}$ is simply the inverse of the matrix of the basis functions. The first two duration components of PBFs, CBFs and OBFs are identical.

To illustrate how we can change the basis functions, we start with a simple example. Consider a two-parameter term structure model as we did in Section 3.4. The duration components K_0 and K_1 at points b_0 and b_1 are given by

$$
\begin{aligned}
K_0 &= -\frac{1}{p}\frac{\partial p}{\partial b_0} = \sum_i c_i t_i \chi_0\, e^{-t_i \sum_j b_j \chi_j} = \langle t\chi_0 \rangle \\
&= \tfrac{1}{2}\langle t\psi_0\rangle - \langle t\psi_1\rangle = \tfrac{1}{2} D_0 - D_1
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
K_1 &= -\frac{1}{p}\frac{\partial p}{\partial b_1} = \sum_i c_i t_i \chi_1\, e^{-t_i \sum_j b_j \chi_j} = \langle t\chi_1 \rangle \\
&= \tfrac{1}{2}\langle t\psi_0\rangle + \langle t\psi_1\rangle = \tfrac{1}{2} D_0 + D_1
\end{aligned} \tag{4.16}
$$

where we have substituted for χ from (3.9) and used (4.3) to replace the brackets with duration components. K_0 is the contribution to the duration of a bond for a unit move of the yield curve at a yield of b_0 while anchoring the yield curve at a yield of b_1. Effectively, K_0 and K_1 are the key rate duration components of the two-parameter term structure model.

The choice of maturity for key rates is up to us. For any given set of key rate maturities, we can find the corresponding key rate duration components.

Term structure key rate duration (TSKRD) components provide an explicit hedge for the exposure of a security or a portfolio. In our framework, at any given key rate maturity, the contributions of all other key rates to duration are zero. This implies that to hedge the curve exposure of a portfolio, we just need to buy or sell the equivalent duration contributions of its key rate components. For example, if the 10-year KRD of a portfolio is 2 years, we can sell 2 years of the duration of the 10-year zero coupon bond to hedge its duration.

Figure 4.2 show the contribution of TSKRD components for a three-parameter term structure model. We chose key rate duration components at the 1-, 5.3-, and 16-year parts of the curve. One interpretation of a TSKRD component is that of a curve that is anchored at all other key interest rates. For example, the 5.3-year component

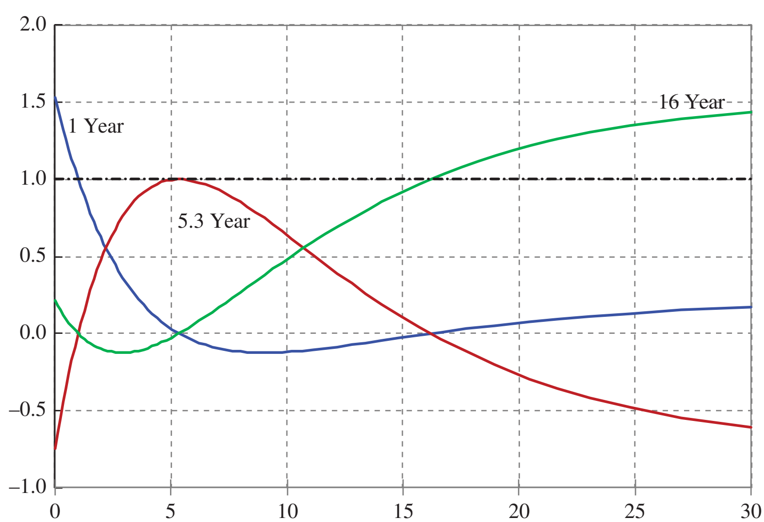


FIGURE 4.2 Key rate contribution to duration, time space

is anchored at 1- and 16-year maturities. The continuity requirements force this curve to move in the opposite directions outside 1–16 years than inside it. This results in the duration contribution of this component being negative for short or long maturities. When a key rate contribution is one, the remaining key rate contributions are zero.

In the conventional KRD models, the impact of a key rate change is localized. This creates yield curve shapes that are not realistic and can result in risk measures that are not accurately captured for a security that matures between two key rates. In our model, the yield curve maintains its continuity while a change to one of the key rates affects the entire yield curve with the maximum impact near the key rate maturity and zero impact at any other key rate point. The sum of all TSKRD contributions is equal to the level duration.

4.6 APPLICATION TO KEY RATE DURATION

In the two-parameter model of the TSIR, if the level and slope durations of a portfolio are 10 and 3 years respectively, the key rate duration components of the model that was developed in Section 4.5, based on (4.15) and (4.16), can be calculated as

$$K_0 = \tfrac{1}{2} D_0 - D_1 = \tfrac{10}{2} - 3 = 2$$
$$K_1 = \tfrac{1}{2} D_0 + D_1 = \tfrac{10}{2} + 3 = 8$$

The maturity of key rates in this case would be at $\psi_1 = \pm\frac{1}{2}\psi_0$ or $\tau = \pm\frac{1}{2}$, $t = 2.2$, $t = 10.7$ years. The contribution to duration of this portfolio at 2.2 and 10.7 years' maturity are respectively 2 and 8 years. Likewise, to hedge this portfolio, we have to sell 2 years of duration using a 2.2-year zero coupon bond and sell 8 years of duration using a 10.7-year zero coupon bond.

In practice, key rate securities are selected on the basis of liquidity or convenience. Assume that the risks of a portfolio are measured by a third party that provides the durations at selected key rates of 3 months, 6 months, 1 year, etc. Table 4.5 shows the weights of each of the key rates along with the CBF duration components of each key rate. The modified durations and level durations of zero coupon bonds are identical by using continuously compounded yield.

We know that we need only five components of the TSIR to capture the risks of a portfolio. We now show how to convert the risks of the portfolio into the selected five key rates. We use the five key rates shaded in Table 4.5 as basis functions for our purpose. We then transpose the matrix and divide all elements of the transposed matrix by the value of the first row, to create a matrix with unit values of the level duration. This step is not necessary, but will result in actual key rate durations rather than the weight of each key rate.

The steps to convert from key rate durations to term structure durations (TSDs) are as follows:

- Identify key rate securities (KRSs) and calculate their conventional TSD and KRD from (4.14). KRSs can be coupon bonds.
- For each KRD of a security, divide it by the KRD of its respective KRS and use the ratio to scale all TSDs of the securities.
- Add TSDs for all KRDs to calculate the total TSD.

TABLE 4.5 Duration components of key rate securities

	Level	Slope	Bend	Cubic	Quartic	Weight
3 Mo	0.238	−0.224	0.182	−0.118	0.040	5
6 Mo	0.545	−0.470	0.267	0.009	−0.283	10
1 Yr	0.980	−0.745	0.154	0.511	−0.931	10
2 Yr	1.942	−1.067	−0.766	1.902	−1.317	10
3 Yr	2.903	−1.056	−2.125	2.580	0.256	10
5 Yr	4.702	−0.300	−4.596	0.801	4.400	10
7 Yr	6.723	1.212	−6.201	−3.494	4.760	10
10 Yr	9.085	3.777	−5.643	−8.406	−1.947	10
15 Yr	12.353	8.367	0.771	−5.001	−8.035	10
20 Yr	15.849	12.559	6.309	1.212	−2.762	10
30 Yr	19.443	16.641	11.289	6.780	2.931	5
Total	6.492	3.049	−0.609	−0.656	−0.437	

For example, suppose that a third party has provided the key rate durations of a security S, and we would like to calculate the corresponding term structure durations. Assume that the third party has used coupon bonds as key securities and the 10-year KRS is a bond with duration of 7.6 years (modified duration). We calculate the TSDs of the key security and find a level duration of 7.9 and slope duration of 2.1. If the 10-year KRD of security S is 3.2 years, then the contribution to the level and slope of the security from this KRD can be calculated as $\frac{3.2}{7.6} \times 7.9$ and $\frac{3.2}{7.6} \times 2.1$ respectively.

If the number of key rates is equal to or greater than the number of parameters in the TSIR model, there will be a unique solution to the conversion. However, if there are fewer key rates than TSIR components, we need to use other techniques (see Section 8.2).

To convert from TSD to TSKRDs, we take the KRS in Table 4.5 and divide each TSD by its level duration and transpose the matrix to arrive at Table 4.6.

TABLE 4.6 Transposed and scaled duration components of key rate securities

	6Mo	2Yr	5Yr	10Yr	30Yr
Level	1.0000	1.0000	1.0000	1.0000	1.0000
Slope	−0.8632	−0.5496	−0.0638	0.4158	0.8559
Bend	0.4904	−0.3944	−0.9774	−0.6212	0.5806
Cubic	0.0166	0.9792	0.1703	−0.9253	0.3487
Quartic	−0.5191	−0.6783	0.9358	−0.2143	0.1507

From (3.22), we can calculate the matrix **M** by inverting this matrix. Multiplying by the vector of total TSDs from Table 4.5, we can calculate the vector of TSKRDs as follows:

$$\begin{bmatrix} 0.3213 & -0.6271 & 0.5135 & -0.3279 & 0.2099 \\ 0.0311 & 0.1413 & -0.4236 & 0.5110 & -0.5590 \\ 0.1975 & -0.2603 & -0.1586 & 0.0490 & 0.6651 \\ 0.1777 & 0.2699 & -0.3213 & -0.4534 & -0.4245 \\ 0.2724 & 0.4762 & 0.3901 & 0.2214 & -0.1085 \end{bmatrix} \times \begin{bmatrix} 6.4923 \\ 3.0485 \\ -0.6094 \\ -0.6556 \\ -0.4374 \end{bmatrix} = \begin{bmatrix} -0.0155 \\ 0.8002 \\ 0.2627 \\ 2.655 \\ 2.7898 \end{bmatrix}$$

Thus, the key durations of the portfolio at key maturities of 6 months, 2 years, ... are respectively −0.0155, 0.8002, ...

We now return to the example in Section 1.4 and calculate the TSKRDs. Knowing the yields at 2 and 18 years of 1% and 4%, we can write the yield curve in KBF terms as

$$\begin{aligned} y &= b_0(\tau - \tau_2) + b_1(\tau - \tau_{18}) \\ b_0 &= \frac{0.04}{\tau_{18} - \tau_2} \\ b_1 &= \frac{0.01}{\tau_{18} - \tau_2} \end{aligned}$$

We construct a table of the risks of the key rates at 2, 10, and 20 years (zero coupon keys), similar to Table 4.6 in PBFs as follows:

Time	2	10	20
	1	1	1
τ	−0.542	0.455	0.851
τ^2	0.294	0.207	0.725

From (3.22) we calculate the transformation matrix by inverting the above matrix of risks as

$$[\mathbf{M}] = \begin{bmatrix} 0.279 & -0.940 & 0.720 \\ 1.168 & 0.782 & -2.529 \\ -0.446 & 0.158 & 1.810 \end{bmatrix}$$

The calculated durations in PBFs of the portfolio A + B in Table 1.1 are

$$[\mathbf{D}] = \begin{bmatrix} 10 \\ 6.724 \\ 6.16 \end{bmatrix}$$

For example, the bend duration of the portfolio is calculated as

$$D_2 = \frac{1}{p}\sum_i c_i e^{-y_i t_i} t\tau^2 = \frac{mv_2 \times 2 \times \tau_2^2 + mv_{18} \times 18 \times \tau_{18}^2}{mv_2 + mv_{18}}$$

$$= \frac{10{,}000{,}000(2 \times (-0.542) \times (-0.542) + 18 \times 0.807 \times 0.807)}{20{,}000{,}000} = 6.16$$

The key rate durations can now be calculated as

$$\left[\mathbf{M}_{ij}\right] \times \left[\mathbf{D}_j\right] = \left[\mathbf{D}_i^{\chi}\right]$$

that is,

$$\begin{bmatrix} 0.279 & -0.940 & 0.720 \\ 1.168 & 0.782 & -2.529 \\ -0.446 & 0.158 & 1.810 \end{bmatrix} \times \begin{bmatrix} 10 \\ 6.724 \\ 6.16 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 1.35 \\ 7.75 \end{bmatrix}$$

The calculated TSKRDs of 0.9, 1.35, and 7.75 years for the 2-, 10-, and 20-year key rates are somewhat different from the calculated values in Section 1.4. We first note that, unlike ordinary key rate durations, the calculated result is independent of how the cash flows are aggregated and the sum of durations is exactly 10 years in this example, as it should be. Additionally, the higher correlations for longer maturity key rates manifest themselves in a higher duration for the 20-year key rate than standard key rate calculations. The decay coefficient is the parameter that facilitates compressing longer dated times to account for higher correlations. If we repeat this exercise and use only two key rates of 2 and 18 years, the resulting key rate durations will be 1 and 9 years respectively. The cash flows that coincide with key rate maturities have no effect on other key rates. In our example, the cash flow at 18 years is not at a key rate and thus it impacts the key rate at the 2-year part of the curve, to maintain continuity of the yield curve, lowering it by 0.1 years.

We took the portfolio of 2- and 18-year treasuries and hedged it using 2-, 10-, and 20 year key rates using standard key rate durations as well as key rates based on the TSIR on a historical basis. Zero coupon Strips were used from 1998 through 2012 and the results were analyzed on a monthly basis. The tracking errors between the standard key rate and term structure based key rate hedges and the portfolio were respectively 63 and 58 bps. Most of the tracking error is due to the changes in cheapness or richness of the key rate securities. The security specific risks cannot be hedged; while they are a source of tracking error and risk, they can also be a source of return, as we will see in the next chapter. If we assume that the spreads of the key rate securities relative to the curve remain unchanged, the tracking error falls by more than 50%.

Due to the higher weights of 2- and 10-year key securities in the standard key rate hedges, the sum of the market values of the hedges was about 4% higher than the portfolio's market value. To balance the market value, a short position of 1-month Libor was added to the hedge positions to match their market value with that of the portfolio. Similar adjustments were made for the TSIR based key rate durations; however, the adjustments were smaller and were positive by about 2%. The market values of term structure hedges are close and usually below the market values of the underlying securities, while the opposite is true for key rate durations. The additional 4% capital that is required for hedging can result in leverage for cash portfolios.

We need at most five TSKRDs to have similar or better tracking error than conventional KRDs for hedging the interest rate exposure of a portfolio, which ordinarily requires about 10 KRDs. In Section 8.3 we will compare KRD and TSD in more detail and show that five duration components of the TSIR offer the same tracking error as 10 KRDs, with significant advantages for performance contribution. We will see in Section 12.1 that the measurement of the KRD can be off by more than 1 year for corporate securities.

The relative performances of the key rate and term structure hedges were very close, and in both cases were about an order of magnitude less than the tracking error.

Alternatively, we can calculate the weight of key rates by interpolation in τ space instead of linear interpolation. Thus, the 20- and 10-year key rate durations of the 18-year treasury will be calculated as

$$D_{20} = \frac{\tau(18) - \tau(10)}{\tau(20) - \tau(10)} 10 = \frac{0.807 - 0.455}{0.851 - 0.455} 10 = 8.89$$
$$D_{10} = 1.11$$

The above analysis was based on the duration of individual zero coupon bonds. For aggregated securities where the duration is overestimated (see Section 1.4 or Section 4.4) in an upward sloping yield curve, the tracking error of the key rate duration hedge is higher than in the above example.

4.7 RISK MEASUREMENT OF A TREASURY INDEX

The term structure risk of a portfolio is calculated by aggregating the market value weighted duration components of all securities. We will now provide analysis of the risks of a custom treasury index.

We created a custom treasury index by equally weighting all outstanding US treasury coupon bonds with a minimum maturity of 1 year, excluding callable bonds. We then calculated the duration components and the continuously compounded yield of the index, as well as the carry (yield plus rolldown of the curve) on a monthly basis. Table 4.7 shows the average annual durations and yields.

Due to the treasury surplus and buy-back program in the US in the late 1990s and early 2000s, the weight of treasuries in the 5-year part of the curve was lowered significantly. In Table 4.7 we can see the bend duration of the index in 2001 at its peak of −0.95, implying the least exposure to the 5-year part of the curve. Subsequent issuance at the long end of the maturity spectrum resulted in significant gains in the level and slope durations as well as large exposures to the cubic and quartic components in the period 2001–2007.

Table 4.8 shows the average duration contributions of the treasury index in different basis functions. For this purpose, we used the natural maturities of 0.19, 1.77, 5.3, 12.1, and 28.5 years.

Table 4.9 shows a sample of duration components for global securities sorted by maturity. The level duration generally increases with maturity; however, for lower

TABLE 4.7 Duration components and yield of an equal weighted treasury index

	1st	2nd	3rd	4th	5th	Carry	Yield
1992	4.46	0.70	−2.05	−0.02	0.21	6.88%	5.98%
1993	4.73	0.94	−1.97	−0.01	0.27	5.79%	5.06%
1994	4.58	0.82	−1.93	0.16	0.26	6.97%	6.51%
1995	4.72	0.97	−1.79	0.18	0.13	6.54%	6.42%
1996	4.75	1.04	−1.68	0.19	0.01	6.43%	6.23%
1997	4.87	1.19	−1.55	0.17	−0.12	6.35%	6.22%
1998	5.26	1.57	−1.34	0.11	−0.33	5.36%	5.34%
1999	5.35	1.71	−1.19	0.04	−0.57	5.81%	5.70%
2000	5.77	2.14	−1.00	−0.22	−0.96	6.07%	6.22%
2001	6.09	2.44	−0.95	−0.56	−1.32	5.15%	4.74%
2002	6.36	2.65	−1.05	−0.95	−1.60	4.78%	4.04%
2003	6.58	2.78	−1.28	−1.39	−1.77	4.19%	3.38%
2004	6.38	2.46	−1.66	−1.45	−1.50	4.48%	3.74%
2005	6.32	2.35	−1.85	−1.49	−1.38	4.38%	4.18%
2006	5.91	1.94	−2.01	−1.27	−1.08	4.79%	4.83%
2007	5.60	1.67	−2.04	−1.09	−0.85	4.64%	4.59%
2008	5.65	1.67	−2.13	−1.04	−0.54	3.71%	3.02%
2009	5.65	1.60	−2.23	−0.86	−0.20	3.39%	2.34%
2010	5.48	1.41	−2.28	−0.60	0.18	3.14%	2.05%
2011	5.52	1.45	−2.22	−0.40	0.44	2.67%	1.65%
2012	5.85	1.76	−1.99	−0.20	0.53	1.73%	1.05%

TABLE 4.8 Average duration components of an equal weighted treasury index

	First	Second	Third	Fourth	Fifth
Cheby	5.52	1.68	−1.72	−0.51	−0.48
Ortho	5.52	1.68	0.09	0.31	−0.03
Poly	5.52	1.68	1.90	1.13	1.15
Exp	5.52	1.92	1.01	0.63	0.43
Key	−0.03	0.88	1.60	2.06	1.01

coupon bonds the maturity is longer than a comparable higher coupon bond. For example, HKD 2.39% 8/20/25 has a level duration of 11.014, while PEN 8.2% 8/12/26 has a duration of 8.683.

The slope duration is negative for short duration bonds, becomes positive at about a level duration of 5 years, and rises after that. The bend duration is positive for very

TABLE 4.9 Duration components of global treasuries, January 3, 2013

Country	Currency	cpn	Maturity	Level	Slope	Bend	Cubic	Quartic
US	USD	0	4/4/13	0.238	−0.224	0.182	−0.118	0.040
JP	JPY	1.3	12/20/14	1.930	−1.069	−0.744	1.890	−1.346
US	USD	0.125	12/31/14	1.978	−1.081	−0.797	1.951	−1.335
AU	AUD	4.75	10/21/15	2.616	−1.060	−1.739	2.430	−0.211
AU	AUD	4.75	10/21/15	2.616	−1.060	−1.739	2.430	−0.211
IL	ILS	6.5	1/31/16	2.737	−0.985	−2.001	2.368	0.311
GB	GBP	8.75	8/25/17	3.919	−0.549	−3.638	1.384	3.116
DK	DKK	4	11/15/17	4.509	−0.395	−4.357	1.050	4.063
US	USD	0.625	11/30/17	4.825	−0.301	−4.770	0.876	4.639
CA	CAD	1.25	3/1/18	4.981	−0.170	−4.932	0.462	4.845
DE	EUR	6.25	1/4/24	8.697	3.692	−4.690	−7.300	−3.108
HK	HKD	2.39	8/20/25	11.014	6.217	−3.378	−9.574	−8.299
PE	PEN	8.2	8/12/26	8.683	4.210	−3.076	−6.052	−4.769
AU	AUD	4.75	4/21/27	10.548	6.080	−2.210	−7.444	−7.804
SE	SEK	2.25	6/1/32	15.559	11.995	4.215	−3.634	−9.687
US	USD	0	11/15/42	29.860	28.629	25.037	19.380	12.126
EU	EUR	2.5	7/4/44	21.397	18.884	13.959	9.545	5.585
CA	CAD	3.5	12/1/45	20.757	18.085	13.000	8.792	5.333
JP	JPY	2	3/20/52	26.042	23.925	19.850	16.441	13.649
GB	GBP	4	1/22/60	20.727	17.875	12.893	9.696	7.879

short duration securities and becomes negative in the middle of the curve, becoming positive at long maturities. Likewise, cubic and quartic components change sign four and five times, respectively.

The slope duration at the long end of the curve is always less than the level duration. Only for derivative securities can the slope duration be higher than the level duration, as is shown in Table 4.3 and Figure 4.1.

CHAPTER 5

Performance Attribution

Performance attribution is a very important and one of the most challenging areas of investment management business. Quite often a portfolio is managed against a benchmark which can be an index or peer group. Relative performance compared to an index is usually a better measure of the capability of a portfolio manager than absolute performance. For example, if the bond market is down 3% and a portfolio manager is down only 2%, he is outperforming the benchmark by 100 bps (1%), which is considered more favorably by his supervisor than a portfolio manager who underperforms by 1% in a market that is up 7%.

The performance of a portfolio is governed by several factors/people:

- The universe of securities or broad maturity/duration ranges of a portfolio is decided by policy or prospectus of the fund or portfolio.
- The relative duration is generally decided by the investment policy committee or the portfolio manager.
- The curve positioning is done by the portfolio manager.
- Security selection, especially for corporate bonds is performed by analysts.
- Trading by traders/portfolio managers.

While the overall performance of a portfolio is measured very easily, the breakdown of individual contributions is very difficult and many investment management companies have devoted considerable resources to drawing up an accurate account of individual contributions. For multi-sector portfolios where derivatives such as options, futures and credit default swaps are used, the task of performance attribution is even more difficult. We will discuss performance attribution of credit securities and derivatives in the future chapters. In this chapter we focus on the performance attribution of treasuries.

The Advanced Fixed Income and Derivatives Management Guide. Saied Simozar

5.1 CURVE PERFORMANCE

The change in the performance of a security due to changes in the TSIR (term change) can be calculated as

$$\Delta p_t = \sum_i c_i e^{-t_i \sum_k (a_k + \Delta a_k)\psi_k} - \sum_i c_i e^{-t_i \sum_k a_k \psi_k} \tag{5.1}$$

After expanding the above equation by Taylor series, we have

$$\Delta p_t = -\sum_i \sum_k \Delta a_k \psi_k c_i t_i e^{-t_i \sum_j a_j \psi_j} + \frac{1}{2} \sum_i \sum_k \sum_l \Delta a_k \Delta a_l \psi_k \psi_l c_i t_i^2 e^{-t_i \sum_j a_j \psi_j} + \ldots \tag{5.2}$$

Substituting from (4.3) and (4.4) and simplifying the result, we can write equation (5.2) as

$$\Delta p_t = -p_t \sum_k \Delta a_k D_k + \frac{1}{2} p_t \sum_k \sum_l \Delta a_k \Delta a_l X_{kl} + \ldots \tag{5.3}$$

Thus, the first order change in the performance of a security or a portfolio of securities due to changes in interest rates is calculated by multiplying the duration components of the security by the changes in the components of the TSIR. Specifically,

$$q_{d,i} = \frac{\Delta p_i}{p} = -D_i \Delta a_i \tag{5.4}$$

where Δp_i is the change in price due to the ith component of the TSIR and D_i and Δa_i are the respective duration and change in the interest rate components and $q_{d,i}$ is the contribution to performance of the ith component of duration. For example, if the slope duration of a security is 5 years and the slope component of the TSIR falls from 0.011 to 0.010, then the expected performance gain from the slope component of the yield curve will be $5 \times 0.001 = 0.5\%$.

The performance contribution from convexity can be calculated by considering all the cross-convexity components. However, in most cases, convexity can be ignored or can be accounted for primarily through the first two components of the TSIR. The general formula to account for convexity is

$$q_{x,ij} = \frac{\Delta p_{ij}}{p} = \frac{\Delta p_{ji}}{p} = \frac{1}{2} X_{ij} \Delta a_i \Delta a_j \tag{5.5}$$

where X_{ij} is the cross-convexity matrix as defined in (4.4) and $q_{x,ij}$ is the contribution to performance of X_{ij}. As a substitute, we can use the following formula for calculating the contribution of convexity to performance:

$$\frac{\Delta p_{\text{convexity}}}{p} = \frac{1}{2} X_{00} (\Delta y_b)^2 \tag{5.6}$$

where y_b is the calculated continuously compounded yield to maturity of the bond or security calculated from

$$\sum_i c_i e^{-y(t)t_i} = \sum_i c_i e^{-y_b t_i} \tag{5.7}$$

5.2 YIELD PERFORMANCE

We now calculate the contribution to performance due to the passage of time. It is standard practice to use the yield of a security and multiply it by time to calculate the contribution of yield to performance. For our analysis, we start by taking the price of a security (4.1) and shift the time slightly:

$$\Delta p = \sum_i c_i \; e^{-y_i(t_i-\Delta t)(t_i-\Delta t)} - \sum_i c_i \; e^{-y_i(t_i)t_i} \approx \sum_i c_i \; e^{-y_i t_i}\left(y_i + t_i \frac{\partial y_i}{\partial t}\right)\Delta t \tag{5.8}$$

The right hand side of (5.8) is derived from Taylor expansion of the left hand side to the first order of Δt and $y_i = y_i(t_i)$. Substituting (2.27) in the above equation leads to

$$q_y = \Delta p = \Delta t \sum_i y_{f,i} c_i \; e^{-y_i t_i} = \left\langle y_{f,i} \right\rangle \Delta t = \theta \Delta t \tag{5.9}$$

Thus, the contribution of yield to performance q_y is proportional to the present value cash flow weighted average of the instantaneous forward rates. In traditional fixed income, where we use a constant yield, the derivative of the yield relative to time is zero, resulting in $\left\langle y_f \right\rangle = y$. In analogy with the options market, we call the yield plus rolldown the θ of a bond.

5.3 SECURITY PERFORMANCE

The next factor to consider for performance is the change in the spread of a security relative to the curve. The components of the TSIR are calculated in such a way as to minimize the yield or price error (the difference between the market price and the calculated price squared) for all securities in the market. Since the fit is not perfect, there will be a difference between the market and the calculated prices of all securities. The calculated price of a security can be written as

$$p_t = \sum_i c_i \; e^{-y(t_i)t_i} \tag{5.10}$$

We can assume that the market price is calculated by a yield curve that has an implied spread compared to the calculated TSIR. Thus,

$$p_m = \sum_i c_i \; e^{-(y(t_i)+s_b)t_i} \tag{5.11}$$

where p_m is the market price and s_b is the implied spread of the bond relative to the TSIR. For example, if the curve does not change but the spread of a security falls, its price increases. We call the change due to such movements the security selection performance. If p_m and p_t are relatively close to each other, as is usually the case, for a level duration of D_0, we can approximately calculate the implied spread as

$$s_b \approx \frac{p_t - p_m}{p_t D_0} \approx \frac{p_t - p_m}{p_m D_0} \tag{5.12}$$

The change in the price due to a change in the spread of the security relative to its curve can be calculated similarly to the calculation that we did for changes in the yield as

$$\Delta p = -\Delta s_b \sum_i c_i\, t_i e^{-(y_i + s_b)t_i} = -D_0 p_m \Delta s_b \tag{5.13}$$

The contribution to performance due to security selection is thus

$$q_b = -D_0 \Delta s_b \tag{5.14}$$

There are convexity components for the security performance, which can be important if a security has a relatively large spread such as we saw in Figure 2.10. For this purpose we rewrite (5.1) to include security spread:

$$\Delta p_m = \sum_i c_i e^{-t_i \sum\limits_k (a_k + \Delta a_k)\psi_k - (s_b + \Delta s_b)t_i} - \sum_i c_i e^{-t_i \sum\limits_k a_k \psi_k - s_b t_t} \tag{5.15}$$

After expansion and some simplification of the above equation, we arrive at

$$\Delta p_m = \Delta p_a - p_m D_0 \Delta s_b + \frac{1}{2} p_m X_{00} (\Delta s_b)^2 + \Delta s_b \sum_k \sum_i \Delta a_k \psi_k c_i t_i^2 e^{-(y_i + s_b)t_i} \tag{5.16}$$

where Δp_a is the change in price due to changes in the components of the TSIR, similarly to (5.3), defined as

$$\Delta p_a = -p_m \sum_k \Delta a_k D_k + \frac{1}{2} p_m \sum_k \sum_l \Delta a_k \Delta a_l X_{kl} + \ldots \tag{5.17}$$

Noting that $\psi_0 = 1$, we can simplify equation (5.16) as

$$\Delta p_m = \Delta p_a - p_m D_0 \Delta s_b + \frac{1}{2} p_m X_{00} (\Delta s_b)^2 + p_m \Delta s_b \sum_k \Delta a_k X_{0k} \tag{5.18}$$

The last two contributions in equation (5.18) are related to the convexity of the spread of the security. The spread of a security is significantly less volatile than the level of interest rates. A rich security or a security that has a negative spread relative to the curve is often very liquid and tends to stay rich; likewise, a cheap security tends to stay cheap. We can ignore the contribution of the convexity of the security to the overall performance and simplify equation (5.18) as

$$\begin{aligned}\Delta p_m &\approx \Delta p_a - p_m D_0 \Delta s_b \\ &= -p_m \sum_k \Delta a_k D_k + \frac{1}{2} p_m \sum_k \sum_l \Delta a_k \Delta a_l X_{kl} - p_m D_0 \Delta s_b\end{aligned} \tag{5.19}$$

The constant spread s_b relative to the calculated TSIR represents the security specific component of our analysis. The spread is a very good measure of richness (negative spread, expensive security) or cheapness (positive spread) of a security. When the spread falls, the security outperforms the curve by an amount equal to $(-\Delta s D_0)$. For example, if the spread of a bond with a duration of 10 years changes from 12 to 7 bps, the expected excess return will be 50 bps.

5.4 PORTFOLIO PERFORMANCE

The performance of a security can be calculated by combining the contributions from duration, convexity, security selection (5.19) and yield (5.9) respectively as follows:

$$\frac{\Delta p_m}{p_m} = -\sum_k \Delta a_k D_k + \frac{1}{2} \sum_k \sum_l \Delta a_k \Delta a_l X_{kl} - D_0 \Delta s_b + \theta \Delta t \tag{5.20}$$

The performance contribution from duration components is calculated by multiplying the duration components and the change in the respective components of the TSIR. With five components of the TSIR, there are 15 independent components of convexity and cross-convexity. For our analysis, we use only the four largest components of convexity, namely X_{00}, X_{01}, X_{11} and X_{02}. The remaining components are related to the volatility of higher order components of the TSIR as well as the respective convexities, which are usually small.

The contribution from yield is calculated by the weighted average carry as defined by (5.9) multiplied by the number of days divided by the number of days in a year. Years that are divisible by 4 are leap years, except for years that are divisible by 100, unless divisible by 400. For example, the year 1900 was not a leap year by but 2000 was. The average number of days in a year will be 365.2425.

The performance of a portfolio can be calculated by maintaining aggregate risks and yield parameters, rather than individual security information. For the duration contribution of the portfolio, we can calculate the aggregate VBP or simply the market value weighted duration components. If a portfolio has N securities, each with a weight or face value w, the aggregated duration contribution will be

$$\Delta M(D_k) = -\Delta a_k \sum_{j=1}^{N} w_j p_{m,j} D_{k,j} \tag{5.21}$$

where ΔM is the change in the market value of the portfolio from changes in the kth component of the TSIR. Convexity, security spread and contribution to carry can also be aggregated the same way. In order to calculate the performance of a treasury portfolio we just need to store the components – five duration, four convexity, one carry and one security spread – on a daily basis, regardless of the number of securities in the portfolio.

We analyzed the performance of a custom treasury index by equally weighting all treasury bonds with a maturity of longer than 1 year from 1992 to 2012 on a daily basis and provide the aggregate annual data in Table 5.1.

The security selection contribution is calculated by multiplying the changes in the spread of securities by their level duration and ignoring the convexity of the security spread. The term structure of rates is calculated in such a way as to create a balance between the positive and negative spreads (see Chapter 20). Therefore, the contribution of security selection to performance should be zero. In Table 5.1, we see a negative performance of security selection averaging about 7 bps (–0.07%) per year.

With the calculated term structure components the market value of the aggregate index matches the actual market value exactly and implies a zero spread for the overall index. In fact, on all days, except for the last business day of the month, the contribution of security selection to performance is zero. Due to index rebalancing, at the end of the month, newly issued bonds enter the index and the bonds with maturities of less than 1 year drop out. As such, either the incoming or outgoing index can be priced exactly. The new bonds, due to high liquidity, tend to be in high demand and rich compared to the curve, and the older bonds tend to be on the cheap side, resulting in the new TSIR having a slightly lower level than if the newly issued bonds were excluded. The implied spread of existing bonds widens slightly with the inclusion of the newly issued bonds, resulting in a slightly negative performance for the existing index. In 2008 and most other years, the entire underperformance of the index, which was 0.07%, was accounted for by the performance of the last business day of the refunding months (February, May, August, November) as the treasury issued new bonds.

The effect of rebalancing is well known in the investment management business; however, we are not aware of any prior calculations that showed the underperformance of the treasury index. In corporate bond or emerging market indexes, sometimes the opposite takes place. A rich security is dropped out of the index, causing the index to outperform the broad market and portfolio managers who had underweighted that security to underperform the index.

The “Total” column in Table 5.1 is the sum of contributions to performance from durations, convexity, carry and security. The error between calculated and realized performance using daily data was less than 1 basis point annually and is shown in the last column labeled “Diff”. The last two rows of the table are the average annual contributions to performance and the annualized standard deviations of those contributions. The steady decline in the volatility of the performance of components is a very attractive feature of our TSIR model. Using other basis functions, such as KBFs, the contribution to performance of successive components is much less regular.

For this analysis, we calculated the TSIR from coupon bonds. For the US and countries where Strips are available, it might be easier to calculate the TSIR from Strips data. For efficient markets, there should not be any difference between coupon bonds or Strips. Coupons can be reconstituted by combining Strips or they can be stripped to the respective zero coupons.

In the fall of 2008, after the Lehman bankruptcy, most treasury coupon bonds outperformed coupon Strips. If we used Strips to calculate the TSIR, we would see a sizable security outperformance relative to the market in 2008 and a relatively negative performance in 2009. Table 5.2 is a reconstruction of Table 5.1, except using coupon Strips to construct the TSIR.

TABLE 5.1 Index performance attribution using coupon bonds for the TSIR

Year	1st	2nd	3rd	4th	5th	Vex	Carry	Security	Market	Total	Diff
1992	1.04%	−0.33%	−0.57%	0.00%	0.00%	0.12%	6.88%	−0.10%	7.06%	7.04%	0.019%
1993	2.53%	0.55%	0.69%	0.01%	0.02%	0.13%	5.79%	−0.07%	9.65%	9.64%	0.006%
1994	−10.69%	0.94%	−0.51%	−0.03%	0.04%	0.26%	6.97%	−0.05%	−3.03%	−3.05%	0.013%
1995	9.00%	0.09%	0.92%	0.04%	−0.02%	0.16%	6.54%	−0.07%	16.67%	16.67%	−0.001%
1996	−2.74%	−0.36%	−0.54%	−0.02%	0.00%	0.22%	6.43%	−0.07%	2.93%	2.93%	0.006%
1997	1.75%	0.68%	0.26%	0.01%	0.01%	0.14%	6.35%	−0.06%	9.12%	9.13%	−0.004%
1998	5.20%	−0.88%	−0.11%	−0.01%	−0.08%	0.20%	5.36%	−0.11%	9.56%	9.57%	−0.004%
1999	−8.16%	0.28%	−0.25%	0.00%	0.01%	0.23%	5.81%	−0.13%	−2.21%	−2.21%	0.003%
2000	5.31%	0.66%	0.56%	−0.02%	0.07%	0.20%	6.07%	−0.08%	12.76%	12.75%	0.000%
2001	9.60%	−6.99%	−1.28%	0.24%	−0.35%	0.36%	5.15%	−0.13%	6.59%	6.59%	0.002%
2002	5.35%	0.53%	0.97%	−0.26%	0.18%	0.36%	4.78%	−0.13%	11.78%	11.78%	0.003%
2003	−1.51%	−0.53%	−0.26%	0.09%	0.12%	0.46%	4.19%	−0.10%	2.45%	2.46%	−0.007%
2004	−4.01%	3.23%	0.48%	−0.02%	0.00%	0.27%	4.48%	−0.07%	4.37%	4.37%	0.000%
2005	−4.41%	3.22%	0.19%	−0.08%	0.06%	0.20%	4.38%	−0.04%	3.53%	3.53%	0.003%
2006	−2.57%	0.42%	0.29%	−0.10%	0.03%	0.12%	4.79%	−0.04%	2.92%	2.92%	−0.001%
2007	5.87%	−1.50%	−0.01%	−0.19%	−0.03%	0.19%	4.64%	−0.06%	8.90%	8.91%	−0.009%
2008	13.11%	−1.95%	−1.40%	0.14%	−0.18%	0.61%	3.71%	−0.07%	13.97%	13.97%	−0.004%
2009	−7.32%	−1.27%	0.72%	−0.08%	0.10%	0.52%	3.39%	−0.12%	−4.04%	−4.06%	0.022%
2010	1.69%	0.21%	0.98%	−0.04%	−0.02%	0.31%	3.14%	−0.03%	6.24%	6.24%	0.000%
2011	5.03%	1.19%	0.48%	0.06%	0.03%	0.40%	2.67%	−0.01%	9.86%	9.86%	0.005%
2012	0.31%	−0.15%	0.05%	0.03%	0.02%	0.20%	1.73%	0.00%	2.17%	2.17%	0.001%
Avg	1.16%	−0.09%	0.08%	−0.01%	0.00%	0.27%	4.92%	−0.07%	6.25%	6.25%	0.002%
Stdev	4.54%	1.03%	0.75%	0.17%	0.14%	0.03%	0.22%	0.04%	5.15%	5.16%	0.005%

TABLE 5.2 Index performance attribution using coupon Strips

Year	1st	2nd	3rd	4th	5th	Vex	Carry	Security	Market	Total	Diff
1992	1.19%	−0.30%	−0.67%	0.00%	0.01%	0.13%	6.90%	−0.19%	7.06%	7.06%	−0.003%
1993	2.71%	0.46%	0.61%	0.00%	0.01%	0.13%	5.81%	−0.09%	9.65%	9.65%	−0.004%
1994	−10.53%	0.95%	−0.51%	−0.03%	0.02%	0.28%	6.99%	−0.19%	−3.03%	−3.03%	−0.007%
1995	8.76%	0.13%	1.08%	0.05%	−0.02%	0.17%	6.56%	−0.03%	16.67%	16.69%	−0.027%
1996	−3.10%	−0.19%	−0.37%	0.00%	0.00%	0.23%	6.44%	−0.07%	2.93%	2.94%	−0.004%
1997	1.67%	0.71%	0.23%	0.01%	0.01%	0.14%	6.35%	0.01%	9.12%	9.14%	−0.019%
1998	5.03%	−0.79%	0.00%	−0.01%	−0.06%	0.21%	5.36%	−0.15%	9.56%	9.59%	−0.027%
1999	−8.02%	0.16%	−0.36%	0.00%	0.02%	0.23%	5.81%	−0.05%	−2.21%	−2.21%	−0.001%
2000	5.48%	0.71%	0.61%	−0.04%	0.08%	0.21%	6.15%	−0.37%	12.76%	12.84%	−0.080%
2001	7.84%	−5.83%	−0.94%	0.10%	−0.17%	0.36%	5.13%	0.11%	6.59%	6.59%	−0.001%
2002	5.85%	0.08%	0.77%	−0.19%	0.11%	0.37%	4.75%	0.05%	11.78%	11.79%	−0.013%
2003	−1.04%	−0.72%	−0.38%	0.15%	0.03%	0.46%	4.16%	−0.21%	2.45%	2.45%	0.003%
2004	−3.54%	2.88%	0.36%	0.05%	−0.04%	0.28%	4.48%	−0.09%	4.37%	4.37%	−0.001%
2005	−4.03%	3.04%	0.07%	−0.02%	0.01%	0.20%	4.39%	−0.11%	3.53%	3.55%	−0.015%
2006	−2.34%	0.39%	0.24%	−0.07%	−0.06%	0.11%	4.78%	−0.13%	2.92%	2.92%	−0.004%
2007	5.13%	−1.23%	0.29%	−0.28%	0.04%	0.19%	4.63%	0.16%	8.90%	8.92%	−0.025%
2008	11.64%	−1.59%	−1.24%	−0.07%	−0.05%	0.67%	3.71%	0.92%	13.97%	13.99%	−0.028%
2009	−5.36%	−1.88%	0.15%	0.21%	0.01%	0.51%	3.41%	−1.07%	−4.04%	−4.03%	−0.013%
2010	1.97%	0.05%	0.75%	0.00%	0.01%	0.31%	3.17%	0.00%	6.24%	6.25%	−0.011%
2011	4.88%	1.30%	0.63%	0.01%	0.00%	0.39%	2.70%	−0.05%	9.86%	9.87%	−0.004%
2012	0.24%	−0.02%	0.13%	0.02%	0.02%	0.20%	1.75%	−0.16%	2.17%	2.18%	−0.006%
Avg	1.16%	−0.08%	0.07%	0.00%	0.00%	0.27%	4.93%	−0.08%	6.25%	6.26%	−0.014%
Stdev	4.63%	1.12%	0.79%	0.18%	0.14%	0.03%	0.22%	0.83%	5.15%	5.16%	0.014%

It is clear from Tables 5.1 and 5.2 that the carry (yield) and the first three duration components of the TSIR account for nearly all the performance contribution of the index. Additionally, the contribution of level duration to performance is largest, followed by slope, bend, etc. This important result almost universally applies to all fixed income portfolios. It emphasizes what we suggested earlier – that the first three components of duration are sufficient for risk measurement and management and performance attribution.

The standard deviation of the security selection, based on the daily data, is very large for the yield curve based on zero coupon bonds. Due to market inefficiencies and volatility of spread between coupon bonds and Strips, the contribution of security selection to performance is very volatile on a daily basis, but due to mean reversion the monthly volatility is significantly less.

The larger tracking error of performance attribution using Strips is due to the convexity effects of the security selection and higher order components of convexity that we have ignored.

Tables 5.1 and 5.2 show the annual performance of the components of the TSIR. It is clear that in most years the contributions of the fourth and fifth components of performance are very small and can be ignored as a first order approximation. The first three components of the TSIR account for more than 98% of the risk of the index for a wide range of decay coefficients. It is instructive to look at the pattern of contribution to performance and risk of different components.

Table 5.3 shows the aggregated historical performance using different decay coefficients. A relatively wide range of decay coefficients (2.16) result in comparable accuracies, provided that they are used in a consistent manner. The table shows the average annual contribution to performance for the components of the TSIR for the period 1992–2012, calculated on a daily basis and aggregated. In the table, decay coefficients ranging from 0.09 to 0.16, corresponding to the point where the slope component of

TABLE 5.3 Decay coefficient and contribution to performance, 1992–2012

Decay	1st	2nd	3rd	4th	5th	Vex	Carr	Securit	Market	Diff
0.09	1.14%	−0.09%	0.10%	−0.01%	−0.01%	0.27%	4.90%	−0.06%	6.25%	0.007%
0.10	1.14%	−0.09%	0.10%	−0.01%	−0.01%	0.27%	4.91%	−0.06%	6.25%	0.006%
0.11	1.15%	−0.09%	0.10%	−0.01%	0.00%	0.27%	4.91%	−0.07%	6.25%	0.005%
0.12	1.15%	−0.09%	0.09%	−0.01%	0.00%	0.27%	4.91%	−0.07%	6.25%	0.004%
0.13	1.16%	−0.09%	0.08%	−0.01%	0.00%	0.27%	4.92%	−0.07%	6.25%	0.002%
0.14	1.17%	−0.09%	0.07%	−0.01%	0.00%	0.27%	4.92%	−0.08%	6.25%	0.000%
0.15	1.17%	−0.09%	0.06%	−0.01%	0.00%	0.28%	4.92%	−0.08%	6.25%	−0.004%
0.16	1.18%	−0.09%	0.05%	−0.01%	0.00%	0.28%	4.92%	−0.08%	6.25%	−0.009%
0.17	1.19%	−0.08%	0.03%	0.00%	0.00%	0.29%	4.92%	−0.07%	6.25%	−0.015%
0.18	1.19%	−0.08%	0.02%	0.00%	0.00%	0.30%	4.92%	−0.07%	6.25%	−0.024%
0.19	1.19%	−0.07%	0.01%	0.00%	0.00%	0.31%	4.92%	−0.07%	6.25%	−0.035%
0.20	1.19%	−0.07%	0.00%	0.00%	0.00%	0.32%	4.91%	−0.07%	6.25%	−0.049%
0.21	1.19%	−0.06%	−0.01%	0.00%	0.00%	0.34%	4.91%	−0.06%	6.25%	−0.068%

TABLE 5.4 Decay coefficient and volatility of performance, 1992–2012

Decay	1st	2nd	3rd	4th	5th	Vex	Carr	Security	Market	Diff
0.09	4.46%	0.37%	1.14%	0.18%	0.17%	0.03%	0.22%	0.03%	5.15%	0.007%
0.10	4.48%	0.54%	1.03%	0.18%	0.17%	0.03%	0.22%	0.03%	5.15%	0.006%
0.11	4.50%	0.70%	0.93%	0.19%	0.16%	0.03%	0.22%	0.03%	5.15%	0.006%
0.12	4.52%	0.87%	0.84%	0.18%	0.15%	0.03%	0.22%	0.04%	5.15%	0.005%
0.13	4.54%	1.03%	0.75%	0.17%	0.14%	0.03%	0.22%	0.04%	5.15%	0.005%
0.14	4.56%	1.19%	0.67%	0.16%	0.12%	0.03%	0.22%	0.04%	5.15%	0.005%
0.15	4.58%	1.35%	0.60%	0.16%	0.10%	0.03%	0.22%	0.04%	5.15%	0.006%
0.16	4.59%	1.51%	0.54%	0.15%	0.09%	0.03%	0.22%	0.04%	5.15%	0.007%
0.17	4.61%	1.69%	0.49%	0.15%	0.08%	0.03%	0.22%	0.04%	5.15%	0.008%
0.18	4.63%	1.87%	0.45%	0.15%	0.07%	0.04%	0.22%	0.04%	5.15%	0.010%
0.19	4.65%	2.07%	0.44%	0.16%	0.08%	0.04%	0.22%	0.04%	5.15%	0.012%
0.20	4.67%	2.30%	0.45%	0.18%	0.09%	0.04%	0.22%	0.04%	5.15%	0.015%
0.21	4.70%	2.57%	0.49%	0.21%	0.10%	0.04%	0.22%	0.04%	5.15%	0.018%

the yield curve (τ) crosses zero in the range of 7.7 to 4.3 years respectively, have average errors of less than 1 basis point per year.

Table 5.4 shows the annualized volatility of contribution to performance for the period 1992–2012. The decay coefficient represents how the cash flows are weighted. Cash flows that are longer than the pivot point (see Section 2.9) contribute positively to the slope and other cash flows contribute negatively. Since a lower decay coefficient implies a longer pivot point, there will be fewer cash flows that are longer than the pivot point and therefore the slope duration will be less. In Table 5.4, as the value of the decay coefficient increases, so will the contribution of slope to the risk. In the middle range of the decay coefficients we can see a steadily decreasing contribution to risks of the components of the term structure combined with very low tracking error.

Table 5.5 compares the performance of different basis functions on an annual basis. The aggregated contributions to performance of durations, for all basis functions, are identical and equal to 1.14%. The difference in performance is only related to the contribution of convexity and security selection. We used only four out of 15 components of convexity to calculate its impact on performance. For Chebyshev basis functions, it captures nearly all the contribution of convexity to performance. For exponential basis functions, those same components (matrix elements 00, 01, 11, 02) significantly overestimate the convexity, and for key basis functions they underestimate convexity. Additionally, none of the basis functions compares favorably to Chebyshev for its successive decline in contribution to performance for each component of the term structure of rates.

The security selection performance was calculated from the spread of the security relative to the first component of the yield curve. For all basis functions except for key, this represents a parallel shift of the yield curve; for key, it represents a parallel shift of

TABLE 5.5 Comparison of aggregated daily performance by basis function, 1992–2012

	1st	2nd	3rd	4th	5th	Vex	Carry	Security	Diff
Cheby	1.16%	–0.09%	0.08%	–0.01%	0.00%	0.27%	4.92%	–0.07%	0.002%
Ortho	1.20%	–0.09%	0.03%	–0.01%	0.01%	0.27%	4.92%	–0.07%	0.006%
Poly	1.32%	–0.09%	–0.17%	–0.01%	0.09%	0.25%	4.92%	–0.07%	0.023%
Exp	1.09%	–0.05%	0.83%	–1.14%	0.40%	0.84%	4.92%	–0.07%	–0.569%
Key	0.00%	0.19%	0.40%	0.39%	0.15%	0.01%	4.92%	–0.01%	0.196%

TABLE 5.6 Comparison of annualized volatility by basis function

	1st	2nd	3rd	4th	5th	Vex	Carry	Security	Diff
Cheby	4.54%	1.03%	0.75%	0.17%	0.14%	0.03%	0.22%	0.04%	0.005%
Ortho	4.94%	0.98%	0.23%	0.13%	0.07%	0.03%	0.22%	0.04%	0.006%
Poly	5.87%	1.31%	2.11%	1.06%	1.54%	0.05%	0.22%	0.04%	0.031%
Exp	4.91%	9.95%	22.47%	23.10%	8.61%	0.13%	0.22%	0.04%	0.116%
Key	0.04%	0.83%	1.71%	2.15%	0.90%	0.00%	0.22%	0.00%	0.048%

the first key rate. Since the average contribution to duration of the first key rate is very small, as can be seen in Table 4.8, the adjustment to the first key rate can be very large and requires security selection convexity to calculate its contribution to performance. Since we are not performing this last step, the security selection performance in key rate basis functions will not be accurate.

Table 5.6 compares the annualized volatility of the performance of each component for different basis functions. Even though the performance volatility of the third component for Chebyshev is comparable to the slope component, due to strong mean reversion of the bend component, the annualized contribution to performance is significantly lower than would be expected from the volatility.

5.5 AGGREGATION OF CONTRIBUTION TO PERFORMANCE

It is a well-known issue that the sum of individual security returns, when compounded and added together, will be different from a portfolio return. The difference is due to cross-terms across time horizons. Here we provide a methodology to calculate the contribution to performance of individual securities so that, when summed, the result is the total portfolio return.

Let q_{ij}, w_{ij} and r_{ij} be respectively the return, weight and contribution to performance of security i in the period from $j-1$ to j. Then

$$r_{ij} = w_{ij} q_{ij} \tag{5.22}$$

Let k be the number of periods. The performance of security i through time k is

$$\tilde{q}_{ik} = \prod_{j=1}^{k} \left(1 + r_{ij}\right) - 1 \tag{5.23}$$

The return of the portfolio of N securities in the period from $j - 1$ to j is given by

$$Q_j = \sum_{i}^{N} r_{ij} \tag{5.24}$$

and the total performance of the portfolio through time k is

$$Q_{0k} = \prod_{j=1}^{k} \left(1 + Q_j\right) - 1 \tag{5.25}$$

The unit value of the portfolio at time k is

$$U_k = \prod_{j=1}^{k} \left(1 + Q_i\right) = \prod_{j=1}^{k} \left(1 + \sum_{i} r_{ij}\right)_j \tag{5.26}$$

It is clear that

$$Q_{0k} \neq \sum_{i} \tilde{q}_{ik} \tag{5.27}$$

To see the effects of the market on performance, consider a portfolio manager who makes an active bet that has an excess return of 10% after 1 day. Starting from a value of \$100, the portfolio's market value will be \$110, while the index will be \$100. Assume that he indexes his portfolio thereafter. If at the end of 1 year, the market appreciates 100%, the market value of the portfolio will be \$220 and that of the market \$200, thus the portfolio manager exhibits an active performance of (220 – 200)/100 = 20%. Likewise, if the market depreciates by 50%, the active performance of the manager will be (55 – 50)/100 = 5%. To accurately calculate performance at the security level, we have to take into account the impact of future market movements and compounding of the original performance.

We can write the performance of the total portfolio as

$$\begin{aligned} Q_{0k} &= \prod_{j=1}^{k} \left(1 + Q_j\right) - 1 = \left(1 + Q_1\right)\left(1 + Q_2\right)\left(1 + Q_3\right)\cdots - 1 \\ &= Q_1\left(1 + Q_2\right)\left(1 + Q_3\right)\cdots + Q_2\left(1 + Q_3\right)\left(1 + Q_4\right)\cdots + Q_3\left(1 + Q_4\right)\left(1 + Q_5\right)\cdots \end{aligned} \tag{5.28}$$

or

$$
\begin{aligned}
Q_{0k} &= Q_1 \frac{(1+Q_1)(1+Q_2)\cdots}{(1+Q_1)} + Q_2 \frac{(1+Q_1)(1+Q_2)(1+Q_3)\cdots}{(1+Q_1)(1+Q_2)} + \cdots \\
&= Q_1 \frac{U_k}{U_1} + Q_2 \frac{U_k}{U_2} + Q_3 \frac{U_k}{U_3} + \cdots \\
&= U_k \sum_j^k \frac{Q_j}{U_j}
\end{aligned} \tag{5.29}
$$

Substituting for Q_j, we have

$$
Q_{0k} = U_k \sum_j^k \frac{Q_j}{U_j} = U_k \sum_j^k \frac{\sum_i^N r_{ij}}{U_j} = U_k \sum_i^N \sum_{j=1}^k \frac{r_{ij}}{U_j} \tag{5.30}
$$

From this equation, we can see that the contribution to performance of security i from the beginning to time k is

$$
r_{i,0k} = U_k \sum_{j=1}^k \frac{r_{ij}}{U_j} \tag{5.31}
$$

The contribution to performance of security i in the interval $k - 1$ to k is

$$
\begin{aligned}
r_{i,0k} - r_{i,0k-1} &= U_k \sum_{j=1}^k \frac{r_{ij}}{U_j} - U_{k-1} \sum_{j=1}^{k-1} \frac{r_{ij}}{U_j} = U_k \frac{r_{ij}}{U_k} + (U_k - U_{k-1}) \sum_{j=1}^{k-1} \frac{r_{ij}}{U_j} \\
&= r_{ij} + (U_k - U_{k-1}) \sum_{j=1}^{k-1} \frac{r_{ij}}{U_j}
\end{aligned} \tag{5.32}
$$

The sum of contributions of each security to performance through period k from (5.31) should be equal to the unit value of the portfolio minus one, that is,

$$
\begin{aligned}
\sum_i r_{i,0k} &= \sum_i^N U_k \sum_{j=1}^k \frac{r_{ij}}{U_j} = U_k \sum_{j=1}^k \frac{1}{U_j} \sum_i^N r_{ij} = U_k \sum_j^k \frac{1}{U_j} \left(\frac{U_j}{U_{j-1}} - 1 \right) \\
&= U_k \sum_j^k \left(\frac{1}{U_{j-1}} - \frac{1}{U_j} \right) = U_k \left(1 - \frac{1}{U_k} \right) = U_k - 1
\end{aligned} \tag{5.33}
$$

CHAPTER 6

Libor and Swaps

The London Inter-Bank Offered Rate (Libor), as its name implies, is the rate that banks charge each other for short term transactions. As such, the implied risk of short term Libor is the credit risk of banks. For example, if an institution deposits $100 million in a 1-week time deposit in a bank, it will receive a rate very close to 1-week Libor. If the bank files for bankruptcy during that week, all the deposit could be lost.

An interest rate swap (IRS) is a transaction where a party receives a fixed rate coupon for the life of the swap and pays floating Libor at predetermined intervals. The value of the floating Libor is usually very close to par (100), but the value of the fixed rate can fall if rates rise or increase if rates fall. A swap transaction is usually considered to be risk-free, since if rates fall, the fixed rate receiver will demand collateral from the floating rate receiver and vice versa. However, since the floating rate is established by Libor, there is an implied banking credit spread for the floating coupon, and since the present value of the fixed rate leg of the swap must be equal to the present value of the future floating rates, the fixed rate must have an embedded banking spread.

The IRS market has become very liquid globally and there are many countries where the swap market is more liquid than in government issued bonds. Considering that an IRS is symmetric and you can take a position in either direction with minimum capital requirements, in recent years long term swap rates have traded at a premium with respect to government rates in many countries, including the US. Eurodollar futures contracts and certificates of deposits are also considered to have similar quality as Libor.

The daily trading volume of swap related transactions is by far the largest segment of the global capital markets and covers the following areas:

- Interest rate swaps and swaptions.
- Asset swaps.
- Over-the-counter forward currency contracts.
- Credit default swaps.
- Inflation swaps.

The Advanced Fixed Income and Derivatives Management Guide. Saied Simozar

A swap transaction is usually cashless, that is, the value of the swap is zero at inception. Like a note or a bond, if interest rates fall, after the initiation of the swap, the value of the fixed leg rises and vice versa. The value of the floating leg is usually very close to par but changes a little if short rates rise or fall significantly. A swap is sometimes a long term contract between two counterparties, and there can potentially be credit related events or default risk during its life. Therefore, before a swap transaction takes place the two counterparties must have a master agreement to govern the mechanics of transaction and its future maintenance. Such agreements have become largely standardized and are called ISDA (International Swaps and Derivatives Association) agreements.

The standardization of the swap agreements has been instrumental in the liquidity of swaps. Nearly all major banks/dealers accept each other's swap agreements. For example, if you have a swap transaction with Bank A, you can sell or assign it to Bank B, as if your swap was with Bank B to begin with. This means that if you get into a long term swap transaction and want to unwind it, you will not be limited to the original party to provide pricing and you can get competitive bids to terminate the agreement by assigning it to a third party.

The introduction of swap futures on many exchanges has increased liquidity and price transparency of IRSs. Futures contracts usually trade with price increments or bid–ask spreads of 1/32 or 0.03 of 100 notional. The bid–ask spreads on over the counter IRSs have tightened significantly in the past several years and in liquid markets they are about 0.5 bps of yield. For example, to trade $20 million of 10-year swaps in the US, the bid–offer coupons can be 2.345%–2.34%, respectively.

If rates fall, the fixed rate receiver (floating rate payer) usually demands collateral for the net market value of the swap to cover it, in case of a credit related event for the floating rate receiver. Likewise, if rates rise, the floating rate receiver (fixed rate payer) will demand collateral for the net value of the swap. The requirement to post collateral as well as the liquidity and security of swaps, especially in the wake of the banking crisis of 2008, where few investors lost money due to bankruptcy or default, has made IRSs very attractive vehicles for fund managers to use for hedging and speculating in the interest rate and currency markets.

At the initiation of a swap transaction, the parties agree on the coupon rate and term of the IRS; market conventions govern the additional details such as the frequency of coupon payment and accrual convention.

The floating coupon of a swap contract is fixed by the organization in charge of maintaining the benchmark rate; this was formerly the British Bankers Association, which published the rate daily at 11:30 a.m. London time, after polling large member banks and averaging their rates after dropping the highest and lowest rates. In the wake the turmoil of 2008 and allegations of collusion among member banks, the governing body for the administration and publication of Libor was changed in January 2014 to NYSE Euronext.

A swap transaction usually settles in two or three business days, and the first coupon of the floating leg will also be fixed. For the US, the floating coupon frequency is generally once every 3 months, but payments are once every 6 months. Two business days before a coupon payment, the following coupon is also fixed, so that accrual for transactions that need to settle after coupon payment can be calculated. Historical floating coupon rates of a swap transaction can thus be derived from historical Libor fixings two or three business days prior to coupon payment depending on the currency.

The coupon rate of fixed rate Libor is set in such a way that the prices of both fixed and floating legs of Libor at initiation would be par (100). At each coupon payment, only the net amount will be paid. For example, if the fixed rate is 5% and the floating rate 3% for semi-annual payments, the fixed rate receiver will receive $(5-3)\frac{6}{12} = 1\%$ of the notional amount at coupon payment date. However, it is possible to agree on a different coupon rate and a premium or discount price for the transaction. One such transaction is the zero coupon swap for the fixed leg. The floating coupon will also accrue and compound until maturity and the net value of the transaction will be settled at maturity.

The popularity of swaps goes beyond investment managers and banks. Many corporations and sovereign countries issue floating rate bonds, with the coupon based on a spread to floating Libor rates. The spread between short term treasury rates and Libor or treasury–eurodollar (TED) spread is often used by central banks as a gauge of the health of the banking system. In the fall of 2008, when US treasury bill rates were at a yield of 0.1%, short term Libor reached rates of more than 4%.

The short end of Libor is influenced by many variables, including global liquidity, the health of the banking system and other cyclical factors in the economy, and tends to be highly volatile. Also, demand for deposits by banks that need to meet end of year reserve requirements for regulatory reasons tends to be higher, leading to higher deposit rates and short term Libor at the end of the year compared to other times.

6.1 TERM STRUCTURE OF LIBOR

Figure 6.1 shows the spot and coupon swap curve along with the corresponding treasury curve. The coupon swap and treasury data are calculated by taking the maturity of the coupon bond and calculating its spread relative to the curve. The spread is simply added to the spot curve to show where the coupon bond would lie relative to the spot (zero coupon) curve.

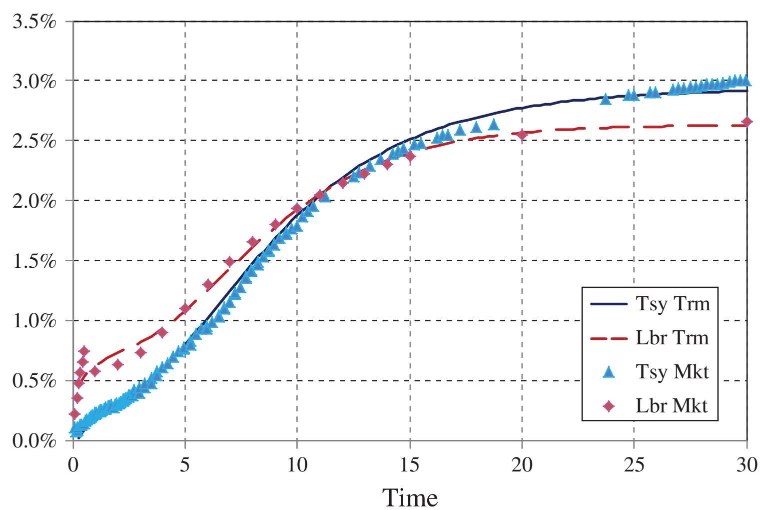


FIGURE 6.1 Term structure of swap curve, May 25, 2012

TABLE 6.1 Selected term structure of swaps, July 30, 2012

Currency	Ticker	Decay	Level	Slope	Bend	Cubic	Quartic
USD	LBR	0.13	1.199%	1.144%	0.201%	0.011%	–0.133%
USD	TSY	0.13	1.088%	1.504%	0.242%	0.011%	–0.133%
EUR	LBR	0.13	1.346%	1.004%	0.113%	–0.103%	–0.127%
EUR	TSY	0.13	0.870%	1.429%	0.271%	–0.103%	–0.127%
JPY	LBR	0.13	0.742%	0.673%	0.335%	0.072%	–0.025%
GBP	LBR	0.13	1.717%	1.091%	0.552%	–0.100%	0.086%
CHF	LBR	0.13	0.600%	0.818%	0.172%		
CAD	LBR	0.13	1.852%	0.838%	0.070%	0.059%	–0.088%
AUD	LBR	0.13	3.644%	0.423%	0.058%		
MXN	LBR	0.13	5.905%	1.759%	0.868%	0.297%	0.188%
ZAR	LBR	0.13	5.822%	1.302%	–0.050%		
CZK	LBR	0.13	1.578%	0.551%	0.228%		
DKK	LBR	0.13	1.356%	1.091%	0.225%	–0.149%	–0.048%
HUF	LBR	0.13	6.459%	–0.463%	0.170%	–0.070%	0.028%
ILS	LBR	0.13	3.261%	1.797%	0.538%		
NOK	LBR	0.13	2.757%	0.653%	0.001%		
NZD	LBR	0.13	3.309%	0.939%	0.030%		
PLN	LBR	0.13	4.465%	–0.201%	0.130%		
SEK	LBR	0.13	2.413%	0.515%	0.463%	0.000%	–0.002%
SGD	LBR	0.13	1.397%	1.233%	0.426%		

Using three parameters, the calculated swap curve matches market rates with an error of about 2–3 bps. The error is significantly larger at the front end of the yield curve where time deposit rates can be volatile. Eliminating short term time deposit rates with maturities of less than 1 year leads to a significant improvement in the fit of the data.

Table 6.1 shows a sample of the term structure of Libor rates (TSLR) for different currencies. Only the first three components of the curve were calculated. For countries where there was a liquid treasury market, the fourth and fifth components of the TSLR were matched to the TSIR. For comparison, the TSIR for the US and the euro region using German government bonds as benchmark are also provided. The slope of the treasury curve in both the US and the euro region is about 35 bps steeper than the respective TSLR.

6.2 ADJUSTMENT TABLE FOR RATES

For risk measurement and management, one or two basis points of error in the TSLR is acceptable; however, for pricing and valuation purposes, more accurate algorithms are needed.

Accurate pricing of all IRSs can be achieved by using a table of adjustments that would enable one to price all swaps exactly. The table is constructed by taking the shortest maturity bond and calculating the yield adjustment that is needed to match its calculated price with the market. Then the yield of the next shortest bond is adjusted, and so on until all swaps are priced exactly, by using the prior adjustments to calculate the market discount rate for each cash flow. For cash flows between two adjustment points, linear interpolation is used.

Using an adjustment table consisting of 24 points is enough to price every swap exactly using the term structure model; this method can be used for treasuries as well. Due to the fragmentation of the treasury market at times of crisis (see Figure 2.10), one should use a set of unbiased treasuries such as coupon Strips which are all fungible. Table 6.2 lists the adjustment table and Table 6.3 provides the valuation parameters for Israeli shekel (ILS) swaps.

The fair price in Table 6.3 is calculated using the TSLR in addition to the adjustment table and "Trm Price" is calculated by using the TSLR only. The spread part of the table represents the relative value or cheapness/richness of individual swaps. Spread relative to "Tsy Mkt" is the spread of the swap relative to the treasury market calculated by using the treasury term structure and adjustment table. The spread relative to "Tsy Trm" is the spread relative to the TSIR and the spread relative to "Lbr Trm" is the spread of the swap relative to the TSLR. We can see from the table that the heavily traded 10-year swap (1-Aug-22) is rich by about 2 bps while the 8-year and 7-year swaps are cheap relative to the curve.

The liquidity and popularity of swaps have led many to consider the swap curve to be the fundamental determinant of interest rates. In many countries, where the government curve is not well established, swaps are the primary definers of interest rates. They can be traded globally without local government interference or barriers. There

TABLE 6.2 Selected adjustment table for TSLR, July 30, 2012

Currency	Ticker	Maturity	Spread
ILS	LBR	0.090	0.0412%
ILS	LBR	0.504	0.0159%
ILS	LBR	0.747	0.0034%
ILS	LBR	0.999	0.0022%
ILS	LBR	1.999	−0.0335%
ILS	LBR	3.003	−0.0245%
ILS	LBR	4.000	0.0026%
ILS	LBR	4.999	−0.0066%
ILS	LBR	5.999	0.0092%
ILS	LBR	6.998	0.0293%
ILS	LBR	8.006	0.0124%
ILS	LBR	9.002	−0.0057%
ILS	LBR	9.999	−0.0203%

TABLE 6.3 Swap valuation table, July 30, 2012

			Price			Spread		
Currency	Cpn	Maturity	Market	Fair	Trm	Tsy Mkt	Tsy Trm	Lbr Trm
ILS	2	1-Aug-13	100	100.00	100.002	−0.003%	−0.003%	0.002%
ILS	2.05	1-Aug-14	100	100.00	99.935	0.036%	0.036%	−0.033%
ILS	2.22	1-Aug-15	100	100.00	99.928	0.001%	0.001%	−0.024%
ILS	2.44	1-Aug-16	100	100.00	100.006	−0.070%	−0.068%	−0.002%
ILS	2.65	1-Aug-17	100	100.00	99.967	−0.182%	−0.184%	−0.007%
ILS	2.89	1-Aug-18	100	100.00	100.043	−0.254%	−0.267%	−0.008%
ILS	3.13	1-Aug-19	100	100.00	100.167	−0.326%	−0.325%	−0.026%
ILS	3.33	1-Aug-20	100	100.00	100.080	−0.409%	−0.395%	−0.011%
ILS	3.51	1-Aug-21	100	100.00	99.967	−0.442%	−0.442%	−0.004%
ILS	3.6775	1-Aug-22	100	100.00	99.859	−0.461%	−0.467%	−0.017%

are several countries where there are two separate markets for swaps, called on-shore and off-shore rates. The off-shore curve tends to be much more transparent and liquid than the on-shore curve which is governed by the level of interest rates as well as capital flow restrictions imposed by the government.

In countries where there is a well-established risk-free government curve such as the US, UK and Japan, swap rates are considered to be Libor based and as such have the implied credit risk of banks. Therefore swaps have a risk premium relative to treasuries and therefore their level depends not only on the level of the risk-free rate, but also on the level of risk premium or spread.

In the past, it was assumed that swap rates could never be below treasury rates in countries with free economies and well-established policies. After all, if the government had unlimited resources to print money and pay off investors, its borrowings were risk-free, but banks did not have such an authority and therefore required a risk premium.

The financial crisis of 2008 changed the notion that swap spreads could not be negative. Many investors or fund managers who needed to extend their duration found swaps very attractive. They were willing to pay a premium and receive fixed rate swaps and still have access to their cash for liquidity reasons or for other investments. The fact that long dated swaps traded through treasuries for most developed countries is also a testament to the economic stability of those countries where investors have confidence in governments not to freeze the collateral of swaps. A swap transaction in developed countries is considered to be a quasi-risk-free transaction.

One can also argue that investors have more confidence in the long run security of a swap contract than long run government's willingness and ability to service its debt. For example, if Congress does not extend the debt ceiling in the US, the treasury can potentially default on its debt.

The phasing out of proprietary desk traders and weakening of hedge funds has also been a large contributor to the negative swap spreads. In the past negative rates would be arbitraged by proprietary desks and in particular by banks.

In the US, where states cannot tax the income of a federal government issued bond, there is an arbitrage possibility by banks if swap spreads are negative or below a certain amount. For example, consider a bank operating in a state that has an annual income of $4 million that is taxable at the rate of 10% at the state level. Assume for simplicity that 5-year treasury bonds have a yield of 5% and 5-year swaps have a yield of 4%. The bank can borrow $100 million from other banks at a rate of 4% and purchase treasury bonds at a yield of 5%. The bank would then have an interest expense of $4 million for the money that it borrowed and will receive $5 million in treasury coupon that is not taxable at the state level. Not only will the bank avoid state income tax, but its income will increase by $1 million with very little duration risk. This arbitrage can take place even if swap rates are the same as treasury rates. It is only when the ratio of swap spread to swap rate is more than or equal to the marginal state tax rate that this arbitrage is not possible.

6.3 RISK MEASUREMENT AND PERFORMANCE ATTRIBUTION OF SWAPS

We assume that treasuries are the fundamental determinant of rates and swaps, and Libor trades at a spread to treasuries. The swap curve can also be represented by a set of basis functions similar to treasury rates (2.18). The TSLR can be written

$$y_{l,i}\left(t_i\right)=\sum_{j=0}^{n-1} a_{l,j}\psi_j(t_i) \tag{6.1}$$

where $y_{l,i}(t_i)$ is the spot calculated yield of Libor at time t_i, $a_{l,j}$ is the jth component of the TSLR, and $\psi_j(t_i)$ is the jth component of the basis function.

With treasury rates as the fundamental driver of interest rates, Libor can be considered to depend on the level of interest rates plus a spread, called swaps spread, Libor spread or TED spread. We write the term structure of swap spread y_{ls} as

$$y_{ls,i}(t_i)=y_{l,i}-y_i=\sum_j (a_{l,j}-a_j)\psi_j(t_i)=\sum_j a_{ls,j}\psi_j(t_i) \tag{6.2}$$

The calculated (term) price of a swap for a security with N cash flows can be written as

$$p_t=\sum_i^N c_i e^{-y_{l,i}(t_i)t_i}=\sum_i^N c_i e^{-t_i\sum\limits_j (a_j+a_{ls,j})\psi_j} \tag{6.3}$$

where $a_{ls,j}$ is the jth component of the term structure of Libor spread. In order to match the price of the security with market prices, we need to add the spread of the security relative to the Libor curve, similar to (5.11), as follows:

$$p_m=\sum_i c_i e^{-t_i\left(s_b+\sum\limits_j (a_j+a_{ls,j})\psi_j\right)} \tag{6.4}$$

Interest rate and swap spread durations will be defined by:

$$D_k = -\frac{1}{p_m}\frac{\partial p_m}{\partial a_k} = \frac{1}{p_m}\sum_i c_i t_i \psi_k e^{-t_i\left(s_b + \sum_j (a_j + a_{ls,j})\psi_j\right)} = \langle t\psi_k(t)\rangle \tag{6.5}$$

$$D_{l,k} = -\frac{1}{p_m}\frac{\partial p_m}{\partial a_{l,k}} = \frac{1}{p_m}\sum_i c_i t_i \psi_k e^{-t_i\left(s_b + \sum_j (a_j + a_{ls,j})\psi_j\right)} = \langle t\psi_k(t)\rangle \tag{6.6}$$

As we see for swaps, interest rate and Libor durations are identical. Nevertheless, we need to calculate both separately, since in a portfolio context there can be many securities with different interest rate and Libor durations and they need to be aggregated separately.

The convexity components of swaps are similar to treasuries; however, there are cross-convexity components which, computationally, can be too resource intensive for the marginal benefits of performance attribution. We have

$$X_{kl} = X_{l,kl} = \frac{1}{p_m}\frac{\partial^2 p}{\partial a_k \partial a_l} = \frac{1}{p_m}\sum_i c_i t_i^2 \psi_k \psi_l e^{-t_i \sum_j a_j \psi_j} = \left\langle t^2 \psi_k(t)\psi_l(t)\right\rangle \tag{6.7}$$

Performance attribution for swaps involves the contribution from changes in interest rates as well as Libor spread and yield of the security. Generalizing (5.20) to include Libor spread, we can write performance as

$$\begin{aligned}\frac{\Delta p_m}{p_m} &= \frac{\Delta p_t}{p_t} - \sum_k \Delta a_{ls,k} D_k \\ &\quad + \frac{1}{2}\sum_k\sum_l \Delta a_{ls,k}\Delta a_l X_{kl} + \frac{1}{2}\sum_k\sum_l \Delta a_{ls,k}\Delta a_{ls,l} X_{kl} \\ &\quad - \Delta s_b D_0 + \theta \Delta t\end{aligned} \tag{6.8}$$

where

$$\Delta p_t = -p_m \sum_k \Delta a_k D_k + \frac{1}{2} p_m \sum_k \sum_l \Delta a_k \Delta a_l X_{kl} + \ldots \tag{6.9}$$

6.4 FLOATING LIBOR VALUATION AND RISKS

The TSLR with associated adjustments can price the swap curve perfectly. We now use the curve to calculate the implied forward floating coupon of a swap. If $c_{f,i}$ is the coupon amount of a time deposit with a face value of 100 initiated at time t_{i-1}, for maturity t_i, then its price at time t_i will be

$$p_i = 100 + c_{f,i} \tag{6.10}$$

The present value of the investment at t_{i-1} and t_i must be equal, leading to

$$\begin{aligned} p_i e^{-(y_i+y_{ls,i}+s_{b,i})t_i} &= 100e^{-(y_{i-1}+y_{ls,i-1}+s_{b,i-1})t_{i-1}} \\ &= (100+c_{f,i})e^{-(y_i+y_{ls,i}+s_{b,i})t_i} \end{aligned} \tag{6.11}$$

Thus, the floating coupon of a Libor bond at time t_i will be given by

$$c_{f,i} = 100(e^{(y_i+y_{ls,i}+s_{b,i})t_i-(y_{i-1}+y_{ls,i-1}+s_{b,i-1})t_{i-1}} - 1) \tag{6.12}$$

Define the market Libor yield, $y_{l,i}$, as the interest rate plus Libor spread plus security spread at a given time:

$$y_{l,i} = y_i + y_{ls,i} + s_{b,i} \tag{6.13}$$

Thus,

$$c_{f,i} = 100(e^{y_{l,i}t_i - y_{l,i-1}t_{i-1}} - 1) \tag{6.14}$$

For a sinking bond, if the remaining principal of a bond is μ_i, we can modify equation (6.14) as

$$c_{f,i} = \mu_i(e^{y_{l,i}t_i - y_{l,i-1}t_{i-1}} - 1) \tag{6.15}$$

This is our generalized formula for the coupon of a security that is based on floating Libor. The price function of a floating bond with N cash flows is thus

$$p_m = \sum_i c_{f,i} e^{-y_{l,i}t_i} \tag{6.16}$$

where the market Libor yield $y_{l,i}$ is equal to the interest rate plus Libor spread plus security spread at a given time.

It is a trivial exercise to show that the price function of a Libor floater will become

$$p_m = \mu_0 e^{-y_{l,0}t_0} = \mu_{t_0=0} \tag{6.17}$$

If the remaining principal of the security is $\mu = 100$, then the present value of a floating Libor bond will be equal to par. There are many corporate securities whose coupons are based on floating Libor and therefore have to be discounted by the corporate curve. For those securities, the price of the floating coupon will not be equal to par.

If the first coupon of the floater has been fixed, the price function will become

$$p_m = (100 + c_1)e^{-y_{l,1}t_1} \tag{6.18}$$

The duration components will be

$$D_k = \frac{1}{p_m}\frac{\partial p_m}{\partial a_k} = t_1\psi_k \tag{6.19}$$

The floating coupon of a bond can contribute to duration risk if the discount function is different from the function that generates the floating coupon. For example, if a corporation issues a bond with a coupon rate that is based on Libor, the impact of the floating coupon on the duration is not zero. In general, the duration of a security that has floating as well as fixed coupon rates is

$$\begin{aligned} D_k &= -\frac{1}{p}\frac{\partial p}{\partial a_k} = -\frac{1}{p}\sum_i^N\left(\frac{\partial c_i}{\partial a_k} + \frac{\partial y_i}{\partial a_k}\right)e^{-y_i t_i} \\ &= -\frac{1}{p}\sum_i^N\left(\frac{\partial c_i}{\partial a_k} + t_i\psi_k\right)e^{-y_i t_i} = -\frac{1}{p}\sum_i^N \frac{\partial c_i}{\partial a_k}e^{-y_i t_i} + \langle t\psi_k(t)\rangle \end{aligned} \tag{6.20}$$

Equation (6.20) is similar to (6.6) plus the contribution to duration of the floating coupon. Note that if the coupon is fixed then $\partial c_i/\partial a_k = 0$. From (6.15),

$$\frac{\partial c_i}{\partial a_k} = \mu\left[t_i\psi(t_i) - t_{i-1}\psi(t_{i-1})\right]e^{y_{l,i}t_i - y_{l,i-1}t_{i-1}} \tag{6.21}$$

Again from (6.15),

$$\mu_i + c_{f,i} = \mu_i e^{y_{l,i}t_i - y_{l,i-1}t_{i-1}} \tag{6.22}$$

Substituting into (6.21), we can then write (6.20) as

$$D_k = -\frac{1}{p}\sum_i^N(\mu_i + c_{f,i})\left[t_i\psi_k(t_i) - t_{i-1}\psi_k(t_{i-1})\right]e^{-y_i t_i} + \langle t\psi_k\rangle \tag{6.23}$$

Equation (6.23) can be further simplified, but then loses its intuitive form, and we will use it to capture durations of floating rate bonds.

6.5 REPO AND FINANCING RATE

The repo or repurchase rate is the interest rate that a borrower of a security pays the lender. A trading desk that owns a treasury bond or treasury bill can lend it and receive cash for trading. The borrower of the security may have sold short the security and need to borrow it on a temporary basis or have short term cash that he needs to invest. A repo is safer than a time deposit, since in case of default the borrower of the security is entitled to sell it and recover his money. The lender receives cash for lending the securities and can invest cash in short term time deposits or other liquid transactions.

Since there is usually a limited supply of a security, if there are too many short sellers, they have to compete to borrow the security and demand a lower interest rate than Libor or overnight rate. If a security is in deep demand by short sellers, the repo rate can become negative, that is, the lender of the security receives both cash and interest for lending his security and can invest cash to earn additional interest. The repo rate can rarely exceed the overnight rate, since lenders have no incentive to lend their security if they cannot earn a higher interest than their cost. If the repo rate is lower than the overnight rate, the security is said to be *on-special*.

Some central banks use the repo or reverse repo as one of the primary tools for monetary operations. Member banks can lend their securities and receive cash to increase liquidity in the system, or the central bank can lend securities and drain liquidity from the banks. When central banks raise the repo rate, borrowing costs for member banks increase and they will be less likely to borrow at higher rates, thereby lowering liquidity in the economy. The reverse repo is the rate at which banks earn interest on their excess liquidity deposited in the central bank.

Financing a position is almost exactly like repo, except that it is usually used to increase leverage. For example, you purchase a security, but you want to borrow funds to pay for its purchase, similar to buying a home and borrowing to pay for it. The rate to finance a position is usually higher than Libor or overnight rate. Lenders usually want some protection for market volatility and therefore finance only part of any purchase. The difference between the market value of a security and the amount of loan that can be borrowed against it is called the “haircut”. For liquid securities, the haircut is usually about 2%, but can vary depending on market volatility and security type.

6.6 STRUCTURAL PROBLEM OF SWAPS

The floating leg of an IRS contract is based on Libor which is based on time deposit rates in a member bank. A time deposit has the credit risk of the bank; if the bank becomes bankrupt, the deposit can be lost. However, all IRSs have zero market value at initiation and have no principal risk. Only when the market moves one leg of an IRS gain value relative to the other. It is the market practice that once a leg of a swap is in-the-money, the holder of that leg will demand collateral from the counterparty. With this practice, the counterparty credit risk of a swap is eliminated and the swap becomes quasi-risk-free. Since the floating leg of a swap has a risk premium relative to treasuries and also due to tax advantages of treasuries in the US, the floating leg of swaps will always be at a premium to short term government bonds or treasury bills. The present value of future floating rates of a swap must have the same value as the fixed rate swap, thus we conclude that fixed rate swaps can never have a yield below treasuries.

In practice, as we see in Figure 6.1, the long end of fixed rate swaps has yields below treasuries. The fact that this is the case points to either IRS inefficiency or the market assuming that the long term counterparty risk is better than the credit rating of the US or other major countries. Figure 6.1 is subject to long term arbitrage, assuming that the treasury rates are indeed risk-free. An asset manager with extra cash to invest can buy long term treasuries and pay fixed rate swap with the same duration and curve risk. Instead of investing the cash at the risk-free rate, he will receive floating Libor with a premium over treasuries and at the long end he pays the fixed rate from the proceeds of the long treasury. The trade is risk-free in the long run and has positive return at both ends of the curve.

Given the current structure of swaps and their dependence on Libor, they will bear the risk premium embedded in Libor and as such will remain subject to long term arbitrage.

This points to a flaw in the structure of IRSs for using Libor to establish the floating coupon of swaps and thus including the risk premium of banks into the swap. The floating rate of an IRS has to be based on a rate that eliminates or mitigates the counterparty or banking credit risk. As such, the repo rate is a much better candidate for establishing the floating leg of an IRS than is Libor. Figure 6.2 shows the average weekly spreads of repo rate and 3-month Libor over 3-month treasury bills.

The spread of overnight repo over treasury bills is close to zero most of the time and is always below the spread of Libor over 3-month treasury bills. Since the repo market is collateralized, it is a better proxy for swaps that are also collateralized and have no principal risk.

Given the popularity of Libor as a benchmark for floating rate notes and bonds, why should a borrower be exposed to the credit risk of banks in addition to its own credit risk? Historically, most central banks have protected large banks from default to ensure the continuation of economic activity. However, in 2008 Lehman was allowed to fail and it led to a widespread banking crisis and widening of banking spreads. Similar widening happened in the 1995–1996 banking crisis in Japan. Since banks are not explicitly or implicitly protected by central banks, the use of Libor is not appropriate as a benchmark for IRSs and other borrowing institutions.

Establishing a new benchmark for floating swaps contracts will not be easy. There are many outstanding long term swap contracts that are based on Libor floaters. One solution for the transition is to initiate the new swaps with repo floaters that will be obtained from member banks, just like Libor. A window can be established in the future where the spread between Libor and repo can be used for a onetime adjustment to long term swaps to be converted to repo based floaters. For example, if a 6-month window is established and the spread of Libor over repo is 10 bps, then the fixed coupon of the swap will be adjusted downward by 10 bps and the swap will be permanently converted to a repo floater. Since swaps are used for hedging and liquidity management, most parties are likely to agree to the transition, knowing that Libor based floater liquidity will likely fall.

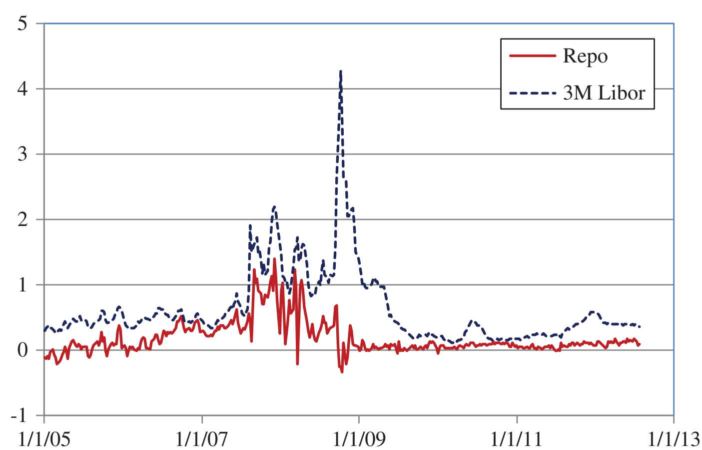


FIGURE 6.2 Spread of repo and Libor over treasury bills

If repo based swaps are adopted, the following list explains how the adjustment should be made to the existing swap contracts:

Interest rate swaps. The fixed coupon of the swap needs to be adjusted based on the spread of Libor and repo.

Inflation swaps. No adjustment is necessary. The fixed leg represents a fixed amount in the future and the floating leg is the cumulative inflation. Both have implicit Libor exposures that cancel each other out.

Asset swaps. Many asset swaps have a spread to floating Libor. Thus, the spread of the floating Libor needs to be adjusted; if the spread is zero, then the floater will receive repo plus the adjustment.

Credit default swaps. Both parties are assumed to receive floating Libor and these cancel each other out. No adjustment is needed.

Currency forwards. No adjustment is necessary as the forward points are already known.

Multi-currency swaps. The adjustment for each currency is based on the spread of Libor minus the repo of that currency.

CHAPTER 7

Trading

In fixed income, trading is an extension of the portfolio management job. A trader has to understand how to identify cheap securities and trade them efficiently but also understand all market conventions and forward markets and be able to manage the overall liquidity of a portfolio. Since most securities in fixed income are very highly correlated, there are numerous ways to structure a portfolio for the same risk profile. From the previous chapters, we know that only three to five parameters of the TSIR are needed to match all the duration components of a portfolio and to manage its risks. With so many securities to choose from in a single currency treasury portfolio, it is the task of the trader to anticipate and maintain sufficient liquidity to manage portfolio durations if need be.

Trading and settlement rules vary by security and currency. For US treasuries, a trade settles the next business day, implying that to buy US treasuries in the secondary market, the price that was agreed upon will be paid in the next business day and the securities will be received at the same time. Most institutions deposit their securities with a custodian that will maintain all the positions and can support transactions. Settlement is usually done through a clearing house that acts as an escrow agent. The buyer sends funds to the clearing house and the seller sends the securities, at which point they are exchanged.

Most securities are settled T+3, that is, three business days after trade date. A trade that is executed on a Thursday is typically not settled until the next Tuesday. For long weekends, another day is added.

In this chapter we will discuss how a trader can add significant value to a portfolio.

7.1 LIQUIDITY MANAGEMENT

As we showed in the previous chapters, only five parameters are necessary to manage the durations of a treasury portfolio. There are numerous combinations of securities that would achieve the desired measure of duration components. A trader has a critical role in understanding the philosophy and style of portfolio managers and anticipating trading requirements. The following is a list of some of the times when a trader needs to maintain or manage a portfolio:

- End of month rebalancing.
- Fund flow.
- Asset allocation.
- Coupon flow.
- Active duration management.
- Derivatives collateral.
- Margin flow.

Most indexes, such as the Barclays Bond Aggregate or Citi's Bond Index, rebalance at the end of the month when cash flows from coupons are reinvested in the index, new securities are added and old ones are dropped from the index. The duration of the index usually increases on the last business day of the month, and portfolio managers who want to maintain a position relative to the index need to rebalance the portfolio. For example, in a refunding month in the US (February, May, August and November), the treasury may issue a new 30-year bond, a new 10-year bond and a new 5-year bond. The entry of these bonds increases the duration of the index at the same time that bonds that have a maturity of less than 1 year are dropped out of the index. For some indexes, such as J. P. Morgan bond indexes, the coupons are reinvested in the index as soon as they are received, but new securities are added and old securities are dropped from the index only at the end of the month. Reinvestment of coupons in the index in the middle of the month causes a jump in the duration of the index. For example, if a bond that pays annual coupon of 7% has a duration of 10 years and a price of 100, its new duration after coupon payment will be

$$\begin{aligned} 107\times 10 &= 7\times 0+\sum_i c_i t_i e^{-y_i t_i} \\ \frac{1}{p}\sum_i c_i t_i e^{-y_i t_i} &= 10.7 \end{aligned} \tag{7.1}$$

Since coupon flows and security changes are known in advance, the shift in the duration of the index is known in advance and most portfolio managers rebalance the portfolio on the last business day of the month.

Fund flow is related to the flow of funds into and out of a portfolio. For mutual funds, there may be periodic distributions, or redemptions by shareholders, or new money that needs to be invested. Periodic distributions are generally known in advance, but redemptions are generally known at most a day in advance. Since it may take up to three business days for trades to settle, a trader needs to maintain a level of cash for redemptions. Typically, portfolios maintain 1–2% cash for this purpose. Institutional accounts usually offer longer lead times for redemptions or withdrawals.

Settlement of foreign securities can be more complicated and, depending on the time horizon, an extra day may be required to settle a trade. For example, selling Japanese government bonds from a portfolio located in the US, during US normal business hours when the Japanese markets are closed, may take an additional day to settle.

Asset allocation can take place passively or actively or both, depending on the product. For example, a balanced fund that is a 50–50 mix of equity and fixed income will require rebalancing to bring it in line with its benchmark. In the periods where equities outperform fixed income, money flows from equities into fixed income and traders need to rescale the portfolio to maintain its structure and integrity. Likewise, in periods where fixed income outperforms equities, traders need to raise the necessary cash to the equity portion of the fund. Such passive rebalancing usually takes place on a monthly or quarterly basis. The asset allocation can also be between two sectors of fixed income. A strategic fund can be a combination of treasuries, emerging markets, high yield, and international securities managed by different portfolio managers. If one of these sectors outperforms others, it will require rebalancing as mandated by product description.

Active asset allocation refers to the active management of the mix between equity and fixed income or two sectors of fixed income. For balanced funds that have a range of allocations, there might be a committee that decides the percentage of each sector or asset class and can change that mix based on their views of the market. Likewise, an investment committee might decide to allocate funds to a sector of fixed income from high yield to treasuries or vice versa.

A trader may be asked to extend or shorten the duration of a portfolio due to active duration management by the portfolio manager. Traditionally, a long duration implied a flattening bet on interest rates, that is, the trader needed to sell some short duration securities and buy long duration securities to get the necessary exposure. With this structure, if the general level of interest rates fell but the slope of the yield curve increased, the portfolio would not get the desired performance gain. With our term structure model this would not happen, as individual components of the yield curve can be separately hedged. Using derivatives or cash securities, we can take an active bet on the level of rates only and hedge the other components. We will discuss this approach in more detail in Chapter 8.

A trader may also be responsible for monitoring the level of cash to meet collateral requirements for derivatives or to pay for margin flow of futures. Depending on a portfolio's mandate, many funds such as mutual funds and pension funds are required to have cash equivalents for deliverable futures such as bond futures. Cash settled futures such as eurodollar futures usually do not require cash collateral.

Traders need to anticipate many of the above cash flows to effectively buy cheap or liquid securities. Liquid securities tend to be expensive and illiquid securities tend to be cheap. For a liquid security such as an on-the-run treasury, the bid–ask spread may be $\frac{1}{64}$ point, while for an illiquid or off-the-run treasury it may be $\frac{1}{16}$ or more.

Table 7.1 shows sample analytics for selected treasury coupon Strips (CS), principal Strips (PS) and coupon bonds (T) in 2012. The fair price is calculated by discounting cash flows by the calculated curve plus adjustments that were covered in Chapter 6, and the model price is calculated by discounting cash flows by the calculated TSIR. Note that the fair price and model price are almost identical, implying that all zero coupon bonds are priced nearly perfectly by the model. The spread to market and spread to curve are representations of the spread of the security relative to the fair price and model price, respectively. The next column, Price R/(C), is the price deviation from the model price. We could also use price difference from the fair price as a measure of value. Rich bonds have a positive Price R/(C), and cheap bonds have a negative price spread. The next two columns are the continuously compounded yield and theta or yield plus rolldown.

Coupon Strips are all fungible, implying that all coupons with the same payment date can be combined together as one security. Principal Strips are, however, not fungible if their maturity falls on the same date. Thus, a coupon bond can be stripped to its individual coupons and principal. If other bonds are also stripped, all the coupons that have the same maturity date can be aggregated. Likewise, if a trader has all the coupons and stripped principal, he can reconstitute the original treasury coupon bond.

There are four securities listed in Table 7.1 with a maturity of 11/15/21: a coupon Strips, a principal Strips, an old 30-year bond that was issued in 1991 and has a remaining maturity of 10 years with a coupon of 7.625, and a recently issued 10-year bond with a coupon of 2 (on-the-run). The principal Strips is more expensive than the coupon Strips by 0.57, even though they have the same maturity and the same exact credit rating. This is about the same amount that the treasury 2, 11/15/21 is rich relative to the curve. When a bond is rich, it is usually the principal Strips that is rich, since all coupons are fungible.

In a normal market, it typically takes about 6 months to a year for an on-the-run treasury to lose its richness and trade with little or no premium. If a trader has to buy a 10-year security for hedging and there is a high possibility that he has to sell the security in the next day or two, it makes sense to buy the on-the-run treasury rather than the cheaper zero coupon Strips. In Table 7.1, if the bid–ask price spread for on-the-run and the Strips are respectively $\frac{1}{64}$ and $\frac{1}{16}$, a round trip trade (a buy and subsequent

TABLE 7.1 Selected treasury bonds, 2012

Ticker	Cpn	Mat	Price	Fair Price	Model Price	Sprd to Mkd bps	Sprd to Crv bps	Price R/(C)	Contd Yld	Theta
CS	0	11/15/21	86.84	86.79	86.74	−1	−1	0.06	1.518	3.294
PS	0	11/15/21	87.41	86.79	86.74	−8	−8	0.62	1.448	3.225
T	2	11/15/21	105.07	104.54	104.49	−6	−7	0.53	1.410	2.937
T	8	11/15/21	158.40	157.79	157.73	−5	−6	0.61	1.300	2.445
CS	0	2/15/22	86.14	86.09	86.01	−1	−2	0.05	1.563	3.344
PS	0	2/15/22	86.61	86.09	86.01	−6	−7	0.52	1.506	3.287
T	2	2/15/22	104.82	104.27	104.19	−6	−7	0.55	1.451	2.959
CS	0	5/15/22	85.40	85.40	85.30	0	−1	(0.01)	1.612	3.392
PS	0	5/15/22	86.00	85.40	85.30	−7	−8	0.59	1.541	3.321
T	1.75	5/15/22	102.24	101.68	101.58	−6	−7	0.56	1.497	3.042
CS	0	11/15/41	44.52	45.43	45.57	7	8	(0.91)	2.763	2.904
PS	0	11/15/41	44.73	45.43	45.57	5	6	(0.70)	2.747	2.888
T	3.125	11/15/41	111.42	112.20	112.37	4	4	(0.78)	2.551	2.694
CS	0	2/15/42	44.14	45.10	45.24	7	8	(0.96)	2.767	2.903
PS	0	2/15/42	44.34	45.10	45.24	6	7	(0.76)	2.752	2.888
T	3.125	2/15/42	111.36	112.22	112.39	4	5	(0.86)	2.556	2.675
CS	0	5/15/42	43.72	44.80	44.94	8	9	(1.08)	2.778	2.909
PS	0	5/15/42	44.00	44.80	44.94	6	7	(0.80)	2.756	2.887
T	3	5/15/42	108.66	109.57	109.74	4	5	(0.91)	2.565	2.700

sell or vice versa) using the on-the-run would cost 0.0156 and for the Strips 0.0625. If we can expect, conservatively, a period of 6 months for the on-the-run to lose half its richness, the loss per day of owning on-the-run is approximately $\frac{0.57}{2\times182} \approx \frac{0.01}{2\times3}$. In order to overcome the transaction cost difference of 0.047, the holding period of the Strips must be longer than $\frac{0.047\times182\times2}{0.57} = 30$ days.

A trader needs to use his knowledge of the portfolio manager's style and the fund's characteristics to anticipate cash flows and decide how to hedge them. Even in a high turnover portfolio, many of the securities do not trade for months or even years at a time. For such core positions, liquidity can be sacrificed for yield. However, a fraction of the portfolio needs to be liquid for hedging purposes. Assuming that a portfolio requires a daily rebalancing of about 0.1 years of duration per day, for a portfolio with a duration of 5 years, the annual turnover will be about $\frac{252\times0.1}{5.0\times2} \approx 2.5$. The additional transaction cost using off-the-run treasuries will be about $2.5\times0.047 \approx 12$bps.

What is interesting in Table 7.1 is the cheapness of the bonds at the long end of the curve. Historically, the bonds at the long end of the curve have been very rich relative to the curve due to convexity premium. However, this cheapness is new and requires an explanation. As we will see in Chapter 10 on convexity adjusted TSIR, convexity is worth more than 1% of return a year at the long end of the curve. For a 30-year zero coupon bond, the convexity is about 900. Assuming an absolute volatility of 60 bps a year, the contribution of convexity to performance can be approximated by $\frac{1}{2}\times900\times0.006^2 = 1.62\%$. Maybe the market is implying that the long term debt dynamic of the US is not sustainable and it requires a yield premium at the long end of the curve. If one believes that the US debt is risk-free at any maturity, the long end of the curve offers very good value.

7.2 FORWARD PRICING

When a security is purchased, it usually settles within one to three business days, at which point cash is paid to the seller and security is transferred to the buyer. This is called a spot trade and the transaction price is called the spot price of the security. It is possible to buy a security for future settlement at a price that is different from the spot price. The seller should be indifferent between, on the one hand, selling the security in the spot market and receiving cash for it and investing the cash in Libor instruments, and, on the other hand, lending it (term repo) and selling it in the forward market. Likewise, the buyer is indifferent between, on the one hand, buying the security, lending it and using the proceeds until the expected forward settle date, and, on the other hand, buying it for forward settlement. In either case, the trader will lay off the risk of the security and receive cash for it, assuming that the buyer does not go into bankruptcy in the interim. For a forward transaction, there are potentially four parameters that need to be considered to calculate the forward price of a security as follows:

- The proceeds of the sale in the forward market, present valued by Libor discounting.
- The potential coupon proceeds between spot and forward dates discounted by Libor.
- The interest earned due to the difference between Libor and the repo rate if the security is on-special.
- The coupon of the security for accrual calculation.

We write these parameters as:

$$
\begin{aligned}
p_{m,d} &= p_{f,d}e^{-y_{l,f}t_f} + \sum_i^{t_i<t_f} c_i e^{-y_{l,i}t_i} + p_{m,d}\left[e^{\left(y_{l,f}-y_{br}\right)t_f} - 1\right]e^{-y_{l,f}t_f} \\
p_{m,d} &= p_m + w_m c \\
p_{f,d} &= p_f + w_f c
\end{aligned}
\tag{7.2}
$$

where $p_{m,d}$ is the dirty market price (price plus accrued interest), $p_{f,d}$ is the dirty forward price (forward price plus accrued interest), $y_{l,i}$ is the Libor yield at time t_i, y_{br} is the repo rate for the security through the forward settlement date, p_m is the clean spot price, p_f is the clean forward price, w_m is the accrual period for market settlement, w_f is the accrual period for forward settlement, and c is the coupon rate. After rearranging the formula, the forward price of a security can be written as

$$
p_f + w_f c = (p_m + w_m c)e^{y_{l,f}t_f}\left[1 - \left(e^{-y_{br}t_f} - e^{-y_{l,f}t_f}\right)\right] - e^{y_{l,f}t_f}\sum_i^{t_i<t_f} c_i e^{-y_{l,i}t_i} \tag{7.3}
$$

If there are no cash flows before the forward date, and the repo rate for the security is the same as Libor, then

$$
p_f + w_f c = (p_m + w_m c)e^{y_{l,f}t_f} \tag{7.4}
$$

For short forward pricing, this can be approximated as

$$
p_f + w_f c = (p_m + w_m c)(1 + y_{l,f}t_f) \tag{7.5}
$$

For example, a 6% US treasury bond, with 91 days of accrual in a coupon period of 182 days, is priced at 105. The forward price for extending the settlement by 2 weeks, given a short term Libor of 0.75%, is

$$
\begin{aligned}
p_f + \frac{91+14}{182}\times\frac{6}{2} &= \left(105 + \frac{91}{182}\times\frac{6}{2}\right)\left(1 + \frac{0.75}{100}\times\frac{14}{360}\right) = 106.5311 \\
p_f &= 104.800
\end{aligned}
$$

The price difference is relatively large; for a \$50 million trade, it amounts to a \$100,000 price reduction, which is often overlooked by traders who request a forward settlement. The difference for corporate bonds can be even larger.

Note that that accrual period and forward settlement time are calculated differently. For accrual period, we use the bond convention (Actual/Actual for the US), and for forward settlement we use the local Libor convention (Actual/360 for the US).

The accrual convention for most corporate bonds is 30/360. For example, if a corporate bond is to settle on February 28 on a Tuesday in a non-leap year at a price of 105, to extend the settlement day by a day will add 3 days to accrual, but only 1 day to financing. In the above example, if the bond was a corporate bond with a 1-day extension from February 28, the price would be

$$
\begin{aligned}
p_f + \frac{91+3}{180}\times\frac{6}{2} &= \left(105 + \frac{91}{180}\times\frac{6}{2}\right)\left(1 + \frac{0.75}{100}\times\frac{1}{360}\right) = 106.5189 \\
p_f &= 104.952
\end{aligned}
$$

Many sell side traders use the spot price for forward settlement. For corporate or emerging markets bonds, this provides a significant advantage to the selling party, since the coupon accrual is often significantly larger than the financing rate. For example, if the coupon rate of the bond is 6% and the financing rate is 1%, the seller would be earning an additional 5% (on a price of 100) for every day the settlement is extended.

7.3 CURVE TRADING

In Chapter 3 it was mentioned that some components of the TSIR have a mean reversion tendency. Here we explore how one can exploit such tendencies for systematic trading strategies. Figure 7.1 shows historical level, slope and bend components of euro based swaps. Euro era monetary policies were largely adopted from the Bundesbank. Thus, to estimate the euro era swap rates before monetary union, German swap rates are used as a proxy for euro swaps. The level has fallen from about 8% in the early 1990s to about 1% currently, and the bend component has hovered around 0% and has a relatively strong mean reversion tendency.

Table 7.2 shows historical statistics for EUR historical swap term structures. The first four rows give the historical maximum, minimum, average, and standard deviations of each component. The current value (as of the last data point) of each component as well as the number of standard deviations from the mean of the current values are also shown. Reversion is the mean half-life of the series in years as explained in Chapter 3, and the final row gives the *t*-statistics for the mean reversion tendency. A *t*-statistic greater than 2 is usually considered to be significant, and the greater the value the higher the statistical significance. The *t*-statistic of 5.44 for the bend component is very significant, implying a mean reversion of 0.26 or about 3 months.

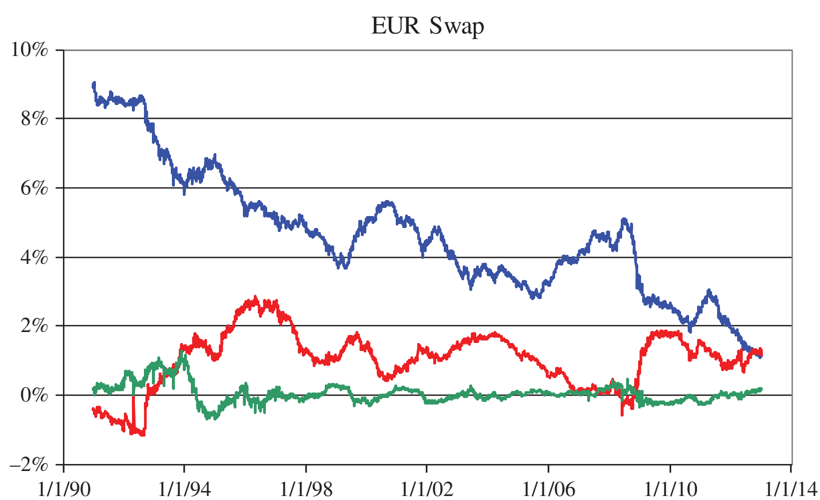


FIGURE 7.1 Historical term structures of euro swaps

TABLE 7.2 Analysis of EUR term structure components

EUR	Level	Slope	Bend
Max	9.03%	2.85%	1.25%
Min	1.06%	−1.18%	−0.70%
Average	4.62%	1.03%	0.05%
Stdev	0.67%	0.69%	0.65%
Current	1.23%	1.29%	0.15%
Sigma's	5.06	−0.38	−0.16
Reversion	5.87	1.67	0.26
T-Stat	1.49	2.34	5.44

TABLE 7.3 EUR swap trade, April 22, 2008

			Libor Duration				
Issue	Face	Price	Level	Slope	Bend	Spread	Theta
EUR-5YR	−1,000,000	100	4.597	−0.330	−4.446	0.022%	4.302%
EUR-6MO	−638,054	100	0.501	−0.438	0.264	0.045%	4.758%
EUR-2YR	1,364,903	100	1.955	−1.070	−0.781	−0.073%	4.113%
EUR-10YR	273,151	100	8.231	3.114	−5.188	−0.022%	4.908%
Total	0.00		0.000	0.000	1.795	−0.078%	−0.384%

We can use the mean reversion as an initial expectation for the time horizon of a trade. Since 1998, the bend component has been in the range of ±0.3%. On April 22, 2008 when the bend component was at 0.25% it appeared that there was an opportunity for a trade. The bend component is the curvature of a parabola that is centered close to 5-year maturity. When the bend component is at the high end of its trading range, it implies that the parabola is very deep at the 5-year part of the curve (concave up). Likewise, when the bend component is negative, the parabola is concave down and the 5-year yield is high.

Since the 5-year swap is very rich, we set up a trade to short the 5-year and go long 2-year and 10-year swaps. For this trade, we need to hedge the level and slope of the curve, so that we are compensated for the richness of the 5-year swap, regardless of the direction of interest rates and the slope of the curve. Table 7.3 shows the size of each swap contract for the trade.

The spread measured relative to the Libor curve is a measure of richness or cheapness of the securities. The total market value of the trade is zero, so the durations, theta (see Section 5.2), and spread are calculated based on the market value of the 5-year swap. The total spread is the expected performance gain of the spread if security spread reverts to zero. We can expect about half retracement of the spread in the 3-month horizon, leading to a cost of $-\frac{0.078 \times 4.597}{2} = -17.9$ bps. We also calculate the

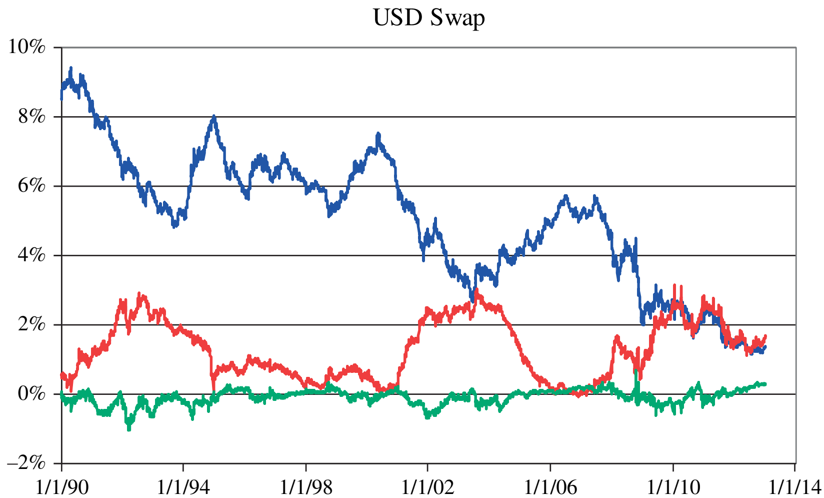


FIGURE 7.2 Historical term structures of USD swaps

theta contribution of $-\frac{38.4}{4} = -9.6$ in the horizon. Thus, the total estimated cost of the trade will be -27.5 bps. The expected gain of the trade will be 50% retracement of the bend in the horizon period, given the mean reversion period of 0.26 years. The expected gain from mean reversion is $1.795 \times \frac{25}{2} = 22.5$ bps. Thus, the expected gain of the trade would be -5 bps, even though the bend component is at an extreme. Knowing the expected value of a trade is a great tool to anticipate returns and to avoid trades that appear reasonable at first sight but end up costing money. In practice this trade would have been profitable by 52 bps thanks to a large change in the bend component that contributed 65 bps and security selection costing only 2.3 bps instead of 18 bps.

Our next analysis is for a similar trade in the US swap market on November 26, 2007 where the bend component was at an extreme value of 0.57%, implying a very rich 5-year swap. Figure 7.2 shows the historical components of the US swap curve.

Table 7.4 shows trade parameters for a trade to sell the 5-year part of the curve and buy the wings. This trade had a positive bend duration, in anticipation of falling bend of the TRLR. The estimated holding cost of the trade for 3 months is expected to be $\frac{-10.2 \times 4.57}{2} - \frac{139.6}{4} = -58$ bps. The expected gain from mean reversion is $1.778 \times \frac{57}{2} = 51$bps. The net expected value of this trade is also negative. Table 7.5 shows the outcome of this trade in 3 months.

Even though the mean reversion had a better performance than we anticipated, it was not enough to overcome negative theta and spread retracement, which were higher than expected.

The purpose of these examples is to show the pitfalls of a trade that appears to be reasonable at first sight but where deeper analysis shows that it may not be a viable trade. We might get lucky with one or two trades, but in the long run, if it does not make sense, it will end up being random noise in a portfolio.

There are times when the headwind against a mean reversion trade is a lot less than the above examples and the trade makes sense. At the end of June 2004, the bend component was -0.47%, not quite at the extreme of the previous example, but the analysis

TABLE 7.4 USD swap trade data, November 26, 2007

			Libor Duration				
Issue	**Face**	**Price**	**Level**	**Slope**	**Bend**	**Spread**	**Theta**
USD-5YR	−1,000,000	100	4.570	−0.325	−4.421	0.028%	4.678%
USD-6MO	−631,906	100	0.498	−0.436	0.264	0.135%	4.348%
USD-2YR	1,355,763	100	1.951	−1.065	−0.784	−0.098%	3.435%
USD-10YR	276,143	100	8.113	3.053	−5.117	−0.014%	4.968%
Total	0.00		0.000	0.000	1.778	−0.102%	−1.396%

TABLE 7.5 USD swap trade performance, November 26, 2007

Issue	Bend	Spread	Theta	Vex	Sum	Market
USD-5YR	1.654%	−0.007%	−1.166%	0.004%	−2.263%	−2.376%
USD-6MO	−0.062%	−0.039%	−0.684%	−0.002%	−1.278%	−1.051%
USD-2YR	−0.398%	−0.329%	1.160%	0.031%	3.548%	3.121%
USD-10YR	−0.529%	0.007%	0.342%	−0.022%	−0.048%	0.218%
Total	0.665%	−0.369%	−0.348%	0.010%	−0.041%	−0.087%

TABLE 7.6 USD swap trade data, June 28, 2004

			Libor Duration				
Issue	**Face**	**Price**	**Level**	**Slope**	**Bend**	**Spread**	**Theta**
USD-5YR	1,000,000	100	4.530	−0.335	−4.369	0.008%	5.547%
USD-6M0	618,168	100	0.504	−0.440	0.264	−0.018%	2.528%
USD-2YR	−1,332,313	100	1.951	−1.067	−0.781	−0.003%	4.465%
USD-10YR	−285,856	100	7.844	2.847	−4.998	−0.015%	6.022%
Total	0.00		0.000	0.000	−1.736	0.016%	−0.561%

was much more supportive of a trade to buy the 5-year part of the curve and short the wings. Table 7.6 shows the parameters of the trade. Note that the bend component is extreme negatively and will likely rise. We therefore need to have a negative bend duration to make money as the bend rises.

The anticipated holding period theta loss is 14 bps and the security spread has an expected return of 3.5 bps. A 50% retracement of the bend component will have a contribution of 42 bps for an expected gain of 32 bps. Table 7.7 shows the actual performance.

TABLE 7.7 USD swap trade performance, November 26, 2007

Issue	Bend	Spread	Theta	Vex	Sum	Market
USD-5YR	1.717%	−0.012%	1.397%	0.052%	4.877%	4.646%
USD-6MO	−0.064%	−0.008%	0.394%	0.000%	0.301%	0.272%
USD-2YR	−0.409%	−0.037%	−1.498%	−0.004%	−2.255%	−2.159%
USD-10YR	−0.562%	−0.011%	−0.434%	−0.065%	−2.467%	−2.309%
Total	0.682%	−0.068%	−0.141%	−0.017%	0.457%	0.449%

7.4 SYNTHETIC SECURITIES

In a single currency or multi-currency portfolio, there are times when you want to express a viewpoint about two different areas of the curve or about two global curves. Such positions can sometimes be constructed by buying or selling securities that mature in the desired areas of the curve. However, specific zero coupon securities are often not available or outright short positions in cash securities are not permitted. Using TSIR or TSLR we can use interest rate swaps to construct such synthetic positions relatively efficiently.

Consider the TSLR for AUD and NZD shown in Figure 7.3 along with the instantaneous forward curves. The solid lines are the spot curves and the spread between the markers and the curve represents the spread of the market data relative to the term structure of the swap curve.

Historically, the AUD and NZD Libor curves have been highly correlated with NZD yields generally at a small premium to AUD yields. The crossing of the two

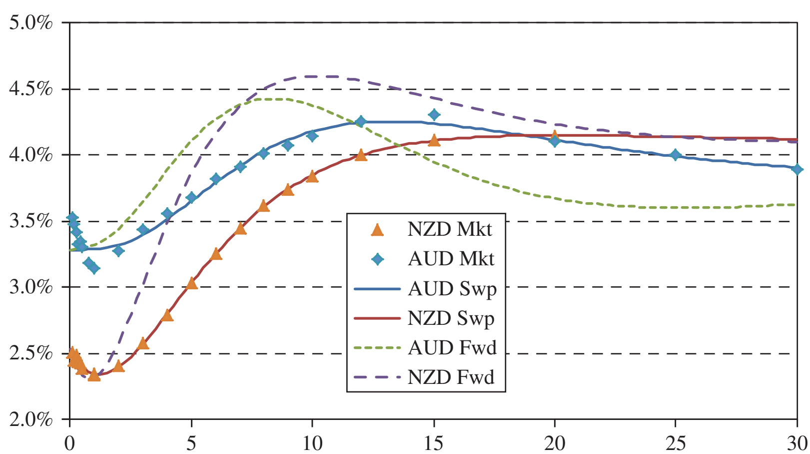


FIGURE 7.3 AUD and NZD swap curves, May 24, 2012

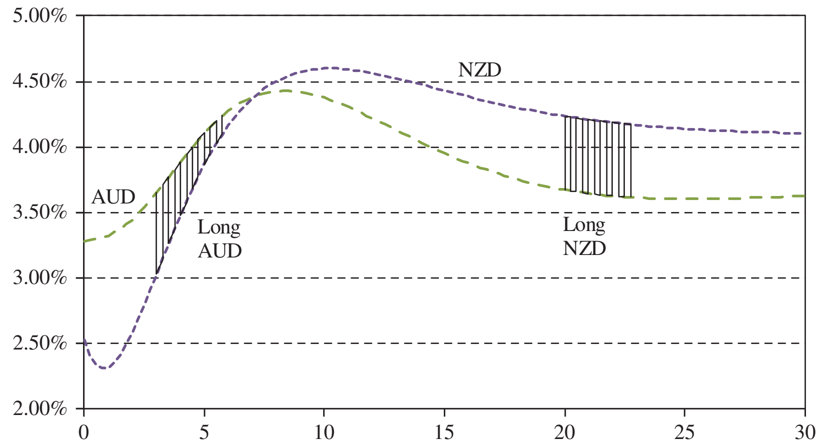


FIGURE 7.4 AUD and NZD instantaneous forward swap curves, May, 24, 2012

spot and forward curves is unusual and suggests a possible trade. Given the dynamic nature of rates, there is no reason why AUD overnight rates will be higher than NZD 2 years in the future and then fall below NZD about 7 years later. Given the historical correlations of the two economies and their rate behavior, one can express a view that the curves will revert to normal and structure a trade that has positive carry in the mean time.

Suppose that we want to create a synthetic security that has a stream of equal cash flows between maturities of 20 and 23 years. We calculate the risks of those cash flows (level, slope and bend) and, knowing that we can represent about 98% of the risks by level, slope and bend components, we can then find a combination of swap coupons that replicates the risks of the stream of cash flows that we are interested in (see Chapter 8 for more details). Linear programming can be used to select securities that maximize or minimize the yield of the replicating risks, depending on whether we want to be long or short the risks, respectively.

Figure 7.4 shows two slices of the forward curves of AUD and NZD swaps that can be used for relative value trading. The durations of a stream of cash flows that mature in the shaded areas on a quarterly basis are shown in Table 7.8.

Having the duration components of the tradable swaps in the market, we can easily find the number of shares of an optimal trade that has the same duration risks as the duration of the stream of cash flows in Table 7.8, (e.g., by using the Solver add-in in Excel). We can then set up a linear programming table that minimizes the number of shares traded as the objective function and in such a way that the net market value exposure to AUD or NZD is zero.

Table 7.8 shows the necessary trades to construct a portfolio with 2000 level VBP (Value of a Basis Point) in each currency. The trade could be structured to have the same VBP in the short end and long end of the curve. However, our trade assumes equal cash flows per unit of time, resulting in about five times more VBP in the long end than in the short end of the curve. The aggregated yield is the duration weighted yield of long and shorts for each currency. The long and short yields are the duration weighted yields of long and short cash flows irrespective of the currency.

TABLE 7.8 Durations of streams of cash flows

Issue	Coupon	Level	Slope	Bend	Exp Yld	Face Amount
AUD 25 Year Lbr	4.08	15.83	12.51	6.28	4.04%	−$1,623,130
AUD 30 Year Lbr	4.00	17.66	14.60	9.04	3.96%	−$112,198
AUD 4 Year Lbr	3.58	3.76	−0.77	−3.40	−3.55%	$378,346
AUD 5 Year Lbr	3.69	4.61	−0.32	−4.47	−3.66%	$1,356,981
NZD 20 Year Lbr	4.07	13.65	9.89	2.66	4.03%	$2,441,539
NZD 1 Year Lbr	2.36	0.99	−0.75	0.15	−2.33%	−$278,909
NZD 6 Year Lbr	3.25	5.49	0.30	−5.32	−3.22%	−$652,618
NZD 7 Year Lbr	3.42	6.27	0.99	−5.73	−3.39%	−$1,510,012
Total VBP AUD		−2000	−2267	−1856	6.98%	$0
Total VBP NZD		2000	2267	1857	8.94%	$0
Long Yield					4.02%	
Short Yield					−2.89%	

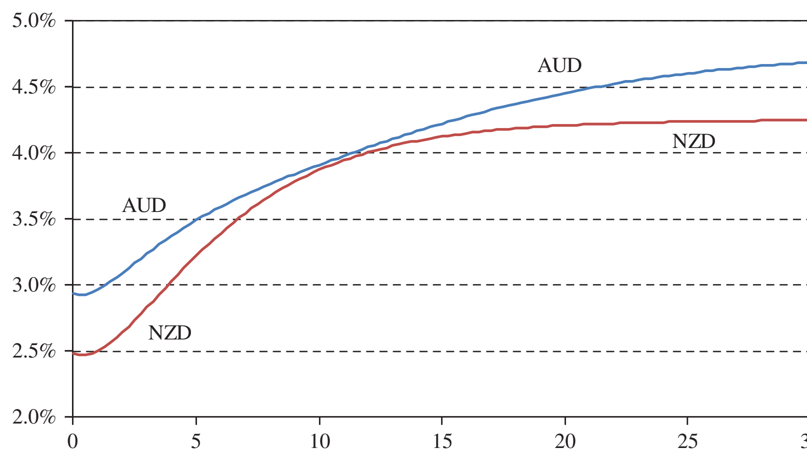


FIGURE 7.5 AUD and NZD swap curves, December, 18, 2012

Figure 7.5 shows the spot swap curves of AUD and NZD about 7 months after the trade. The curves have normalized to a large extent, implying that our trade must have worked very well.

Table 7.9 provides a summary of the trade result about 7 months after initiation. The trade resulted in a profit of $133,000 with the original scaling of 2000 VBP, implying a favorable change in relative rates of about 133,000/2000 = 66.5 bps. While opportunities like this do not present themselves on a frequent basis, relatively

TABLE 7.9 Summary of trade result, December 18, 2012

Issue	Dirty Price	Face	Market Value	Coupon Payment
AUD 25 Year Lbr	95.24	−$1,623,130	−$1,545,887	−$33,112
AUD 30 Year Lbr	92.06	−$112,198	−$103,294	−$2,241
AUD 4 Year Lbr	101.35	$378,346	$383,441	$6,777
AUD 5 Year Lbr	101.60	$1,356,981	$1,378,636	$25,062
NZD 20 Year Lbr	100.90	$2,441,539	$2,463,586	$49,716
NZD 1 Year Lbr	100.02	−$278,909	−$278,970	−$3,284
NZD 6 Year Lbr	100.30	−$652,618	−$654,589	−$10,589
NZD 7 Year Lbr	100.41	−$1,510,012	−$1,516,169	−$25,821
Total			$126,754	$6,508

large and highly successful trades can be structured when the opportunity is available. A spread trade like this has significantly lower risk than outright duration bets and a much higher success rate.

Interestingly, Figure 7.5 provides another opportunity for a low risk and high success rate trade. It can be structured in two ways:

1. Buying AUD barbell and selling NZD bullet. This trade takes advantage of the higher AUD yields at the short and long ends of the curve and similar yields in the 10-year part of the curve. The trade has positive carry and is likely to have positive performance as the curves normalize.
2. Buying AUD butterfly and selling NZD butterfly. A long butterfly trade is buying the long and short ends of the curve and selling the middle part of the curve. A long butterfly trade is the same as a long bend duration trade. If long and short rates outperform the midrange rates, the trade makes money and thus has significantly less risk and volatility than an outright duration trade. The risk of the suggested trade is even lower than a normal butterfly trade, since AUD and NZD rates are highly correlated. The trade offers positive carry, which means that time is on our side, and the longer it takes for the trade to work, the more carry we will collect. Even if the trade goes against us, the loss will be mitigated by the positive carry.

7.5 REAL TIME TRADING

In a dynamic market, prices change constantly. From the time a trade idea is analyzed till the trade is executed, the market has often moved. The price or yield change of a non-liquid security is often linked to the price or yield of a benchmark security. These links are not always accurate, and more sophisticated means of assessing the price change of a security are necessary. For example, the price of a long zero coupon bond cannot be accurately benchmarked against a coupon bond, and the price of a floating rate emerging country bond cannot be compared to the price of a fixed rate asset.

The US treasury benchmark securities or the so-called on-the-run treasuries are generally not a good representative of the US yield curve, since they are often on-special in the repo market. When a bond is on-special, it can be financed at a lower rate than the deposit rate. For example, if the deposit rate is 5% and a security is on-special at 2%, the holder of the security can borrow cash at a rate of 2% by lending his security. The borrowed cash can then be invested at a cash rate of 5%. Since on-special securities have an extra source of income for their holders, they tend to have a lower yield and/or a higher market price.

We can often use on-the-run treasuries to estimate the change in the shape of the yield curve, provided that a benchmark security's repo rate does not change significantly. We can write the change in the price of a benchmark security from the changes in the components of the TSIR as

$$\Delta p_j = -p_j \sum_i D_{ij} \Delta a_i \tag{7.6}$$

where D_{ij} is the ith duration component of the benchmark bond j and Δa_i is the change in the ith component of the TSIR. It is straightforward to construct a spreadsheet with live data feeds and calculate the price change of on-the-run treasuries in real time. Knowing their duration components, we can calculate the changes in the first three components of the TSIR that would best replicate the respective price change of on-the-run treasuries. We can then link the changes in the components of the TSIR to the duration components of other securities and calculate their expected price change due to the treasury market using the above equation. This process works universally for all securities, including treasuries, corporates, high yield, emerging markets, and mortgages.

Calculating the real time market impact of a security can provide for a much more efficient way of trading in fixed income. By taking into account the movement of the market, a portfolio manager can better structure a relative value trade that is to a large extent independent of the direction of the market.

CHAPTER 8

Linear Optimization and Portfolio Replication

The objective of portfolio optimization in fixed income is to identify a handful of securities in order to provide the optimum (maximum) property that we are interested in. We can require that the optimum portfolio be the one with the highest yield, the highest carry, the highest convexity or the highest spread. The process of finding the optimum solution is called linear programming (LP).

We will provide a simple example of LP in this chapter, but refer the reader to one or more of very excellent textbooks on this subject, including those by Vanderbei [8] and by Gass [9], for more details.

Assume that we want to construct a portfolio with a market value of \$1 million and we have two securities, A and B, to choose from. The portfolio can accept a cash balance. If x and y are the market values of A and B respectively, in units of millions of dollars, we have

$$x + y \leq 1 \tag{8.1}$$

In LP this is called a constraint. Suppose that the duration and carry of A and B are respectively 3 and 12 and 6% and 4%, and the acceptable duration range is 4.5–6 years. Thus,

$$\begin{aligned} 3x + 12y &\leq 6 \\ 3x + 12y &\geq 4.5 \end{aligned} \tag{8.2}$$

The policy also limits the percentage of security B to 30% of the portfolio or \$0.3 million. This constraint can be written as

$$y \leq 0.3 \tag{8.3}$$

In linear programming the values of all variables are zero or positive, just as we want in this exercise. Given the above constraints, we want to construct a portfolio with the highest possible carry. The carry of the portfolio is

$$Z = 6\%x + 4\%y \tag{8.4}$$

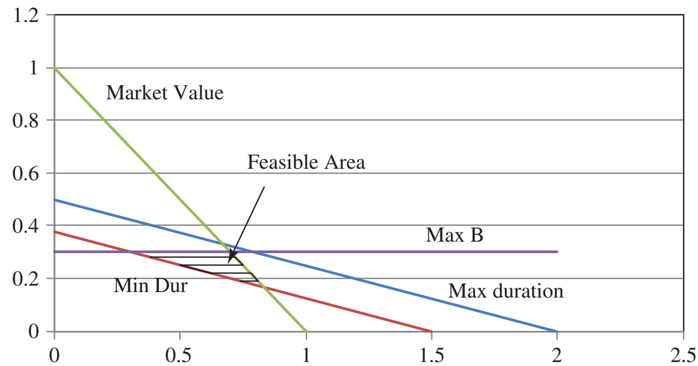


FIGURE 8.1 Portfolio optimization example

Z is called the objective function, and our goal is to maximize the objective function in such a way that it meets all the constraints. We solve this problem graphically by changing all inequalities to equalities and drawing all the lines and finding the area that meets all the constraints. Figure 8.1 shows all the constraints.

Any point inside the shaded area meets all the constraints and is called a feasible solution. In LP, the optimum point is always on a vortex of the feasible solution area. If we examine the three corners of the feasible area triangle, we find the optimum solution to be

$$\begin{aligned} (x,y) &= \left(\tfrac{5}{6}, \tfrac{1}{6}\right) \\ Z &= 5.667\% \end{aligned} \tag{8.5}$$

If the carry of A and B were respectively 3% and 6%, instead of 6% and 4% as discussed above, then the optimum solution would be at a different vortex, namely,

$$\begin{aligned} (x,y) &= (0.7, 0.3) \\ Z &= 3.9\% \end{aligned} \tag{8.6}$$

With two variables in an LP problem, we can use graphical method to find the optimum solution. For every additional variable, we add a dimension to the problem; when there are 150 or more bonds, we need a computer program to solve for the optimum solution. Sometimes there is no feasible solution. In the above example, if the policy requirement for bond B were a maximum of 15%, there would be no feasible solution. Therefore, it is crucial to have constraints that are reasonable and logical, since for a large number of variables it can be very difficult to figure out which constraints make the solution non-feasible.

We demonstrated in Chapter 5 that the performance of a portfolio can be measured by its carry, duration, and convexity components. The contribution of convexity to performance is small, and for a portfolio that has a balanced risk profile the convexity of the portfolio will not be that different from the benchmark. Thus, to replicate an index, we have to match the duration components of the portfolio with those of the index. The first three components are usually enough as we demonstrated, but five may be used, that is,

$$\sum_j w_j D_{i,j} p_j = MD_i \tag{8.7}$$

where D_i is the ith duration component of the index, $D_{i,j}$ is the ith duration component of security j, M is the market value, p_j is the price of security j, w_j is the weight or number of units of security j. To add the first component of convexity, we can also require

$$\sum_j w_j X_{00,j} p_j = MX_{00} \tag{8.8}$$

If there are 150 bonds in an index, there are more than 500 million ways of selecting five different bonds to match the durations. There is an infinite number of ways that more than five bonds can be selected and weighted to match all the duration components. To avoid leverage, we must also make sure that the market value of all bonds will be less than or equal to the total cash available. Thus,

$$\sum_j w_j p_j \le M \tag{8.9}$$

Equation (8.9) suggests that the portfolio need not be fully invested in securities and can accept a cash balance. This process can also be used for active portfolio management as well. For example, to have a long duration of 0.5 years relative to the index, we can use the following constraint:

$$\sum_j w_j D_{0,j} p_j = M(D_0 + 0.5) \tag{8.10}$$

Unlike traditional duration measurements, where a long duration implies a flattening position, we can manage individual components separately.

If we use (8.7) and (8.9) as constraints in an LP optimization and match the first three duration components, LP software will choose three or four bonds for index replication. To force zero cash balance, we must change inequality (8.9) to equality. The objective function can be written as one that would maximize the yield of the portfolio:

$$Z = \sum_j w_j D_{0j} p_j y_j = MD_0 y \tag{8.11}$$

y_j is the yield of security j. The yield of a portfolio is calculated by duration market value weighting its components; see Chapter 1 for a derivation of the portfolio yield.

This constraint works only if the level duration is set to a constant value. If we allow the level duration to have an acceptable range, then the objective function generally forces the maximum allowable duration in a steep yield curve. However, optimizing carry which is market value weighted will result in a solution that is not a strong function of the duration range. To optimize on the carry of the portfolio, θ, we use

$$Z = \sum_j^N w_j p_j \theta_j = M\theta, \tag{8.12}$$

where θ_j is the carry of security j and N is the number of securities in the portfolio. Using the TSIR to optimize or hedge a portfolio offers the following advantages over methods relying on correlation matrices or principal components analysis:

- There is no need for historical correlation matrices to calculate the principal components for a country where the information is not available.
- The consistency and comparability of duration components across different currencies are guaranteed if the same decay coefficient is used for all currencies.
- One can hedge two of the first three components and take an active view on a third component of the TSIR.
- Unlike equities where correlation matrices are very important, the risk parameters of a treasury portfolio can easily be quantified by three to five parameters. Further diversification will not lower the tracking error materially. The term structure components can be very efficiently used in LP.
- The model can be very easily applied to multi-currency portfolios and credit portfolios.

8.1 PORTFOLIO OPTIMIZATION EXAMPLE

Our analysis in Chapter 5 showed that the first three duration components of the TSIR can account for nearly all the performance of a treasury index. Thus, a portfolio with identical level, slope, and bend duration components as a benchmark is expected to replicate the return of the benchmark. This technology requires only four bonds to create a fully invested index replicating portfolio.

To create the index replicating portfolio, LP optimization was used to create a portfolio from the universe of bonds that matched the first three or five duration components of the index and maximized the yield. The custom treasury index from Section 5.4 for the benchmark was used and the universe consisted of the index and coupon Strips. To eliminate pricing errors, the yield of selected bonds in the portfolio was compared with the yield of other bonds with similar maturity.

Tables 8.1 and 8.2 summarize the performance results for the optimized portfolio and the benchmark using five and three term structure components, respectively. The optimization was carried out by maximizing yield, carry (yield plus rolldown), and carry duration and the universe of securities was either coupon bonds or coupons as well as Strips.

Recall that the carry is simply the change in the price of a security per unit of time: from (5.9),

$$-\frac{\partial p}{\partial t} = \sum_{i}^{n} c_i e^{-y_i t_i} \left(y_i + t_i \frac{\partial y_i}{\partial t} \right) \tag{8.13}$$

where n is the number of cash flows of the bond. The minus sign implies that as time moves forward, the time to maturity lessens. The parameter $t_i \, \partial y_i / \partial t$ is simply the change in the yield per unit time times duration of the cash flow. Thus, carry is a combination of yield plus the impact of change in yield as the security rolls down the curve. We also performed optimization using carry duration as a proxy for carry plus convexity.

TABLE 8.1 Performance of index replicating portfolio using five components, 1992–2012

	Index	Coupons			Cpns & Strips		
		Yield	Carry	Mixed	Yield	Carry	Mixed
Return	6.23%	6.79%	6.83%	6.62%	7.31%	7.31%	7.26%
Volatility	4.88%	4.86%	4.87%	4.89%	4.85%	4.84%	4.83%
Relative Return		0.56%	0.60%	0.39%	1.08%	1.08%	1.03%
Tracking Error		0.26%	0.25%	0.21%	0.48%	0.48%	0.47%
IR		2.16	2.43	1.87	2.26	2.25	2.20
8/31/08–12/31/08		–0.21%	–0.25%	0.16%	–0.39%	–0.75%	–0.60%
12/31/08–4/30/09		1.76%	1.65%	1.04%	2.26%	2.42%	1.43%

The index performance in Tables 8.1 and 8.2 is slightly different from that in Table 5.1, in that on a few occasions securities that had suspect pricing and were picked by the optimizer were eliminated from the universe to ensure that performance gains were realistic. Carry duration optimization is referred to in the Table as "Mixed".

During times of crisis, such as after the Lehman bankruptcy in September 2008, coupon bonds had a premium relative to Strips. A portfolio optimized using coupon bonds had a huge divergence in performance from a portfolio optimized from zero coupon bonds.

Using five components for the optimization clearly has a better information ratio, but not a better return. Most of the higher tracking error of the three components model is from periods of turbulence where diversification along the yield curve improves the tracking error.

Our optimized portfolio outperformed the index by about 50–100 bps per year, depending on the method, with a tracking error of about half the return. The information ratio calculated by dividing the relative return by the tracking error is around 2. Overall, optimizing carry had the best return as well as information ratio.

TABLE 8.2 Performance of index replicating portfolio using three components, 1992–2012

	Index	Coupons			Cpns & Strips		
		Yield	Carry	Mixed	Yield	Carry	Mixed
Return	6.23%	6.74%	6.87%	6.57%	7.33%	7.35%	7.21%
Volatility	4.88%	4.91%	4.91%	4.89%	4.90%	4.87%	4.91%
Relative Return		0.51%	0.64%	0.34%	1.10%	1.12%	0.98%
Tracking Error		0.29%	0.33%	0.27%	0.68%	0.58%	0.69%
IR		1.76	1.91	1.27	1.63	1.94	1.42
8/31/08–12/31/08		–0.48%	–0.38%	–0.43%	–1.12%	–1.48%	–1.08%
12/31/08–4/30/09		1.06%	1.69%	0.83%	2.18%	2.50%	2.17%

Transaction costs for the portfolio optimization process were ignored, but the composition of an optimized portfolio does not change significantly from one month to the next. Assuming a quarterly optimization of the portfolio and a transaction cost of about 2.5 bps per optimization, the overall information ratio would fall a little but would still be very respectable. If the transaction costs are included in the optimization process and transaction frequency is increased, a higher information ratio is achievable. We cover transaction costs in actual portfolio optimization in Chapter 19.

8.2 CONVERSION TO AND FROM CONVENTIONAL KRD

In Section 4.5 we showed how to convert from key rate duration (KRD) to term structure duration (TSD). If the number of key rates is more than the number of parameters in the TSIR model, there will be a unique solution to the conversion. However converting from TSD to a conventional KRD may not always lead to a unique solution, since no more than five TSIR components are needed (constraints), while there may be ten or more KRDs. We can create a unique solution to the conversion by using LP.

For securities, conversion of TSD to KRD is simple, since the primary risk of the security is an interpolation of its two nearest key rate securities (KRSs). We can require the objective function of the LP to maximize the weights of these two KRSs.

In order to calculate the KRD of a portfolio, it is probably best to write an objective function that maximizes its yield. Let n_t be the number of TSIR components, and n_k be the number of conventional KRSs. Let w_i be the weight of the ith KRS. Let ψ_{ji} be the jth TSIR duration component of the ith KRS, similar to (3.18). Let D_i be the ith KRD of the security or portfolio, and K_i be the ith KRD of the KRS. Here are the formulas to convert from KRD to TSD and vice versa, assuming that the number of key rates is more than the number of components of the TSIR. To convert from KRD to TSD:

$$\sum_{i}^{n_k} w_i \psi_{ji} = D_j \qquad j = 1, \ldots, n_t \tag{8.14}$$

To convert from TSD to KRD, we use the following constraints:

$$\sum_{i}^{n_k} w_i = 1 \tag{8.15}$$

$$\sum_{i}^{n_k} w_i \psi_{ji} = D_j \qquad j = 1, \ldots, n_t \tag{8.16}$$

The objective function for portfolios is to maximize the yield:

$$\sum_{i}^{n_k} w_i y_i \psi_{0i} = \max \tag{8.17}$$

The objective function for securities with non-contingent cash flows is

$$w_k + w_{k+1} = \max \tag{8.18}$$

where w_k and w_{k+1} are weights of key rates that have respectively closest shorter and longer maturity than the security.

Linear programming will result in a solution only if all the parameters are positive. To allow for negative values, we can add, negative duration components of the key rates to the constraints:

$$\begin{aligned} &\sum_{i}^{2n_k} w_i \psi_{ji} = D \qquad j = 1, \dots, n_t \\ &\underset{i>n_k}{\psi_{ji}} = -\psi_{j(i-n_k)} \end{aligned} \tag{8.19}$$

If all the components of the KRD thus obtained are positive, we can then try to find a solution by maximizing the weight of two KRDs that are closest to the level duration of the portfolio.

8.3 KRD AND TERM STRUCTURE HEDGING

Most portfolios that are hedged using key rate durations use about ten of the following key rates: 6 months, 1, 2, 3, 4, 5, 7, 10, 15, 20, and 30 years. One of these, usually 6 months, 4 or 15 years, is dropped from the list. As we have seen in previous chapters only five components of the TSD are sufficient for hedging. If there are 10 KRDs, there is an opportunity to structure an optimized hedge and add return to the portfolio. Our objective will then be to select five key rates in such a way as to maximize or minimize yield or carry of the hedge. For example, if a portfolio is long duration and we want to hedge it by selling KRSs, our objective will be to sell the securities in such a way to minimize their expected return. Likewise, to add duration to a portfolio, we select securities that will maximize the expected return given all other constraints.

To minimize the effect of the richness of on-the-run bonds, we created a portfolio of equal weights of off-the-run treasuries with the maturities 1.5, 2.5, 3.5, 5.5, 7.5, 9, 14, 18, and 24 years. If there were two securities with identical maturity, the security that was issued most recently was selected. Since the exact maturities were not available, the closest maturity to the above list was selected on the last business day of every month. The KRD of the portfolio was measured using the following KRSs: 6 months, 1, 2, 3, 5, 7, 10, 15, 20, and 30 year zero coupon bonds. We then constructed

TABLE 8.3 Performance of hedging methods, 1998–2012

Hedge	Excess Return	Tracking Error
KRD	0.23%	0.47%
Yield Maximized	0.24%	0.62%
Yield Minimized	−0.03%	0.43%
Carry Maximized	0.35%	0.53%
Carry Minimized	−0.12%	0.46%

the portfolio, its hedge using the KRSs and four optimized hedges using TSDs. There was one maximization and one minimization each for yield and carry. Table 8.3 shows annualized tracking errors and performance of the different hedging methods. If the market value of the hedges was larger or smaller than the market value of the portfolio, a 1-month time deposit was purchased or sold respectively, to bring the market values of the hedges in line with the portfolio.

While all tracking errors are comparable, it is clear that optimization on carry offers the best performance, as it did in Tables 8.1. Using ten KRDs does not offer an advantage on tracking error over five components of the TSIR. It requires twice as many trades and its performance may not be desirable. For example, if the portfolio is long due to overweight in corporate securities, KRD hedging can be costly as the hedge has positive performance. Using TSIR durations and optimization not only requires fewer transactions, but also can add value to the portfolio. Even though we used off-the-run treasuries, many coupon bonds can be at a premium price compared to zero coupon bonds due to liquidity and deliverability in bond futures. This is a primary reason why the KRD hedge outperformed the portfolio.

CHAPTER 9

Yield Volatility

Quantifying yield volatility is the key ingredient for pricing securities with options, including callable and puttable bonds and swaptions. A swaption is an option on a forward swap contract, and if exercised, it will become an interest rate swap. Swaptions are often quoted as the price of a receiver (call) or a payer (put) with time to expiration of the option followed by the maturity of the underlying swap. For example, buying a 3-month by 10-year at-the-money forward receiver implies paying the premium for the right to buy a 10-year swap in 3 months at the implied 3-month forward 10-year swap rate.

The premium is in units of the currency, but the strike is in market yield. At the expiration of the option, if the market yield is lower than the strike yield, a receiver swap can be exercised and the swap can be sold for a premium. There is a very active market in trade swaptions with varying option expiration dates, up to 10 years and many different maturity dates in USD, EUR, GBP and JPY.

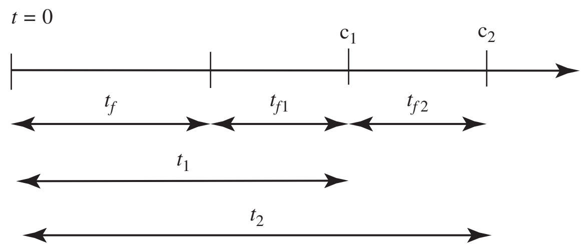
Most options for equity or currency markets have terms of less than 1 year and the implied volatility of the option can be calculated from its price or vice versa. Additionally, the nature of the underlying equity or currency does not change in the course of the option. Fixed income options, on the other hand, can be considerably more complicated for the following reasons:

- Bond options can have a very long expiration date. A 30-year bond that is callable in 10 years is an option with an expiration date of 10–30 years in the future.
- As a bond gets closer to maturity, its remaining life and volatility change as its price converges to par at maturity.
- The interest rate that is used to discount the future price of the option is itself dependent on the path of interest rates.
- There are additional complications for options on credit securities that will be discussed in full in Chapter 15.

9.1 PRICE FUNCTION OF YIELD VOLATILITY

Analysis of volatility for bonds leads to the term structure of yield volatility surface. The term structure of yield volatility is dependent upon the time to expiration of the option as well as the time to the maturity of the underlying security or bond.

Before attempting to calculate the term structure of volatility, let us examine a security with two cash flows c_1 and c_2 at times $t_f + t_{f1}$ and $t_f + t_{f1} + t_{f2}$.



The forward price function of the security at some future time t_f will be

$$p = c_1 e^{-y_{f1} t_{f1}} + c_2 e^{-(y_{f1} t_{f1} + y_{f2} t_{f2})} \tag{9.1}$$

where y_{f1} is the forward rate and t_{f1} is the forward time to the first cash flow of the security and y_{f2} is the forward yield for the second cash flow. If y_f and t_f are the yield and time to the forward date and y_1 and y_2 are the yields to the first and second cash flows respectively, then

$$y_1 t_1 = y_f t_f + y_{f1} t_{f1} \tag{9.2}$$

and

$$y_2 t_2 = y_f t_f + y_{f1} t_{f1} + y_{f2} t_{f2} \tag{9.3}$$

As we can see, yields have linear relationships, unlike volatility which has a quadratic relationship. To analyze volatility, we note that, by definition, the yield volatility is equal to the standard deviation of the relative changes in yield over a period of time. If y_i is a sequence of yields at different times, the square of its standard deviation is

$$\sigma^2 = \sum_{i=1}^{n} \left(\ln\left(\frac{y_{i+1}}{y_i}\right) \right)^2 = \sum_{i=1}^{n} \left(\ln\left(1 + \frac{\Delta y_i}{y_i}\right) \right)^2 \approx \sum_{i=1}^{n} \left(\frac{\Delta y_i}{y_i} \right)^2 \tag{9.4}$$
$$\Delta y_i = y_{i+1} - y_i$$

If the yield change is measured on a daily basis and there are 252 trading days in a year, then the square of annualized volatility is equal to the square of the expected standard deviation of the yield change after 1 year. If we denote the volatility by v, then

$$v^2 = \sum_{i=1}^{252} \left(\frac{\Delta y_i}{y_i} \right)^2 \tag{9.5}$$

To calculate the square of the standard deviation of the yield change after 6 months, we need to use half as many observations. Thus, the square of the standard deviation of the yield change is proportional to time times the square of volatility:

$$\sigma(t)^2 = v^2 t \tag{9.6}$$

If the change in yield after a short time δt, is δy, we can write

$$\begin{aligned} \sigma(t)^2 &= v^2 \delta t = \frac{(\delta y)^2}{y^2} \\ \delta y &= y v \sqrt{\delta t} \end{aligned} \tag{9.7}$$

We can now evaluate the expected changes in the yields in (9.2) and (9.3) after a short period of time δt. If v_1 and v_2 are the respective volatilities for y_{f1} and y_{f2} and ρ is the correlation between the two rates, assuming that the yield to the forward date does not change, the expected change in the spot yield to the cash flow dates after a time δt will be

$$\delta y_1 = y_{f1} v_1 \sqrt{\delta t} \tag{9.8}$$

$$\delta y_2 = \frac{\sqrt{(y_{f1} v_1 t_{f1})^2 + (y_{f2} v_2 t_{f2})^2 + 2\rho y_{f1} v_1 t_{f1} y_{f2} v_2 t_{f2}}}{t_{f1} + t_{f2}} \sqrt{\delta t} \tag{9.9}$$

If the correlation coefficient is not 1, the result will be complex non-linear equations and we will not cover it here. Going forward we will assume that the correlation coefficient is always 1, unless it is mentioned explicitly, as in corporate bonds. However, the term structure of volatility will match all market volatilities and thus has implied correlations embedded in it.

From the definition of modified duration (1.8) or continuously compounded duration (1.18), we know that

$$p + \Delta p = p - p D_m \Delta y_m = p - p D \Delta y \tag{9.10}$$

Making the substitution for the change in yield from (9.7) results in

$$\begin{aligned} p + \Delta p &= p - p D_m \Delta y_m = p - p D \Delta y \\ &= p - p D_m y_m v_m \delta t = p - p D y v \delta t \end{aligned} \tag{9.11}$$

where D_m and y_m are the modified duration and market yield of the security, v_m is the implied market volatility of the security, and D, y, and v are the duration, yield, and volatility in the continuously compounded framework. As we learned in Chapter 1, we can convert the market yield to the continuously compounded yield as follows,

$$y = m \ln\left(1 + \frac{y_m}{m}\right) \tag{9.12}$$

where m is the coupon frequency. We can calculate the relationship between the continuously compounded volatility and market volatility as follows:

$$y+\delta y = y + yv\sqrt{\delta t} = n\ln\left(1+\frac{y_m+\delta y_m}{m}\right) = n\ln\left[\left(1+\frac{y_m}{m}\right)\left(1+\frac{\delta y_m}{n(1+y_m/m)}\right)\right]$$
$$= n\ln\left(1+\frac{y_m}{m}\right) + n\ln\left(1+\frac{\delta y_m}{n(1+y_m/m)}\right) \approx y + \frac{\delta y_m}{1+y_m/m} = y + \frac{y_m v_m\sqrt{\delta t}}{1+y_m/m} \tag{9.13}$$

or

$$yv = \frac{y_m v_m}{1+y_m/m} \tag{9.14}$$

If v_i is the forward spot volatility associated with y_i for a cash flow at time t_i, we can write the change in the price function of a security after time δt as

$$\sum_i c_i e^{-(y_i+y_iv_i\sqrt{\delta t})t_i} = p - \sum_i c_i e^{-y_it_i} y_i v_i t_i \sqrt{\delta t} = p - pDyv\sqrt{\delta t} \tag{9.15}$$

$$pDyv = \sum_i c_i y_i v_i t_i e^{-y_it_i} \tag{9.16}$$

From (6.15), we can similarly derive the contribution of a floating Libor rate to the volatility price as

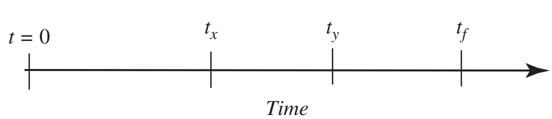
$$p_i t_i y_i v_i = \mu(y_{i-1}v_{i-1}t_{i-1}e^{-y_{i-1}t_{i-1}} - y_i v_i t_i e^{-y_it_i}) \tag{9.17}$$

Equation (9.16) is the price function of volatility in terms of its component volatilities using a correlation coefficient of one.

9.2 TERM STRUCTURE OF YIELD VOLATILITY

Our model for volatility is similar to the term structure of rates model, with the exception that volatility is a surface and depends on both time to maturity and time to expiration of the underlying option. Instead of modeling the relative yield volatility, we model the absolute yield volatility $y_i v_i$ in the forward space. Consider a segment of the yield curve that spans from time t_x to t_f. We define the time to the mid-point of the segment as

$$t_y = \frac{t_x+t_f}{2} \tag{9.18}$$



We define the term structure of Libor volatility (TSLV) or term structure of yield volatility (TSYV) as the forward volatility of the segment with mid-point t_y and expiration time t_x:

$$y(t_x, t_y)v(t_x, t_y) = \sum_k e_k \psi_{vk}(t_x, t_y) \tag{9.19}$$

where ψ_{vk} is the kth component of the basis function of the TSYV. ψ_{vk} depends on both the expiration time of the option t_x and the time to the mid-point of the forward segment t_y. For simplicity, we express the dependence of the volatility to the expiration time in polynomial basis functions as a function of t_x. The first three components will be

$$\psi_{v,0} = 1, \quad \psi_{v,1} = \theta, \quad \psi_{v,2} = \theta^2 \tag{9.20}$$

with

$$\theta = 1 - 2e^{-\beta t_x} \tag{9.21}$$

β is the associated decay coefficient for time to expiration of volatility, analogous to (2.16).

The dependence of volatility to the mid-point of the forward segment t_y in PBFs, using three components $\psi_{y,l}$, can be written as

$$\psi_{y,0} = 1, \quad \psi_{y,1} = \tau, \quad \psi_{y,2} = \tau^2 \tag{9.22}$$

Equation (9.19) can now be written as

$$y(t_x, t_y)v(t_x, t_y) = \sum_{k,l} b_{kl} \theta^k \tau^l \tag{9.23}$$

In this formulation, high order components contribute to the short end of the volatility surface. For example, the contribution of $e^{-8\beta t_x}$ in $\theta^8 = (1 - 2e^{-\beta t_x})^8$ decays very rapidly with time. For simplicity, we will use fast decaying components to capture the effects of short term volatility, in order to minimize the number of components that are necessary to replicate market volatility. We define the absolute term structure of volatility (yv) as follows:

$$y(t_x, t_y)v(t_x, t_y) = b_0 + b_1\tau_1 + b_2\tau_1^2 + b_3\theta_1 + b_4\theta_1^2 + b_5\tau_1\theta_1 + b_6\theta_2 + b_7\theta_2\tau_2 \tag{9.24}$$

The following definitions and decay coefficients are used:

$$\tau_1 = 1 - 2e^{-\alpha_1 t_y}, \quad \alpha_1 = 0.13 \tag{9.25}$$

$$\tau_2 = 1 - 2e^{-\alpha_2 t_y}, \quad \alpha_2 = 0.60 \tag{9.26}$$

$$\theta_1 = 1 - 2e^{-\beta_1 t_x}, \quad \beta_1 = 0.25 \tag{9.27}$$

$$\theta_2 = 1 - 2e^{-\beta_2 t_x}, \quad \beta_2 = 1.4 \tag{9.28}$$

Here α_1 is the decay coefficient for the term structure of rates, β_2 is the characteristic decay coefficient for the time to the expiration of the option, α_2 is the decay coefficient cross-component of short term expiration time and mid-forward point, and β_2 is the characteristic decay coefficient for short term components of the expiration time.

Figure 9.1 shows the cross-sections of the volatility surface for different expiration dates using the above formulation. At the front end of the curve, where interest rates are close to zero (0.25%) the volatility is very high and options with short maturity of the underlying rarely trade. For example, the market for 6-month options on a 1-year underlying forward bond is very illiquid and trading costs of such options are

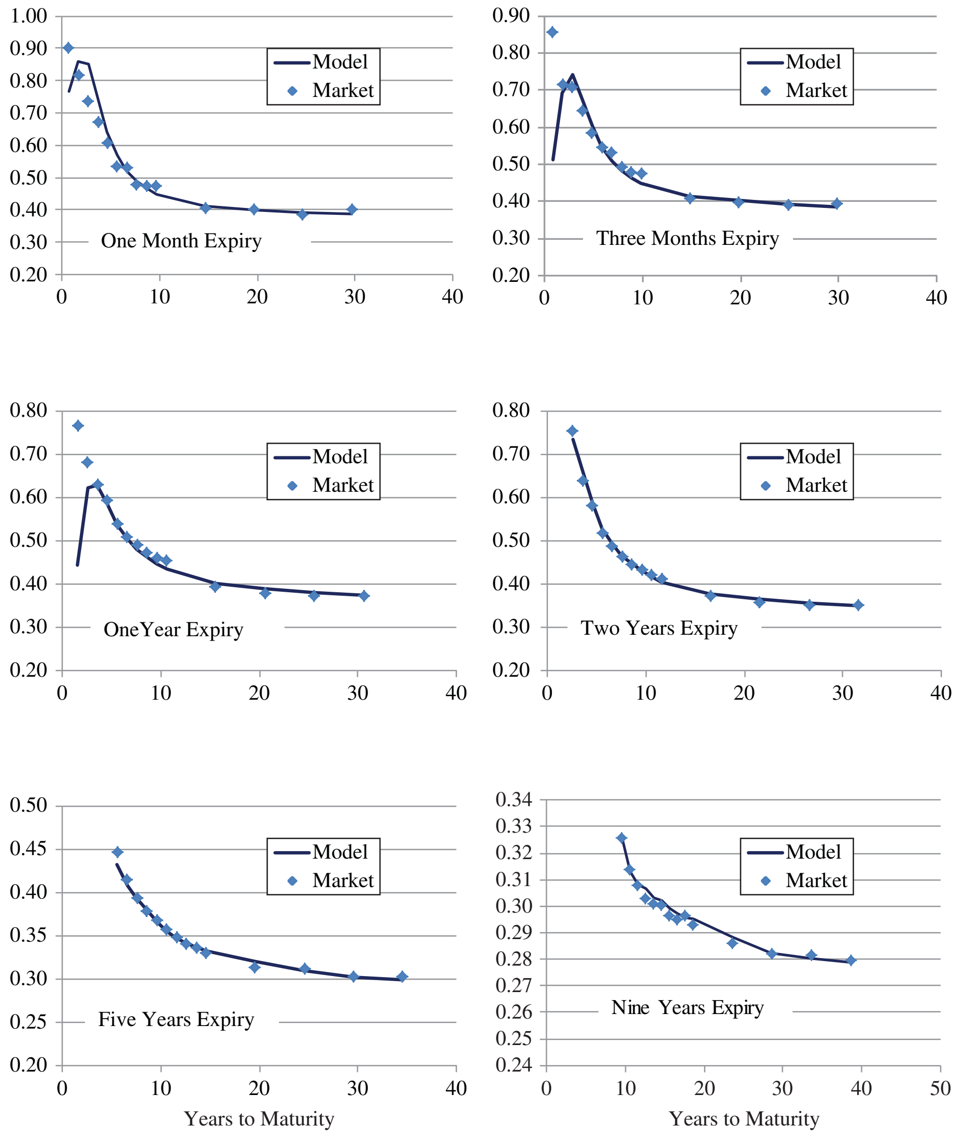


FIGURE 9.1 Selected cross-sections of relative Libor volatility, June 30, 2012

typically several percent. Quite often there are kinks and irregularities in the early part of the short expiration–short maturity volatility surface. For final maturity of 2 years or more, the maximum error of our model is usually less than 0.5% volatility, which is well within transaction costs. However, for calculating option premiums, it is not the relative volatility that needs to be accurate; it is the basis point volatility or absolute volatility which drives the price change on the underlying security. Our volatility price function (9.16) optimizes absolute volatility. Figure 9.2 shows selected cross-sections of the absolute volatility in percentage interest rates. For example, a volatility of 0.5 means an expected volatility of 0.5% per year.

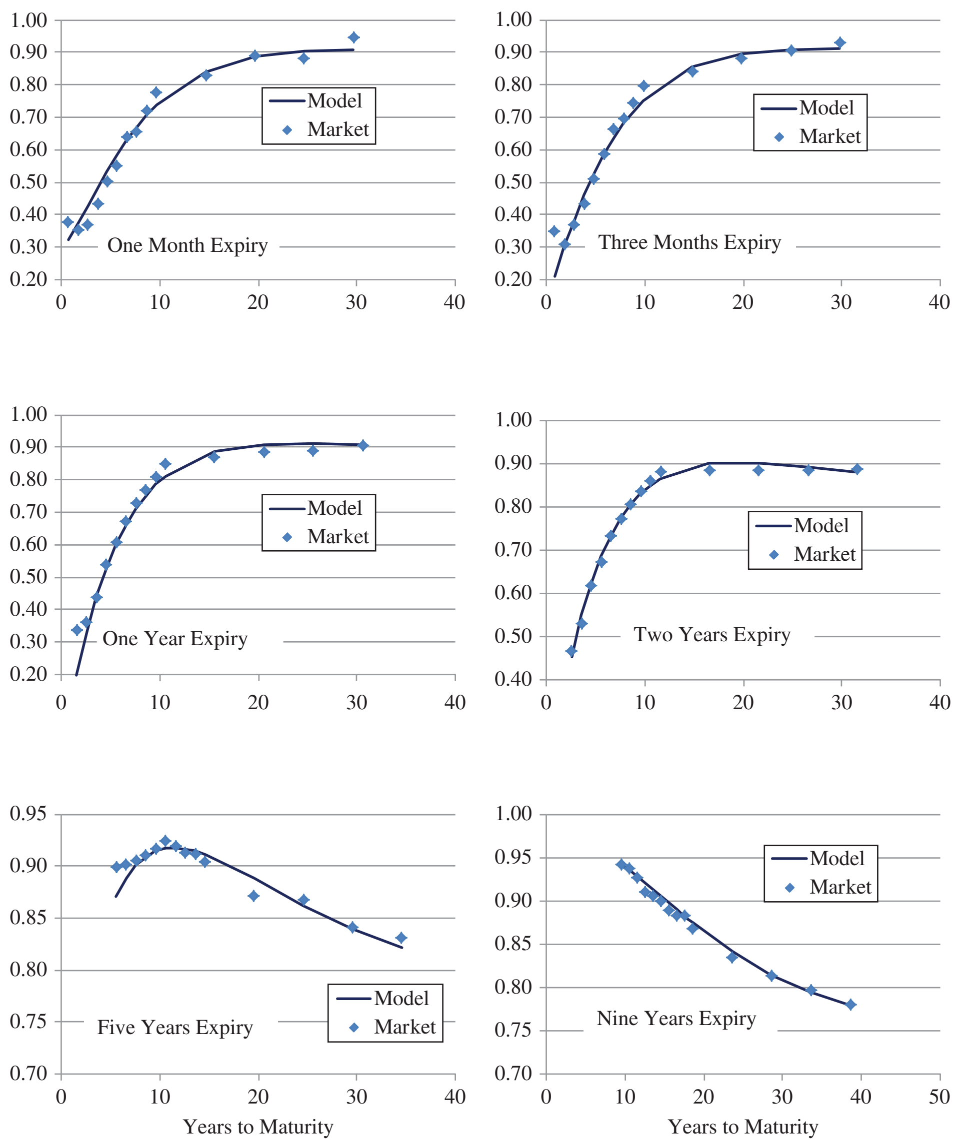


FIGURE 9.2 Selected cross-sections of absolute Libor volatility, June 30, 2012

The historical components of the TSLV have relatively strong correlations with each other, implying that a reduced set of coefficients is possible. Table 9.1 shows the correlation coefficients of the TSLV.

One way to find a reduced set of coefficients is through principal components analysis. Table 9.2 is the table of eigenvectors and eigenvalues of such an analysis. The eigenvalues will be the weights of the each vector or component of the reduced set of coefficients. The coefficients of each vector or column of the table are normalized, that is, the sum of squares of the coefficients is 1. The last row is the corresponding eigenvalues or the historical weight of each vector.

TABLE 9.1 Correlations of historical components of TSLV, 2000–2012

Vol.	bl	b2	b3	b4	b5	b6	b7	b8
1.323%	1	−0.837	0.865	−0.887	0.657	−0.944	0.362	−0.573
1.166%	−0.837	1	−0.663	0.636	−0.457	0.852	−0.537	0.674
0.509%	0.865	−0.663	1	−0.693	0.434	−0.827	0.260	−0.368
0.630%	−0.887	0.636	−0.693	1	−0.834	0.815	−0.144	0.509
0.274%	0.657	−0.457	0.434	−0.834	1	−0.541	−0.090	−0.406
2.479%	−0.944	0.852	−0.827	0.815	−0.541	1	−0.403	0.538
2.484%	0.362	−0.537	0.260	−0.144	−0.090	−0.403	1	−0.831
2.706%	−0.573	0.674	−0.368	0.509	−0.406	0.538	−0.831	1

TABLE 9.2 Principal components of historical components of TSLV, 2008–2012

bl	b2	b3	b4	b5	b6	b7	b8
0.205	−0.041	−0.046	0.016	−0.191	0.558	−0.051	0.009
−0.220	−0.035	0.069	−0.274	−0.155	0.013	−0.179	0.358
0.457	−0.050	−0.442	−0.832	0.140	−0.332	−0.011	−0.387
−0.397	0.248	−0.118	0.063	−0.396	−0.132	−0.342	−0.805
−0.724	−0.945	−0.878	0.472	0.865	0.714	−0.905	0.229
0.049	0.177	0.007	0.002	0.089	0.058	−0.111	0.050
−0.105	0.013	0.049	−0.080	0.082	0.218	0.073	−0.105
0.070	−0.091	0.102	−0.013	0.033	−0.030	−0.109	−0.094
5.378	1.438	0.715	0.228	0.141	0.056	0.029	0.015

The weight of the fourth (sixth) vector is only 4% (1%) of the weight of the first vector based on historical data. In efficient markets, the bid–ask spread of volatility is usually more than 2%, implying that the first four principal components should provide reasonable accuracy for most calculations. Using the first five principal components should provide the necessary accuracy for nearly all applications.

9.3 VOLATILITY ADJUSTMENT TABLE

We can create a table of volatility adjustments that would enable us to price nearly all volatilities exactly. The table is constructed by taking the shortest expiration and maturity point on the volatility surface and calculating the absolute yield volatility adjustment that is needed to match the calculated and market volatilities. Then the adjustment for the next shortest maturity for the same expiration time is calculated, and we continue until all the adjustments for a given expiration time are calculated. The same process is repeated for the next expiration date and continues until all volatilities are priced.

Currently there are 13 expiration months and 14 maturities in the US market. We can select about ten expiration times and ten maturities to calculate the adjustment table or we can simply price all of them using a 13 × 14 matrix. Table 9.3 shows a sample adjustment table for the above volatility term structures. Nearly all adjustments are less than 2 bps of absolute volatility per year. Table 9.4 shows the market volatility, calculated volatility ("Fair") using the adjustment table and the model volatility without adjustment.

With the adjustment table of about 100 adjustments, nearly all volatilities can be calculated within 0.5 bps of market volatility well within transaction costs of such trades. For expiration and volatilities that are not in the adjustment table, for example to calculate the adjustment for an expiration of 2.5 years and maturity 13 years, we take a rectangle on the volatility surface that surrounds the point of interest. In this example, the rectangle will be at expirations of 2 and 3 years and maturities of 10 and 15 years. We first calculate the adjustment for expiration of 2.5 years with a maturity of 10 years by interpolating between the adjustments at 2 and 3 years. We carry out the same process at 15 years maturity and then interpolate between 10 and 15 years to calculate the adjustment for 2.5 × 13 years volatility. These adjustments are usually very small but allow us to price the whole volatility surface nearly exactly.

TABLE 9.3 Adjustment table for US swap volatility, June 30, 2012

Expiration Years	Maturity Years	Adjustment Volatility
2.00	1.0	0.0123%
2.00	2.0	–0.0195%
2.00	3.0	–0.0074%
2.00	4.0	–0.0147%
2.00	5.0	–0.0027%
2.00	7.0	–0.0004%
2.00	10.0	0.0174%
2.00	15.0	–0.0186%
2.00	20.0	–0.0202%
2.00	30.0	0.0147%
3.00	1.0	–0.0188%
3.00	2.0	–0.0048%
3.00	3.0	–0.0017%
3.00	4.0	0.0040%
3.00	5.0	–0.0096%
3.00	7.0	–0.0044%
3.00	10.0	0.0001%
3.00	15.0	–0.0109%
3.00	20.0	–0.0043%
3.00	30.0	0.0134%

TABLE 9.4 Market, fair, and model volatilities, June 30, 2012

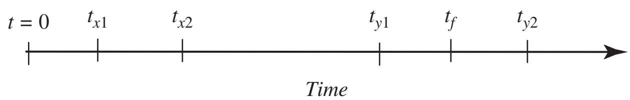
Expiration	Maturity	Market	Fair	Model
2.00	1.0	0.754	0.754	0.734
2.00	2.0	0.639	0.639	0.663
2.00	3.0	0.579	0.579	0.586
2.00	4.0	0.517	0.517	0.529
2.00	5.0	0.488	0.488	0.490
2.00	6.0	0.464	0.464	0.465
2.00	7.0	0.445	0.445	0.446
2.00	8.0	0.433	0.432	0.430
2.00	9.0	0.420	0.420	0.415
2.00	10.0	0.412	0.412	0.405
2.00	15.0	0.372	0.371	0.378
2.00	20.0	0.358	0.358	0.365
2.00	25.0	0.351	0.353	0.355
2.00	30.0	0.351	0.351	0.348
3.00	1.0	0.612	0.611	0.629
3.00	2.0	0.549	0.548	0.552
3.00	3.0	0.496	0.496	0.497
3.00	4.0	0.463	0.463	0.461
3.00	5.0	0.434	0.434	0.439
3.00	6.0	0.420	0.417	0.421
3.00	7.0	0.404	0.404	0.406
3.00	8.0	0.394	0.390	0.392
3.00	9.0	0.387	0.381	0.382
3.00	10.0	0.375	0.375	0.375
3.00	15.0	0.350	0.349	0.353
3.00	20.0	0.340	0.340	0.342
3.00	25.0	0.334	0.333	0.332
3.00	30.0	0.330	0.330	0.327

9.4 FORWARD AND INSTANTANEOUS VOLATILITY

Equation (9.24) calculates the absolute yield volatility of a forward yield starting at t_x and ending at t_f for volatility between time 0 and t_x. Now let us calculate the volatility of a forward line segment that starts at a forward time other than t_x.

Assuming $v(t_{x1}, t_{x2}, t_{y1}, t_{y2})$ to be the relative yield volatility of a segment of the yield curve between t_{y1} and t_{y2} and expiration time interval between t_{x1} and t_{x2}, such that $t_{x1} \leq t_{x2} \leq t_{y1} \leq t_{y2}$, we define the absolute yield volatility w, as

$$w(t_{x1}, t_{x2}, t_{y1}, t_{y2}) = v(t_{x1}, t_{x2}, t_{y1}, t_{y2}) y(t_{y1}, t_{y2}) \tag{9.29}$$



If we represent the forward yield between time t_x and t_{y2} as $y(t_x, t_{y2})$ then

$$y(t_x, t_{y2}) = \frac{y(t_x, t_{y1})(t_{y1} - t_x) + y(t_{y1}, t_{y2})(t_{y2} - t_{y1})}{(t_{y2} - t_x)} \tag{9.30}$$

The term structure of volatility (9.16) provides the volatility for the segment with expiration time t_x, maturity t_{y2} and mid-point to maturity t_f and we denote it by $W(t_x, t_{y2})$:

$$y(t_x, t_f) v(t_x, t_f) = w(0, t_x, t_x = t_{y1}, t_{y2}) = W(t_x, t_{y2}) \tag{9.31}$$

The change in the yield of the segment (t_{x2}, t_{y2}) after time δt is equal to $w(0, t_{x2}, t_{x2}, t_{y2})\sqrt{\delta t}$. Likewise, the change in the yield of the segment (t_{x2}, t_{y1}) will be $w(0, t_{x2}, t_{x2}, t_{y1})\sqrt{\delta t}$. If the absolute volatility of the segment (t_{y1}, t_{y2}) in the interval $(0, t_{x2})$ is $w(0, t_{x2}, t_{y1}, t_{y2})$, then from (9.30), assuming perfectly correlated movements of the yield curve, we have

$$\begin{aligned} &w(0, t_{x2}, t_{x2}, t_{y2})(t_{y2} - t_{x2})\sqrt{\delta t} \\ &\quad = w(0, t_{x2}, t_{x2}, t_{y1})(t_{y1} - t_{x2})\sqrt{\delta t} + w(0, t_{x2}, t_{y1}, t_{y2})(t_{y2} - t_{y1})\sqrt{\delta t} \end{aligned} \tag{9.32}$$

or

$$\begin{aligned} W(t_{x2}, t_{y2})(t_{y2} - t_{x2})\sqrt{\delta t} &= W(t_{x2}, t_{y1})(t_{y1} - t_{x2})\sqrt{\delta t} \\ &\quad + w(0, t_{x2}, t_{y1}, t_{y2})(t_{y2} - t_{y1})\sqrt{\delta t} \end{aligned} \tag{9.33}$$

Thus,

$$w(0, t_{x2}, t_{y1}, t_{y2}) = \frac{W(t_{x2}, t_{y2})(t_{y2} - t_{x2}) - W(t_{x2}, t_{y1})(t_{y1} - t_{x2})}{(t_{y2} - t_{y1})} \tag{9.34}$$

If the correlation between the segments is not one, we replace equation (9.32) with a triangular equation of the form

$$y = \sqrt{x_1^2 + x_2^2 + 2\rho x_1 x_2} \tag{9.35}$$

(assuming a log-normal distribution of rates) where x_1 and x_2 are the two terms on the right hand side of equation (9.32) and ρ is the correlation coefficient.

If the absolute volatility of the line segment (t_{y1}, t_{y2}) in the time interval $(0, t_{x1})$ is v_1 and in the interval (t_{x1}, t_{x2}) is v_2, these volatilities by definition will be uncorrelated since they are in two non-overlapping time intervals. The volatility in the period $(0, t_{x2})$ can be calculated by adding the square of the standard deviations, similarly to the derivation of (9.6):

$$v_3^2\, t_{x2} = v_1^2\, t_{x1} + v_2^2 (t_{x2} - t_{x1}) \tag{9.36}$$

Making the substitutions $w(0,t_{x2},t_{y1},t_{y2})$ for v_3, $w(0,t_{x1},t_{y1},t_{y2})$ for v_1 and $w(t_{x1},t_{x2},t_{y1},t_{y2})$ for v_2, we arrive at

$$w(t_{x1},t_{x2},t_{y1},t_{y2}) = \sqrt{\frac{\left(w(0,t_{x2},t_{y1},t_{y2})\right)^2 t_{x2} - \left(w(0,t_{x1},t_{y1},t_{y2})\right)^2 t_{x1}}{t_{x2}-t_{x1}}} \tag{9.37}$$

Equation (9.37) is the absolute yield volatility of the forward segment in the interval (t_{y1},t_{y2}) and implies that there is no serial correlation between yield volatility, that is, the volatility in the interval $(0,t_{x1})$ is independent of the volatility in the interval (t_{x1},t_{x2}). Since all segments of volatility in (9.37) refer to the (t_{y1},t_{y2}) segment, the absolute volatility can be converted to relative volatility by dividing both sides of the equation by the forward yield of the line segment, leading to

$$v(t_{x1},t_{x2},t_{y1},t_{y2}) = \sqrt{\frac{\left(v(0,t_{x2},t_{y1},t_{y2})\right)^2 t_{x2} - \left(v(0,t_{x1},t_{y1},t_{y2})\right)^2 t_{x1}}{t_{x2}-t_{x1}}} \tag{9.38}$$

To calculate the volatility of the instantaneous forward rates between time 0 and t_{x2}, we can calculate the limit of (9.34) as t_{y2} approaches t_{y1}:

$$\begin{aligned} w(0,t_{x2},t_{y1},t_{y1}) &= \frac{\partial}{\partial t_{y1}}\left(W(t_{x2},t_{y1}-t_{x2})(t_{y1}-t_{x2})\right) \\ &= W(t_{x2},t_{y1}) + \frac{W(t_{x2},t_{y1})}{\partial t_{y1}}(t_{y1}-t_{x2}) \end{aligned} \tag{9.39}$$

The instantaneous volatility of the instantaneous rates can be calculated by inserting the above equation into (9.37) and allowing t_{x2} approach t_{x1} Thus,

$$\begin{aligned} w(t_{x1},t_{x1},t_{y1},t_{y1}) &= \sqrt{\frac{\partial\left[\left(w(0,t_{x1},t_{y1},t_{y1})\right)^2 t_{x1}\right]}{\partial t_{x1}}} \\ &= \sqrt{\left(w(0,t_{x1},t_{y1},t_{y1})\right)^2 + 2t_{x1}\frac{\partial w(0,t_{x1},t_{y1},t_{y1})}{\partial t_{x1}} w(0,t_{x1},t_{y1},t_{y1})} \end{aligned} \tag{9.40}$$

For a credit bond with a spread s_i, credit volatility of $v_{c,i}$ and floating coupon defined by (12.18), the volatility price function can be written as

$$\begin{aligned} pDyv = &\sum_i c_i(y_i+s_i)v_{c,i}t_i e^{-(y_i+s_i)t_i} \\ &- p_r\sum_i (y_i v_i t_i - y_{i-1}v_{i-1}t_{i-1})e^{-y_{i-1}t_{i-1}-s_i t_i} \end{aligned} \tag{9.41}$$

CHAPTER 10

Convexity and Long Rates

In the previous chapters we explained the performance contribution of a security or a portfolio of securities on the basis of movements in the components of the TSIR. We now revisit this issue and analyze the impact of convexity on the shape of the yield curve.

At very long maturities, the value of τ in the term structure of rates approaches unity and long spot rates approach a constant value called the consol rate, that is,

$$y\big|_{t\to\infty} = y\big|_{\tau\to 1} = a_0 + a_1 + a_2 + a_3 + a_4 + \dots \tag{10.1}$$

There has been much research and debate on the subject of long term rates. In the 1990s the long end of the zero coupon curve in the US had a negative slope, but lately this has changed. Dybvig, Ingersoll and Ross [10] proved that long forward and zero coupon rates can never fall. This implies that in an arbitrage- and friction-free environment, long rates can only rise. Before we derive our analytical shape of long rates, we will prove that long rates cannot change at all!

10.1 THEOREM: LONG RATES CAN NEVER CHANGE

We will base our proof on the following assumptions:

- Forward rates must be finite and can never be negative in any time interval.
- There is an endless availability of risk-free zero coupon maturities that can be borrowed or lent without transaction costs.
- There cannot be any arbitrage opportunity, that is, there cannot be an expected positive return without the possibility of negative return.
- Long term rates are continuous and asymptotically approach a constant value and will do so in all future paths.
- Long rates can change over time.

Consider three portfolios of zero coupon bonds in an initial state such that

$$Q_a = N_a e^{-y_{1,a} t_a}, \quad Q_b = N_b e^{-y_{1,b} t_b}, \quad Q_c = N_c e^{-y_{1,c} t_c} \tag{10.2}$$

where $y_{1,a}$ is the yield and t_a is the time to maturity of portfolio Q_a, and similarly for the other two portfolios. The face amounts N_a, N_b, and N_c are selected such that

$$\begin{aligned} Q &= Q_a - Q_b + Q_c = 0 \\ \frac{\partial Q}{\partial y} &= N_a t_a e^{-y_1, a t_a} - N_b t_b e^{-y_{1,b} t_b} + N_c t_c e^{-y_{1,c} t_c} = 0 \end{aligned} \tag{10.3}$$

We choose the maturities and the number of shares such that

$$\begin{aligned} & t_a = T - \theta, \quad t_b = T, \quad t_c = T + \theta \\ & N_b e^{y_{1,b} t_b} = 1 \\ & N_a e^{y_{1,a} t_a} = N_c e^{y_{1,c} t_c} = \tfrac{1}{2} \end{aligned} \tag{10.4}$$

Assume that at very long rates, yields approach a terminal value asymptotically as follows:

$$y_1 (a)\Big|_{t_a \to \infty} \approx y_\infty + \frac{d_1}{t_a^n} \tag{10.5}$$

where d is a finite number and n is the power of asymptotic approach of the yield to its terminal value. The forward yield between points a and b is

$$f_{ab} = \frac{y_b t_b - y_a t_a}{t_b - t_a} \tag{10.6}$$

Substituting for the yield from (10.5), we obtain

$$f_{ab} = y_\infty + \frac{d}{t_b - t_a} \left(\frac{1}{t_b^{n-1}} - \frac{1}{t_a^{n-1}} \right) \tag{10.7}$$

The forward rates can become unbounded at very long times if the asymptotic power $n < 1$. Thus,

$$y(t)\Big|_{t \to \infty} \approx y_\infty + \frac{d}{t^n}, \quad n \geq 1 \tag{10.8}$$

Assume that the portfolio in (10.3) will evolve to a new state after a short time Δt with a new terminal rate that is shifted by δ. The new portfolio can be written as

$$Q_2 = N_a e^{-y_{2,a}(t_a - \Delta t)} - N_b e^{-y_{2,b}(t_b - \Delta t)} + N_c e^{-y_{2,c}(t_c - \Delta t)} \tag{10.9}$$

At very long rates, the asymptotic approach can be written as

$$\begin{aligned} y_2(t) &\approx y_{2,\infty} + \frac{d_2}{t^n} = y_{2,\infty} + \varepsilon_2 \\ y_{2,\infty} &= y_{1,\infty} + \delta \end{aligned} \tag{10.10}$$

Thus, the new state of the portfolio can be written as:

$$\begin{aligned} Q_2 = {} & N_a e^{-(y_{1,\infty}+\delta+\varepsilon_{2,a})(T-\theta-\Delta t)} - N_b e^{-(y_{1,\infty}+\delta+\varepsilon_{2,b})(T-\Delta t)} \\ & + N_c e^{-(y_{1,\infty}+\delta+\varepsilon_{2,c})(T+\theta-\Delta t)} \end{aligned} \tag{10.11}$$

For small values of δ, ε and Δt, we can expand the above equation using Taylor series. The expansion for the portfolio Q_a is

$$\begin{aligned} Q_{2,a} - Q_{1,a} = {} & N_a e^{-(y_{1,\infty}+\delta+\varepsilon_{2,a})(T-\theta-\Delta t)} - N_a e^{-(y_{1,\infty}+\varepsilon_{1,a})(T-\theta)} = \\ & N_a e^{-(y_{1,\infty}+\varepsilon_{1,a})(T-\theta)} \times \\ & \Bigg[1 + (y_{1,\infty} + \delta + \varepsilon_{2,a})\Delta t + \frac{1}{2}(y_{1,\infty} + \delta + \varepsilon_{2,a})^2 \Delta t^2 \\ & - (\delta + \varepsilon_{2,a} - \varepsilon_{1,a})(T-\theta) + \frac{1}{2}(\delta + \varepsilon_{2,a} - \varepsilon_{1,a})^2 (T-\theta)^2 + \cdots - 1\Bigg] \\ = {} & \frac{1}{2}\Bigg[1 + (y_{1,\infty} + \delta + \varepsilon_{2,a})\Delta t + \frac{1}{2}(y_{1,\infty} + \delta + \varepsilon_{2,a})^2 \Delta t^2 \\ & - (\delta + \varepsilon_{2,a} - \varepsilon_{1,a})(T-\theta) + \frac{1}{2}(\delta + \varepsilon_{2,a} - \varepsilon_{1,a})^2 (T-\theta)^2 + \cdots - 1\Bigg] \end{aligned} \tag{10.12}$$

with similar expressions for portfolios Q_b and Q_c. Using (10.3) and (10.4), after some simplification and allowing for $Q_1 = 0$, we find

$$\begin{aligned} Q_2 = {} & \frac{\Delta t}{2} d_2 \left(1 + y_{2,\infty}\Delta t\right)\left[\frac{1}{(T-\theta)^n} - \frac{2}{T^n} + \frac{1}{(T+\theta)^n}\right] \\ & + \frac{d_2^2 \Delta t^2}{2}\left[\frac{1}{(T-\theta)^{2n}} - \frac{2}{T^{2n}} + \frac{1}{(T+\theta)^{2n}}\right] \\ & - \frac{d_2 - d_1}{2}\left[\frac{1}{(T-\theta)^{n-1}} - \frac{2}{T^{n-1}} + \frac{1}{(T+\theta)^{n-1}}\right] \\ & + (d_2 - d_1)\delta\left[\frac{1}{(T-\theta)^{n-2}} - \frac{2}{T^{n-2}} + \frac{1}{(T+\theta)^{n-2}}\right] \\ & + \frac{(d_2 - d_1)^2}{2}\left[\frac{1}{(T-\theta)^{2n-2}} - \frac{2}{T^{2n-2}} + \frac{1}{(T+\theta)^{2n-2}}\right] \\ & + \delta^2\theta^2 + \cdots \end{aligned} \tag{10.13}$$

With finite values of d_1,d_2, y, θ and small but finite values of δ and Δt, the limiting values of (10.13) will be governed by T and n for very large values of T. The first and second lines of the above equation tend to zero for large values of T. The third line can be expanded using Taylor series in θ:

$$
\begin{aligned}
&-\frac{d_2-d_1}{2}\left[\frac{(T^2-T\theta)^{n-1}-2T^{2n-2}+(T^2-T\theta)^{n-1}}{(T^3-T\theta^2)^{n-1}}\right] \\
&=-\frac{d_2-d_1}{2}\left[\frac{(n-1)(n-2)T^{2n-4}\theta^2+\cdots}{(T^3-T\theta^2)^{n-1}}\right]\to\sim\left(\frac{1}{T^{n+1}}\right)
\end{aligned}
\tag{10.14}
$$

This will tend to zero for large values of T. Similarly, the fourth and fifth lines tend to zero as T^{-n} and T^{-2n}, respectively. It can easily be shown that for all components of the expansion of $(\varepsilon_{2,a}-\varepsilon_{1,a})(T-\theta-\Delta t)$, the highest order component in T will cancel out and they will all converge to zero with power of $1/T^n$ or faster. This is shown as

$$
\begin{aligned}
&\frac{(d_2-d_1)^m}{m!}\left[\frac{1}{(T-\theta)^{m(n-1)}}-\frac{2}{T^{m(n-1)}}+\frac{1}{(T+\theta)^{m(n-1)}}\right] \\
&=\frac{(d_2-d_1)^m}{m!}\left[\frac{\big((T+\theta)T\big)^{m(n-1)}-2\big(T^2-\theta^2\big)^{m(n-1)}+\big((T-\theta)T\big)^{m(n-1)}}{\big((T^2-\theta^2)T\big)^{m(n-1)}}\right]
\end{aligned}
\tag{10.15}
$$

In a Taylor series expansion of the numerator, the highest order contribution of T is cancelled, leading to convergence. The last line of (10.13) is finite and always positive. This implies that if interest rates change, in a frictionless environment, portfolio Q will always have positive return and therefore it is an arbitrage. To avoid arbitrage, at least one of our assumptions at the beginning of this theorem has to be invalid.

The assumption that long zero rates approach a constant value is invalid. There is no rationale why long zero rates need to be constant. The rational assumption should be that the expected return of long term rates needs to be constant to prevent arbitrage. It is the return that can be arbitraged not the yield, and the yield does not need to approach a constant value.

10.2 CONVEXITY ADJUSTED TSIR

In the previous section it was proved that, assuming constant long rates, a portfolio can be constructed with positive return that is proportional to the square of the change in long rates and has zero risk. We know that the second order effect is generally due to convexity, and we will try to quantify this effect in this section.

Assume that at the very long end of the curve, the expected return of long rates and future forward rates will be equal. The key to understanding the behavior of long rates is that, in order to prevent arbitrage, the rate of returns must approach a constant value. Consider the price of a long zero coupon rate that pays off one unit at maturity as follows:

$$p = e^{-y_s T} \tag{10.16}$$

Here T is the long term maturity of the zero coupon bond and y_s is the spot yield. After a time Δt, if the yield rate has changed by Δy, we can write the portfolio value as

$$p(\Delta t) = e^{-(y_s+\Delta y)(T-\Delta t)} = e^{-y_s T} e^{-(\Delta y T - y_s \Delta t - \Delta y \Delta t)} \tag{10.17}$$

We expand the second exponent in Taylor series to obtain

$$\begin{aligned} p(\Delta t) = e^{-y_s T} \Big(1 - \Delta y T + y_s \Delta t + \Delta y \Delta t + \tfrac{1}{2}(\Delta y T)^2 \\ + \tfrac{1}{2}(y_s \Delta t)^2 - y_s T \Delta y \Delta t + \cdots\Big) \end{aligned} \tag{10.18}$$

Noting that duration and convexity are equal to T and T^2 respectively, equation (10.18) can be modified to

$$\begin{aligned} p(\Delta t) = p(0)\Big[1 + y_s \Delta t - \Delta y D + \tfrac{1}{2} X(\Delta y)^2 + \Delta y \Delta t \\ + \tfrac{1}{2}(y_s \Delta t)^2 - y_s D \Delta y \Delta t + \cdots\Big] \end{aligned} \tag{10.19}$$

where D and X are the spot duration and convexity, respectively. After simple rearrangements and substitutions, (10.19) can be written as

$$\frac{\Delta p}{p} = y_s \Delta t - D\,\Delta y + \frac{1}{2} X\, {y_s}^2 \left(\frac{\Delta y}{y_s}\right)^2 + \cdots \tag{10.20}$$

If v_y is the annualized relative yield volatility, we can write the expected value of the expression in parentheses in (10.20) as

$$\left(\frac{\Delta y}{y_s}\right)^2 \to v_y^2 \Delta t \tag{10.21}$$

We denote the extra return due to convexity as y_x and we call it the convexity yield. Thus, the change in price due to the convexity component of (10.20), Δp_x, can be written as

$$\frac{\Delta p_x}{p} = \frac{1}{2} X\, {y_s}^2\, {v_y}^2\, \Delta t = y_x\, \Delta t \tag{10.22}$$

Substituting into (10.20) leads to

$$\begin{aligned} \frac{\Delta p}{p} &= (y_s + \tfrac{1}{2} X y_s^2 v_y^2)\Delta t - D\,\Delta y_s + \cdots \\ &= (y_s + \tfrac{1}{2} T^2 y_s^2 v_y^2)\Delta t - T\,\Delta y_s + \cdots \end{aligned} \tag{10.23}$$

The expected return of the portfolio is equal to the return due to yield ($y_s \Delta t$), duration ($-D\Delta y$), and convexity ($\tfrac{1}{2} X y_s^2 v_y^2 \Delta t$). The contribution of convexity to return is proportional to time and provides extra positive return in addition to return due to yield. For a given y_s, the more convex a portfolio the higher the expected return will be. If long rates are constant, we can buy a more convex portfolio and short a less convex portfolio and have positive return without risk. This is the basis for the previous theorem. To prevent arbitrage, the extra short term return realized through the convexity should be offset by lower yield of high convexity securities. We must assume that long term return expectations are constant, instead of constant long term yields. For a given yield, y_s, we require

$$\lim_{t\to\infty}\left(y_s + \tfrac{1}{2}Xy_s^2 v_y^2\right) = \text{const} \tag{10.24}$$

Thus, the overall return expectation is calculated by adding the convexity yield to the spot yield y_s, that is,

$$y_t = y_s + y_x = y_s + \tfrac{1}{2}Xy_s^2 v_y^2 \tag{10.25}$$

where y_t is the calculated or term structure yield of a security, equal to its expected rate of return, and y_s is the market spot yield. v_y is the annualized volatility of the zero coupon bonds and should be written as $v_y(t)$. However, for simplicity and to focus on the contribution of convexity at the long end of the yield curve, we will use a flat term structure of volatility. The effect of convexity on the short end of the yield curve, dominated by short term interest rate expectations, is generally small.

The historical annualized relative yield volatility, v_y, can be calculated from

$$v_y^2 = \frac{N_B}{N}\sum_{i=1}^{N}\left[\ln\left(\frac{y_{s,i+1}}{y_{s,i}}\right)\right]^2 \tag{10.26}$$

where N_B is the number of business days in a year and N is number of observations. The observed yield curve in the market is the spot yield y_s in (10.25), while the term structure model provides the calculated yield, y_t. The convexity of a zero coupon bond in the continuously compounded framework is equal to t^2. We can thus transform (10.25) and derive the arbitrage-free spot yield as a function of the calculated yield as follows:

$$y_t = y_s + \frac{1}{2}t^2 y_s^2 v_y^2 \tag{10.27}$$

$$y_s = \frac{\sqrt{1 + 2y_t t^2\, v_y^2} - 1}{t^2\, v_y^2} \tag{10.28}$$

Equation (10.28) or (10.27) represents the convexity adjusted TSIR model. These equations imply that, in an efficient and arbitrage-free environment, the short term return expectations of a duration adjusted long zero coupon bond is constant. Equation (10.27) represents our return and yield-volatility relationship and will be the basis for constructing the term structure of credit spreads (Chapter 12) and inflation expectations (Chapter 11).

The convexity adjusted instantaneous forward rate y_{fs} can be calculated from the expected forward rate of return y_{ft} given by (2.27) as follows:

$$y_{ft} = y_t + t\frac{\partial y_t}{\partial t} = \frac{\partial}{\partial t}(y_t t) \tag{10.29}$$

Substituting from (10.27) leads to

$$\begin{aligned} y_{ft} &= \frac{\partial}{\partial t}(y_s t) + \frac{\partial}{\partial t}\left(\frac{1}{2}y_s^2 v^2 t^3\right) \\ &= y_{fs} + \frac{1}{2}\frac{\partial}{\partial t}\left[(y_s t)^2 v^2 t\right] \\ &= y_{fs} + v^2 t(y_s t)\frac{\partial}{\partial t}(y_s t) + \frac{1}{2}y_s^2 v^2 t^2 \\ &= y_{fs} + v^2 t(y_s t)y_{fs} + \frac{1}{2}y_s^2 v^2 t^2 \end{aligned} \tag{10.30}$$

$$y_{fs} = \frac{y_{ft} - \frac{1}{2}y_s^2 v_y^2 t^2}{1 + y_s v_y^2 t^2} \tag{10.31}$$

The contribution of convexity to the yield of a long maturity zero coupon bond can be quite significant. For example, assuming a yield volatility of $v_y = 0.08$ and a yield of $y_t = 0.06$, the convexity adjustment for the yields of a 25-year and a 30-year zero coupon bond will be 58.6 and 78.4 bps, respectively. This implies a drop of $78.4 - 58.6 = 19.8$ bps in the spot yield curve between 25- and 30-year maturities due to convexity.

Equation (10.25) could also be represented in terms of absolute yield volatility; however, empirical evidence suggests that absolute yield volatility is higher in higher interest rate environments and vice versa. For a constant absolute yield volatility, the contribution of convexity to the return of very long maturity zero coupon bonds could be so large as to make the implied spot yield in (10.25) negative.

If we assume that v_y is a constant or approaches zero at an asymptotic rate that is slower in order of magnitude than $1/t$, then the long term spot (10.28) can be written as

$$y_s(t \to \infty) \to \frac{\sqrt{2y_t}}{t v_y} \tag{10.32}$$

The instantaneous forward rates (10.31) can be written as

$$y_{fs} = \frac{y_{ft} - \frac{1}{2}y_s^2 v_y^2 t^2}{1 + y_s v_y^2 t^2} = \frac{y_t + \frac{\partial y_t}{\partial t}t - \frac{1}{2}{y_s}^2 v_y^2 t^2}{1 + y_s v_y^2 t^2} = \frac{\frac{\partial y_t}{\partial t}t + y_s}{1 + y_s v_y^2 t^2} \tag{10.33}$$

after substituting from (10.27) for y_t. We can see from (2.29) that all derivatives of $\dfrac{\partial y_t}{\partial t}$ have factor $(1-\tau)$ which goes to zero exponentially at large values of t. Thus, (10.33) can be simplified as

$$y_{fs}(t \to \infty) \to \frac{1}{v^2\, t^2} \tag{10.34}$$

The above analysis suggests that the spot yields of long zero coupon bonds approach zero with the inverse of time to maturity. There is some evidence in the market to support this model. The US treasury Strips yield curve has had a downward slope for maturities longer than 25 years for most of the time since early 1990s to mid-2000s.

To accurately include convexity in the TSIR, we have to use (10.27) by substituting for y_t with (2.18) and minimize the error between y_s and spot market yields. We can also use v_y as an adjustable parameter to calculate the market implied yield volatility.

The new price function and spot duration components (4.3), adjusted for convexity, will take the following general form:

$$p = \sum_i c_i e^{-y_s t_i} \tag{10.35}$$

$$\begin{aligned} D_k &= -\frac{1}{p}\frac{\partial p}{\partial a_a} = \frac{1}{p}\sum_i c_i \frac{\partial y_s}{\partial a_k} e^{-y_s t_i} \\ &= \frac{1}{p}\sum_i c_i \frac{\partial y_s}{\partial y_t}\frac{\partial y_t}{\partial a_k} e^{-y_s t_i} \\ &= \frac{1}{p}\sum_i c_i \frac{\partial y_s}{\partial y_i} \psi_k e^{-y_s t_i} \end{aligned} \tag{10.36}$$

From (10.27),

$$\frac{\partial y_s}{\partial y_t} = \frac{1}{1 + y_s v_y^2 t^2} \tag{10.37}$$

$$D_k = -\frac{1}{p}\frac{\partial p}{\partial a_k} = \frac{1}{p}\sum_i \frac{c_i t_i \psi_k}{1 + y_s v_y^2\, t_i^2} e^{-y_s t_i} = \left\langle \frac{t\psi_k(t)}{1 + y_s v_y^2\, t^2} \right\rangle \tag{10.38}$$

where y_s is calculated from (10.28) and y_t is calculated from (2.18), that is,

$$y_t = \sum_j a_j \psi_j \tag{10.39}$$

There are two ways to define the price sensitivity with respect to volatility. The traditional way is to calculate the derivative with respect to v_y. This is widely used in options calculations and is called vega (ϖ):

$$\varpi = \frac{1}{p}\frac{\partial p}{\partial v_y} = \frac{1}{p}\sum_i \frac{c_i t_i y_s^2 v_y t_i{}^2}{1 + y_s v_y^2 t_i{}^2} e^{-y_s t_i} = \left\langle \frac{y_s^2 v_y\, t^3}{1 + y_s v_y^2\, t^2} \right\rangle \tag{10.40}$$

The second method, which we will call the duration of volatility (D_v), is obtained by calculating the derivative of price with respect to the square of v_y:

$$D_v = -\frac{1}{p}\frac{\partial p}{\partial (v_y^2)} = -\frac{1}{2v_y}\frac{1}{p}\frac{\partial p}{\partial v_y} = -\frac{1}{2}\left\langle \frac{y_s^2\, t^3}{1 + y_s v_y^2\, t^2} \right\rangle \tag{10.41}$$

The duration of volatility has the same units as ordinary duration.

The vega of a 30-year zero coupon bond with a volatility of 9% and yield of 5% is about 4.5. This implies a price impact of 45 bps for a change in volatility of 0.001. While the yield volatility changes significantly in the options markets, its impact on the price of zero coupon bonds has been relatively smaller than is implied by our calculation, on a historical basis.

10.3 APPLICATION TO CONVEXITY

Figure 10.1 shows the calculated convexity adjusted TSIR as well as market yields of coupon Strips. The implied yield volatility for Figure 10.1 is 0.0975 for May 28, 1999, implying a yield volatility of about 60 bps for a spot yield of 6.2%. This appears to be a fair value for volatility; however, most of the time the implied volatility has been much lower than the observed or option implied volatility. As we mentioned in Chapter 7, as of 2012, the long end of the curve had a yield premium instead of a discount. One interpretation that can be given to the positive slope at the long end of the curve is a risk premium as investors question the sustainability of the US debt dynamic. It can also be due to market inefficiency.

The downward slope of the yield curve for maturities longer than 20 years is directly related to the volatility effect. Figure 10.2 shows the same curve without convexity adjustment. The average yield error of the convexity adjusted TSIR is 0.016%, and of

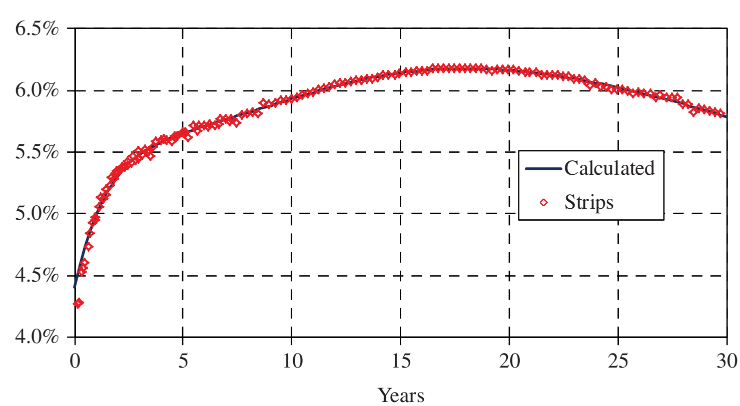


FIGURE 10.1 Convexity adjusted yield curve, May 28, 1999

the non-convexity adjusted curve 0.055%. Each curve requires only five parameters for fitting. Figure 10.1 requires four components of the TSIR plus one volatility, while Figure 10.2 requires five components of the TSIR.

The implied yield volatility of 0.0596 on May 29, 1998 compared to 0.0975 on May 28, 1999, meant a lower downward slope on the former date. Thus, even though the yields of 20-year zero coupon bonds were 5.90% and 6.17% on the former and latter dates respectively, the implied zero coupon yield for a maturity of 30 years for both cases was 5.8%. Figure 10.3 shows the implied 100-year treasury curves for May 1999 as well as for May 1998.

We will call the lower yield of coupon Strips at the long end of the treasury market the convexity premium. Implied volatility as well as the historical data provided in Table 10.2 suggest that the contribution of convexity to performance was about 2% per year, and thus the historical convexity premium was very cheap, disappearing completely as of 2012.

Table 10.1 shows the components of the TSIR with and without volatility. The standard deviation of the difference between the calculated and the market yield is

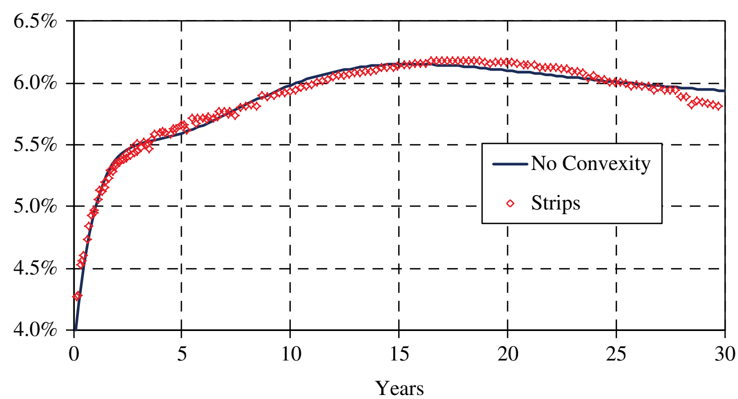


FIGURE 10.2 Yield curve without convexity adjustment, May 28, 1999

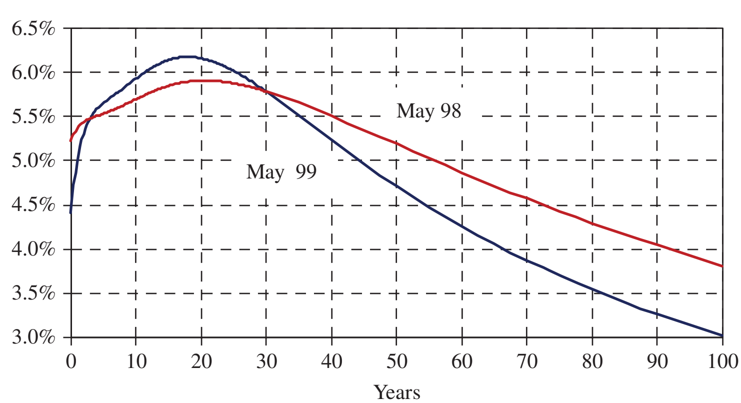


FIGURE 10.3 Convexity adjusted long zero curves

much lower for the volatility (convexity) adjusted yield curve. The quartic component of the TSIR was calculated to be zero in both cases, implying that using four components of the TSIR plus implied volatility provided a much better representation of the yield curve than using five components of the TSIR.

Table 10.2 shows the annualized daily market returns as well as the contribution from convexity and security selection components for arguably the most convex security in fixed income, namely a 30-year treasury coupon Strips.

The first row in the table represents the aggregate contributions of level, slope, bend, cubic, and quartic duration components on a daily basis as well as the annualized standard deviation of those contributions.

The security contribution is derived from the level duration times the change in the spread of the security relative to the curve. The curve was not adjusted for convexity and only bonds with a maturity less than 22 years were used for calculating the curve. The security selection contributed –0.34% on an annual basis. It is instructive to look at the very large volatility of the security selection attribution relative to its return. This is due to the very large duration of the security and the change in the shape of the yield curve at the long end as well as the relative position of the security compared to the yield curve.

The yield contribution is calculated from yield plus rolldown on the curve. Since the changes in the durations and spread correspond to the two yield curves at two

TABLE 10.1 Components of the TSIR

	Level	Slope	Bend	Cubic	Quartic	Vol.	Error
May 98, Vex	5.681%	0.502%	0.124%	0.083%	0.000%	5.962%	0.016%
May 98, No Vex	5.530%	0.401%	–0.077%	0.069%	–0.087%		0.027%
May 99, Vex	5.798%	1.255%	0.083%	0.226%	0.000%	9.754%	0.016%
May 99, No Vex	5.422%	0.878%	–0.396%	0.138%	–0.204%		0.055%

TABLE 10.2 Return attribution of coupon Strips 2/15/2027, 1997–2012

	Annualized Return	Annualized Stdev
Curve	5.79%	19.73%
Security	–0.34%	9.93%
Yield	5.55%	0.73%
Convexity-4	2.04%	0.30%
Convexity-15	1.94%	0.27%
Total-4	13.04%	19.81%
Total-15	12.94%	19.81%
Market	12.95%	19.82%
Market - Total-4	–0.09%	0.12%
Market - Total-15	0.01%	0.05%

different dates, the rolldown is already incorporated into the components of the curve, and thus the yield has to be instantaneous forward yield (2.27).

Convexity-4 consists of the four largest components of the convexity, namely X_{00}, X_{01}, X_{11}, and X_{02}. Convexity-15 is the sum of all 15 components of the convexity. A 5×5 convexity matrix is symmetric and has five diagonal and ten off-diagonal elements, leading to 15 separate components.

Total-4 (Total-15) is the sum of performance attributions from curve, yield, security selection, and four (all) components of convexity.

The Market – Total-4 and Market – Total-15 rows show the difference between performance attribution and market performance on an annual basis as well as the associated annualized standard deviations.

Note that convexity and security selection performance are relatively large due to the long duration of the security. The table shows that even if all components of convexity are used, there is about 5 bps of tracking error between market and calculated performance. Convexity captures second order effects of interest rate movements; this tracking error is due to higher order components.

The average convexity for the study period of this security was about 500, which is about 20 times the convexity of a typical portfolio of bonds that has 5 years of duration. Since the tracking error of our performance attribution relative to the market performance is related to convexity and higher order terms, we expect that the difference between our performance attribution and the market on an annual basis to be about 20 times less than the tracking error for the zero coupon treasury. Thus, for a typical portfolio, we can expect the annualized error using four components of convexity to be less than 1 basis point.

10.4 CONVEXITY BIAS OF EURODOLLAR FUTURES

The effect of convexity on yield and return that was discussed for long zero coupon bonds is much more evident for eurodollar futures contracts (EDFC). EDFCs are very liquid and have virtually zero convexity. The yield premium that is demanded by EDFC investors compared to a hedging portfolio is relatively large and is well known in the marketplace.

An EDFC contract is based on a forward 90-day time deposit with a notional value of \$1 million. If c is the (biased) forward time deposit coupon rate, we can write the future value of a time deposit as

$$1{,}000{,}000\left(1+\frac{cn_e}{100N_y}\right)=1{,}000{,}000\left(1+\frac{y_e n_e}{N_y}\right) \tag{10.42}$$

where n_e is the number of days to the maturity of the time deposit (90 for the US) for the underlying EDFC, N_y is the number of days in a year for calculation purpose (360 for the US) and y_e is the forward yield of the time deposit for a stated contract. An EDFC has zero market value at initiation. The only source of return for the contract are changes in interest rates.

In order to analyze EDFCs, we will first build a replicating portfolio that has the same interest rate sensitivity as the EDFC contract. The price of an EDFC is stated as

$$p=100-c=100(1-y_e) \tag{10.43}$$

Since the price of a portfolio of one EDFC can be written as

$$Q_m = A + By_e \tag{10.44}$$

B is selected in such a way that the change in the price of the contract for a change of 1 basis point in the rate is equal to \$25. Thus,

$$\frac{\partial Q_m}{\partial y_e} = B = -1{,}000{,}000\,\frac{n_e}{N_y} = -250{,}000 \tag{10.45}$$

The value of an EDFC at initiation is zero, that is, there is no cash requirement to buy or sell an EDFC. Only margin money is required to be posted to the exchange. For an initial yield of y_0, an EDFC can be written as

$$Q_m = 1{,}000{,}000\,\frac{n_e}{N_y}(y_0 - y_e) \tag{10.46}$$

Since the swap curve is based on forward Libor rates, we can construct a replicating portfolio for an EDFC as

$$Q_r = -C_1 e^{-y_1 t_1} + C_2 e^{-y_2 t_2} \tag{10.47}$$

where t_1 and t_2 are the beginning and ending times of the forward time deposit. At initiation, the values of the coefficients in equation (10.47) are selected in such a way that the value of the portfolio is zero and its risk is the same as the respective EDFC:

$$C_1 e^{-y_1 t_1} = C_2 e^{-y_2 t_2} \tag{10.48}$$

To convert from (10.42) to our continuously compounded notation, we can write

$$y_f t_f = y_2 t_2 - y_1 t_1 = y t_f = \ln\left(1 + \frac{y_e n_e}{N_y}\right) \tag{10.49}$$

The sensitivity of our replicating portfolio to changes in rates is

$$\frac{\partial Q_r}{\partial y_e} = \frac{\partial Q_r}{\partial y}\frac{dy}{dy_e} \tag{10.50}$$

with

$$\frac{dy}{dy_e} = \frac{1}{t_f}\frac{n_e}{N_y + y_e n_e} \tag{10.51}$$

After simplification and making the necessary substitutions using (10.48), we can calculate the coefficients of (10.47) as

$$\begin{aligned} C_1 &= 250{,}000\,\frac{N_y + y_e n_e}{n_e}\,e^{y_1 t_1} \\ C_2 &= 250{,}000\,\frac{N_y + y_e n_e}{n_e}\,e^{y_2 t_2} \end{aligned} \tag{10.52}$$

The term structure risks of EDFC will therefore be

$$\frac{\partial Q_r}{\partial a_k} = -250{,}000\frac{N_y + y_e n_e}{n_e}\left[t_2\psi_k(t_2) - t_1\psi_k(t_1)\right] \tag{10.53}$$

Since an EDFC is based on a 90-day time deposit, the level duration will be closer to $\frac{90}{365.25} = 0.246$. Our simplified version of the risk in (4.9) which stated that the level duration was 0.25 years did not include the effect of market convention accruals.

The expected change in the price of an EDFC is from the change in interest rates only. The price sensitivity of an EDFC from (10.46) based on market convention yield is

$$\frac{\partial Q_m}{\partial y_e} = -1{,}000{,}000\frac{n_c}{N_y} = -250{,}000 \tag{10.54}$$

To calculate the sensitivity of the replicating portfolio to changes in interest rates we perform a Taylor series expansion of the replicating portfolio (10.47) as

$$Q_r = Q_{r0} + \frac{\partial Q_r}{\partial y}\Delta y + \tfrac{1}{2}\frac{\partial^2 Q_r}{\partial y^2}(\Delta y \tag{10.55}$$

or

$$Q_r = 250{,}000\frac{N_y + y_e n_e}{n_e}\left[-t_f\Delta y + \tfrac{1}{2}(t_2^2 - t_1^2)(\Delta y)^2 \cdots\right] \tag{10.56}$$

While the first order interest rate sensitivity of the replicating portfolio matches that of the EDFC, the replicating portfolio has a convexity component which is always positive. We can write the contribution of convexity to the return of the portfolio as

$$Q_x = V\,\tfrac{1}{2}(t_2^2 - t_1^2)(\Delta y)^2 \tag{10.57}$$

where

$$V = 250{,}000\frac{N_y + y_e n_e}{n_e} \tag{10.58}$$

Due to the convexity, the replicating portfolio is superior to the EDFC. As we explained earlier, the contribution of convexity to return is like additional yield. To avoid arbitrage, the yield of an EDFC must be adjusted to balance the convexity advantage of the replicating portfolio. For a change in time of Δt, the expected change in $(\Delta y)^2$ can be written as

$$\left\langle (\Delta y)^2 \right\rangle = (y_f v_f)^2\,\Delta t \tag{10.59}$$

where v_f is the forward rate volatility. The contribution to convexity can be written as

$$Q_{\text{vex}} = V\,\tfrac{1}{2}\,t_f(t_1 + t_2)(y_f v_f)^2\,\Delta t \tag{10.60}$$

The total earnings of the replicating portfolio due to convexity through the initiation time of the forward time deposit will be

$$E_x = 250{,}000\,\frac{N_y + y_e n_e}{n_e}\,t_f \int_0^{t_1}\left(t + \frac{t_f}{2}\right)(y_f v_f)^2\,dt \tag{10.61}$$

To prevent arbitrage, the expected return of EDFC must be equal to the expected earning of the replicating portfolio. Therefore, the EDFC requires an additional yield to be fairly compensated due to the lack of convexity.

Substituting

$$I = \int_0^{t_1}\left(t + \frac{t_f}{2}\right)(y_f v_f)^2\,dt \tag{10.62}$$

EDFCs require an additional expected yield equal to

$$y_x = \frac{N_y(N_y + y_e n_e)}{4n_e^2}\,t_f I \tag{10.63}$$

The expected price function of the EDFC (10.43) can be written as

$$p = 100(1 - y_e - y_x) \tag{10.64}$$

where y_e is the unbiased expected forward yield of the EDFC calculated from

$$1 + \frac{y_e n_e}{N_y} = e^{y_2 t_2 - y_1 t_1} \tag{10.65}$$

For constant absolute yield volatility, (10.62) can be simplified as

$$I = \tfrac{1}{2}\,t_1(t_1 + t_f)(y_f v_f)^2 \tag{10.66}$$

Therefore, the convexity bias in yield of the Eurodollar contracts must be equal to

$$y_x = \frac{N_y(N_y + y_u n_c)}{8n_c^2}\,t_1(t_1 + t_f)(y_f v_f)^2 \tag{10.67}$$

The implied constant yield volatility can be calculated from the market price of the EDFC and the forward implied yield, if forward volatilities are not available:

$$v_f = \frac{2n_c}{y_f}\sqrt{\frac{2(y_m - y_e)}{N_y(N_y + y_u n_c)t_1(t_1 + t_f)}} \tag{10.68}$$

For the US, the following conventions apply:

$$N_y = 360$$
$$n_c = 90$$
$$y_u = \frac{r_u}{100}$$
$$r_u = 100 - p$$

where p is the price of the EDFC. Table 10.3 shows market and calculated parameters for Eurodollar futures.

The table starts with the fifth active quarterly contract. The fair price is the calculated price based on swap curve as well as the term structure of volatility. TED is the calculated treasury–eurodollar spread. The unbiased yield is the calculated forward market yield without convexity adjustment. The implied convexity yield is the difference between market yield and unbiased yield, and the fair convexity yield is the calculated convexity yield of the EDFC based on (10.62). From the table we can see that the contribution of convexity to the price of EDFC is less than 1 basis point for the first 15 contracts (EDH15).

At the long end of the EDFC, the implied convexity contribution is significantly less than the calculated convexity and is a representation of the market inefficiency. Not only is the TED spread negative, implying that swaps are rich relative to treasuries, but also EDFCs are rich to the swap curve. With many hedge funds and proprietary desks out of business there are few mechanisms through which the inefficiencies can be arbitraged.

Many portfolio managers, endowments, insurance companies, and pension funds that need to fund long term liabilities find EDFCs very suitable for hedging. Since EDFCs are not deliverable and are cash settled at the expiration of the contracts, most investors are not required to collateralize them and can acquire them with small margin requirements. Bond futures, on the other hand, are required to be collateralized with cash.

Figure 10.4 shows the swap and treasury curves that were used for the calculation. The calculated yield of nearly all swap contracts is within a fraction of a basis point

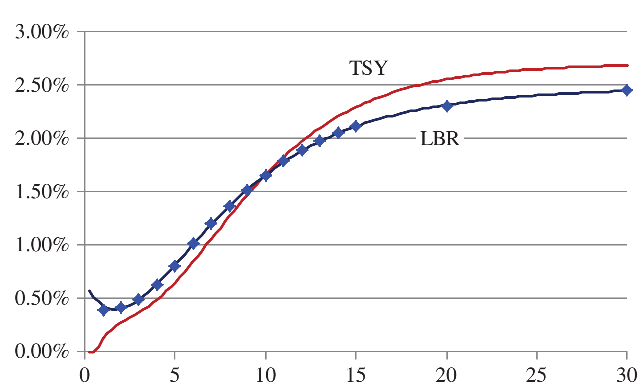


FIGURE 10.4 Treasury and swap curves for calculations of EDFC, July 30, 2012

TABLE 10.3 Eurodollar futures contracts, July 30, 2012

				Yield			
Contract	Market Price	Fair Price	TED	Market	Unbiased	Implied Vex	Fair Vex
EDU13	99.585	99.631	0.078%	0.415%	0.369%	0.046%	0.001%
EDZ13	99.555	99.576	0.118%	0.445%	0.423%	0.022%	0.001%
EDH14	99.530	99.499	0.118%	0.470%	0.499%	−0.029%	0.002%
EDM14	99.495	99.484	0.119%	0.505%	0.513%	−0.008%	0.003%
EDU14	99.445	99.476	0.106%	0.555%	0.519%	0.036%	0.004%
EDZ14	99.375	99.393	0.103%	0.625%	0.600%	0.025%	0.007%
EDH15	99.315	99.299	0.008%	0.685%	0.692%	−0.007%	0.010%
EDM15	99.220	99.186	0.040%	0.780%	0.801%	−0.021%	0.014%
EDU15	99.105	99.064	0.078%	0.895%	0.917%	−0.022%	0.019%
EDZ15	98.970	98.950	0.080%	1.030%	1.025%	0.005%	0.025%
EDH16	98.845	98.833	0.073%	1.155%	1.135%	0.020%	0.032%
EDM16	98.710	98.685	0.109%	1.290%	1.274%	0.016%	0.041%
EDU16	98.570	98.522	0.172%	1.430%	1.426%	0.004%	0.052%
EDZ16	98.415	98.397	0.154%	1.585%	1.539%	0.046%	0.064%
EDH17	98.280	98.274	0.130%	1.720%	1.651%	0.069%	0.075%
EDM17	98.145	98.136	0.202%	1.855%	1.777%	0.078%	0.087%
EDU17	98.025	98.003	0.329%	1.975%	1.897%	0.078%	0.101%
EDZ17	97.895	97.887	0.314%	2.105%	1.998%	0.107%	0.114%
EDH18	97.800	97.775	0.295%	2.200%	2.099%	0.101%	0.126%
EDM18	97.705	97.685	0.248%	2.295%	2.172%	0.123%	0.143%
EDU18	97.620	97.602	0.196%	2.380%	2.240%	0.140%	0.158%
EDZ18	97.530	97.505	0.168%	2.470%	2.321%	0.149%	0.174%
EDH19	97.475	97.416	0.140%	2.525%	2.398%	0.127%	0.187%
EDM19	97.415	97.370	0.056%	2.585%	2.428%	0.157%	0.201%
EDU19	97.360	97.324	−0.037%	2.640%	2.454%	0.186%	0.221%
EDZ19	97.300	97.252	−0.072%	2.700%	2.511%	0.189%	0.237%
EDH2O	97.270	97.179	−0.105%	2.730%	2.563%	0.167%	0.258%
EDM20	97.235	97.106	−0.263%	2.765%	2.618%	0.147%	0.277%
ED U20	97.195	97.034	−0.561%	2.805%	2.667%	0.138%	0.299%
EDZ20	97.150	96.977	−0.611%	2.850%	2.706%	0.144%	0.317%
EDH21	97.125	96.931	−0.660%	2.875%	2.743%	0.132%	0.326%
EDM21	97.100	96.861	−0.562%	2.900%	2.791%	0.109%	0.348%
EDU21	97.070	96.791	−0.308%	2.930%	2.837%	0.093%	0.372%
EDZ21	97.030	96.738	−0.326%	2.970%	2.864%	0.106%	0.398%
EDH22	97.015	96.701	−0.341%	2.985%	2.889%	0.096%	0.410%
EDM22	96.985	96.607	−0.373%	3.015%	2.959%	0.056%	0.434%

TABLE 10.4 Euribor futures contracts, July 30, 2012

				Yield			
Contract	Market Price	Fair Price	TED	Market	Unbiased	Implied Vex	Fair Vex
ERU13	99.695	99.436	0.730%	0.305%	0.563%	−0.258%	0.001%
ERZ13	99.635	99.425	0.631%	0.365%	0.573%	−0.208%	0.002%
ERH14	99.575	99.401	0.598%	0.425%	0.596%	−0.171%	0.003%
ERM14	99.500	99.311	0.628%	0.500%	0.685%	−0.185%	0.004%
ERU14	99.425	99.195	0.683%	0.575%	0.799%	−0.224%	0.006%
ERZ14	99.335	99.110	0.688%	0.665%	0.881%	−0.216%	0.009%
ERH15	99.245	99.014	0.668%	0.755%	0.974%	−0.219%	0.013%
ERM15	99.125	98.900	0.632%	0.875%	1.083%	−0.208%	0.017%
ERU15	98.995	98.777	0.617%	1.005%	1.201%	−0.196%	0.022%
ERZ15	98.840	98.656	0.596%	1.160%	1.316%	−0.156%	0.028%
ERH16	98.700	98.530	0.570%	1.300%	1.435%	−0.135%	0.035%
ERM16	98.545	98.366	0.541%	1.455%	1.591%	−0.136%	0.043%
ERU16	98.405	98.182	0.534%	1.595%	1.765%	−0.170%	0.053%
ERZ16	98.255	98.043	0.499%	1.745%	1.894%	−0.149%	0.063%
ERH17	98.135	97.915	0.462%	1.865%	2.014%	−0.149%	0.071%
ERM17	98.020	97.838	0.574%	1.980%	2.079%	−0.099%	0.083%

of the market yields, therefore the inefficiencies mentioned are real and not an artifact of the calculations.

In Table 10.4 we see that Euribor futures contracts are also trading very rich to the calculated curve and all have negative implied convexity yield. The richness of EDFCs is evident nearly in all actively traded currencies and is a sign of market inefficiencies in this sector of the market.

CHAPTER 11

Real Rates and Inflation Expectations

With the issuance of the first inflation linked (IL) bond in the US in January 1997, a new era began for trading and hedging real yields and inflation expectations in what are commonly known as treasury inflation protected securities (Tips). Since then there has been an explosion of inflation linked securities and inflation swaps that trade globally. The UK, Canada, Australia and a few other countries had issued IL bonds before the US treasury.

For a real bond, the stated real coupon is multiplied by the accumulated inflation factor from the issuance date of the bond to be converted to nominal coupon and paid out. For example, if the annualized coupon of an IL bond is 3% and the cumulative inflation factor at the time of the coupon payment is 1.3, then the amount of coupon payment will be $3 \times 1.3 = 3.9$ divided by the coupon frequency. For the US, the inflation factor used is the non-seasonally adjusted headline CPI. Most inflation reports are frequently adjusted as more accurate information is collected and seasonal patterns shift due to weather or globalization. However, to avoid confusion, the US treasury has agreed not to revise the reported headline CPI that is used for Tips indexation. Inflation adjustment would create chaos in the markets as investors who have traded their bonds could demand higher payments if prior inflation reports are adjusted higher.

11.1 TERM STRUCTURE OF REAL RATES

One can calculate the term structure of real rates (TSRR) and the term structure of inflation expectations (TSIE) by comparing the yields of IL and nominal bonds. Treasury issued inflation bonds have the same quality as regular treasuries and are considered to have risk-free interest and principal payments.

We define the TSRR y_r and the TSIE y_n in the same manner as we defined the TSIR (2.18):

$$y_r = \sum_{j=0}^{n-1} a_{r,j}\psi_j \tag{11.1}$$

The Advanced Fixed Income and Derivatives Management Guide. Saied Simozar

$$y_n = \sum_{j=0}^{n-1} a_{n,j}\psi_j \tag{11.2}$$

where $a_{r,j}$ and $a_{n,j}$ are the coefficients of the jth component of the spot real rate and inflation expectation, respectively. Before analyzing the TSRRs, we explore the stability of real rates and inflation expectations.

11.2 THEOREM: REAL RATES CANNOT HAVE LOG-NORMAL DISTRIBUTION

The relationship between spot rates and instantaneous forward rates is given by (2.27). Analogously to (10.27), for a given relative real yield volatility of $v_{y,r}$, the calculated TSRR of real rate of return $y_{t,r}$ and spot real yield $y_{s,r}$ are related to each other by

$$y_{t,r} = y_{s,r} + \tfrac{1}{2} y_{s,r}^2 v_{y,r}^2 t^2 \tag{11.3}$$

Equations (10.27) and (11.3) imply log-normal distribution of volatilities for the movements of interest rates and real rates, respectively. A log-normal distribution can never be zero. The US Treasury has guaranteed that for US real bonds, the real principal of bonds are guaranteed at par if there is deflation between the issue date of a bond and its maturity, thereby providing a floor of zero real yields for long term Tips. We will now analyze the implications of log-normal distributions for real bonds.

Long term nominal yields cannot become negative, since investors have the option of storing their cash at zero rates with almost zero carrying cost. Therefore, the zero rate acts as a barrier that an investor would have no incentive to cross. If nominal rates become negative, investors will store their cash in safe deposit boxes and avoid negative interest rates, thus negative nominal rates are not sustainable. We can make the same argument about credit spreads as well. As the level of credit spreads falls to zero, the incentive to buy credit sensitive securities rather than risk-free securities would fall to zero.

Our argument for log-normal distribution of real rates is not as strong as for nominal rates since there is no real or inflation protected currency which guaranties investors of zero real returns in a deflationary environment. However, we believe that the introduction of IL bonds where the principal is protected against deflation and other natural inflation hedges, are likely to provide a floor for negative long term real rates in deflationary and inflationary environments.

It appears that we cannot use a log-normal distribution for the TSIE, since there are no natural barriers at zero inflation rate. In the recent history of Western economies, there have been periods of positive as well as negative inflation rates. While the absolute volatility of inflation tends to be higher during high inflation periods, log-normal distribution alone cannot account for that. One can attempt a combination of normal and log-normal distribution functions for inflation expectations. We will first analyze the mathematical implications of the normal distribution function for inflation.

The historical annualized inflation volatility can be calculated from

$$w_n^2 = \frac{N_B}{N} \sum_{i=1}^{N} \left(y_{s,n,i+1} - y_{s,n,i}\right)^2 \tag{11.4}$$

where N_B is the number of business days in a year and N is the number of data points. For the TSIE $y_{t,n}$, spot inflation expectations $y_{s,n}$, and nominal government yield $y_{t,g}$ we have the following identities:

$$y_{t,g} = y_{t,r} + y_{t,n} \tag{11.5}$$

$$y_{s,g} = y_{s,r} + y_{s,n} \tag{11.6}$$

Identity (11.5) states that the return expectation of a long bond is equal to the sum of return expectations of a real bond and inflation for the same maturity. Identity (11.6) states that the spot nominal rate is equal to the spot real rate plus spot inflation expectation. Violation of either of these identities will result in arbitrage opportunities.

Expanding the price function of a nominal zero coupon bond in Taylor series in terms of changes in real and inflation yields, similar to (10.18), leads to

$$p = e^{-y_{s,g}t} = e^{-y_{s,r}t - y_{s,n}t} \tag{11.7}$$

After time Δt the expected price is

$$\begin{aligned} p + \Delta p &= e^{-(y_{s,g}+\Delta y_{s,g})(t-\Delta t)} \\ &= e^{-(y_{s,r}+\Delta y_{s,r})(t-\Delta t)-(y_{s,n}+\Delta y_{s,n})(t-\Delta t)} \end{aligned} \tag{11.8}$$

Expanding this equation in Taylor series, similarly to (10.18), and noting that Δt is negative results in

$$\begin{aligned} \Delta p &= -\frac{\partial p}{\partial t}\Delta t + \frac{\partial p}{\partial y_{s,g}}\Delta y_{s,g} + \frac{1}{2}\frac{\partial^2 p}{\partial y_{s,g}^2}(\Delta y_{s,g})^2 \\ &= -\frac{\partial p}{\partial t}\Delta t + \frac{\partial p}{\partial y_{s,r}}\Delta y_{s,r} + \frac{\partial p}{\partial y_{s,n}}\Delta y_{s,n} + \\ &\quad + \frac{1}{2}\frac{\partial^2 p}{\partial y_{s,r}^2}(\Delta y_{s,r})^2 + \frac{1}{2}\frac{\partial^2 p}{\partial y_{s,n}^2}(\Delta y_{s,n})^2 + \frac{\partial^2 p}{\partial y_{s,r}\partial y_{s,n}}\Delta y_{s,r}\Delta y_{s,n} + \cdots \end{aligned} \tag{11.9}$$

After converting the changes to volatilities as in (10.20) and some simplification we arrive at

$$\begin{aligned} y_{t,g} &= y_{s,g} + \tfrac{1}{2}v_{y,g}^2 y_{s,g}^2 t^2 \\ &= y_{s,r} + \tfrac{1}{2}v_{y,r}^2 y_{s,r}^2 t^2 + y_{s,n} + \tfrac{1}{2}w_{y,n}^2 t^2 + \rho_{rn} y_{s,r} v_r w_{y,n} t^2 \end{aligned} \tag{11.10}$$

where ρ_{rn} is the correlation coefficient between $y_{s,r}$ and $y_{s,n}$, and $w_{y,n}$ is the absolute volatility of inflation expectations. Substituting (11.6) in the above equation and rearranging the parameters slightly leads to

$$v_y^2\, y_{s,g}^2 - v_r^2\, y_{s,r}^2 = w_{y,n}^2 + 2\rho_{rn} y_{s,r} v_r w_{y,n} \tag{11.11}$$

As $t \to \infty$, the left hand side of the above equation approaches zero since both y_s and $y_{s,r}$ approach zero. Therefore, after some simplification, we have

$$w_{y,n}(t \to \infty) = -2\rho_{rn} y_{s,r} v_r = 0 \tag{11.12}$$

Thus, assuming a normal distribution for inflation expectations and a log-normal distribution for real rates has led to (11.12), implying that long term volatility of inflation should approach zero. Since this is not a logical conclusion, our original assumption is incorrect. Normal and log-normal distributions cannot coexist. To restore arbitrage neutrality and mathematical stability, we have to assume that long term inflation expectations are also log-normally distributed.

Log-normal distribution for inflation implies that long term inflation expectations cannot cross the zero rate without creating an arbitrage opportunity. Using log-normal distribution for inflation, the return expectation of inflation and spot inflation can be written as

$$y_{t,n} = y_{s,n} + \tfrac{1}{2} v_{y,n}^2 y_{s,n}^2 t^2 \tag{11.13}$$

Equation (11.10) changes as follows:

$$\begin{aligned} y_{t,g} &= y_{s,g} + \tfrac{1}{2} v_{y,g}^2\, y_{s,g}^2\, t^2 \\ &= y_{s,r} + \tfrac{1}{2} v_{y,r}^2\, y_{s,r}^2 t^2 + y_{s,n} + \tfrac{1}{2} v_{y,n}^2 y_{s,n}^2 t^2 + \rho_{rn} y_{s,r} y_{s,n} v_{y,g} v_{y,r} t^2 \end{aligned} \tag{11.14}$$

We can rewrite (11.14) for arbitrage-free spot yields, using (10.27), (11.3) and (11.13), as

$$y_t = y_{t,r} + y_{t,n} + \rho_{rn} y_{s,r} y_{s,n} v_r v_n\, t^2 \tag{11.15}$$

Thus, depending on the sign of the correlation coefficient between real yields and inflation rates, the return expectation of a nominal bond can be greater than, equal to or less than the sum of return expectations of its components.

If we assume that inflation expectations are log-normally distributed, stability is restored to (11.14). However, given that there are no barriers to the inflation rate crossing zero and becoming negative, we have to assume that inflation does not have a log-normal distribution. Our original assumption of a log-normal distribution for real yields has to be invalid. Assuming a normal or log-normal distribution for real rates and/or inflation expectations leads to mathematical or practical inconsistencies and is thus unacceptable. We can only conclude that inflation expectations and real rates are correlated in such a way that their sum (i.e., nominal rates) will be log-normally distributed but neither is normally or log-normally distributed.

Most options formulas for bonds use log-normal distributions to calculate the forward distribution of rates and implied prices. Based on this analysis, one has to question the validity of pricing models for options on IL bonds.

There are other econometric barriers for very high real rates. Real rates serve as a proxy for the productivity of an economy. If real rates are very high, companies that borrow at nominal rates need to have productivity gains that match or exceed the real rates for their profitability to stay the same or increase. In the history of the Western and emerging economies there are numerous instances of inflation and nominal rates running out of control, in some cases exceeding 1000% per year. However, there are no documented cases of real rates adjusted for inflation staying at high levels for a long period of time. In fact, to combat inflation, central banks often raise short rates above inflation to slow down economic activity and inflation.

At very high rates of inflation, one has to adjust real rates by compounding inflation. For example, if the inflation expectation for the next 12 months is 1000% and nominal rates are 1020%, a nominal investment of \$100 will have a nominal return of \$1020. At today's prices, the value of \$1020 will be \$102 and thus inflation adjusted real rate will be , $\frac{102-100}{100} = 2\%$ which is significantly different from the $1020\% - 1000\% = 20\%$ which is obtained by subtracting inflation from nominal rates.

11.3 INFLATION LINKED BONDS

Accurate analysis of IL bonds is among the most complicated of all non-contingent securities. Table 11.1 lists the timeline of dates to analyze for IL bonds.

Since inflation is not known instantaneously, all inflation linked products are priced based on an inflation lag. For most markets the inflation lag is about 2 months; however, for some older securities, particularly UK inflation linked gilts, the inflation lag is about 8 months. Australia and New Zealand report inflation on a quarterly basis and inflation accrual is based on the average of the prior two quarters.

For every cash flow of an IL bond, there is an inflation reference point. The inflation reference point is the method used to calculate inflation compensation for the bond holder.

For the US market, the inflation reference is calculated by the linear interpolation of the realized non-seasonally adjusted inflation of the reference month (2 months prior to the date in question). For example, to calculate an inflation reference for April 10, suppose that the inflation factor was 120 at the end of January and 120.36 at the end of February. Inflation for the month of February, which is 2 months before April, is $120.36/120.00 - 1 = 0.3\%$. April has 30 days and it is assumed that inflation accrues linearly, therefore for April 10, the inflation reference point will be

$$f = 120.0 + \frac{120.36 - 120.00}{30} 10 = 120.12$$

The last known nominal date for a bond is the date before which all cash flows are nominal with known factors. For example, if the inflation report for February is released on March 20, then on March 21 we know the inflation factor for all cash flows through the end of April. If an IL bond has a coupon payment on April 15, we can calculate the inflation reference point for April 15 and divide it by the inflation reference point for the dated date of the bond. This ratio is the inflation compensation for the coupon payment of the bond on April 15.

TABLE 11.1 Timeline for cash flow analysis of inflation linked bonds

A	Inflation reference point for the start of interest accrual date (dated date)
B	Interest accrual date (dated date)
C	Inflation reference point for settle date
D	Settle date
E	Last known nominal date; cash flows before this date are nominal with known factors
F	Last nominal date for which inflation has been realized but not reported. Cash flows up to this date are nominal with unknown factors
G	Partially nominal date – cash flows before this date are partially nominal and partially real
H	Inflation lag; date at which a coupon goes nominal
I	Coupon payment
J	…
L	Inflation lag; reference point for maturity date
M	Maturity date

There is also an accumulated but not accrued inflation factor from the reference inflation date to the trade date. For example, for a bond traded on April 9 and settled on April 10, the last known inflation is end of February report, but inflation is only accrued for one third of the month of April. The accumulated inflation through the end of February is already known, and on trade date the inflation for March and the first nine days of April have already taken place and accumulated but not reported or factored into the invoice price.

On March 10, before the inflation report for February, the cash flows of a bond that pays coupon in April will be nominal, but the factor is not known. If there is a spike in inflation in March, the April cash flows will not benefit from it. One can estimate the expected inflation using market consensus or from historical inflation for February to estimate the impact of inflation on April cash flows.

Likewise, the reference inflation for the month of May is March. The inflation factor for a bond that pays a coupon at the end of May depends on the level of inflation in March. On March 15, about half of the inflation for the month has been realized but not reported. Thus, a cash flow at the end of May is $\frac{15}{31} = 48.4\%$ nominal and 51.6% real. The adjustment factor for such a bond will be based on 48.4% of the expected inflation in the month of March. Similarly, a cash flow on May 12 will be scaled by $\frac{12}{31} \times \frac{15}{31} = 18.7\%$ of the expected March inflation rate. For the accurate analysis of partially nominal cash flows, we must break a cash flow into two components: one discounted by the nominal curve and one by the real curve.

The adjustment for Australian real bonds is significantly more complicated. For example, on April 10, before the first quarter inflation report, all the cash flows in the third quarter are nominal. Additionally, cash flows in the fourth quarter have nominal components as half of the first quarter inflation will be realized in the factor for the fourth quarter. Moreover, the contribution of inflation in the first ten days of April will be fully realized in the first quarter of the following year.

The price of an IL bond can be written as

$$p = \sum_{i=0}^{N} c_i e^{-y_{r,in} t_{in} - (y_i t_i - y_{in} t_{in})} \tag{11.16}$$

where $t_i\,(t_{in})$ is the time to cash flow i (minus an inflation lag – for an inflation lag of 2 months, $t_{in} = t_i - \frac{2}{12}$), $y_i\,(y_{in})$ is the nominal yield at time $t_i\,(t_{in})$, $y_{r,in}$ is the real yield at time t_{in}, and c_i is cash flow i, $i=1, \ldots, N$.

For accurate analysis of IL bonds, the dirty price (price plus accrued interest) as well as all cash flows must be multiplied by an inflation reference factor and an accumulated but not accrued inflation factor. The real duration components of a real bond are

$$D_{r,j} = \frac{1}{p} \sum_{i=0}^{N} c_i t_{in} \psi_j e^{-y_{r,in} t_{in} - (y_i t_i - y_{in} t_{in})} \tag{11.17}$$

where ψ_j is the jth component of the basis function for time t_{in}. The nominal duration components of a real bond are defined as

$$D_{n,j} = \frac{1}{p} \sum_{i=0}^{N} c_i (t_i \psi_i - t_{in} \psi_{jn}) e^{-y_{r,in} t_{in} - (y_i t_i - y_{in} t_{in})} \tag{11.18}$$

Therefore, every inflation linked security has exposure to nominal rates. The larger the inflation lag $(t_i - t_{in})$, the larger the exposure to nominal rates.

In order to calculate the term structure of real rates, we first need to calculate the real cash flows of real bonds. A real cash flow is a cash flow that will be compensated at the stated real coupon plus inflation factor and thus will be discounted by real rates only. We first need to subtract the present value of all cash flows that are on or before the last nominal date from the market price of the real bond. Define the real cash flows of the bond as

$$c_{in} = c_i e^{-(y_i t_i - y_{in} t_{in})} \tag{11.19}$$

Substituting from (11.19) into (11.16), we calculate the real price p_r as

$$p_r = \sum_{i=0}^{N} c_{in} e^{-y_{r,in} t_{in}} \tag{11.20}$$

Equation (11.20) defines the implied real cash flows of a real bond with all the nominal components taken out and can be used to calculate the TSRR.

Figure 11.1 shows the TSIR and the TSRR along with the traded securities on July 30, 2012, and Figure 11.2 shows the TSIE on the same day. Since inflation linked securities were issued in the US in 1997, the TSRR has had a narrow range for the most part and structure of nominal rates and inflation expectations were very similar. Since 2008, the TSRR has varied significantly and the TSIE has been much more stable.

What is most interesting about the TSRR is the negative spot real rates for maturities shorter than 15 years. The TSIE looks like a normal yield curve, gradually sloping upward and leveling off at long maturities, and it appears to be a driver of real rates. It is more likely that market participants have a relatively simple inflation expectation model that is used to derive the relative complex shape of the real rates.

The level of real rates is an indication of growth expectations in an economy and is a reflection of real demand for money. When the productivity of capital is high, companies are looking to borrow, even at high real rates, and use the capital for production or service sector investments that will have high rates of return. When productivity is low or investment risk is high, or there are excess savings, companies are not willing to borrow for risk of failure and lenders are willing to lend their money for negative real

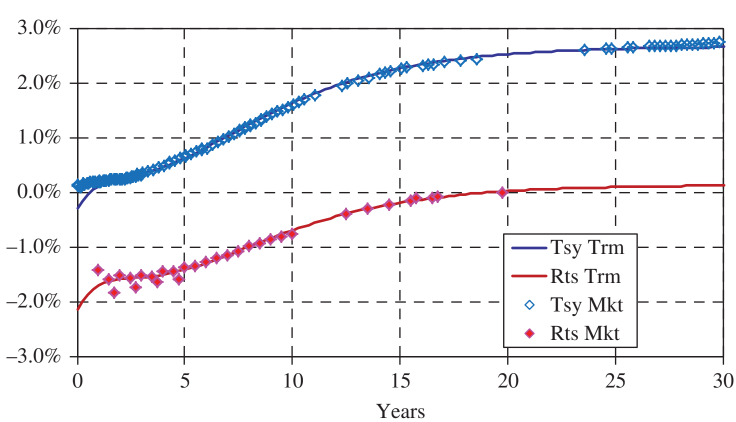


FIGURE 11.1 Spot real (Rts) and nominal (Tsy) rates, July 30, 2012

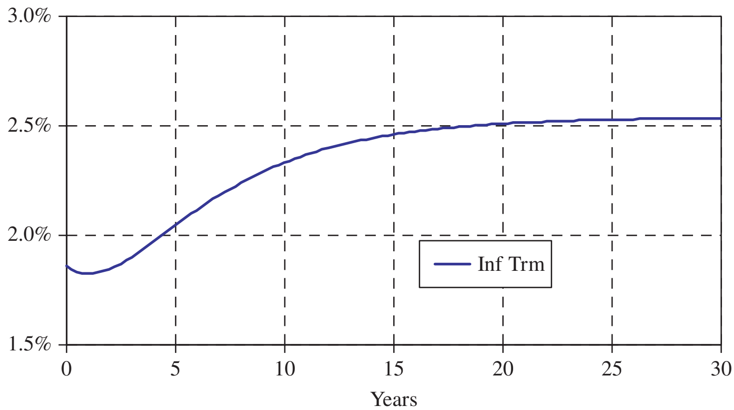


FIGURE 11.2 Term structure of inflation expectations, July 30, 2012

rates instead of investing it in risky ventures. Figure 11.1 is an indication that investors are willing to accept guaranteed negative real returns rather than investing their money in ventures.

Market conventions for accrual and trading are country dependent for IL bonds. In the UK, Australia, New Zealand, and Sweden the trading price includes the inflation factor. In the US, which is modeled after Canadian markets, and the euro zone the trading price does not include the inflation factor. For these markets, the trading price is multiplied by the inflation factor to calculate the invoice price.

Table 11.2 shows a sample of IL bonds from selected countries. Unlike nominal bonds, adding the accrued interest to the price of the bond does not give the invoice price, and even for countries such as Sweden where the price includes the factor, the accrual needs to be multiplied by the inflation factor.

Since each cash flow is discounted by both real and nominal bonds, the calculated spread is relative to both curves. The spread relative to market is the best estimate of the spread using the adjustment table for the market price of bonds. In the table there are two bonds with relatively large spreads that require explanation, namely, EUR 1.75% 4/15/20 and USD 0.125% 4/15/17. Germany does not have a large IL bond market and the EUR curve is derived from the French IL bond market with the addition of the credit spread of France relative to Germany (see Section 12.6), and thus the calculated spread relative to the market is not necessarily very accurate. For US 0.125 4/15/17, the richness is due to the very low coupon of the bond as will be discussed in the next section.

Table 11.3 shows the yield and interest rate durations of our sample IL bonds. Since all cash flows of a real bond are combinations of real as well as nominal accrual periods, the calculated yield of a real bond is a composite yield that approximates real yield for long maturity bonds. For short maturity bonds, the yield could be very misleading. In order to calculate the accurate real yield of a real bond, we need to strip the contribution of nominal cash flows. This is accomplished using equation (11.20).

The nominal duration of some IL bonds is very significant and needs to be properly calculated in any portfolio that invests in them. Older UK gilt stocks have an inflation

TABLE 11.2 Price and spreads for selected IL bonds, July 30, 2012

Security			Price					Spread	
Crncy	Cpn	Maturity	Mkt	Accr	Invoice	Fair	Model	Mkt	Curve
AUD	4	8/20/20	195.36	0.82	195.36	195.37	195.33	0.000%	–0.003%
AUD	3	9/20/25	140.74	0.36	140.74	140.75	140.69	0.000%	–0.004%
CAD	3	12/1/36	165.86	0.52	197.58	165.88	165.57	0.000%	–0.010%
EUR	1.75	4/15/20	119.17	0.53	128.66	118.03	118.48	–0.133%	–0.080%
GBP	1.25	11/22/27	123.34	0.24	153.87	123.35	124.36	0.000%	0.058%
GBP	4.125	7/22/30	315.08	0.11	315.27	315.08	309.72	0.000%	–0.122%
SEK	3.5	12/1/28	192.19	2.35	195.07	192.21	192.64	0.000%	0.017%
USD	2	1/15/14	104.36	0.09	129.92	104.37	104.45	–0.003%	0.053%
USD	2	7/15/14	106.58	0.09	129.45	106.58	106.71	–0.001%	0.064%
USD	2.375	1/15/17	116.51	0.11	132.82	116.42	116.69	–0.022%	0.034%
USD	0.125	4/15/17	106.78	0.04	107.67	105.86	106.10	–0.186%	–0.139%
USD	0.625	7/15/21	113.45	0.03	115.24	113.45	113.21	0.000%	–0.024%
USD	0.125	1/15/22	108.21	0.01	109.77	108.24	107.74	0.002%	–0.047%
USD	3.875	4/15/29	167.33	1.14	235.02	167.33	167.61	0.000%	0.012%
USD	3.375	4/15/32	167.85	1.00	218.30	167.87	167.19	0.000%	–0.026%

TABLE 11.3 Yield and interest rate durations for selected IL bonds, July 30, 2012

Security			Yield		Tsy Duration		
Crncy	Cpn	Maturity	Theta	Exp	Level	Slope	Bend
AUD	4	8/20/20	0.736%	0.334%	0.506	0.422	–0.122
AUD	3	9/20/25	0.894%	0.568%	0.592	0.633	0.525
CAD	3	12/1/36	0.445%	0.202%	0.167	0.177	0.202
EUR	1.75	4/15/20	–0.204%	–0.840%	0.167	0.152	–0.034
GBP	1.25	11/22/27	0.840%	–0.267%	0.167	0.197	0.228
GBP	4.125	7/22/30	0.370%	–0.286%	0.664	0.707	0.737
SEK	3.5	12/1/28	0.177%	0.035%	0.167	0.179	0.189
USD	2	1/15/14	–1.401%	–1.373%	0.167	–0.063	–0.147
USD	2	7/15/14	–1.403%	–1.370%	0.167	–0.033	–0.199
USD	2.375	1/15/17	–1.060%	–1.355%	0.167	0.076	–0.213
USD	0.125	4/15/17	–1.112%	–1.507%	0.167	0.093	–0.206
USD	0.625	7/15/21	0.411%	–0.833%	0.167	0.177	0.048
USD	0.125	1/15/22	0.540%	–0.775%	0.167	0.187	0.083
USD	3.875	4/15/29	0.407%	–0.183%	0.166	0.178	0.183
USD	3.375	4/15/32	0.299%	–0.109%	0.166	0.180	0.202

lag of 8 months, while the more recently issued bonds have a lag of 2 months. We can see that the nominal level duration of US Tips is about 0.167 years or 2 months. However, there is more information in the nominal slope duration of IL bonds, since it identifies to a large extent the average position of nominal cash flows on the curve. Recall that the ratio of slope duration to level duration of a zero coupon bond is

$$\frac{D_1}{D_0} = \tau = 1 - 2e^{-\alpha t} \tag{11.21}$$

Thus, when the ratio of nominal slope to level is about 1, the contribution of nominal duration is at the long end of the curve, and when it is negative it is at the front end of the curve. Unlike slope duration of zero coupon bond, the nominal slope duration of a real bond is similar to the slope duration of a eurodollar futures contract and can be larger than its level duration. We can see in Table 11.3 that bonds that mature in 2014 have negative slope durations, while bonds that mature in the late 2020s have a slope duration that is comparable to the level duration.

Table 11.4 lists the real duration components of the selected IL bonds. The interpretation of a real duration for an IL bond is the same as the nominal duration for a nominal bond. The credit duration is the duration of the security to the issuer, or the treasury in this case. For each cash flow, part of the exposure is from real rates and part from nominal rates. However, the exposure to credit is at all times, thus the credit duration is the sum of nominal and real durations.

At the portfolio level, nominal, real, and credit durations are aggregated separately and hedged separately as well.

TABLE 11.4 Real and credit durations for selected IL bonds, July 30, 2012

Security			Real Duration					Credit Duration		
Crncy	Cpn	Maturity	Level	Slope	Bend	4th	5th	Level	Slope	Bend
AUD	4	8/20/20	6.528	1.347	−5.695	−3.808	3.533	7.031	1.772	−5.819
AUD	3	9/20/25	10.597	5.848	−3.454	−9.164	−7.642	11.186	6.484	−2.931
CAD	3	12/1/36	18.782	15.843	9.383	2.533	−3.803	18.949	16.021	9.585
EUR	1.75	4/15/20	7.115	1.634	−6.223	−4.546	3.831	7.282	1.786	−6.256
GBP	1.25	11/22/27	13.880	9.595	−0.120	−9.199	−12.961	14.046	9.791	0.107
GBP	4.125	7/22/30	13.454	9.476	1.156	−6.183	−10.235	14.113	10.188	1.889
SEK	3.5	12/1/28	13.145	9.009	0.250	−7.395	−10.946	13.312	9.188	0.439
USD	2	1/15/14	1.275	−0.883	−0.052	0.953	−1.266	1.442	−0.947	−0.199
USD	2	7/15/14	1.757	−1.033	−0.541	1.665	−1.411	1.924	−1.066	−0.740
USD	2.375	1/15/17	4.106	−0.641	−3.870	1.793	3.280	4.273	−0.565	−4.083
USD	0.125	4/15/17	4.529	−0.493	−4.419	1.450	4.101	4.696	−0.399	−4.625
USD	0.625	7/15/21	8.567	3.025	−6.348	−7.509	0.868	8.734	3.202	−6.300
USD	0.125	1/15/22	9.244	3.705	−6.254	−8.715	−0.773	9.411	3.892	−6.171
USD	3.875	4/15/29	13.176	9.065	0.457	−6.978	−10.585	13.342	9.243	0.640
USD	3.375	4/15/32	15.470	11.869	4.033	−3.718	−9.492	15.636	12.049	4.235

11.4 SEASONAL ADJUSTMENTS TO INFLATION

In the US the most closely watched measures of inflation are the core inflation or the personal consumption expenditures deflator which are seasonally adjusted and exclude the volatile food and energy components. However, nearly all global IL bonds trade on the basis of headline inflation. Headline inflation is subject to strong seasonal patterns that can impact the price and rate of return of short term IL bonds. For example, the headline inflation in January in the US is very high compared to other months, and a bond that matures in April will have the full influence of January inflation, while a bond that matures in January will not.

There is a wealth of information on calculating seasonal factors. Both the Federal Reserve and the European Central Bank have supported the development of standardized procedures for seasonal adjustments. The methods developed by the US Census Bureau [11] and Tramo-Seats [12] are very widely used for seasonal adjustments.

Our objective is to develop seasonal adjustments with low computational overhead that can be used to estimate the impact of seasonality on cash flows. Calculation of IL bonds requires a historical table of inflation to be able to calculate an inflation reference index for a bond that was issued in the past. The historical inflation index in the table can be used for seasonal analysis as well.

Table 11.5 shows a sample of historical headline inflation indexes for the US. We first calculate the average annualized inflation rate for every month of the last data available. For example, for January 2012,

$$\frac{1}{12}\ln\left(\frac{226.655}{168.8}\right) = 0.02456$$

TABLE 11.5 Sample US headline inflation index

	Jan	Feb	Mar	Apr	May	Jun
2000	168.8	169.8	171.2	171.3	171.5	172.4
2001	175.1	175.8	176.2	176.9	177.7	178
2002	177.1	177.8	178.8	179.8	179.8	179.9
2003	181.7	183.1	184.2	183.8	183.5	183.7
2004	185.2	186.2	187.4	188	189.1	189.7
2005	190.7	191.8	193.3	194.6	194.4	194.5
2006	198.3	198.7	199.8	201.5	202.5	202.9
2007	202.4	203.5	205.4	206.7	207.9	208.4
2008	211.1	211.7	213.5	214.8	216.6	218.8
2009	211.143	212.193	212.709	213.24	213.856	215.693
2010	216.687	216.741	217.631	218.009	218.178	217.965
2011	220.223	221.309	223.467	224.906	225.964	225.722
2012	226.655	227.663	229.392	230.085	229.815	229.478

We now remove the variations in the average inflation that are due to the first or last months. The difference between the average for each month and the average of all months is the adjustment required to make sure seasonality is not affected by a spike in inflation in any given month. Thus, the average annual inflation for the 12 years ending in January 2012 as calculated above is 0.02456. Likewise, we calculate the average annual inflation for the 12 months ending in February 2012 and so on. In Table 11.5, the average inflation for all months is 0.02392. We want to change the inflation index values in such a way that the average annual inflation for all the months of 2012 will be equal to 0.02392. All index values in the month of January are scaled using the formula

$$I_{\text{year,scaled}} = I_{\text{year}} e^{(2012-\text{year})(0.02456-0.02392)} = I_{\text{year}} e^{(2012-\text{year})0.00064}$$

The resulting index will result in the same average inflation for all annual rates ending in all months of 2012. Monthly inflation rates can be calculated by simply dividing the index of one month by the prior. For example, the monthly inflation for April 2003, using the values in Table 11.5, is equal to $\frac{183.8}{184.2} - 1 = -0.22\%$. In practice, the calculation should be done from the scaled indexes, resulting in $\frac{185.13}{185.74} - 1 = -0.33\%$. Once the average monthly inflations are calculated, the average inflation for each month is calculated.

By subtracting the average of the monthly averages from each monthly average, the seasonal adjustment can be calculated. Figure 11.3 shows the seasonal monthly inflation rates for the periods 1991–2012 and 2001–2012. The biggest difference between the two ranges is in the January adjustment. Since the sum of monthly adjustments is zero, the seasonal adjustments for all months of 1991–2012 period other than January fall below those of 2001–2012 range.

To correct for unusual monthly spikes in the data, we can remove the maximum and minimum inflation rates for every month in the data series and then calculate the seasonal patterns. The change that this refinement would make to the US data is negligible, but it is a good practice to do it for future events.

Figure 11.4 shows the standard deviation of monthly CPI in the US for every year since 1981. It appears that globalization has had a major impact on the seasonality of inflation. The most volatile components of headline CPI are food and energy.

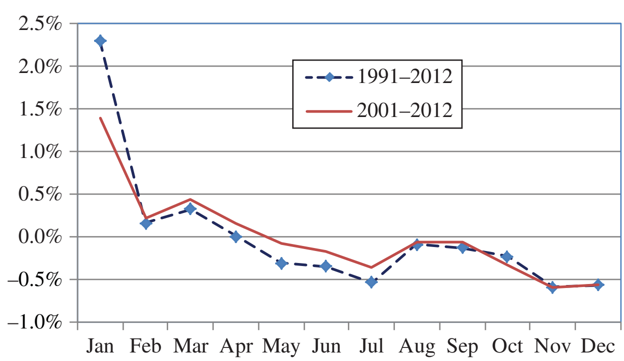


FIGURE 11.3 Average monthly inflation rates

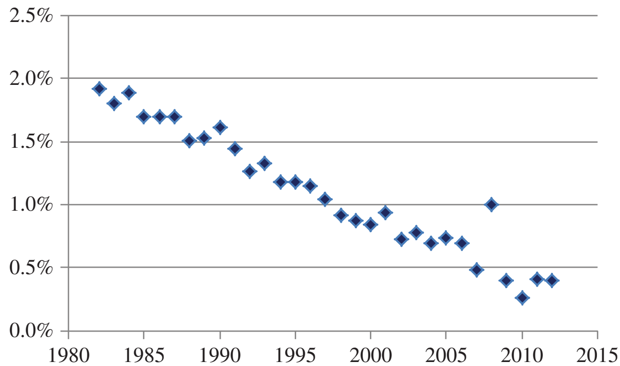


FIGURE 11.4 Standard deviation of monthly inflation in the US

Imports of produce from global sources and in particular from the southern hemisphere in the winter months and less reliance on heating oil have had a major impact on the price stability of food and energy and variations of monthly CPI. The seasonal variation of inflation is not likely to go to zero, but more likely it will level off at the current levels.

The monthly volatility of the seasonal inflation from Figure 11.3 for 2001–2012 is 0.535%. Since the volatility of monthly inflation has been steadily falling, it is reasonable to assume that we should use volatility other than historical volatility. If we use the average standard deviation for the last 5 years minus the highest and lowest, the volatility will be 0.402%. We can then scale the monthly seasonal pattern by the ratio $\frac{0.402}{0.535}$ and use it for future seasonal factors. Table 11.6 lists the calculated seasonal factors for the US CPI.

TABLE 11.6 Seasonal factors for US CPI

	Unadjusted Seasonal	Adjusted Seasonal	Adjusted Factor
Jan	1.389%	1.042%	1.01048
Feb	0.228%	0.171%	1.00171
Mar	0.439%	0.330%	1.00330
Apr	0.163%	0.122%	1.00123
May	−0.079%	−0.059%	0.99941
Jun	−0.165%	−0.124%	0.99876
Jul	−0.359%	−0.269%	0.99731
Aug	−0.067%	−0.050%	0.99950
Sep	−0.067%	−0.050%	0.99950
Oct	−0.326%	−0.245%	0.99756
Nov	−0.600%	−0.450%	0.99551
Dec	−0.557%	−0.418%	0.99583
Stdev	0.535%	0.401%	

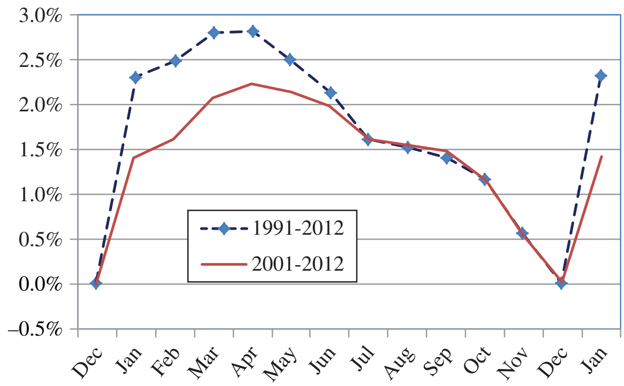


FIGURE 11.5 Cumulative seasonal inflation adjustment for US

The cumulative inflation adjustment on a monthly basis for the US is shown in Figure 11.5. The adjustment is the highest in April and lowest in December. Therefore cash flows for the end of June will get the highest seasonal adjustment of roughly 2.2% and for January about 0.86%, given that there are 2 months of inflation lag compared to end of December. Therefore, bonds that mature in July usually have a lower real yield, and bonds that mature in January have a higher real yield. There is no calendar seasonality for the issuance of IL bonds. For example, if the treasury issues a bond in January and another in July, assuming that the real yields do not change, both bonds will have the same coupon and real yields. However, by the time the July bond is issued, the January bond has accumulated the seasonal high inflation accrual, while the July bond is at seasonal low point of inflation accrual. Going forward, the July bond will have an implied high inflation period in the last 6 months before maturity and thus will have a lower implied yield.

In order to calculate the seasonal adjustment, we have to use the difference between the cumulative seasonal inflation adjustment (CSIA) between the last day that inflation is available and the reference inflation point of a cash flow. For example, for a cash flow on January 15, the reference inflation index is November 15. If the last reported inflation index is for the end of June, then we subtract the CSIA for November 15 (the average of the October and November indexes) from the CSIA for June. The exponent of that value is the factor by which the cash flow for January 15 will be multiplied.

The factor to adjust the real rates covers a longer period than the last day for which an inflation reference is available. For example, if inflation for April is reported on May 15, and the inflation lag is 2 months, then for a trade on May 5 the last inflation reference is for May 30, while the inflation reference through the end of June has already taken place but not yet been reported. We need to make an adjustment to the cash flows by estimating the inflation for the month of April, for example, by averaging the previous 5 years of historical data, and use that factor to scale the cash flows.

Table 11.7 shows the yields of short maturity Tips on July 31, 2012. We can see that the unadjusted yields of bonds maturing in 2014 are –0.978%, –1.009%, and –1.315% for maturity in the months of January, April, and July, respectively. A similar pattern is repeated for 2015 and 2017 as well. For 2016, the April maturity has a

TABLE 11.7 Yield of short maturity Tips, July 31, 2012

Cpn	Maturity	Price	Unadj Yield	Adjusted Yield	Spread
0.625	4/15/13	100.539	−0.139%	−1.699%	−0.364%
1.875	7/15/13	102.773	−1.011%	−1.110%	0.268%
2	1/15/14	104.371	−0.978%	−1.313%	0.091%
1.25	4/15/14	103.895	−1.009%	−1.657%	−0.245%
2	7/15/14	106.582	−1.315%	−1.369%	0.050%
1.625	1/15/15	107.047	−1.196%	−1.398%	0.033%
0.5	4/15/15	104.742	−1.219%	−1.630%	−0.195%
1.875	7/15/15	109.785	−1.360%	−1.399%	0.037%
2	1/15/16	111.555	−1.262%	−1.408%	0.024%
0.125	4/15/16	105.184	−1.238%	−1.538%	−0.113%
2.5	7/15/16	115.531	−1.314%	−1.346%	0.069%
2.375	1/15/17	116.523	−1.223%	−1.336%	0.051%
0.125	4/15/17	106.789	−1.271%	−1.507%	−0.141%
2.625	7/15/17	120.051	−1.282%	−1.308%	0.039%
1.625	1/15/18	115.742	−1.162%	−1.255%	0.041%
1.375	7/15/18	115.938	−1.199%	−1.220%	0.017%
2.125	1/15/19	121.383	−1.067%	−1.145%	0.032%

slightly higher yield, which may be due to pricing inefficiency or more likely is a coupon effect, which we will explain shortly.

The market seasonal price adjustment is significantly different for April maturity bonds, probably due to the market participants using a longer history for seasonal adjustment. We can see in Figure 11.3 that if we use the period 1991–2012 for seasonal adjustment, the January adjustment is much larger, and this will impact the April maturity bonds directly, since the inflation factor for these bonds is the reference CPI in mid-February.

Given that there is no standard method to estimate seasonal adjustments, there will always be some dispersion in the pricing of Tips, and that is where a trader can add value by correctly anticipating seasonal adjustments. Real yields and TSRR are also calculated using seasonal adjustments, and therefore there is no unique TSRR. The calculated TSIR, TSRR, and TSIE in Figures 11.1 and 11.2 are based on seasonal adjustments explained above using inflation data up to June 2012. It is remarkable that the implied TSIE is so smooth, considering all the adjustment.

The US Treasury has guaranteed that if there is deflation, the principal of Tips will be protected. Bonds that have a very low coupon, such as 0.125% 4/15/2016, will have a guaranteed price of 100 at maturity if there is deflation. However, bonds that have higher coupon and/or accumulated inflation can lose the accumulated inflation and therefore have a potentially significant downside during periods of deflation. Thus, lower coupon securities have a better downside protection, given that there is an implied put option for all bonds at par. The lower downside protection for lower coupon bonds should translate into a premium price or lower market yield.

Theoretically speaking and based on long term data, investors usually demand compensation for inflation. During periods of high inflation, nominal yields tend to be high, and as inflation falls, so do nominal yields. Historically, inflation falls the most during recoveries, not during recessions. Quite often this takes portfolio managers by surprise, since bond yields fall the most in recoveries as well, while most investors expect to see rates rise in a recovery. Real yields tend to be much less volatile than nominal yields in periods of changing inflation.

Thus, it is logical to expect that nominal yields follow the path of inflation or expected inflation. During the winter months, when seasonal inflation is highest, nominal rates should be higher than in the summer, when seasonal inflation is falling. In practice, the short end of nominal rates is a smooth curve that is dominated by bank lending and borrowing and the Federal Reserve's Fed Funds rate and does not react to inflation seasonality. Therefore, there has to be an adjustment to the short term real yields in such a way that the sum of inflation and real rates equals short term nominal rates.

Market yield for a real bond is not a clearly defined quantity, since a portion of the cash flow of a bond is nominal. Depending on the shape of TSIR and TSRR, the contribution of nominal cash flows to the price of an IL bond can vary. The best measure of value for IL bonds is the spread relative to the curve, which must be a constant for both nominal and real cash flows relative to the respective curves.

11.5 INFLATION SWAPS

Inflation swaps are instruments that can be used for direct trading of CPI. Like Tips and other IL bonds, inflation swaps trade based on non-seasonally adjusted headline CPI. Inflation swaps trade on the basis of inflation rate and at the present time they are all zero coupon swaps. For a notional face value N_z, coupon rate (market inflation rate) of r, compounding frequency of m and maturity of z years, the fixed swap receiver will at maturity be entitled to

$$Q_{z,\mathrm{fix}} = N_z\left(1+\frac{r}{m}\right)^{mz} \tag{11.22}$$

The floating swap receiver will be entitled to

$$Q_{z,\mathrm{flt}} = N_z\,\frac{I_{r,z}}{I_{r,0}} \tag{11.23}$$

where $I_{r,z}$ and $I_{r,0}$ are the inflation reference indexes at the maturity and at the initiation of the swap, respectively. The values of the fixed and floating legs of a swap are identical at the initiation of the swap. Similarly to IL bonds, with inflation swaps there is an inflation lag which is 2 months in nearly all traded contracts.

Unlike most interest rate swaps, where the floating rate receiver has very little or no interest rate risk, in inflation swaps the primary risk is in the floating inflation component. The fixed rate receiver is basically a zero coupon swap that has no direct sensitivity to inflation rates. However, if inflation rates fall, the floating leg will underperform the fixed leg and the long receiver will net the difference between the market value of the fixed and the floater. Just like ordinary interest rate swaps, the long inflation receiver will outperform the floating leg if implied rates fall.

The present value or face of the fixed and floating legs of an inflation swap are calculated by discounting the maturity value by the Libor rate. If t_z is the time to maturity of the swap, $y_{s,g}$, $y_{s,r}$, and $y_{s,l}$ are the treasury (government), real treasury, and Libor yields, then the present value of the swap is

$$p_{v,\text{fix}} = Q_{z,\text{fix}} e^{-y_{s,l} t_z} \tag{11.24}$$

The spot inflation yield $y_{s,n}$ is the implied inflation rate and is equal to the treasury rates minus real rates. It can be written as

$$y_{s,n} = (y_{s,g} - y_{s,r}) = \frac{1}{t_z} \ln\left(\frac{I_{r,n}}{I_{r,0}}\right) \tag{11.25}$$

The amount that will be received at the maturity of an inflation swap is equal to the accumulated inflation through the inflation reference point which is generally lagged by 2 months. Thus, the future value will be

$$p_{f,\text{flt}} = Q_{z,\text{flt}} e^{(y_{s,gn} - y_{s,rn}) t_{zn}} \tag{11.26}$$

where t_{zn} is the time to the inflation reference point for the maturity (usually 2 months before maturity), $y_{s,gn}$ is the nominal treasury (government) yield at the final inflation reference point, and $y_{s,rn}$ is the real treasury yield at the final inflation reference point. The subscript n refers to inflation. For time, it refers to the inflation reference point for a given cash flow. The principal at maturity needs to be discounted by Libor to calculate the present value of the inflation swap, that is,

$$p_{v,\text{flt}} = Q_{z,\text{flt}} e^{-y_{s,l} t_z - (y_{g,rn} - y_{s,gn}) t_{zn}} \tag{11.27}$$

Thus, the floating leg of an inflation swap is a function of treasury, real, and Libor rates and all durations need to be accounted for correctly. This formulation works for floating coupon inflation swaps as well. We just need to provide the summation for all cash flows as follows:

$$p_{v,\text{flt}} = \sum_i c_i e^{-y_{s,l} t_i - (y_{s,rn} - y_{s,gn}) t_{in}} \tag{11.28}$$

In the foregoing we drop the subscript s for spot rates, since it applies to all rates. The Libor yield is equal to the treasury rate plus Libor spread. Thus, nominal, real, and Libor duration risks of floating zero coupon inflation swaps are respectively

$$D_{g,k} = [t_z \psi_k(t) - t_{zn} \psi_k(t_{zn})] e^{-y_l t_z - (y_{rn} - y_{gn}) t_{zn}} \tag{11.29}$$

$$D_{r,k} = t_{zn} \psi_k(t_{zn}) e^{-y_l t_z - (y_{rn} - y_{gn}) t_{zn}} \tag{11.30}$$

$$D_{l,k} = t_z \psi_k(t) e^{-y_l t_n - (y_{rn} - y_{gn}) t_{zn}} \tag{11.31}$$

For a floating coupon inflation swap, the implied floating coupon for a period ending at t_i is, from (6.15),

$$c_{f,i} = \mu_i \left(e^{(y_{g,in} - y_{r,in})t_{in} - (y_{g,i-1,n} - y_{r,i-1,n})t_{i-1,n}} - 1 \right) \tag{11.32}$$

where μ is the principal balance at the start of the period. To calculate the duration risks of floating coupon inflation swaps, we need to add the floating risk to the durations, similar to the adjustments in (6.23):

$$D_k = -\frac{1}{p}\frac{\partial p}{\partial a_k} = -\frac{1}{p}\sum_i^N \left(\frac{\partial c_{f,i}}{\partial a_k} + \frac{\partial y_i}{\partial a_k} \right) e^{-y_i t_i} \tag{11.33}$$

$$\frac{\partial c_{f,i}}{\partial a_{g,k}} = (y_{g,in}t_{in} - y_{g,i-1,n}t_{i-1,n})(\mu_i + c_{f,i}) \tag{11.34}$$

$$\frac{\partial c_{f,i}}{\partial a_{r,k}} = (y_{r,in}t_{in} - y_{r,i-1,n}t_{i-1,n})(\mu_i + c_{f,i}) \tag{11.35}$$

$$D_{g,k} = \frac{1}{p}\sum_i \left\{ c_{f,i}\left[t_i\psi_k(t) - t_{in}\psi_k(t_{in})\right] + \frac{\partial c_{f,i}}{\partial a_{g,k}} \right\} e^{-y_l t_i - (y_{r,in} - y_{g,in})t_{in}} \tag{11.36}$$

$$D_{r,k} = \frac{1}{p}\sum_i \left\{ c_{f,i}t_{in}\psi_k(t_{in}) + \frac{\partial c_{f,i}}{\partial a_{r,k}} \right\} e^{-y_l t_i - (y_{r,in} - y_{g,in})t_{in}} \tag{11.37}$$

$$D_{l,k} = \frac{1}{p}\sum_i c_{f,i}t_i\psi_k(t)e^{-y_l t_i - (y_{r,in} - y_{g,in})t_{in}} \tag{11.38}$$

If the floating coupon has a fixed component, it is simply added to the coupon in the summation.

Table 11.8 shows the risks of selective zero coupon inflation swaps. Note that the Libor duration is equal to the sum of treasury and real durations for the floating leg of the swap. If the Libor spread changes, both the floating and fixed legs of the swap react similarly, therefore the swap has no exposure to the Libor spread at initiation. As the swap matures and treasury or real rates change, one leg of the swap can develop a slightly different exposure to Libor than the other. If treasury rates rise, the fixed leg will underperform. If real rates rise and all other rates stay unchanged, the implied inflation rate will fall and the floating rate will underperform the fixed leg. Thus the long swap is positively exposed to the inflation rate. If implied inflation falls the fixed swap outperforms the floating leg and vice versa.

The inflation swap market is not fully developed and there are small but measurable differences between implied inflation from treasury–Tips spread and inflation swaps. Figure 11.6 shows the implied and market inflation rates. The difference between the calculated and market rates is about 5–10 bps, which is close to transaction costs.

TABLE 11.8 Risks of selected inflation swaps, July 31, 2012

		Treasury			Real			Libor		
Type	Maturity Years	Level	Slope	Bend	Level	Slope	Bend	Level	Slope	Bend
Fixed	2	2.00	−1.09	−0.82				2.00	−1.09	−0.82
Fixed	5	5.00	−0.23	−4.97				5.00	−0.23	−4.97
Fixed	7	7.00	1.36	−6.46				7.00	1.36	−6.46
Fixed	10	10.00	4.54	−5.86				10.00	4.54	−5.86
Fixed	15	15.00	10.73	0.36				15.00	10.73	0.36
Fixed	20	20.00	17.03	9.00				20.00	17.03	9.00
Fixed	30	30.00	28.78	25.24				30.00	28.78	25.24
Float	2	0.17	0.03	−0.35	1.83	−1.12	−0.47	2.00	−1.09	−0.82
Float	5	0.17	0.21	−0.22	4.83	−0.44	−4.75	5.00	−0.23	−4.97
Float	10	0.17	0.31	0.32	9.83	4.23	−6.18	10.00	4.54	−5.86
Float	15	0.17	0.30	0.54	14.83	10.42	−0.18	15.00	10.73	0.36
Float	20	0.17	0.27	0.52	19.83	16.76	8.48	20.00	17.03	9.00
Float	30	0.17	0.21	0.34	29.83	28.57	24.89	30.00	28.78	25.24

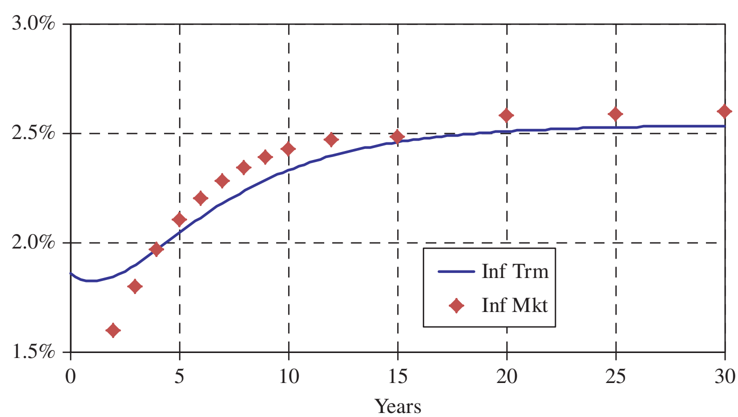


FIGURE 11.6 Implied and market inflation rates, July 31, 2012

CHAPTER 12

Credit Spreads

In Section 5.3 we discussed the concept of spread relative to the TSIR. In this chapter we will quantify the meaning of yield spread and develop valuation, hedging and risk measurement methodologies for spread products.

12.1 EQUILIBRIUM CREDIT SPREAD

So far, our analysis has focused on risk-free non-contingent cash flow bonds where our interpretation of non-contingency is related to embedded options in a bond. Thus, we did not include callable treasury bonds in our analysis, even though they are risk-free. We now turn attention to option-free or bullet risky assets and argue that, in an efficient market, yield spread is associated with default risk.

The price of a risk-free asset with a cash flow of c at time t is, from (10.35),

$$p_t = ce^{-y_s t} \tag{12.1}$$

We assume that investors are indifferent between the following two scenarios:

- A risk-free cash flow of $\rho(t)$ at time t.
- A unit cash flow with a probability of $\rho(t)$ and no cash flow with a probability of $1 - \rho(t)$ at time t.

We can write the price p_r of a risky bond as

$$p_r(t) = \rho(t)p_t(t) = \rho(t)c\,e^{-y_s(t)t} \tag{12.2}$$

where $\rho(t)$ is the probability of no default or the survival probability by time t. If the expected instantaneous default rate $\eta(t)$ is known at time t, the change in survival probability $\Delta\rho(t)$ between t and $t + \Delta t$ is given by

$$\Delta\rho(t) = -\rho(t)\eta(t)\Delta t \tag{12.3}$$

The negative sign in (12.3) signifies the decline of survival probability with time. The total survival probability can be calculated as

$$\begin{aligned}\frac{\Delta\rho(t)}{\rho(t)} &= -\eta(t)\Delta t \rightarrow \frac{d\rho(t)}{\rho(t)} = -\eta(t)dt \\ \int_0^t \frac{d\rho(t)}{\rho(t)} &= -\int_0^t \eta(t)dt\end{aligned} \tag{12.4}$$

leading to

$$\rho(t) = e^{-\int_0^t \eta(t)dt} \tag{12.5}$$

We now define the spot default rate $s_{s,c}(r)$ of a credit security as the average probability of default in the period $(0,t)$, such that

$$\begin{aligned}s_{s,c}(t) &= \frac{\int_0^t \eta(t)dt}{t} \\ s_{s,c}(t)t &= \int_0^t \eta(t)dt = \int_0^t s_{f,c}(t)dt\end{aligned} \tag{12.6}$$

The definitions of spot default rate $s_{s,c}(t)$ and instantaneous default rate $s_{f,c}(t)$ or $\eta(t)$ are analogous to the definitions of spot yield $y(t)$ and instantaneous forward rate $y_f(t)$ in (2.27). The spot default rate is equal to the time average of the instantaneous default rates. If the default rate is a constant in time, then

$$s_{s,c}(t) = s_{f,c}(t) \tag{12.7}$$

With (12.6), the survival probability (12.5) can be written as

$$\rho(t) = e^{-s_{s,c}(t)t} \tag{12.8}$$

Substituting $\rho(t)$ from (12.8) into (12.2) leads to

$$p_r = c\,e^{-[y_s(t)+s_{s,c}(t)]t} \tag{12.9}$$

In (12.9) the implied default rate has been very conveniently translated into a yield spread over risk-free rate. In other words, in an efficient market, the yield premium of a risky asset is equal to its default rate, provided that the recovery value of the defaulted security is zero. Since the implied default rate or the yield spread is not a constant, we refer to it as the term structure of credit spread (TSCS).

The contribution of spread convexity to the spread yield is similar to the contribution of convexity to the spot yield (10.27). The implied spread $s_{t,c}$ is related to the market spot spread $s_{s,c}$ and the relative spread volatility v_c and relative treasury (government) volatility v_g by

$$\begin{aligned}y_{t,g} + s_{t,c} &= y_{s,g} + s_{s,c} + \tfrac{1}{2} y_s^2 v_g^2\, t^2 + \tfrac{1}{2} s_{s,c}^2\, v_c^2\, t^2 + y_{s,g} s_{s,c}\, v_g v_c\, t^2 \xi_{gc} \\ s_{t,c} &= s_{s,c} + \tfrac{1}{2} s_{s,c}^2\, v_c^2\, t^2 + y_{s,g} s_{s,c}\, v_g v_c\, t^2 \xi_{gc}\end{aligned} \tag{12.10}$$

where ξ_{gc} is the correlation between the treasury (government) rate and default rate of the credit security. This equation suggests that the convexity adjusted spread yield should fall for very long maturities if the correlation coefficient is positive or if the volatility of credit spread is higher than that of treasury rates. In practice, the contribution of convexity to yield spread is not observable at the present time since the TSCS is not very well developed and risky credits rarely issue long maturity bonds. The supply of super-long maturity bonds (~100 years maturity) is too limited to draw any conclusions about the convexity contribution to the TSCS at the present time. Spreads can be much more volatile than interest rates for most assets; therefore, spread convexity can be very important, even for medium term maturities.

12.2 TERM STRUCTURE OF CREDIT SPREADS

The TSCS can be written in a manner similar to the TSIR (10.39) as

$$s_{t,c} = \sum_j b_{c,j}\psi_j \tag{12.11}$$

where ψ_j is the jth basis function of the yield spread and $b_{c,j}$ is its coefficient. The components of the TSCS can be calculated for issuers of liquid bonds with different maturities and identical seniority. The convexity adjusted price of a risky security (12.9) can be written as

$$p_r = \sum_i c_i\, e^{-[y_s(t_i)+s_{s,c}(t_i)]t_i} = \sum_i c_i\, \rho_i e^{-y_s(t_i)t_i} \tag{12.12}$$

Using the definition of the effective cash flow, $c_{e,i}$

$$c_{e,i} = c_i\rho_i \tag{12.13}$$

The price of a risky bond becomes like the price of a treasury bond,

$$p_r = \sum_i c_{e,i}e^{-y_s(t_i)t_i} \tag{12.14}$$

12.3 RISK MEASUREMENT OF CREDIT SECURITIES

The spread duration and convexity components of credit securities can be calculated in a similar fashion to those of treasuries. Assuming no recovery, from (12.12) we can derive

$$D_{c,k} = -\frac{1}{p_t}\frac{\partial p}{\partial b_{c,k}} = \frac{1}{p_t}\sum_i c_i t_i \psi_k e^{-(y_i+s_i)t_i} = \left\langle t\psi_k(t)\right\rangle \tag{12.15}$$

$$X_{c,kl} = \frac{1}{p_t}\frac{\partial^2 p}{\partial b_{c,k}\partial b_{c,l}} = \frac{1}{p_t}\sum_i c_i t_i^2 \psi_k \psi_l\, e^{-(y_i+s_i)t_i} = \left\langle t^2\psi_k(t)\psi_l(t)\right\rangle \tag{12.16}$$

The spread curve is calculated by subtracting the term structure of the treasury curve from the term structure of credit rates (TSCR). One should use extreme care to ensure that the spread curve is compatible with the treasury curve. They must both have the same decay coefficient and use the same set of basis functions. With this framework, the term structure of the credit spread will be

$$s_c(t) = \sum_j (a_{c,j} - a_j)\psi_j = \sum_j b_{c,j}\psi_j = y_c(t) - y(t) \tag{12.17}$$

where $a_{c,j}$ are the coefficients of the TSCR. If there is only one security, the calculated spread curve will be parallel to the treasury curve, implying that only the level of the credit curve is different from the treasury curve; all other components will be identical to the treasury curve. If there are two or more securities of the same credit issuer with identical seniority, we can calculate a slope for the credit curve as well. The remaining components must be equal to those of the treasury curve; they cannot be set to zero. If there are enough bonds by the same issuer, the bend component can also be calculated; however, fourth and fifth order components are almost never necessary. It is best to match them with the treasury curve. The algorithm for calculating the term structure of the credit curve must therefore allow for matching two or more components to those of the respective treasury in the currency of issuance.

The cheapness or richness of a security or bonds of an issuer is measured relative to the level of the TSCS which is the first component of the term structure. Care has to be taken when using key basis functions (KBFs), since spread cannot be measured relative to the key rates. If we match three of the components of the TSCR with the treasury curve and calculate a credit curve based on the first two key rates, the calculated credit curve will be unrealistic. The KBF is the natural basis function for risk measurement, but is very poor for valuation. Valuation requires a set of basis functions such that the spread relative to the first basis function is a representative of the cheapness or richness of the security. In the KBF, the components are localized and its use for valuation causes more problems than it solves. It is best to use the CBF for the credit, Libor and all other securities and convert the duration components to KBF after they are calculated. While the KBF is a great tool for hedging interest rate and credit exposure of a portfolio, it is a very poorly constructed methodology for calculating spreads.

12.4 CREDIT RISKS EXAMPLE

The conventional duration and KRD of credit securities can be further complicated by the slope of the credit curve, in addition to the slope of the treasury curve. To illustrate this point, Table 12.1 lists a few bonds of the Ford Motor Company along with their respective conventional as well as term structure durations. The conventional and term structure durations of comparable treasuries are also shown in the last two columns.

The yield and duration columns are based on the continuously compounded yield of the securities. The level duration is the correct way of calculating the duration by discounting the cash flows using their respective discount yield, while the conventional duration discounts all cash flows by the same market yield. Therefore, the difference

TABLE 12.1 Comparison of duration components of credit securities, July 30, 2012

			Credit				Comparable Tsy	
Coupon	Maturity	Yield	Duration	Level	Slope	Bend	Duration	Level
8	6/1/14	1.72%	1.73	1.73	−1.01	−0.55	1.69	1.69
12	5/15/15	3.10%	2.44	2.44	−1.02	−1.56	2.36	2.36
6.5	8/1/18	4.06%	5.12	5.15	0.16	−4.91	5.08	5.08
9.375	3/1/20	5.34%	5.65	5.69	0.90	−4.90	5.57	5.56
9.215	9/15/21	4.94%	6.57	6.60	1.80	−4.88	6.55	6.54
8.875	1/15/22	5.36%	6.91	6.94	2.04	−4.91	6.19	6.16
7.125	11/15/25	5.77%	8.95	8.89	4.36	−3.18	8.55	8.47
7.5	8/1/26	5.93%	9.24	9.16	4.64	−2.85	9.12	9.01
6.625	2/15/28	5.66%	9.91	9.76	5.51	−1.79	9.02	9.00
6.375	2/1/29	5.74%	10.62	10.42	6.15	−1.23	10.47	10.30
9.3	3/1/30	5.95%	9.87	9.63	5.49	−1.24	9.12	9.01
7.75	6/15/43	6.82%	12.75	11.80	7.99	2.10	11.14	10.95
7.4	11/1/46	6.11%	13.95	12.80	9.12	3.29	13.04	12.76
9.98	2/15/47	6.80%	12.48	11.45	7.72	1.98	11.14	10.95
7.7	5/15/97	6.87%	14.53	12.72	8.97	3.25	12.42	12.15

between the duration and level is a measure of the accuracy of the duration calculation for hedging. At low durations, the two duration measures are very close or identical. However, at longer maturities they start to diverge. In the conventional duration measure, the long dated cash flows are discounted by the same average yield as short dated cash flows instead of by their spot yield. Due to the steepness of the credit curve as well as the treasury curve, the long dated cash flows must be discounted by a higher yield than short dated cash flows and thus their contribution to duration should be lower than the conventional duration calculation. Likewise, the short dated cash flows are discounted by a higher yield than their spot yield, resulting in a lower contribution to duration. The net result is that for long duration securities a gap develops between the term structure duration and the conventional duration.

If a treasury portfolio that is managed against a benchmark is constructed by using coupon securities, the error in measurement of the duration of securities in the portfolio and the benchmark offset each other to a large extent. For credit securities, the gap in durations can be too large to be canceled by hedging. In the last row of Table 12.1, the gap between the conventional duration and term structure duration is about 1.8 years (14.53 − 12.72), while comparable treasuries that can be used for hedging have a gap of only 0.27 years (12.42 − 12.15). Thus, hedging the interest rate duration of the security by a comparable treasury or by using KRD will leave a duration mismatch of more than 1.5 years.

The duration mismatch is most prominent in high yield and emerging markets securities. It is further compounded at times of market selloffs. As spreads widen, the hedging mismatch increases to the detriment of a portfolio. Since the duration is overestimated, the amount of treasuries that need to be sold to hedge the interest rate exposure is also overestimated. The result is a portfolio that is underweight the interest rate duration at a time that interest rates are falling and spreads are widening. Without calculating the TSCR, the calculated durations and respective hedges cannot be trusted when spreads are wide.

12.5 FLOATING RATE CREDIT SECURITIES

The floating coupon of a bond is usually based on a liquid high quality short term reference index such as 3-month treasury bill or 6-month Libor. Floating rate bonds have almost zero duration if the issuer is also the issuer of the reference index. For example, the duration components of floating Libor bond in a swap transaction are zero before the first coupon is fixed, and therefore its price is always equal to par.

The implied floating coupon of a bond from time t_{i-1} to t_i is, from (6.15),

$$c_{f,i} = 100\left(e^{y(t_i)t_i - y(t_{i-1})t_{i-1}} - 1\right) = 100\left(e^{y_i t_i - y_{i-1}t_{i-1}} - 1\right) \tag{12.18}$$

where y_i is the spot yield of the floating benchmark, which is usually Libor or treasury bill. The price of this security is

$$p = \sum_i c_{f,i} e^{-y_i t_i} + 100 e^{-y_m t_m} \tag{12.19}$$

where t_m is the time to final maturity and the last cash flow is the principal payment. Substituting for c_j from (12.18) and using an initial time of zero, it is a trivial exercise to show that (12.19) is always equal to 100 and is independent of the yield level. Therefore, the duration components of a floating bond that is discounted by the curve that generates its forward coupons are always zero.

In general, a cash flow c_j is equal to the sum of contributions from the constant (fixed) rate coupon $c_{c,i}$, floating rate coupon for the interval $c_{f,i}$, and principal payment $c_{p,i}$ for a sinking or capitalizing bond. Thus

$$c_i = c_{c,i} + c_{f,i} + c_{p,i} \tag{12.20}$$

If μ_i is the remaining principal amount of a floating rate coupon payment, the floating rate part of the cash flow can be written as

$$c_{f,i} = \mu_i \left(e^{y_{s,i}t_i - y_{s,i-1}t_{i-1}} - 1\right) \tag{12.21}$$

The price function will take the following form:

$$p = \sum_i (c_{c,i} + c_{f,i} + c_{p,i}) e^{-(y_i + s_i)t_i} \tag{12.22}$$

In (10.38) we calculated the duration components of a bond with fixed cash flows. In order to calculate the duration components of a floating rate bond, we need to calculate the sensitivity of the expected future coupons with respect to the TSIR, that is, equation (6.23), which is the duration risks of floating Libor without convexity adjustment. With convexity adjustment, the basis functions are simply divided by $1 + y_s v_y^2 t_i^2$. We also need to calculate the sensitivity to volatility. The risk measures are thus

$$\frac{\partial c_{f,i}}{\partial a_k} = \mu_i \left(\frac{t_i \psi_{k,i}}{1 + y_{s,i} v_{y,i}^2 t_i^2} - \frac{t_{i-1} \psi_{k,i-1}}{1 + y_{s,i-1} v_{y,i-1}^2 t_{i-1}^2} \right) e^{y_{s,i} t_i - y_{s,i-1} t_{i-1}} \tag{12.23}$$

$$\frac{\partial c_{f,i}}{\partial v_y} = \mu_i \left(\frac{y_{s,i}^2 v_{y,i} t_i^3}{1 + y_{s,i} v_{y,i}^2 t_i^2} - \frac{y_{s,i-1}^2 v_{y,i-1} t_{i-1}^3}{1 + y_{s,i-1} v_{y,i-1}^2 t_{i-1}^2} \right) e^{y_{s,i} t_i - y_{s,i-1} t_{i-1}} \tag{12.24}$$

Thus, the duration components and vega of a risky bond can be written as

$$D_k = \frac{1}{p_r} \sum_i \left[\frac{c_i t_i \psi_{k,i}}{1 + y_s v_y^2 t_i^2} - \frac{\partial c_{f,i}}{\partial a_k} \right] \rho(t_i) e^{-y_{s,i} t_i} \tag{12.25}$$

$$\varpi = \frac{1}{p_r} \sum_i \left[\frac{c_i y_s^2 v_s t_i^3}{1 + y_s v_s^2 t_i^2} + \frac{\partial c_i}{\partial v_y} \right] \rho(t_i) e^{-y_{s,i} t_i} \tag{12.26}$$

Since the value of near term cash flows is a higher percentage of the price of a risky bond, the present value of long term cash flows and hence the convexity contribution falls exponentially with time. Ignoring the effect of volatility on the term structure of credit spreads, the interest rate duration of a floating rate credit bond can be simplified as

$$D_k = \frac{1}{p_r} \sum_i \left(c_i t_i \psi_{k,i} - \frac{\partial c_{f,i}}{\partial a_k} \right) \rho(t_i) e^{-y_{s,i} t_i} \tag{12.27}$$

where

$$\frac{\partial c_{f,i}}{\partial a_k} = \left(t_i \psi_{k,i} - t_{i-1} \psi_{k,i-1} \right) (\mu_i + c_{f,i}) \tag{12.28}$$

The credit duration of a floating rate bond does not depend on the floating rate coupon and is simply equal to

$$D_{c,k} = \frac{1}{p_r} \sum_i c_i t_i \psi_{k,i} \rho(t_i) e^{-y_{s,i} t_i} \tag{12.29}$$

If the floating coupon of a credit security is a function of real rates such as floating coupon of an inflation swap, then the real and nominal durations can be calculated by substituting (11.34) and (11.35) for (12.28). If the discount function is based on real rates, the necessary inflation lag adjustment must be made to the discount function.

12.6 TSCS EXAMPLES

Many corporations or sovereign nations issue bonds with different seniority levels. In case of default one or more bonds will get preferential treatment over other bonds. This is even true for the US government. In general, the principal of a bond is presumed to have seniority over its coupons. Additionally, there are many government agencies that issue bonds with different levels of seniority. For each level of seniority of an issuer's bonds, there is a different TSCS. Most credit agencies such as Moody's, S&P, and Fitch assign credit ratings to the bonds of an issuer based on the financials of the issuing entity and its capital structure as well as the position of the bond in the capital structure.

The government of Brazil has issued two sets of US dollar denominated bonds. The first set of these bonds, called Brady bonds, is the byproduct of restructured defaulted debt, which used to be owed to US banks. Another set of dollar denominated bonds issued by Brazil are called eurobonds, which were issued through competitive bidding in the capital markets.

Brazil has issued several US dollar denominated eurobonds across the maturity spectrum which can be used to calculate the TSCR. By comparison with the US curve, the TSCS can be calculated.

In Figure 12.1 the diamonds represent the relative yield of bonds compared to the TSCR for Brazil. Bonds that are below the Brazil curve have a lower yield than calculated and are rich (expensive) and bonds that are above the curve are cheap. Two legacy Brady bonds trade at a yield premium to the rest of the market even though they appear to have the same credit rating.

The forward curve in Figure 12.1 is the implied instantaneous spread or default rate (12.6). Thus, the market expects the credit quality of Brazil to deteriorate to a default rate of more than 6% per year in 12 years (assuming no recovery). This has interesting implications for portfolio management in an efficient market environment where there are no economic or other barriers to buying or selling securities. For example, consider a portfolio that is constructed by buying 100 million of the Brazil with a maturity of 10 years and selling equal market value of Brazil 5-year and hedging the US interest rate exposure of the trade. The implied forward spread of Brazil 5-year in the future would be about 2.8%, which is generous given the present levels.

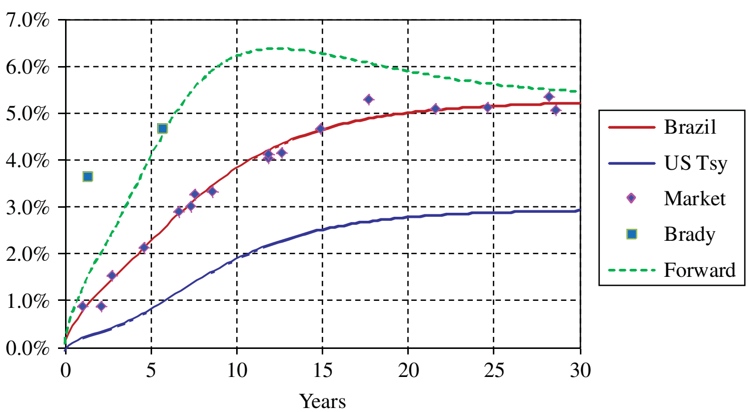


FIGURE 12.1 Credit spread of Brazil, May 25, 2012

TABLE 12.2 Term structure of Brazil, May 25, 2012

	Level	Slope	Bend	Cubic	Quartic
TSY	1.26%	1.56%	0.28%	−0.04%	−0.10%
BRAZIL	2.63%	2.63%	0.16%	−0.04%	−0.10%
SPREAD	1.37%	1.07%	−0.12%	0.00%	0.00%

TABLE 12.3 Term structure of European credit spreads, May 25, 2012

	Level	Slope	Bend	4th	5th
Germany	0.805%	1.339%	0.279%	−0.104%	−0.128%
Spreads					
Austria	0.345%	0.508%	−0.127%	0.000%	0.000%
Belgium	0.833%	0.775%	−0.253%	0.000%	0.000%
Spain	4.852%	0.198%	−0.845%	0.000%	0.000%
Finland	0.021%	0.270%	−0.282%	0.000%	0.000%
France	0.505%	0.625%	−0.087%	0.000%	0.000%
Ireland	3.489%	0.634%	−1.613%	0.000%	0.000%
Italy	4.137%	0.641%	−0.775%	0.000%	0.000%
Netherlands	0.119%	0.212%	−0.214%	0.000%	0.000%
Portugal	6.725%	1.649%	−3.165%	0.000%	0.000%

The yield spread of the Brazil A bond with a maturity of a little over 5 years is about 220 bps over the Brazil curve and appears to be very cheap, compared to other bonds, even though it is a legacy Brady bond.

Table 12.2 shows the parameters of the Brazil curve as well as the US curve and the resulting spread. Only the first three components were calculated; the remaining two parameters were matched to the treasury curve.

The TSCS applies to sovereign countries that issue a bond in a currency that they cannot print and thus are subject to default risk. For example, Latin American countries issuing bonds denominated in USD and all Euro countries have credit risks. In the euro zone, Germany is considered to be the most creditworthy nation and the spreads of all other countries are measured relative to Germany's.

Table 12.3 shows the term structure components of Germany and the spreads of a few euro zone countries relative to Germany's. The fourth and fifth components of the TSCS were matched to those of the German curve, so only three components were independently fitted to the data.

Figure 12.2 shows the TSIR and TSRR for Germany and the TSCR and TSRC (Term Structure of Real Credit) for France. The top curve is the calculated TSCR (Crd Trm) along with the actively traded bonds (Crd Mkt) for the French treasury market. The next curve is the German government curve which is a proxy for the euro zone treasury

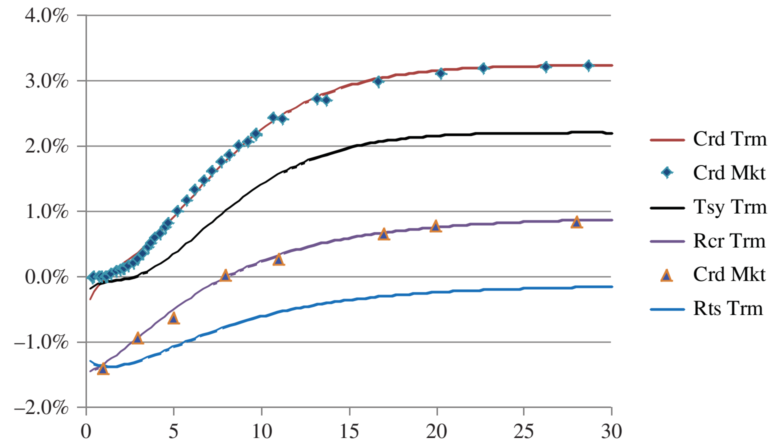


FIGURE 12.2 Term structures of rates in France and Germany, July 31, 2012

market (Tsy Trm). The third curve down is the term structure of real credit for France (Rcr Trm) as well as the French real treasury market (Crd Mkt). The bottom curve is the TSRR for the euro zone (Rts Trm).

France has an active real rates market (IL bonds) while Germany does not. In order to calculate the TSRR for Germany, we calculated the TSRR for France and subtracted the spread curve of French nominal treasuries relative to the German curve from the French real rates. This method works if both countries have similar inflation rates or use the same inflation index for IL bonds. If inflation rates are different and IL bonds use different inflation measures, then there will be no relationship between the spreads of nominal bonds and real bonds.

12.7 RELATIVE VALUES OF CREDIT SECURITIES

Measuring the spread of a security relative to the TSIR has many advantages over the traditional method of measuring the spread relative to a benchmark or an on-the-run treasury. Since on-the-run treasuries are sometimes on-special in the repo market, the spread tends to be exaggerated. For example, consider a security S whose spread of 80 bps is measured against treasury T. Assume that T is on-special and its yield is 20 bps below the curve. If T converges to the curve, while the spread of the credit security is unchanged, the spread of S falls to 60 bps relative to T. One would get the impression that S has outperformed credit securities that are measured against different treasuries. In practice, T has underperformed the curve by 20 bps and all other securities have maintained their relative valuations. Additionally, it is not always possible to compare the cash flows and maturity of a spread bond with an on-the-run treasury. The best measure of relative value for any security is its spread relative to its own curve.

Table 12.4 lists a number of relevant analytics for a few credit issues. This table and the calculated values require some explanation. The first two bonds are dollar denominated bonds issued by the Republic of Panama which has issued several dollar

TABLE 12.4 Analytics for selected credit securities, July 31, 2012

	Issue		Price		Spread		Treasury Duration			Libor Duration			Credit Duration				Yield	
Issuer	**Cpn**	**Maturity**	**Mkt**	**Model**	**Curve**	**Tsy Mkt**	**Level**	**Slope**	**Bend**	**Level**	**Slope**	**Bend**	**Level**	**Slope**	**Bend**	**Theta**	**Mkt**	**Expnt1**
PA Republic	8.1250	4/28/34	142.00	155.68	0.761%	2.733%	11.70	7.85	1.36				11.70	7.85	1.36	5.04%	4.95%	4.89%
PA Republic	6.7000	1/26/36	141.75	136.35	−0.289%	1.658%	13.52	9.84	3.32				13.52	9.84	3.32	4.08%	3.95%	3.92%
026609AC1	7.2500	3/1/23	143.00	143.91	0.079%	1.042%	7.85	3.01	−4.65				7.85	3.01	−4.65	3.84%	2.58%	2.56%
345370BJ8	8.8750	1/15/22	125.49	128.79	0.370%	4.011%	6.96	2.06	−4.93				6.96	2.06	−4.93	6.05%	5.40%	5.32%
AE Emirate	0.9034	12/7/16	87.00			4.031%	−0.17	0.01	0.26	−4.41	0.61	4.32	4.23	−0.60	−4.05	5.19%	4.57%	4.54%
Co Republic	2.0920	11/16/15	100.50	102.20	0.527%	1.797%	0.05	−0.05	0.05	−3.12	0.94	2.58	3.16	−0.99	−2.53	3.28%	2.17%	2.16%

denominated bonds and therefore its TSCR can be calculated. From the TSCR we can calculate the fair or model price as well as the spread of each bond relative to its curve. Since both bonds have fixed coupons, they have the same treasury and credit durations. One of the bonds is very cheap and one rich relative to Panama's credit curve. The spread relative to the credit curve is a much better measure of the cheapness or richness of a bond than yield or spread relative to the treasury curve.

The last two bonds in the table have floating coupons based on Libor rates. If the Libor spread widens, the future coupons of the bonds will increase without directly changing the credit curve and therefore the price of the bond increases. The Libor duration of floating credit bonds is thus negative. There are not enough bonds for AE Emirate to calculate the TSCR and therefore no spread to curve and model prices were calculated.

12.8 PERFORMANCE ATTRIBUTION OF CREDIT SECURITIES

We can write the general price of a credit security as

$$p_m = \sum_i c_i e^{-(y_i + s_{c,i} + s_b)t_i} \tag{12.30}$$

where $s_{c,i}$ is the spread curve of the credit, and s_b is the spread of the security (bond) relative to its curve. The spread of the security is calculated as before in such a way that the discounted value of the cash flows will match the market price of the security. The change in the performance can be calculated as

$$\begin{aligned}\frac{\Delta p_m}{p_m} = &-\sum_k D_{g,k}\Delta a_{g,k} - \sum_k D_{l,k}\Delta a_{l,k} - \sum_k D_{r,k}\Delta a_{r,k} \\ &-\sum_k D_{c,k}\Delta a_{c,k} - D_{c,0}\Delta s_b\end{aligned} \tag{12.31}$$

where the subscript g is for government (treasury), l is for Libor, r is for real, c is for credit, and b is for the bond (security). In a complex portfolio, performance attribution can only be done through the decomposition of risk components. Table 12.5 shows a sample from an emerging markets portfolio report including performance attribution. This table is relatively detailed and requires explanation for the interpretation and derivation of some of the fields.

At the beginning of the month the market value of the benchmark (JP Morgan EMBI+) is scaled to match the market value of the portfolio. Throughout the month the market values are allowed to move independently based on performance and the difference between the two is the relative performance of the portfolio. The calculated performance for the portfolio and the benchmark are in cells V6 (=532.28 bps) and V10 (=445.95 bps), respectively. Therefore, the portfolio has outperformed the benchmark by 86 bps in the month. The interest rate (treasury), Libor, and credit durations of the portfolio and benchmark are in rows 4–6 and 8–10 for the beginning of the month, the previous day, and the last day. There is no exposure to Libor for the portfolio or the benchmark. The term structure of rates for treasury and Libor are in rows 12–14.

TABLE 12.5 Emerging markets portfolio report

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V
1						Duration											Spread to					
2								Treasury				Libor			Credit		TSY	Curve	Theta	Yield	Perf., bps	
3		MV				*Level*	*Slope*	*Bend*	*Cubic*	*Quartic*	*Level*	*Slope*	*Bend*	*Level*	*Slope*	*Bend*					Actual	
4	6/29/12	169,459,160				8.05	3.78	-2.10	-3.09	-1.90				8.05	3.78	-2.10	4.04%	0.39%	6.76%	5.96%		
5	7/30/12	178,023,957		Portfolio		8.23	3.99	-1.97	-3.09	-2.00				8.23	3.99	-1.97	3.60%	0.37%	6.14%	5.41%		505.42
6	7/31/12	178,479,143				8.23	3.99	-1.96	-3.09	-1.98				8.23	3.99	-1.96	3.60%	0.37%	6.13%	5.38%		532.28
7																						
8	6/29/12	169,459,160				7.71	3.47	-1.86	-1.98	-0.58				7.71	3.47	-1.86	3.04%	-0.05%	5.46%	4.91%		
9	7/30/12	176,466,656		Index		7.91	3.68	-1.74	-1.97	-0.65				7.91	3.68	-1.74	2.70%	-0.07%	4.98%	4.45%		413.52
10	7/31/12	177,016,198				7.92	3.69	-1.74	-1.97	-0.66				7.92	3.69	-1.74	2.69%	-0.07%	4.95%	4.42%		445.95
11																						
12	6/29/12					1.20%	1.57%	0.22%	0.02%	-0.15%	1.32%	1.20%	0.16%								Port	Index
13	7/30/12			Curve		1.09%	1.50%	0.24%	0.01%	-0.13%	1.24%	1.06%	0.34%							Rate	142.01	135.30
14	7/31/12					1.06%	1.49%	0.25%	0.01%	-0.13%	1.22%	1.04%	0.34%							Libor	0.00	0.00
15																				Credit	325.81	251.56
16				Performance bps, Portfolio		107.30	30.75	6.21	-5.36	3.11				244.19	70.46	11.16	322.32	11.97	54.63	Scrty	11.97	12.90
17				Performance bps, Index		103.46	28.67	5.52	-3.37	1.00				156.63	86.67	8.27	253.80	12.90	44.12	Carry	54.63	44.12
18																						
19																				Total	534.42	443.88
20																						
21			Market Value			Spread Duration					Spread to Treasury				Spread Curve						Sector Perf.	
22		Port	Index	Net		Port		Net			Port	Index	Net		BOM		NOW				Port	Index
23	Sector					*Level*	*Slope*	*Level*	*Slope*						*Level*	*Slope*	*Level*	*Slope*				
24	AR	7,259,423	3,630,429	3,628,993		0.33	0.16	0.15	0.07		11.51%	10.49%	1.02%		12.9%	-4.8%	12.57%	-4.05%			8.00	2.34
25	BG	0	813,116	-813,116		0.00	0.00	-0.01	0.00		0.00%	2.38%	-2.38%		3.20%	0.00%	2.38%	0.00%			0.00	1.01
26	BR	12,327,122	21,881,914	-9,554,792		0.68	0.39	-0.41	-0.17		1.91%	1.34%	0.57%		1.31%	0.73%	1.21%	0.41%			16.08	36.40
27	CO	1,215,203	9,856,440	-8,641,238		0.09	0.07	-0.40	-0.19		1.54%	1.39%	0.15%		1.47%	0.79%	1.39%	0.64%			2.05	10.92
28	EC	0	370,962	-370,962		0.00	0.00	-0.01	0.00		0.00%	8.80%	-8.80%		9.32%	0.00%	8.80%	0.00%			0.00	0.50
29	HU	0	3,670,920	-3,670,920		0.00	0.00	-0.15	-0.06		0.00%	4.82%	-4.82%		5.66%	-0.58%	4.99%	-0.15%			0.00	8.12
30	HR	0	2,507,751	-2,507,751		0.00	0.00	-0.09	-0.02		0.00%	4.86%	-4.86%		5.55%	-0.54%	4.86%	-0.05%			0.00	6.11
31	ID	2,448,455	11,617,732	-9,169,277		0.12	0.06	-0.40	-0.18		2.23%	2.14%	0.09%		2.35%	0.24%	2.09%	0.09%			5.22	21.27
32	MX	7,865,466	24,373,790	-16,508,324		0.84	0.70	-0.39	0.04		2.08%	1.41%	0.68%		1.50%	0.52%	1.41%	0.16%			28.74	41.22
33	PA	0	5,528,237	-5,528,237		0.00	0.00	-0.30	-0.17		0.00%	1.61%	-1.61%		1.57%	0.91%	1.61%	0.62%			0.00	5.96
34	PE	731,070	7,378,005	-6,646,934		0.07	0.06	-0.41	-0.26		1.55%	1.08%	0.47%		1.20%	0.35%	1.27%	-0.26%			2.23	15.38
35	PH	6,510,350	16,983,213	-10,472,863		0.39	0.23	-0.48	-0.23		1.83%	1.65%	0.18%		1.85%	0.27%	1.60%	0.15%			14.40	27.94
36	TR	27,794,557	23,484,837	4,309,720		0.74	0.00	-0.29	-0.48		2.60%	2.56%	0.04%		2.77%	0.41%	2.52%	0.08%			31.59	48.14
37	UA	1,007,300	2,331,171	-1,323,871		0.00	0.00	-0.05	0.00		7.88%	8.15%	-0.28%		7.95%	-1.82%	7.28%	-1.12%			1.12	2.22
38	VE	24,750,975	17,153,151	7,597,824		0.72	0.13	0.14	-0.06		11.31%	10.35%	0.97%		8.69%	1.89%	8.46%	1.84%			61.98	30.03
39	ZA	0	0	0		0.00	0.00	0.00	0.00		0.00%	0.00%	0.00%		0.00%	0.00%	0.00%	0.00%			0.00	0.00
40	RU	83,733,926	25,434,527	58,299,398		4.26	2.21	3.42	2.00		2.71%	2.00%	0.71%		2.20%	0.93%	1.80%	0.78%			224.07	50.55
41	UST	0	0	0		0.00	0.00	0.00	0.00		0.00%	0.00%	0.00%								0.00	0.00
42	Cash	2,835,295	0	2,835,295																		

For example, the change in level of rates in the month was $1.20\% - 1.06\% = 14$ bps. The contribution to the performance of level duration for the portfolio would thus be approximately $14 \times 8.05 = 112$ bps. The accurate contribution in cell F16 (=107.3) is in basis points.

The recovery value of all bonds is assumed to be zero and there are no floating rate bonds in the portfolio or the benchmark, thus the overall credit durations of the portfolio and benchmark are equal to the interest rate durations. The performance of each sector or country, however, needs to be calculated by multiplying the change in the credit term structure of the respective credit and credit durations. The overall performance of credit curve is calculated in rows 16–17 and columns N–P. The level and slope of the spread curve at the beginning of the month and the last day are shown in columns O–R and rows 24–40. The credit performance of each sector including contributions from curve, are shown in columns U–V and from row 24.

The overall performance of each category is in column U–V and rows 13–19. For example, the rate performance of 142.01 bps for the portfolio is the sum of performance from level, slope, etc. We can also see that the security selection performance for the portfolio is $(0.388\% - 0.372\%) \times 8.05 = 13$ bps. The calculated performance of the portfolio and the index is about 2 bps different from the market value performance calculation.

The performance of each security is calculated based on its exposure to all applicable curves plus security selection and theta or carry (yield plus rolldown). We can also estimate the total performance of the portfolio and benchmark by multiplying the change in spread to the treasury times the level duration. For the portfolio, the contribution of credit is $(4.04 - 3.60) \times 8.05 = 352$ bps. Add to this the interest rate contribution of 142 bps and the yield (not carry) contribution of 48 bps, and the total will be 542 bps which is close to the total performance.

The small errors in performance attribution are due to rounding errors, ignoring convexity, and pricing inefficiencies where some bonds are not priced every day which could distort credit curves. Overall the accuracy is excellent.

12.9 TERM STRUCTURE OF AGENCIES

The US government agencies are among the largest issuers of debt in the world. There has been much debate about the implied guarantee of the US government of the debt issued by agencies. The general market consensus before the Lehman bankruptcy was that the US government had an implicit guarantee and their short term debt traded at very tight spreads relative to US treasuries. The US government was forced to bail out both Fannie Mae and Freddie Mac after the Lehman bankruptcy.

Nonetheless, the debt of both agencies trades at very tight spreads to the US government debt and their liquidity is very high. The term structure of agency rates (TSAR) can be calculated similarly to the TSCR. Given the slight perceptions of the debt of different agencies, one TSAR can be calculated for each agency.

Table 12.6 is a sample of the term structure of agency spreads. There is very little differentiation between the four largest agencies in pricing and yield as they all enjoy the same implicit backing from the government. The positive slope of the credit spread points to increasing spread with maturity. At the front of the curve the spread is equal to level minus slope plus bend (see equation (2.22)) and thus the spreads are very tight to the treasury market.

TABLE 12.6 Term structure of agency spreads, July 30, 2012

Name	Ticker	S & P	Moodys	Level	Slope	Bend
Federal Home Loan Bank	FHLB	AGN	AGN	0.42%	0.40%	0.13%
Freddie Mac	FHLMC	AGN	AGN	0.35%	0.40%	0.07%
Fannie Mae	FNMA	AGN	AGN	0.36%	0.54%	−0.09%
Tenn Valley Authority	TVA	AGN	AGN	0.48%	0.47%	0.16%

12.10 PERFORMANCE CONTRIBUTION

Performance attribution is the business of finding out the sources of risk and return in a portfolio relative to a benchmark or in absolute terms. Performance contribution is the process of relating the sources of return to respective teams or individuals who contribute to the management of a portfolio. These two processes are related but are not the same. In fixed income, there are two ways to allocate risk to a portfolio: by market value or by (spread) duration. The interplay of these two allocation paradigms has often meaningful and sometimes large consequences for performance contribution and the overall performance of a portfolio. Performance contribution is strongly dependent on investment process.

Let us analyze a hypothetical example to delineate the interplay of risk allocation by market value and spread duration on a portfolio. Consider a fixed income benchmark with two issuers. The performance of the portfolio managers is measured by sector/issuer allocation, while the performance of analysts is measured by the change in spread of their securities. Issuer A has a very steep credit curve that becomes steeper after 1 year, and issuer B has an inverted credit spread curve that inverts further after 1 year. Investment policy requires neutral sector/issuer contribution to duration. Table 12.7 shows the weight and performance of each issuer after 1 year.

TABLE 12.7 Performance contribution example

	Index		Port	
	Issuer A	Issuer B	Issuer A	Issuer B
Mv Weight	40%	60%	70%	30%
Duration Contribution	2	3	2	3
Spread	4%	3%	2.50%	1.50%
Spread Change	0	0	−0.15%	−0.15%
Excess Perf. Sprd	1.60%	1.80%	1.75%	0.45%
Sprd Change Perf.	0.00%	0.00%	0.30%	0.45%
Sector Performance	1.60%	1.80%	2.05%	0.90%
Portfolio Performance	3.40%		2.95%	

The average duration of all the bonds by issuers A and B is 5 years. The market value weight of the index for issuers A and B is 40% and 60% and the contributions to duration are 2 and 3 years, respectively. However, for the portfolio, the analyst for issuer A selects a bond with a lower duration than the average and requires 70% of the portfolio to get the required 2 years of spread duration. Likewise, the analyst for issuer B selects a higher duration bond than average and requires only 30% of the market value to achieve the required contribution to duration of 3 years.

After 1 year, the spread of issuer A becomes steeper, while the average spread stays unchanged. Thus, its contribution to the performance of the index is zero. However, since the security that the analyst selected was at the front of the curve, its spread fell by 15 bps. Analyst A is credited with a performance contribution of $15 \times 2 = 30$ bps. Likewise, the spread curve of security B flattens and the spread of the security that analyst B selected at the long end of the credit curve falls by 15 bps even though the overall spread of B stayed unchanged. Analyst B is credited with $15 \times 3 = 45$ bps of performance. The excess return of the benchmark relative to treasuries is $(40\% \times 4) + (60\% \times 3) = 340$ bps. Since the portfolio manager overweighted the higher returning issuer, he will be credited with a performance contribution of $(70\% \times 4) + (30\% \times 3) = 370$ bps or 30 bps of excess return. In practice, since allocation to A was at the short end of the steep curve, the yield of A was only 2.5% and the performance of A after 1 year was $70\% \times 2.5 = 175$ bps plus 30 bps for spread change, a total of 205 bps. The performance of B in the portfolio was 90 bps. The total performance of the portfolio was 295 bps, which is 45 bps below the benchmark return. So why did the portfolio underperform the benchmark while the analysts and portfolio manager were all credited with positive contributions? The simple answer is inconsistent metrics and incentives for performance contribution.

The key to performance contribution is its additive property; the contributions of analysts, portfolio managers, and the chief investment officer have to sum to the performance of the portfolio, just as performance attribution does. The performance targets that are set for analysts and portfolio managers have to be practical, achievable, curve neutral, and, as far as possible, non-directional. Most importantly, the incentive structure of all involved parties has to be aligned and measureable with the performance of the portfolio. Let us review the implications of these objectives in practice.

If the performance of analysts is measured by spread change, it can lead to suboptimal sector selection, as we saw in the previous example. The most stable and predictable source of return in a portfolio is the yield or carry adjusted for default. In the long run, the performance is dominated by yield and has to be incorporated into performance contribution if analysts are to pay attention to it. If we use excess return (return adjusted by treasury return) as an incentive for performance contribution, where both yield and spread change contribute to performance, higher market value allocation to a sector will increase its excess return at the expense of other sectors. In the above example, analyst A had excess return of 175 bps from spread which was more than the excess return of issuer A in the index by 15 bps, even though analyst A used a very low yielding security. In fact, if excess return is the yardstick for measuring performance contribution, seasoned analysts tend to overweight their sectors as safe bets to achieve positive contribution.

One way to measure the performance contribution of analysts is by measuring the performance of issuers (or names) that they select to outperform their sectors similarly to the way equity analysts pick names. If a credit index such as the corporate part of the Barclays Aggregate Bond Index has 700 issuers in it, a portfolio manager may need only 100 names to replicate it. If there are 12 sectors in the index and there are four analysts, each analyst may be assigned three sectors to cover. Each analyst needs to provide about ten or more names that he has the most confidence in. The portfolio manager can pick from all the names that analysts have provided to construct the portfolio. He can choose not to include some names in the portfolio or to overweight or underweight some sectors, but those are the contribution risks that the portfolio manager takes, not the analysts.

The performance contribution of analysts in each sector will be measured by assigning equal market value weighting to all the picked names in a sector and subtracting the sector return from those names. For example, if healthcare is 12% of the index with an excess return of 160 bps, and an analyst covering healthcare has selected 15 names with average excess returns of 190 bps, the analyst will be credited with a performance contribution of $(190 - 160) \times 12\% = 3.6$ bps for the entire portfolio. The portfolio manager can in fact use the analyst's recommendation and realize the same return for the portfolio. However, if the portfolio manager decides not to use some of those issues, that is the risk that he takes. In Chapter 19 we will explain how the portfolio manager can take the analysts' recommendations and construct the portfolio.

The performance contribution of a portfolio manager will be measured by the excess return of the asset class minus the excess return of the analysts. Transaction costs, which can be high for credit portfolios, need also to be taken into account and there has to be a disincentive for excessive name changing and trading recommendations. Transaction costs can be estimated by the price or yield spread that a market maker makes for a security and can be charged equally to the performance of analysts and portfolio manager.

A chief investment officer or investment policy committee can change the allocation of asset classes, be responsible for the overall duration of the portfolio allocation to out-of-benchmark asset classes such as currencies, emerging markets, and high yields, and his performance can be measured by the excess returns of these decisions.

In general, performance contribution is a very important yet sensitive and somewhat imprecise science. It cannot be performed in fixed income without accurate security level analytics to decompose the respective returns of all securities, sectors, and asset classes. More importantly, performance contribution has to be aligned with investment process and how risk is allocated and measured. Many of the off-the-shelf packages for fixed income fail to understand the interplay of yield, spread, duration, and spread duration for performance contribution.

12.11 PARTIAL YIELD

Yield is a very useful measure of value for a fixed income security. However, for complex securities yield can sometimes lose its intended meaning and usefulness. Consider a UK IL gilt stock with 8 months of inflation lag and 1 year to maturity. The yield of such a security is a composite value of 4 months of real yield and 8 months of nominal yield. Likewise, a corporate bond with floating Libor coupon has exposure to interest rates (treasury), Libor, and credit rates. What is the meaning of yield for such a security? Spread is a much better measure of value for such a security. Now consider a 10-year

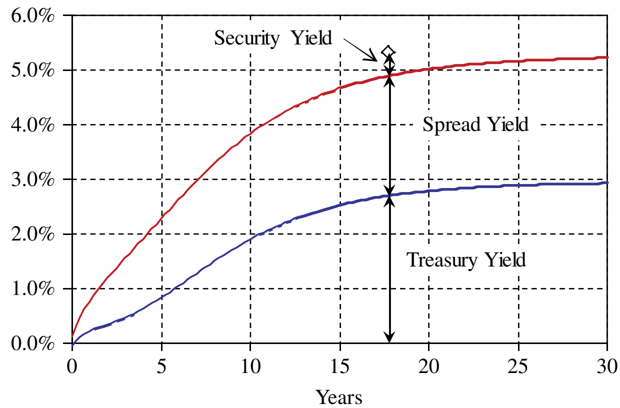


FIGURE 12.3 Contribution to partial yield

corporate bond that has 30% probability of default in its lifetime, where 40% of the principal value can be recovered upon default. For such a security, even the spread is not a good measure of value if it is not adjusted by default probability.

We introduce the concept of partial yield as the weighted contribution of all component cash flows that make up the price of a security over its life. Figure 12.3 shows the treasury and credit curve as well as the spread of a security. To calculate the partial credit yield of the security, we multiply the spread of the security over the treasury curve at every cash flow and duration weight the present value of all those cash flows and divide the final result by the duration market value of the security. The advantage of duration weighting is that, for bonds with no recovery, the sum of partial yields will be very close to the calculated yield of the security. For bonds with implied recovery, the contribution of recovery to the price is discounted by Libor and the bond will have a Libor partial yield.

In Table 12.8 partial yields of selected securities are shown along with continuously compounded yield calculated based on price. The penultimate column (Sum) is the sum of the four partial yields of a security plus security specific spread contribution to yield. For most bonds the sum of partial yields and the calculated yield based on market price are within 1 basis point of each other. For IL bonds the calculated yield is slightly different from the market convention yield and is based on full inflation accrual. For example, for the US on July 31, 2012, the inflation for the month of June has been released and the inflation accrual is known through the end of August. The market convention is to use the reference inflation for August 1 for invoice price calculation.

We assumed that German Bunds represent the equivalent of treasury quality in EUR denominated bonds and all other countries are credit issuers. The term structure of real rates for Italy is calculated as

$$y_{c,r} = y_r + (y_{c,t} - y_t) = y_r + s \tag{12.32}$$

where $y_{c,r}$ and $y_{c,t}$ are the real and nominal yield of a credit issuer (Italy) and y_r and y_t are the real and nominal yield of the treasury issuer (Germany).

TABLE 12.8 Partial yields of selected securities, July 31, 2012

						Partial Yield						
Sctr	Cntry	Cpn	Matur	Price	Sprd	Tsy	Real	Libor	Crdit	Secr	Sum	Yield
Tsy	US	0	8/15/22	84.97	−0.015%	1.638%	0.000%	0.000%	0.000%	−0.015%	1.623%	1.623%
Tsy	US	1.75	5/15/22	102.55	−0.075%	1.538%	0.000%	0.000%	0.000%	−0.075%	1.462%	1.463%
Tsy	US	6.25	8/15/23	147.55	−0.072%	1.609%	0.000%	0.000%	0.000%	−0.072%	1.537%	1.542%
Tsy	DE	1.75	7/4/22	104.38	−0.054%	1.321%	0.000%	0.000%	0.000%	−0.054%	1.267%	1.269%
IL	US	0.13	7/15/22	108.70	−0.066%	0.057%	−0.718%	0.000%	0.000%	−0.066%	−0.727%	−0.727%
IL	IT	2.35	9/15/19	83.86	0.827%	0.054%	−0.875%	0.000%	4.954%	0.827%	4.961%	4.960%
IL	GB	1.88	11/22/22	127.75	0.165%	0.055%	−0.967%	0.000%	0.000%	0.165%	−0.747%	−0.747%
Corp	US	7.25	3/1/23	143.00	0.079%	1.515%	0.000%	0.000%	0.962%	0.079%	2.557%	2.563%
Corp	US	8.88	1/15/22	125.49	0.370%	1.320%	0.000%	0.000%	3.624%	0.370%	5.315%	5.321%
Swap	BR	0	1/3/22	41.96	−0.006%	9.218%	0.000%	0.000%	0.000%	−0.006%	9.212%	9.212%
Swap	HU	6.73	8/2/22	100.00	−0.009%	7.149%	0.000%	−0.624%	0.000%	−0.009%	6.517%	6.517%

The partial yield for the security, "Secr", applies to all cash flows that have the credit risk of the security and is the same as the spread of the security relative to its credit curve for nearly all bonds.

The partial yield is a much better indicator of value than market conventional yield for complex securities since it decomposes the yield into its respective components.

CHAPTER 13

Default and Recovery

Default and recovery are the ever-present risks of investing in credit securities. While most companies do their best to prevent default, it is sometimes inevitable that a company is forced to default due to factors outside its control. Default can take place for one of the following reasons:

- *Inability to pay*. A company can run out of cash due to poor management, a downturn in economy or risk aversion by suppliers. In 2008, some retailers filed for bankruptcy due to the withdrawal of supplier lines of credit.
- *Unwillingness to pay*. Sovereign countries can default on their debt simply because they choose to do so. Creditors have great difficulty freezing the assets of the sovereign nations or forcing them to pay their debt and often are forced to negotiate a reduction in principal of the debt.
- *Unsustainable dynamics*. While a company can have enough cash and income to service its debt, it may have low sales or high interest payments that would eventually drain its cash position and it can file for bankruptcy to protect its business.

Most companies borrow or draw on their line of credit before filing for bankruptcy, since getting financing would be very difficult afterwards. If a company has a high burn rate, the bonds can appreciate after default and bankruptcy filing due to a higher expected recovery value. Sometimes, creditors try to force a company into bankruptcy to protect their assets.

13.1 RECOVERY, GUARANTEE AND DEFAULT PROBABILITY

It is standard practice to calculate the spread of a security assuming that the recovery value of a defaulted security is zero. In most cases the recovery value is not zero and a fraction r_i of the remaining principal, μ_i, of the bond at time t_i can be recovered upon default. The present value of a security that defaults at time t_i is

$$p_c(t_i) = r_i\mu_i\, e^{-y_s(t_i)t_i} \tag{13.1}$$

Equation (13.1) assumes that the recovery value is discounted by treasury rate. It is more appropriate to discount the recovery value by Libor:

$$p_c(t_i) = r_i \mu_i \, e^{-y_{l,s}(t_i)t_i} \tag{13.2}$$

Theoretically, if $r(t_i)$ is the instantaneous recovery rate, the value of recovery at periodic coupon intervals i will be equal to

$$r_i = e^{y_{l,s}(t_i)t_i} \int_{t_{i-1}}^{t_i} r(t) e^{-y_{l,s}(t)t} dt \tag{13.3}$$

The default probability in a period leading to time t_i is the difference between survival probabilities at the beginning and end of the period. If s_i is the spread of a security relative to the treasuries at time t_i, then from (12.8) we have

$$\rho_{i-1} - \rho_i = e^{-s_{i-1}t_{i-1}} - e^{-s_i t_i} = \left(e^{-s_{i-1}t_{i-1}+s_i t_i} - 1\right)\rho_i \tag{13.4}$$

For sovereign nations that cannot be forced into bankruptcy and there is significant uncertainty about their willingness to pay or the amount that can be recovered in case of default, a guarantee may be demanded from investors at the time of issuance of the debt. Some of the debt of South American and Eastern European countries in the 1980s was restructured under the Brady plan; it was subsequently referred to as Brady bonds. Most Brady bonds had principal and/or rolling interest guarantees (RIGs). The principal guarantee was usually a US Treasury zero coupon security and the RIG consisted of high quality deposits with third parties. The RIG would be used only in case of default and it typically covered two or more coupon payments. If there was a default, the RIG would be used to make coupon payments until it run out. However, if the sovereign country made a coupon payment, the RIG would be rolled forward. The RIG was like a partial put option on the security.

For securities that have a principal guarantee and/or RIG, the recovery applies to the exposure of the security to the credit adjusted by the value of guarantees at the time of default. Thus, if $g_{c,j,i}$ is the guaranteed coupon cash flow at time t_i due to default in the period $(t_{j-1,} t_i)$, then the guaranteed portion of the coupon cash flow is equal to the default probability in all the prior periods times the respective guarantee amount, namely,

$$g_{c,i}(t_i) = \sum_{j=1}^{i} \left(\rho_{j-1} - \rho_j\right) g_{c,j,i} \tag{13.5}$$

Near the maturity date, the coupon guarantee will take the form

$$g_{c,i}(t_i) = \sum_{j=j_g}^{n} \rho_j g_{c,j,i}, \quad j \geq j_g \tag{13.6}$$

where n is the number of cash flows of the bond and j_g is the index number for the date at which the RIG will be equal to or larger than the sum of remaining coupons through maturity date. For example, if a bond has a coupon guarantee of \$8 with semi-annual coupon payments of \$4 per par value, then, if the bond has not defaulted 1 year prior to maturity, the last two coupons will be paid from the guarantee. At that point the cash flows need to be discounted by Libor or treasury rates depending on the quality of the guarantee. Usually, the principal guarantee is treasury quality but the coupon guarantee is a deposit in a bank or escrow company and is of Libor quality. Each guarantee has to be discounted by its respective discount function.

We define the value of the guaranteed portion of the principal cash flow as

$$g_{p,i}(t_i) = \sum_{j=1}^{i} \left(\rho_{j-1} - \rho_j\right) g_{p,j,i} \tag{13.7}$$

The total guaranteed cash flow from default in the past is

$$\begin{aligned} g_{t,i} &= g_{c,i} + g_{p,i} \\ g_{t,i,j} &= g_{c,i,j} + g_{p,i,j} \end{aligned} \tag{13.8}$$

In order to analyze bonds with principal and/or coupon guarantees for default and recovery, we need to calculate the guaranteed portion of the cash flows and subtract it from the outstanding principal to calculate the credit risk of the security. If there is default in the interval leading to t_i cash flow, the present value at time t_i of all future cash flows from guarantees will be

$$p_{g,i} = e^{y_i t_i} \sum_{j=i+1}^{n} g_{t,i,j} e^{-y_j t_j} \tag{13.9}$$

This is provided that the guaranteed cash flows are discounted at treasury rates. If some of them are discounted by Libor, we have to modify equation (13.9) to adjust for the discount function. The recovery of a defaulted bond with implied guarantees applies only to the credit portion of the cash flows. Thus, the amount of credit that is subject to recovery is equal to $\mu_i - p_{g,i}$, of which only $r_i(\mu_i - p_{g,i})$ can be recovered. Thus, the sum of recovery and guarantee from default in the period (t_{i-1}, t_i) is

$$p_{g,i} + r_i(\mu_i - p_{g,i}) = r_i \mu_i + (1 - r_i) p_{g,i} \tag{13.10}$$

where $r_i \mu_i$ is the recovery amount, $p_{g,i}$ is the present value of all the guarantees at time t_i, and $r_i p_{g,i}$ is the amount that is not subject to recovery because it is guaranteed.

There is no contribution to recovery from default in the prior periods. In this formulation, the guaranteed amount is scaled by $(1 - r_i)$, instead of adjusting the recovery amount by the guarantee. Thus, the effective cash flow (12.13) of a bond with guarantees and recovery potential is given by

$$c_{e,i} = c_i \rho_i + g_{t,i} + \left(\rho_{i-1} - \rho_i\right) p_{g,i} + \left(\rho_{i-1} - \rho_i\right) r_i \left(\mu_i - p_{g,i}\right). \tag{13.11}$$

Here we have multiplied the cash flow c_i by the survival probability ρ_i. $g_{t,i}$ denotes the cash flows from prior defaults, and $p_{g,i}$ the present value of all guarantees. The term $\rho_{i-1} - \rho_i$ is the default probability in the period (t_{i-1}, t_i) that will result in the recovery r_i for the portion of the credit exposure that is not guaranteed $\left(\mu_i - p_{g,i}\right)$ plus guarantees $p_{g,i}$.

If there are no guarantees, that is, if $g_{t,i}$ is zero in (13.9), equation (13.11) can be rewritten as

$$c_{e,i} = \left(r_i c_{f,c,i} + c_i\right) e^{-s_{s,c,i} t_i} \tag{13.12}$$

where

$$c_{f,c,i} = \left(e^{s_{s,c,i}t_i - s_{s,c,i-1}t_{i-1}} - 1\right)\mu_i = \frac{(\rho_{i-1} - \rho_i)}{\rho_i}\,\mu_i \tag{13.13}$$

is the implied forward spread coupon of the risky bond in the interval leading to the ith cash flow.

The generalized price of a risky bond is the present value sum of cash flows in (13.11). After some simplifications and noting that the guarantee is scaled by $(1 - r_i)$, it can be written as

$$p_m = \sum_i \left(c_i\rho_i + g_{t,i} + c_{f,c,i}\rho_i r_i + (1 - r_i)\frac{c_{f,c,i}}{\mu_i}\,\rho_i p_{g,i} \right) e^{-y_i t_i} \tag{13.14}$$

This leads to the redefinition of $c_{e,i}$ in (13.11) as

$$c_{e,i} = c_i\rho_i + g_{t,i} + c_{f,c,i}\rho_i r_i + (1 - r_i)\frac{c_{f,c,i}}{\mu_i}\,\rho_i p_{g,i} \tag{13.15}$$

Let us define the constant recovery rate r_c as a recovery rate that calculates the market price of a security from its credit term structure, that is,

$$p_m = \sum_i \left(c_i\rho_i + g_{t,i} + c_{f,c,i}\rho_i r_c + (1 - r_c)\frac{c_{f,c,i}}{\mu_i}\,\rho_i p_{g,i} \right) e^{-y_i t_i} \tag{13.16}$$

r_c is a measure of the cheapness or richness of a security and applies to all securities even if they have zero expected recovery rate. If r_c is higher than the expected recovery rate, the security is rich; if it is lower, the security is cheap.

Assuming no cash flow guarantee, the market price can be simplified as

$$p_m = p_{nr} + \sum_i r_i c_{f,c,i} e^{-(y_{s,i} + s_{s,c,i})t_i} = \sum_i (c_i + r_i c_{f,c,i})\rho_i e^{-y_{s,i}t_i} \tag{13.17}$$

where p_{nr} is the price with no recovery of the security given in (12.12). The constant recovery rate is equal to the present value of the recovery in the future, which is equal to default probability in a period multiplied by the recovery amount and discounted to the present time. Thus,

$$p_m - p_{nr} = 100 r_c \sum_i (\rho_{i-1} - \rho_i) e^{-y_{s,i}t_i} = r_c \sum_i c_{f,c,i}\rho_i e^{-y_{s,i}t_i} \tag{13.18}$$

With guarantee, the constant recovery rate can be calculated as

$$\begin{aligned} p_m - p_{nr} - \sum_i \left(g_{t,i} + \frac{c_{f,c,i}}{\mu_i}\,\rho_i p_{g,i} \right) e^{-y_{s,i}t_i} \\ = r_c \sum_i \left(c_{f,c,i} - \frac{c_{f,c,i}}{\mu_i}\,p_{g,i} \right) \rho_i e^{-y_{s,i}t_i} \end{aligned} \tag{13.19}$$

If the principal guarantee is discounted by treasury rate and the coupon guarantee and recovery rates are discounted by Libor, we need to modify the summation for guarantee in (13.9) as

$$p_{g,i} = e^{y_i t_i} \sum_{j=i+1}^{n} g_{p,i,j} e^{-y_j t_j} + e^{y_{l,i} t_i} \sum_{j=i+1}^{n} g_{c,i,j} e^{-y_{l,j} t_j} \tag{13.20}$$

For bonds where there is no guaranteed principal and coupon, we can simplify (13.16) to

$$p_m = \sum_i c_i \rho_i e^{-y_i t_i} + \sum_i c_{f,c,i} \rho_i r_i e^{-y_{l,i} t_i} \tag{13.21}$$

Defining the effective recovery rate r_{ie} as

$$\begin{aligned} r_i e^{-y_{l,i} t_i} &= r_{ie} e^{-y_i t_i} \\ r_{ie} &= r_i e^{-(y_{l,i} - y_i) t_i} \end{aligned} \tag{13.22}$$

we can write (13.21) as

$$p_m = \sum_i (c_i + c_{f,c,i} r_{ie}) e^{-(y_i + s_i) t_i} \tag{13.23}$$

The effective cash flow for bonds that have guarantees but no recovery can be simplified as

$$c_{e,i} = c_i \rho_i + g_{t,i} \tag{13.24}$$

Here, we need not concern ourselves with future guarantees, since all future guarantees will be accounted for at their scheduled time. The effective cash flow depends only on the spread. However, for floating coupon bonds, the amount of available guarantee depends on the coupon rate and is not fixed.

13.2 RISK MEASUREMENT WITH RECOVERY

For bonds with recovery value but no guarantees, which applies to most credit securities, the price function from (13.12) is

$$p_m = \sum_i \left(r_i c_{f,c,i} + c_i \right) e^{-y_{s,i} t_i - s_{s,c,i} t_i} \tag{13.25}$$

From (13.13) we can write

$$\frac{\partial c_{f,c,i}}{\partial a_{c,k}} = (t_i \psi_{k,i} - t_{i-1} \psi_{k,i-1}) e^{t_i s_{s,c,i} - t_{i-1} s_{s,c,i-1}} \mu_i \tag{13.26}$$

We can use (13.26) to develop an analytic equation for the duration and convexity of bonds with recovery value. In practice it is more convenient to calculate the price of such securities and shift the spread curve to calculate the duration and convexity components. This method can be used even for the most complex securities involving guarantee, recovery value, and floating coupon. If $p_m(\Delta s_{a,k})$ is the price of the security with a shift in the kth component of the spread curve, then

$$D_{c,k} = -\frac{1}{p_c}\frac{p_m(\Delta s_{a,k}) - p_m(-\Delta s_{a,k})}{2\Delta s_{a,k}} \tag{13.27}$$

$$\begin{aligned} X_{c,kl} = \frac{1}{p_c}\Bigg(& \frac{p_m(\Delta s_{a,k} + \Delta s_{a,l}) - p_m(-\Delta s_{a,k} + \Delta s_{a,l})}{4\Delta s_{a,k}\Delta s_{a,l}} \\ & - \frac{p_m(\Delta s_{a,k} - \Delta s_{a,l}) - p_m(-\Delta s_{a,k} - \Delta s_{a,l})}{4\Delta s_{a,k}\Delta s_{a,l}}\Bigg) \end{aligned} \tag{13.28}$$

The generalized effective cash flows of a bond with guarantee, recovery, and floating rate cash flows in (13.15) will take the form

$$c_{e,i}(s) = (c_{c,i} + c_{f,i} + c_{p,i})\rho_i + r_{g,i}(s) + c_{f,c,i}\rho_i(r_i + r_{g,i}(y)) \tag{13.29}$$

where $r_{g,i}(s)$ is the effective guarantee from prior defaults and is only a function of spread s,

$$r_{g,i}(s) = \sum_{j=1}^{i}\left(\rho_{j-1} - \rho_j\right) g_{t,i,j} \tag{13.30}$$

and $r_{g,i}(y)$ is the effective present value of recovery adjusted future guarantees and is only a function of forward yield,

$$r_{g,i}(y) = \frac{\left(1 - r_j\right)}{\mu_i} e^{y_i t_i} \sum_{j=i}^{n} g_{t,i,j} e^{-y_j t_j} \tag{13.31}$$

The price function of bonds with recovery and guarantee can be calculated by substituting for $c_{e,i}$ in (12.14). Thus, the duration components and vega of a risky bond, can be written as

$$D_k = \frac{1}{p_r}\sum_i\left[\frac{c_{e,i}(s)t_i\psi_{k,i}}{1 + y_s v_y^2 t_i^2} - \frac{\partial c_i}{\partial a_k}\rho(t_i) + c_{f,c,i}\rho_i\frac{\partial r_{g,i}(y)}{\partial a_k}\right]e^{-y_{s,i}t_i} \tag{13.32}$$

$$\varpi = \frac{1}{p_r}\sum_i\left[\frac{c_{e,i}(s)y_s^2 v_s\, t_i^3}{1 + y_s v_y^2 t_i^2} + \frac{\partial c_i}{\partial v_y}\rho(t_i) + c_{f,c,i}\rho_i\frac{\partial r_{g,i}(y)}{\partial v_y}\right]e^{-y_{s,i}t_i} \tag{13.33}$$

with

$$\frac{\partial r_{g,i}(y)}{\partial a_k} = \frac{t_i\psi_k r_{g,i}(y)}{1 + y_s v_y^2 t_i^2} - \frac{(1 - r_i)}{\mu_i} e^{y_i t_i}\sum_{j=1}^{n}\frac{g_{t,i,j}t_j\psi_{k,j}}{1 + y_s v_y^2 t_j^2}e^{-y_j t_j} \tag{13.34}$$

$$\frac{\partial r_{g,i}(y)}{\partial v_y} = r_{g,i}(y)\frac{y_s^2 v_s\, t_i^3}{1 + y_s v_y^2 t_i^2} - \frac{(1 - r_i)}{\mu_i} e^{y_i t_i}\sum_{j=1}^{n}\frac{g_{t,i,j}y_s^2 v_s t_j^3}{1 + y_s v_y^2 t_j^2}e^{-y_j t_j} \tag{13.35}$$

Since the value of near term cash flows is a higher percentage of the price of a risky bond, the curve exposure tends to increase with the bond spread. The slope and bend components of duration for bonds with a yield spread of more than 5%, such as some high yield bonds, are quite significant.

Likewise, we can calculate the spread duration components and spread vega by differentiating the price function relative to the components of the TSCS. The result will be somewhat similar to (13.32) and (13.33) if y is replaced by s. If the correlation between the treasury rate and credit spread is zero, using the interest rate dependent guarantee cash flows, the spread duration and spread vega can be calculated as

$$D_{c,k} = \frac{1}{p_r}\sum_i\left[\frac{c_{e,i}(y)t_i\psi_k}{1+s_{s,c}v_c^2t_i^2} - \frac{\partial c_{f,c,i}}{\partial a_{c,k}}(r_i + r_{g,i}(y))\rho_i - \frac{\partial r_{g,i}(s)}{\partial a_{c,k}}\right]e^{-y_{s,i}t_i} \tag{13.36}$$

$$\varpi_c = \frac{1}{p_r}\sum_i\left[\frac{c_{e,i}(y)t_i\psi_k s_{s,c}^2 v_c\, t_i^3}{1+s_{s,c}v_c^2t_i^2} + \frac{\partial c_{f,c,i}}{\partial v_c}(r_i + r_{g,i}(y))\rho_i + \frac{\partial r_{g,i}(s)}{\partial v_c}\right]e^{-y_{s,i}t_i} \tag{13.37}$$

where $s_{s,c}$ is the spot spread of the bond, v_c is the spread volatility of credit security, and

$$\frac{\partial c_{f,c,i}}{\partial a_{c,k}} = \left[\frac{t_i\psi_{k,i}}{1+s_{s,c,i}v_{c,i}^2t_i^2} - \frac{t_{i-1}\psi_{k,i-1}}{1+s_{s,c,i-1}v_{c,i-1}^2t_{i-1}^2}\right]e^{s_{s,c,i}t_i - s_{s,c,i-1}t_{i-1}} \tag{13.38}$$

$$\frac{\partial c_{f,c,i}}{\partial v_c} = \left[\frac{s_{s,c,i}^2 v_{c,i}t_i^3}{1+s_{s,c,i}v_{c,i}^2t_i^2} - \frac{s_{s,c,i-1}^2 v_{c,i-1}t_{i-1}^3}{1+s_{s,c,i-1}v_{c,i-1}^2t_{i-1}^2}\right]e^{s_{s,c,i}t_i - s_{s,c,i-1}t_{i-1}} \tag{13.39}$$

$$\frac{\partial r_{g,i}(s)}{\partial a_{c,k}} = \sum_{j=1}^{i}\left[\frac{t_j\psi_{k,j}\rho_j}{1+s_{s,c,j}v_{c,j}^2t_j^2} - \frac{t_{j-1}\psi_{k,j-1}\rho_{j-1}}{1+s_{s,c,j-1}v_{c,j-1}^2t_{j-1}^2}\right]g_{t,i,j} \tag{13.40}$$

$$\frac{\partial r_{g,i}(s)}{\partial a_{c,k}} = \sum_{j=1}^{i}\left[\frac{s_{s,c,j}^2 v_{c,j}t_j^3\rho_j}{1+s_{s,c,j}v_{c,j}^2t_j^2} - \frac{s_{s,c,j-1}^2 v_{c,j-1}t_{j-1}^3\rho_{j-1}}{1+s_{s,c,j-1}v_{c,j-1}^2t_{j-1}^2}\right]g_{t,i,j} \tag{13.41}$$

The derivatives of $r_{g,i}(s)$ and $r_{g,i}(y)$ relative to the term structure of spread and interest rates are long and not very practical. It is best to calculate the risks when guarantee and recovery are both present by equations (13.27) and (13.28).

Most spread curves are not very well developed and the convexity adjusted spread curve cannot be calculated at the present time. Ignoring the convexity adjustment, the duration components of the credit curve can be calculated as

$$D_{c,k} = \frac{1}{p_r}\sum_i\left[c_{e,i}(y)t_i\psi_k - \frac{\partial c_{f,c,i}}{\partial a_{c,k}}(r_i + r_{g,i})\rho_i - \frac{\partial r_{g,i}(s)}{\partial a_{c,k}}\right]e^{-y_{s,i}t_i} \tag{13.42}$$

$$\frac{\partial c_{f,c,i}}{\partial a_{c,k}} = \left[t_i\psi_{k,i} - t_{i-1}\psi_{k,i-1}\right]e^{s_{c,c,i}t_i - s_{c,c,i-1}t_{i-1}} \tag{13.43}$$

$$v_c = \varpi_s = 0, \quad y_c = y_s, \quad s_{c,c,i} = s_{s,c,i} \tag{13.44}$$

In investment management and trading, it is standard practice to use a recovery value r_i of zero for a credit bond. Since the recovery value is rarely zero, its estimation, for example on the basis of an issuer's assets, can be very valuable for bonds that trade at significant discount to par. When the recovery value is incorporated into the pricing of an issuer's bonds, the TSCS becomes equal to the term structure of default probability (TSDP). Without the recovery value, a credit spread is just a number that provides a measure of value but cannot be compared with securities that have different recovery values. The incorporation of recovery value into the calculation of default probability allows for much better comparison of two securities.

The recovery value is like a put option on a security. As the default probability increases, the present value of the recovery increases and provides downside protection on the price of a security. For a given price of a security, the higher the default probability, the higher the implied recovery has to be and vice versa. Thus, as the recovery value increases, for a given price, the implied default probability increases as well. The spread duration of a security can be significantly lower for high recovery securities than would otherwise be expected if the recovery value is subtracted from the price of the bond. For example, if the spread duration of a bond with zero recovery is 7 years and the present value of recovery with a recovery value of 50 is 30 for a par bond, the spread duration will be significantly less than $\frac{100-30}{100} \times 7 = 4.9$ years.

Tables 13.1 and 13.2 show selected analytics and TSCS for two corporate bonds and a Brady bond issued by the Dominican Republic that has a principal guarantee and \$6 of RIG for a face value of \$100. The coupon of the Brady bond is based on 6-month Libor with a spread of 0.8125. The RIG covers several years of coupon guarantee at the prevailing Libor rates. There is a significant amount of information in these tables that requires explanation. There are four rows of data for the Ford Motor Company bond, four rows for the IBM bond and three rows for the Dominican Republic Brady bond.

The Ford bond has progressively higher recovery values. When the recovery value is zero, the credit durations are the same as treasury durations. However, with increasing recovery value, the credit durations fall well below treasury durations and the security develops exposure to Libor due to the discounting of recovery value by Libor. Also due to potential default and recovery, the cash flows are realized sooner and the treasury duration falls as well. The level of TSCS or TSDP will also rise with increasing recovery value. With the inclusion of recovery value, the meaning of yield and durations will be completely distorted and the spreads and the term structure exposure will become much more important for valuation and risk measurement.

For the IBM bond, assuming a recovery value of 50%, the calculated default rate close to maturity is about 2.6% per year for a survival probability of 82.7%. The present value of recovery will be 5.63 for a par value of 100. If recovery value is more than 50%, the implied default rate would have to be higher for a given price of the security. Based on historical data, the implied survival probability of 82.7% at 50% recovery is considerably lower than one would expect. Therefore, either the recovery value is lower than 50%, implying a lower default probability for the same market price, or the bond price is cheap. We will show in Section 15.9 that corporate bond prices are not efficiently priced. Analysis of recovery value provides for a much better decision-making about the spread of corporate bonds than simply using the spread of the security.

The last three rows of data for the Brady bond use different assumptions for discounting the guaranteed principal and RIG. The first row of data for Dominican Republic discounts all the guarantees by treasury rates, the second discounts RIG by Libor and the third discounts all guarantees by Libor. Due to the guarantees, the levels of treasury durations are positive in all three cases. Recall from Table 12.4 that floating rate bonds usually have small treasury durations. When the principal is of Libor quality, since the

TABLE 13.1 Selected analytics with recovery or guarantee, July 31, 2012

Issue			Price		Spread		Treasury Duration			Libor Duration			Credit Duration			Recovery & Guarantee		
Issuer	Cpn	Maturity	Mkt	Model	Curve	TsyMkt	Level	Slope	Bend	Level	Slope	Bend	Level	Slope	Bend	Implied	Pv	PvGuar
FordMo.	8.875	1/15/22	125.49	128.41	0.337%	4.011%	6.78	1.93	−4.81				6.78	1.93	−4.81	0.00	0.00	0.00
FordMo.	8.875	1/15/22	125.49	127.75	0.349%	5.123%	6.52	1.73	−4.69	0.40	0.04	−0.34	5.04	1.28	−3.62	25.00	9.64	0.00
FordMo.	8.875	1/15/22	125.49	126.66	0.272%	7.071%	6.10	1.39	−4.48	1.03	0.08	−0.86	3.34	0.65	−2.44	50.00	25.05	0.00
FordMo.	8.875	1/15/22	125.49	127.40	0.339%	5.737%	5.91	1.24	−4.38	1.30	0.10	−1.09	2.82	0.47	−2.07	57.84	31.94	0.00
IBM	8.375	11/1/19	143.56	145.50	0.229%	0.929%	5.78	0.82	−5.13				5.78	0.82	−5.13	0.00	0.00	0.00
IBM	8.375	11/1/19	143.56	145.30	0.258%	1.164%	5.74	0.80	−5.09	0.06	−0.00	−0.05	4.57	0.57	−4.04	25.00	2.11	0.00
IBM	8.375	11/1/19	143.56	144.94	0.278%	1.555%	5.68	0.76	−5.03	0.16	−0.01	−0.14	3.37	0.33	−2.95	50.00	5.63	0.00
IBM	8.375	11/1/19	143.56	144.55	0.254%	1.959%	5.57	0.70	−4.93	0.34	−0.02	−0.30	2.29	0.11	−1.98	72.66	11.69	0.00
DO Republic	1.3125	8/30/24	97.00	102.48	6.388%	11.363%	5.02	3.82	−0.02	−5.93	−2.16	3.93	0.69	0.11	−0.54	0.00	0.00	83.22
DO Republic	1.3125	8/30/24	97.00	102.51	6.414%	11.382%	5.03	3.82	−0.02	−5.65	−2.10	3.72	0.70	0.11	−0.54	0.00	0.00	83.24
DO Republic	1.3125	8/30/24	97.00	103.36	7.731%	12.694%	5.42	4.12	−0.14	4.64	3.95	0.42	0.63	0.10	−0.49	0.00	0.00	84.30

TABLE 13.2 Partial yield and TSCS, July 31, 2012

Issue			Partial Yield				Yield		TSCS			Rcvr
Issuer	Cpn	Maturity	Tsy	Lbr	Crd	Sec	Sum	Calc.	Level	Slope	Bend	Implied
Ford Mo.	8.875	1/15/22	1.33%	0.00%	3.68%	0.34%	5.35%	5.32%	3.92%	3.54%	−0.14%	0
Ford Mo.	8.875	1/15/22	1.29%	0.01%	3.40%	0.30%	4.99%	5.32%	4.84%	4.28%	−0.26%	25.00
Ford Mo.	8.875	1/15/22	1.23%	0.02%	2.71%	0.11%	4.07%	5.32%	6.69%	5.85%	−0.42%	50
Ford Mo.	8.875	1/15/22	1.20%	0.03%	2.28%	−0.03%	3.48%	5.32%	7.73%	6.78%	−0.46%	57.84
IBM	8.375	11/1/19	1.00%	0.00%	0.72%	0.25%	1.97%	1.90%	1.70%	1.91%	0.18%	0
IBM	8.375	11/1/19	1.00%	0.00%	0.68%	0.28%	1.95%	1.90%	1.88%	2.02%	0.15%	25.00
IBM	8.375	11/1/19	0.99%	0.00%	0.64%	0.29%	1.92%	1.90%	2.21%	2.19%	0.08%	50
IBM	8.375	11/1/19	0.98%	0.01%	0.64%	0.20%	1.82%	1.90%	2.86%	2.47%	−0.11%	72.66
DO Republic	1.3125	8/30/24	1.89%	0.00%	0.77%	0.88%	3.54%	2.94%	6.35%	2.64%	0.89%	0
DO Republic	1.3125	8/30/24	1.89%	−0.10%	0.77%	0.89%	3.44%	2.94%	6.35%	2.64%	0.89%	0
DO Republic	1.3125	8/30/24	1.89%	−0.08%	0.65%	0.90%	3.36%	2.94%	6.35%	2.64%	0.89%	0

Libor spot curve is below the treasury curve at the point of maturity of the bond, the value of the principal guarantee is higher and the value of the credit must be lower to compensate for the constant price of the security. To estimate how much the fair model price of the security increases by discounting the principal guarantee by Libor, we note that the spread to curve of the security is wider by $7.73 - 6.39 = 1.34\%$ for Libor discounting. Multiplying this widening by the average credit duration of $\frac{0.69+0.63}{2}$, we find a credit value that is lower by 0.88%. This value is the same change in the value of the guarantee that is captured by the difference in model prices of the two scenarios, namely, $103.36 - 102.48 = 0.88$. When the principal guarantee of the bond is of Libor quality, the Libor duration of the bond becomes positive from negative.

13.3 PARTIAL YIELD OF COMPLEX SECURITIES

In Figure 12.3 we showed schematically how different components of yield are decomposed to calculate partial yields. In some cases, such as real bonds and bonds with recovery, that decomposition is not always straightforward.

For an inflation linked treasury bond, all cash flows are discounted by treasury rates. Part of the discounting term is by real rates and part by nominal rates. To calculate the contribution of each to partial yield, we decompose each cash flow into two: one with a maturity at the expiration of the real rate (inflation reference point for the cash flow), and one starting at that point and running through the date of the actual cash flow. We then scale each yield by its respective level duration. From (11.16), we calculate the partial real yield of an IL bond as

$$\begin{aligned} y_r &= \frac{\sum_{i=0}^{n} c_i t_{in} y_{r,in} e^{-y_{r,in}t_{in} + y_{in}t_{in} - y_i t_i}}{\sum_{i=0}^{n} c_i t_{in} e^{-y_{r,in}t_{in} + y_{in}t_{in} - y_i t_i} + \sum_{i=0}^{n} c_i (t_i - t_{in}) e^{-y_{r,in}t_{in} + y_{in}t_{in} - y_i t_i}} \\ &= \frac{\sum_{i=0}^{n} c_i t_{in} y_{r,in} e^{-y_{r,in}t_{in} + y_{in}t_{in} - y_i t_i}}{\sum_{i=0}^{n} c_i t_i e^{-y_{r,in}t_{in} + y_{in}t_{in} - y_i t_i}} \end{aligned} \tag{13.45}$$

where the subscript n refers to the inflation reference point for a cash flow. The nominal partial yield is

$$y_g = \frac{\sum_{i=0}^{n} c_i (t_i - t_{in}) y_i e^{-y_{r,in}t_{in} + y_{in}t_{in} - y_i t_i}}{\sum_{i=0}^{n} c_i t_i e^{-y_{r,in}t_{in} + y_{in}t_{in} - y_i t_i}} \tag{13.46}$$

The partial credit yield of a security with recovery and/or guarantee is calculated using (13.15) for cash flow as:

$$y_c = \frac{\sum_i c_i t_i s_{s,c,i} e^{-(y_{s,i} + s_{s,c,i})t_i}}{\sum_i c_i t_i e^{-(y_{s,i} + s_{s,c,i})t_i}} \frac{D_{c,0}}{D_t} \tag{13.47}$$

where

$$D_t = \frac{1}{p_m} \sum_i c_{e,i} t_i e^{-y_{s,i} t_i} \tag{13.48}$$

in which $c_{e,i}$ is given by (13.15). The partial Libor yield of a security with recovery and/or guarantee is given by the cash flows that are discounted by Libor as

$$y_l = \frac{\sum_i c_{c,c,i} t_i \left(r_i + r_{g,i}(y)\right) s_{s,l,i} e^{-(y_{s,i}+s_{s,c,i})t_i}}{\sum_i c_{n,c,i} t_i \left(r_i + r_{g,i}(y)\right) e^{-(y_{s,i}+s_{s,c,i})t_i}} \frac{D_{l,0}}{D_t} \tag{13.49}$$

For securities with recovery or guarantee, the recovery is like a put option on the security. If the spread rises, the value of the security falls and the likelihood of default increases, leading to a higher recovery value that partially compensates for the spread widening. Therefore, the spread duration of a security with recovery is less than one without recovery.

In order to calculate the partial yields of a security with recovery, assume that each cash flow is a separate security and we want to calculate the yield of a portfolio comprising all the cash flows as securities. For simplicity, consider a portfolio whose cash flows depend on interest rates plus spread. We define the partial yield of the credit as the equivalent yield that the same set of cash flows would have without recovery rate. Using (13.12), we write the market value of the cash flows as

$$\begin{aligned} \sum_i c_{e,i} e^{-y_i t_i} &= \sum_i \left(r_i c_{f,c,i} + c_i\right) e^{-(y_i+s_{s,c,i})t_i} \\ &= \sum_i c_i e^{-(y_i+s_c)t_i} \end{aligned} \tag{13.50}$$

Subtracting the right hand side from the left and expanding the equation using Taylor series to first order leads to

$$\begin{aligned} &\sum_i e^{-(y_i+s_{s,c,i})t_i} \left[r_i c_{f,c,i} + c_i(1 - e^{-(s_c - s_{s,c,i})t_i})\right] = 0 \\ &s_c \approx \frac{\sum_i \left(c_i s_{s,c,i} t_i - r_i c_{f,c,i}\right) e^{-(y_i+s_{s,c,i})t_i}}{\sum_i c_i t_i e^{-(y_i+s_{s,c,i})t_i}} \end{aligned} \tag{13.51}$$

In Table 13.2, the sum of partial yields falls with increasing recovery rate. As the recovery rate increases, the default probability rises and the likelihood of earlier realization of recovery increases. Given the slope of the treasury and credit curves, the overall yield falls as the cash flows are realized in a shorter time horizon which can be seen from the level duration. For example, if default is likely in the next year, there is a high likelihood that recovery which is discounted by the front end of Libor will be realized leading to a very low yield. The sum of partial yields is a better representation of the yield of a security than the market yield which is invariant under all default and recovery scenarios.

13.4 FORWARD COUPON

The forward coupon is the implied coupon rate that an issuer needs to pay if it issued a bond for forward settlement. The implied forward coupon rate is used to calculate the call or put probability for bonds with call/put provisions and therefore it can happen in the distant future. If p_f is the forward price, $c_{p,i}$ the principal payment at time t_i, $c_{g,i}$ the guarantee potion of cash flow at time t_i, t_f the forward time, w_i the weight of coupon at t_i, w_a the accrual weight of the coupon, and c_0 is the current coupon of the security, then the forward price function can be written as

$$\begin{aligned} p_f + w_a c_0 = e^{y_{i,f} t_f} \sum_i c_{g,i} e^{-y_{l,i} t_i} + e^{y_{c,f} t_f} \sum_i c_{p,i} e^{-y_{c,i} t_i} \\ + c_0 e^{y_{c,f} t_f} \sum_i w_i e^{-y_{c,i} t_i} \end{aligned} \tag{13.52}$$

If c_f is the expected forward coupon, then

$$\begin{aligned} 100 + w_a c_f = e^{y_{l,f} t_f} \sum_i c_{g,i} e^{-y_{l,i} t_i} + e^{y_{c,f} t_f} \sum_i c_{p,i} e^{-y_{c,i} t_i} \\ + c_n e^{y_{c,f} t_f} \sum_i w_i e^{-y_{c,f} t_i} \end{aligned} \tag{13.53}$$

Therefore,

$$c_f = c_0 \frac{100 - e^{y_{l,f} t_f} \sum_i c_{g,i} e^{-y_{l,i} t_i} - e^{y_{c,f} t_f} \sum_i c_{p,i} e^{-y_{c,i} t_i}}{p_f - e^{y_{l,f} t_f} \sum_i c_{g,i} e^{-y_{l,i} t_i} - e^{y_{c,f} t_f} \sum_i c_{p,i} e^{-y_{c,i} t_i}} \tag{13.54}$$

13.5 CREDIT DEFAULT SWAPS

A credit default swap (CDS) is a swap transaction that transfers the risk of default for a credit security from the buyer of protection to the seller of protection in exchange for a predetermined periodic payment from the buyer to the seller for the duration of the contract. If the underlying security defaults, the buyer delivers defaulted bonds to the seller and receives the par amount for the face value of the bonds. If no default takes place for the duration of the contract, the periodic payments that the seller receives will be all profit.

Consider a corporation, XYZ, issuing a floating rate bond with a coupon of Libor plus a spread at par (price of 100). An investor will be indifferent between investing his money in floating Libor, on the one hand, and buying XYZ bond and protection for the duration of the bond, on the other hand. If there is default, the buyer simply delivers the defaulted bond and receives his original par investment. If there is no default, the buyer will receive Libor plus spread and pays the protection premium. If protection premium is equal to the spread of the bond over Libor, then there is no arbitrage. Therefore, for a bond that is priced at par, the CDS premium is equal to the credit spread of the bond over Libor.

For a bond that trades at a discount to par, the premium for its protection is higher than the spread. If the premium is equal to the spread, you can buy the bond and buy protection at the same time to hedge the credit risk. If there is no default, the extra spread that is earned will pay for protection of the bond. However, in the event of a default, you profit by delivering the bond that you had purchased at less than par and receive the par amount for it.

Implied in a CDS contract is a floating Libor contract for both buyer and seller, in addition to the premium. The seller of protection will receive floating Libor plus the periodic premium and pay floating Libor. The two floating Libor payments cancel each other and the seller receives the net premium payment. However, for analysis, we need to include the floating Libor in the cash flows.

Assuming $s(t)$ to be the term structure of credit spread of the bond with recovery value r_i at t_i, we can use (13.23) to calculate the price of a bond as

$$p_b = \sum_i (c_i + r_{ie} c_{f,c,i}) e^{-(s_{s,i} + y_{s,i}) t_i} \tag{13.55}$$

Likewise, we can calculate the price of a CDS as

$$p_{\text{cds}} = \sum_i (c_{c,i} + y_{f,l,i} + r_{ie} c_{f,c,i}) e^{-(s_{s,c,i} + y_{s,i}) t_i} \tag{13.56}$$

where $y_{f,l,i}$ is the forward Libor rate in the interval between $i - 1$ and i, and $c_{c,i}$ is the CDS periodic premium. Forward periodic treasury and Libor rates are similar to forward spread (12.6) and are given by

$$y_{f,i} = \left(e^{y_{s,i} t_i - y_{s,i-1} t_{i-1}} - 1\right) p_{p,i} \tag{13.57}$$

$$y_{f,l,i} = \left(e^{y_{s,l,i} t_i - y_{s,l,i-1} t_{i-1}} - 1\right) p_{p,i} \tag{13.58}$$

If an issuer has issued many bonds, equation (13.55) can be used to calculate the TSCS or term structure of default rates, assuming that the recovery rate is known. Substituting the calculated TSCS and the recovery rate into (13.56), the price of CDS can be calculated for known periodic premiums. Likewise, knowing the price, the periodic premiums can be calculated. However, if the periodic premium and the price of CDS are known, we can use (13.55) and (13.56) to calculate the implied recovery rate in an iterative way. For liquid securities where CDS prices are available for a range of maturities, the term structure of recovery rate can be estimated as well.

For a reasonably accurate estimate of the market implied recovery rate, the bonds of an issuer must have accurate pricing as well as coupon diversity to allow for recovery

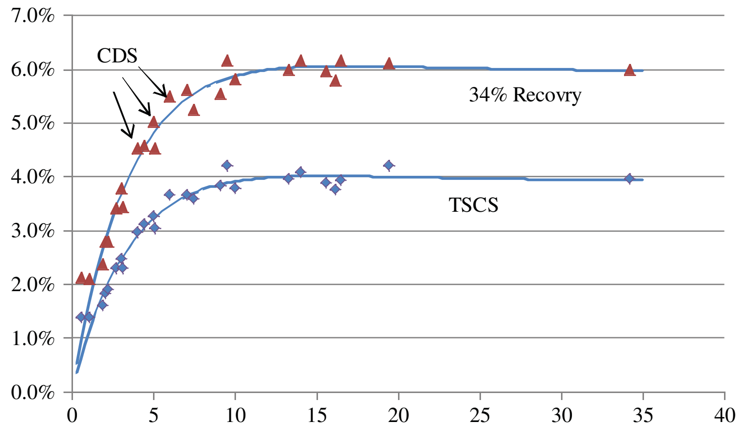


FIGURE 13.1 TSCS and TSDP for Ford Motor Co., July 31, 2012

differentiation. Consider a bond manager intending to get exposure to the bonds of an issuer. He has the option of buying a 10-year bond that is trading at a price of $115 or selling 10-year protection on the issuer. Suppose that in case of default the recovery value will be 82% of par and the protection premium is $2 per year. Assuming fair pricing of the protection and a default probability of x in the first year, the expected gain with no default $(1-x)c_c$ must be equal to the expected loss:

$$(1-x)c_c = 100x(1-r), \quad r = 0.82,\ c_c = 2,\ x = 10\%$$

For the alternative of purchasing the bond where the default loss will be significantly more, the spread must be as follows:

$$(1-0.1)s_c \times 100 = (115-82)\times 0.1, \quad s_c = 3.67\%$$

Thus, for a bond that is priced higher than par, the spread must be higher than the equivalent CDS spread, and bonds that are trading at a discount require a lower spread than CDS for fair compensation.

Figure 13.1 shows the TSCS for Ford Motor Co. with 0% and 34.5% recovery rates, respectively. The 34.5% recovery rate is the optimized recovery rate that would most closely price all bonds. It is probably a good estimate of the recovery value, and the curve represents the TSDP. If an analyst estimates the recovery rate to be 50%, then we can see a much larger differentiation between different bonds and CDS rates. At 50% recovery rate, CDS rates are about 40 bps cheap relative to the curve. Most cash bonds that are trading at premium prices have lower spreads than warranted if default is a possibility.

Incorporation of recovery value into the spread curve of securities, especially high yield bonds where the default probability is higher, is an indispensable tool for bond traders and portfolio managers. The insight that can be obtained by the analysis of a bond using different recovery scenarios can be very valuable for relative value trading and portfolio positioning. Recovery of a defaulted bond is a long process and sometimes an issuer can take several years to exit bankruptcy. The recovery value is the price that the bond is expected to trade at when a company files for bankruptcy.

The market convention for evaluating a CDS is to use a recovery value of zero. The price function of a CDS using TSCR with zero recovery can be written as

$$p_{\text{cds}} = \sum_i (c_{c,i} + y_{f,l,i}) e^{-(s_{s,c,i} + y_{s,i}) t_i} \tag{13.59}$$

For securities that have a high default probability and where the spread is very high, most market participants price the CDS at a spread of 5% over Libor and pay a cash amount to the seller of protection. For example, if the spread for a 10-year CDS is 12%, then

$$100 = \sum_i \left(\frac{12}{m} + y_{f,l,i} \right) e^{-(s_b + y_{s,i}) t_i} \tag{13.60}$$

where m is the coupon frequency of the CDS. From this equation we solve for the spread of the security relative to the treasury curve and then calculate the price of the CDS at a spread of 5% as

$$p = \sum_i \left(\frac{5}{m} + y_{f,l,i} \right) e^{-(s_b + y_{s,i}) t_i} \tag{13.61}$$

The calculated price is now 60.45, implying that the seller of protection will receive 39.55 for 100 par value of the CDS and will also receive a premium of 5% on an annual basis. However, in the event of default, the seller of protection will have to pay a par amount of 100 and will receive the defaulted bonds.

CHAPTER 14

Deliverable Bond Futures and Options

Bond futures are exchange traded futures contracts where at expiration, and during a delivery period, the seller can deliver from a basket of bonds (deliverable basket) to the buyer and receive the price of the futures contract multiplied by a scaling factor plus accrued interest. The scaling factor, called the conversion factor, provides a mechanism to create a contract where most of the bonds in the basket have similar prices, even though their coupons and market prices can be significantly different.

The introduction of bond futures in 1977 was a financial engineering masterstroke and it radically changed the liquidity, transparency, pricing efficiency, and hedging capability of bond market investors. The contracts have become very popular and have been replicated in most major bond markets, including the euro zone, Japan, UK, and Canada. Prior to the introduction of bond futures, secondary markets in bonds were through over-the-counter desks and lacked pricing uniformity and transparency. US and EUR 10-year bond futures trade with a bid–ask price spread of about 0.016 per 100 of face value and average around 1 million contracts with a notional amount of more than $100 billion a day in trading.

The deliverable basket of bonds includes all the bonds that meet certain maturity, coupon, and issuer criteria established by the exchange where the bond futures are traded. For example, the deliverable basket for US bond futures that trade on the Chicago Board of Trade includes all the bonds with a minimum maturity of at least 15 years from the first day of the delivery month, with coupon rates larger than zero, that are issued by the US Treasury. If the bond is callable, the earliest call date must be longer than 15 years.

The face value or the notional amount of each contract is established by the exchange. For US bond futures the notional amount is $100,000, while for 2-year futures the notional amount is $200,000.

US bond futures contracts have a period for the delivery of bonds by the short seller to the buyer of the bond futures. The short notifies the buyer of the delivery any day during the delivery period, which is usually 1 month. Some bond futures such as Japanese government bond futures have a fixed delivery date that is only 1 day.

Most bond futures contracts have an associated option with a different expiration date than the contract expiration of the futures. After calculating the price volatility of the options, the price volatility of the underlying contract at the expiration date of the option can be calculated by interpolation or extrapolation of the option volatility. We will first derive a simple options pricing formula that will be used in evaluating bond futures and, in a later chapter, for swaptions and bond options.

14.1 SIMPLE OPTIONS MODEL

Our derivation of the option's pricing formula is based on the random walk process without the use of differential equations. For more detailed option pricing formulas including barrier, digital, and all exotic options see Haug [13].

Consider a security with an initial price p_0 that can change in very small relative steps of δ such that

$$\frac{p-p_0}{p_0}=\delta\approx\ln\left(\frac{p}{p_0}\right) \tag{14.1}$$

In this model the price changes are relative. For example, if the price is for a stock and each share of the stock splits into two, then the absolute changes in price would be half of the pre-split change, but the relative price change would be the same. After a very large number of steps N, the number of ways that M of those steps will be up and $N-M$ will be down is given by

$$\rho=\binom{N}{M}=\frac{N!}{M!(N-M)!} \tag{14.2}$$

For very large numbers, the distribution will be centered close to its maximum likelihood. The highest likelihood of the above distribution will be the same as the highest likelihood of its logarithm. The factorials can be expanded using the gamma function expansion as follows:

$$\begin{aligned}
&\Gamma(N)=(N-1)!\\
&\ln(\Gamma(N+1))\approx N\ln(N)-N-\frac{1}{2}\ln\left(\frac{N}{2\pi}\right)+\ldots\\
&Z=\ln(\rho)\approx(N-1)\ln(N-1)-(M-1)\ln(M-1)-(N-M-1)\ln(N-M-1)+\ldots\\
&\frac{\partial Z}{\partial M}=\frac{\partial\ln(\rho)}{\partial M}=0\\
&M_0\approx\frac{N}{2}
\end{aligned} \tag{14.3}$$

We now expand the distribution around its most likely state using Taylor series:

$$\begin{aligned}
Z&=Z_0+\frac{\partial Z}{\partial M}\left(M-\frac{N}{2}\right)+\frac{1}{2}\frac{\partial^2 Z}{\partial M^2}\left(M-\frac{N}{2}\right)^2+\ldots\\
&=N\ln(2)-\frac{2}{N}\left(M-\frac{N}{2}\right)^2+\ldots
\end{aligned} \tag{14.4}$$

Thus, the probability distribution can be written as

$$\rho \approx Ae^{-\frac{2}{N}\left(M-\frac{N}{2}\right)^2} \tag{14.5}$$

where A is a constant to ensure that the sum of all cases in the distribution is unity. After N steps, if each step is δ, the net difference between up and down steps will be

$$x = (2M - N)\delta = \ln\left(\frac{p}{p_0}\right) \tag{14.6}$$

where p is the price after N steps and p_0 is the original price. Making the substitution in (14.5) results in

$$\rho \approx Ae^{-x^2/2N\delta^2} \tag{14.7}$$

If there are n_0 steps per unit of time, for example, if the expected daily volatility is δ, then $n_0\delta^2$ will be the annualized volatility of price, denoted by v_p. For a given time t,

$$N\delta^2 = t(n_0\delta^2) = tv_p^2 \tag{14.8}$$

Hence,

$$\rho \approx Ae^{-x^2/2v_p^2t} \tag{14.9}$$

We require that the sum of all probabilities be unity; this is called normalization:

$$\int_{-\infty}^{\infty} \rho dx = 1, \quad \rho = \frac{1}{\sqrt{2\pi t v_p}} e^{-x^2/2v_p^2t} \tag{14.10}$$

Equation (14.10) is the standard normal distribution function. If the step up was slightly larger than the step down or vice versa, the distribution would be drifting toward higher prices. In fact, for stocks this is the case. For example, suppose a stock price is at 100 and with 50% probability goes to 110 or 90. In the next step it can go up another 10% to 121 or down 10% to 81, and the average of these outcomes would not be 100 but 101. This is the case because the price cannot go below zero, but there is no upside to it. In the above equation, if x is replaced with $x - \mu$, the normalization requirement will still be met and it will be called a drifted distribution. The center of the distribution is simply shifted to $x = \mu$. For stock prices, the arbitrage-free requirement must also be met which states that the average price of a stock at some time in the future based on its distribution must be equal to the forward price of the stock – that is, for a drifted distribution,

$$p_f = \frac{1}{\sqrt{2\pi t v_p}} \int_{-\infty}^{\infty} pe^{-(x-\mu)^2/2v_p^2t} dx, \quad x = \ln\left(\frac{p}{p_0}\right), p = p_0e^x \tag{14.11}$$

From (14.6), we can calculate p as a function of x, noting that p_0 is the expected forward price and is equal to p_f. Thus,

$$p_f = \frac{1}{\sqrt{2\pi t v_p}} \int_{-\infty}^{\infty} p_f e^{x-(x-\mu)^2/2v_p^2t} dx \tag{14.12}$$

From (14.12), the value of μ can be calculated analytically. The result is

$$\mu = -\frac{v_p^2 t}{2} \tag{14.13}$$

The probability distribution function (14.6) in extended form can be written as

$$\rho(p) = \frac{1}{\sqrt{2\pi t} v_p} e^{-\left(\ell n\left(p/p_f\right)+v_p^2 t/2\right)^2 \Big/ 2v_p^2 t} d \ln(p) \tag{14.14}$$

For a security that does not pay dividend, the forward price can be calculated from the current price as

$$p_f = p_0 e^{r_t} \tag{14.15}$$

where r is the risk-free short term rate. Equation (14.14) is the Black–Scholes probability distribution of forward prices and can be used to calculate all other option properties. Having calculated the probability distribution of forward prices, we can calculate the forward price of a call or put option for a strike price of p_x as follows:

$$\begin{aligned} p_{\text{call}} &= \frac{1}{\sqrt{2\pi t} v_p} \int_{x_x}^{\infty} (p - p_x) e^{(x-\mu)^2/2v_p^2 t} dx \\ p_{\text{put}} &= \frac{1}{\sqrt{2\pi t} v_p} \int_{-\infty}^{x_x} (p_x - p) e^{(x-\mu)^2/2v_p^2 t} dx \end{aligned} \tag{14.16}$$

where x_x is the value of x at the strike price. The future value of the option must be discounted by the prevailing interest rate to the present time. We will cover bond options in detail in Chapter 15.

Equation (14.14) is arbitrage-free for stocks, that is, the expected future price of the stock over all possible forward paths of price movements is equal to the forward price. If we use the same methodology for bond options, assuming that yields have log-normal distribution, then the expected future yield over all possible paths will be equal to its forward yield. However, since the price–yield relationship of a bond is not linear, the expected future price of the bond will not be equal to its forward price if we use (14.13).

For bond options the drift has to be calculated in such a way that the expected future price over all possible paths of interest rates will be equal to the forward price. The drift value cannot be calculated analytically and must be calculated by iteration.

14.2 CONVERSION FACTOR

The conversion factor is equal to the price of a bond based on a notional yield, called the conversion yield, divided by 100. Most bond/note futures contracts adjust (round) the maturity date to a monthly or quarterly multiple of the first delivery date. For example, US bond futures contracts adjust the maturity of the bond to the quarterly date of the contract delivery date. US 10-year note futures round the maturity date to the monthly date of the first delivery date of the contract. For example, for September 2012 futures, a bond with a maturity of 8/15/2029 is rounded to 6/4/2029 for the calculation of the conversion factor only. Likewise, for 10-year note futures, a note with a maturity of 5/15/2019 will be rounded to 5/4/2019.

Let us calculate the conversion factor for the Treasury 6.125% 8/15/2029 for September 2012 delivery (USU12). The first delivery date for this contract is September 4, 2012. The rounded maturity date is 6/4/2029. The fractional period to the next coupon is

$$f_a = \frac{12/4/2012 - 9/4/2012}{12/4/2012 - 6/4/2012} = 0.49727$$

Given the conversion yield of 0.06 (6%), the price plus accrued of this security is calculated as

$$p + a = \sum_{i=0}^{33} \frac{\frac{6.125}{2}}{(1+\frac{0.06}{2})^{i+f_a}} + \frac{100}{(1+\frac{0.06}{2})^{33+f_a}}$$

where 34 (0 to 33 inclusive) is the number of semi-annual coupon payments between 9/4/2012 and maturity date. Accrued interest is

$$a = \frac{6.125}{2}(1 - f_a)$$

The conversion factor for this security is thus

$$CF = \frac{p}{100} = \frac{101.298}{100} = 1.01298$$

14.3 FUTURES PRICE ON DELIVERY DATE

On the delivery date, the seller can deliver the notional amount for each contract from the basket of deliverable bonds to the buyer and receive a cash amount equal to the futures price times the conversion factor for the bond plus accrued interest. For example, if the futures price is 125, for each contract, the seller can delivery \$100,000 face value of 6.125% 8/15/2029 and receive a price of 125 × 1.01298 or deliver 4.5% 2/15/2036 and receive a price of 125 × 0.81316 for the bonds. The seller is incentivized to provide the bonds that are cheapest to deliver (CTD). To avoid arbitrage, on the delivery date, the futures price must be equal to the price of the CTD bond divided by its conversion factor. If there are n bonds in the basket with forward prices $p_{f,i}$ and conversion factors $f_{c,i}$, the exchange traded futures price will be equal to

$$p_x = \min_{i=1}^{n} \left(\frac{p_{f,i}}{f_{c,i}} \right) \tag{14.17}$$

14.4 FUTURES PRICE PRIOR TO DELIVERY DATE

The arbitrage-free futures price prior to the delivery date is equal to the weighted average price of the bonds in the basket divided by the respective conversion factor times the probability that that bond becomes the cheapest to deliver.

Consider a futures contract where there is only one deliverable bond. We can write

$$v_p = D_f y_f v_y \tag{14.18}$$

where v_y is the yield volatility, v_p the price volatility, D_f the forward duration, and y_f the forward yield.

Knowing the price volatility of a security from the options market, we can calculate the yield volatility of the forward security. Using a log-normal distribution for the forward paths of interest rates, we calculate the forward price from the TSIR of a security from (14.11) as

$$p_{f,t} = \frac{1}{\sqrt{2\pi t}v_y} \int p_f(y) e^{-(\ln(y)-\ln(y_0)-\mu)^2/2v_y^2 t} d\ln\left(y\right) \tag{14.19}$$

where μ is the drift and $p_f(y)$ is the price of the bond with a yield of y and y_0 is the current forward yield. The volatility process is based on the log-normal distribution of the yield, but the arbitrage-free requirement for all future paths of interest rates depends on price. Therefore, due to convexity, the value of μ cannot be calculated analytically and must be calculated by iteration. This equation is the basis for arbitrage-free requirement of the forward pricing of a bond, that is, the weighted sum of all forward prices must be equal to the market forward price of the bond.

For numerical calculations, it is computationally much less expensive to calculate the distribution function than to calculate the prices. We therefore calculate an array of prices and use a drifted unit normal distribution function to calculate the drift. A drifted unit normal distribution (DUND) function is defined as

$$\mathrm{DUND} = \frac{1}{\sqrt{2\pi}} e^{-(x-\zeta)^2/2} \tag{14.20}$$

The value of ζ is varied until the following equation is satisfied:

$$p_{f,t} = \frac{1}{\sqrt{2\pi}} \int p_f(y) e^{-(x-\zeta)^2/2} dx \tag{14.21}$$

where ζ is the unitized drift given by

$$\zeta = \frac{\mu}{v_y\sqrt{t}} = \frac{\mu}{\sigma_y} \tag{14.22}$$

$$\sigma_y^2 = v_y^2 t. \tag{14.23}$$

To calculate the applicable yield for a given value of the DUND, we use the transformation

$$\frac{\ln(y)-\ln(y_0)}{\sigma_y} = x \Rightarrow y = y_0 e^{\sigma_y x} \tag{14.24}$$

Alternatively, we can use a unit normal distribution (UND) function and include the drift in the yield calculation as

$$y = y_0 e^{\sigma_y (x-x_0)} \tag{14.25}$$

In practice, it is most convenient to use either a UND or a DUND and calculate the yield factor from (14.24) or (14.25). All calculations for pricing of bond futures are performed in the forward space; therefore we will drop the forward subscript from all variables for convenience.

Knowing the price volatility of the bond futures contract, we calculate the yield volatility and drift of all bonds in the basket. To calculate the forward price of a basket of bonds we will first need to find the aggregate yield volatility and drift of the basket. For each bond i in a basket of deliverable bonds we can write

$$v_{p,i} = D_i y_i v_{y,i} \tag{14.26}$$

where $v_{p,i}$ and $v_{y,i}$ are price and yield volatilities of security i respectively. Then

$$p_i = \frac{1}{\sqrt{2\pi t}v_{y,i}} \int p_i(y) e^{-(\ln(y)-\mu_i)^2/2v_{y,i}^2 t}\, d\ln(y) \tag{14.27}$$

Suppose that we know the probability w_i that bond i becomes deliverable. Thus,

$$\sum w_i = 1 \tag{14.28}$$

Denote the yield volatility of the basket by $v_{y,x}$, the price volatility of the basket by $v_{p,x}$, the forward duration of the basket by D_x, and the forward yield of the basket by y_x. Thus,

$$D_x = \sum_i^n w_i D_{f,i} \tag{14.29}$$

$$y_x = \frac{\sum_i^n w_i D_{f,i} y_{f,i}}{D_x} \tag{14.30}$$

$$v_{p,x}^2 = \sum_i^n w_i v_{y,i}^2 D_{f,i}^2 y_{f,i}^2 \tag{14.31}$$

In (14.31) we can assume that the yield volatility of all the bonds in the basket is the same in order to estimate the yield volatility of the basket from the price volatility of the future.

In markets where the Libor term structure of volatility is available, we can assume that the yield volatility of the treasury bonds is proportional to the yield volatility of a comparable Libor security; the proportionality factor r will be the ratio of the treasury bonds' volatility relative to Libor. This provides a better market driven approximation to the yield volatility of the basket of bonds. If the yield volatility of a comparable Libor bond is $v_{l,i}$, we can write the yield volatility of the respective treasury bond as

$$v_{y,i} = v_{l,i} r \tag{14.32}$$

Substituting from (14.32) into (14.31), we can calculate the volatility ratio r if we know the weights of each bond in the basket:

$$r^2 = \frac{v_{p,x}^2}{\sum_i^n w_i v_{l,i}^2 D_{f,i}^2 y_{f,i}^2} \tag{14.33}$$

In this equation, the price volatility of bond futures can be calculated from the options markets by interpolating between options that expire before and after the futures expected delivery date. We can now create a log-normal distribution defined by its drift μ_x and yield volatility $v_{y,x}$ calculated as follows:

$$\mu_x = \sum_{i}^{n} w_i \mu_i \tag{14.34}$$

$$v_{p,x}^2 = v_{y,x}^2 D_x^2 y_x^2 \tag{14.35}$$

$$v_{y,x}^2 = \frac{r^2 \sum_{i}^{n} w_i v_{l,i}^2 D_i^2 y_i^2}{D_x^2 y_x^2} \tag{14.36}$$

We define beta for each bond as the ratio of the yield volatility of the bond relative to the basket:

$$\beta = \frac{v_{y,i}}{v_{y,x}} = \frac{v_{l,i}}{v_{y,x}} r \tag{14.37}$$

The log-normal distribution density function will be

$$d\,\rho(y) = \frac{1}{\sqrt{2\pi t} v_{y,x}} e^{-(\ln(y)-\mu_x)^2/2v_{y,x}^2 t} d\ln(y) \tag{14.38}$$

$$\int_0^y d\rho(y) = 1 \tag{14.39}$$

The probabilistic futures price will be calculated as

$$p_x = \frac{1}{\sqrt{2\pi t} v_{y,x}} \int_{y=0}^{\infty} \min\left(\frac{p_i(\beta_i y)}{f_{c,i}}\right) e^{-(\ln(y)-\mu_x)^2/2v_{y,x}^2 t} d\ln(y) \tag{14.40}$$

where $\min\left(p_i(\beta_i y)/f_{c,i}\right)$ is the minimum value of all bonds in the basket at a yield of y. The next step is to calculate the weights of each bond in the distribution. We calculate (14.19) through (14.36) assuming equal weight for all bonds in the basket. From the resulting distribution, we calculate the CTD at each value of the forward yield. This will give us the likely deliverables at every forward yield, hence the new weight for each deliverable. We use the new weights in the equations and recalculate all the parameters. Generally, after two or three iterations the solution converges and the weights do not change.

The weight of each bond in the basket is calculated as

$$w_j = \frac{1}{\sqrt{2\pi t} v_{y,x}} \int e^{-(\ln(y)-\mu_x)^2/2v_{y,x}^2 t} d\,\ell n(y) \tag{14.41}$$

for all y such that

$$\frac{p_j(\beta_j y)}{f_{c,j}} = \min\left(\frac{p_i(\beta_i y)}{f_{c,i}}\right)$$

Once the solution stabilizes and all the weights are calculated, risk parameters and valuations of futures contracts can be calculated as well.

14.5 EARLY VERSUS LATE DELIVERY

All the calculations in the previous section were based on the price of bonds on delivery date. One has to take the market price of bonds and calculate the forward prices using repo or Libor rates using (7.3) or (7.4). In an upward sloping yield curve, the farther the delivery date, the lower the price of the bond will be.

We can think of a bond as two securities: one with a maturity date equal to the delivery date and one using the proceeds of the maturity of the first bond and maturing at the usual maturity of the original bond. If y_1, y_2, D_1, D_2 are the respective yield and durations of the two securities and y and D are the yield and duration of the original security, given that the present values of all are the same,

$$D_2 \approx D - D_1 \tag{14.42}$$

We can use (1.27) to estimate y_2 as

$$y_2 = \frac{yD - y_1 D_1}{D_2} = \frac{yD - y_1 D_1}{D - D_1} \tag{14.43}$$

When $y_1 < y$, the longer D_1 is, the higher y_2 will be and vice versa. A higher y_2 implies a lower forward price and more profit for the short seller of the bond future. Therefore, in an upward sloping yield curve, delivery takes place on the last delivery date, and, in an inverted yield curve, delivery takes place on the first delivery date.

14.6 STRIKE PRICES OF THE UNDERLYING OPTIONS

Since the seller of futures contract has the option to deliver any eligible bond in the basket at the delivery time, he is long an option to deliver the CTD bond. If the CTD bond changes, the seller will exercise his option. Therefore, the strike price of this option is at a forward yield point where the CTD changes. There can be multiple strike prices if several bonds become CTD at different forward rates.

The call strike price is the price at which the CTD changes at higher prices (lower yields) and the put strike price is for a change in the CTD at lower prices. If the current CTD bond is designated with subscript d, the call and put values of the delivery option are given by

$$C = \frac{1}{\sqrt{2\pi t}v_{y,x}} \int_{y=0}^{y_0} \left[\frac{p_d(\beta_d y)}{f_{c,d}} - \min_i^{N_B} \left(\frac{p_i(\beta_i y)}{f_{c,i}} \right) \right] e^{-(\ln(y)-\mu_x)^2/2v_{y,x}^2 t} \, d\ln(y) \tag{14.44}$$

$$P = \frac{1}{\sqrt{2\pi t}v_{y,x}} \int_{y_0}^{\infty} \left[\frac{p_d(\beta_d y)}{f_{c,d}} - \min_i^{N_B} \left(\frac{p_i(\beta_i y)}{f_{c,i}} \right) \right] e^{-(\ln(y)-\mu_x)^2/2v_{y,x}^2 t} \, d\ell\mathrm{n}(y) \tag{14.45}$$

14.7 RISK MEASUREMENT OF BOND FUTURES

All the calculations for the futures contracts in the previous section were performed in the forward time space at the optimal delivery point of the futures contract. Knowing the market price of bonds in the basket, the forward prices can be calculated from (7.3). Let us assume that there is only one deliverable bond in the basket. An investor can either buy the bond future, which only requires posting collateral for margin movement, or buy the underlying bond for forward settlement and retain access to his cash. The forward price of the bond excludes cash flows that will take place between now and the delivery date. If p_v is the present value of cash flows that will occur after the delivery date, from (7.4), assuming that the bond is discounted by repo rate y_{br}, we can write the forward price as

$$p_f = -w_f c + (p_v + w_m c)e^{y_{br}\, t_f} \tag{14.46}$$

where w_f and w_m are the accrual period for forward and market settlement, respectively. Instead of discounting by Libor, we have to discount the forward price by the repo rate. The futures price based on this bond will be

$$p_x = \frac{-w_f c + (p_v + w_m c)e^{y_{br}\, t_f}}{f_c} \tag{14.47}$$

Given that the repo duration is a relatively small part of the risk of futures and the accrual contribution to the price is small, we can simplify the above equation as

$$p_x \approx \frac{p_v}{f_c} e^{y_{br}\, t_f} \tag{14.48}$$

If the repo rate is the same for all bonds in the basket, the forward compounding of the repo factor will be the same for all bonds and the futures price can be written as

$$p_x = e^{y_{br} t_f} \int \ldots d\ln(\tag{14.49}$$

The repo duration components of the futures will thus be

$$D_{x,p} = -t_f \psi_k(t_f) \tag{14.50}$$

Therefore, futures have a negative duration exposure to short repo rates. Since the repo competes with Libor and the Fed Funds rate, futures generally have short exposure to short rates. In times of crisis, such as in the wake of the Lehman bankruptcy, repo and Libor rates have diverged significantly.

Interest rate duration components of futures cannot be calculated analytically since there is an option embedded in the futures contracts. If rates rise, bonds with the longest duration underperform other bonds and become cheapest to deliver, and when rates fall, bonds with shortest duration underperform other bonds and become cheapest to deliver. In other words, the weight of bonds in the basket changes as interest rates move and we have to calculate the effect of that weight change on duration of the futures. Due to the optionality of futures, some futures have negative convexity.

Duration and convexity components are calculated by shifting the level of interest rates and calculating its impact on price using (13.27) and (13.28), respectively. Since bond futures have zero market value, it is more appropriate to calculate the VBP for each contract than duration components. The futures price is only a reference value for calculating margin movements and no money changes hands when a futures contract is bought or sold.

14.8 ANALYTICS FOR BOND FUTURES

There is a security spread duration component associated with futures contracts as well. The exposure of the futures contract to the spread of the CTD security relative to the TSIR is slightly different from the duration of the futures. The duration components are based on the forward price of the security but the spread of the security will impact the current price of the security and thus has a slightly longer duration.

Tables 14.1, 14.2 and 14.3 show options, valuations, and risk analytics for a few global futures contracts, respectively. The futures prices and ticker symbols are from Bloomberg. Given the extremely low level of interest rates compared to the notional coupon of the futures, the call options have almost no value. For futures with a longer delivery date, the value of the option increases. For example, TYZ12 expires 3 months after TYU12 and its put option is accordingly more valuable.

TABLE 14.1 Futures options analytics, July 31, 2012

Security				Volatility		Option Value		Strike Price	
Ticker	Crncy	Size	Last Trade	Price	Yield	Call	Put	Call	Put
TUU12	USD	200,000	09/28/12	0.62%	177.14%	0.000	0.000	110.85	108.85
FVU12	USD	100,000	09/28/12	2.38%	119.37%	0.000	0.000	127.19	118.12
TYU12	USD	100,000	09/19/12	4.86%	89.35%	0.000	0.016	141.47	126.51
TYZ12	USD	100,000	12/19/12	4.87%	84.51%	0.000	0.219	141.71	125.49
USU12	USD	100,000	09/19/12	10.65%	49.33%	0.000	0.002	176.02	128.51
OEU12	EUR	100,000	09/06/12	3.98%	281.95%	0.000	0.036	129.73	122.06
RXU12	EUR	100,000	09/06/12	8.59%	101.01%	0.000	0.002	155.57	128.85
UBU12	EUR	100,000	09/06/12	23.61%	70.11%	0.000	0.016	177.70	115.81
JBU12	JPY	100 Mil	09/10/12	2.22%	76.58%	0.000	0.000	147.39	133.89
GU12	GBP	100,000	09/26/12	6.68%	52.46%	0.000	0.364	131.77	121.00
CNU12	CAD	100,000	09/19/12	10.74%	88.26%	0.000	0.001	154.47	117.08
YMU12	AUD	100,000	09/17/12	1.29%	50.07%				
XMU12	AUD	100,000	09/17/12	1.31%	42.21%				
ZYU12	NZD	100,000	09/12/12	0.63%	25.64%				
ZTU12	NZD	100,000	09/17/12	0.66%	18.87%				

TABLE 14.2 Futures valuations analytics, July 31, 2012

	Price					Spread	Yield	
Ticker	Market	Fair	Model	Recons.	Notional	To Curve	Exp	Theta
TUU12	110.30	110.49	110.45	110.49	220,609	−0.018%	0.197%	0.405%
FVU12	124.78	124.96	124.85	124.75	124,781	−0.022%	0.480%	1.133%
TYU12	134.66	134.95	134.56	134.60	134,656	−0.048%	0.905%	2.129%
TYZ12	133.64	133.90	133.42	133.50	133,641	−0.058%	0.984%	2.235%
USU12	151.03	151.74	150.98	150.73	151,031	−0.045%	1.997%	2.722%
OEU12	127.75	127.88	128.05	127.85	127,750	0.029%	0.317%	1.187%
RXU12	144.57	144.75	144.35	144.70	144,570	−0.035%	1.086%	2.542%
UBU12	137.08	137.21	136.83	137.17	137,080	−0.016%	2.025%	2.154%
JBU12	144.01	144.11	144.06	144.08	144,010,000	−0.006%	0.435%	1.260%
GU12	121.88	121.70	121.87	121.69	121,880	0.017%	1.524%	2.811%
CNU12	138.94	139.14	138.58	139.11	138,940	−0.052%	1.594%	2.330%
YMU12	97.48	97.42	97.71	0.00	109,995	−0.002%	2.568%	2.776%
XMU12	97.00	97.04	97.25	0.00	125,704	−0.001%	2.938%	3.521%
ZYU12	97.61	97.06	97.34	0.00	116,148	−0.003%	2.920%	3.541%
ZTU12	96.62	96.59	96.83	0.00	138,927	−0.002%	3.381%	3.932%

TABLE 14.3 Futures risk analytics, July 31, 2012

	Tsy Duration					Repo Duration			Spread Dur
Ticker	Level	Slope	Bend	Cubic	Quartic	Level	Slope	Bend	Level
TUU12	1.77	−0.97	−0.75	1.88	−1.46	−0.09	0.09	−0.08	1.86
FVU12	4.17	−0.52	−4.16	1.84	3.48	−0.09	0.09	−0.08	4.26
TYU12	6.13	1.06	−5.89	−2.62	4.51	−0.13	0.13	−0.12	6.27
TYZ12	6.35	1.70	−6.06	−3.50	3.77	−0.38	0.35	−0.24	6.70
USU12	10.99	6.76	−1.70	−6.93	−8.13	−0.13	0.13	−0.12	11.09
OEU12	4.52	−0.21	−4.60	0.93	4.17	−0.10	0.10	−0.09	4.65
RXU12	7.90	2.71	−5.97	−6.39	0.75	−0.09	0.09	−0.08	7.93
UBU12	16.92	14.08	7.27	1.65	−3.38	−0.09	0.09	−0.08	17.11
JBU12	6.72	1.43	−6.31	−3.69	4.47	−0.10	0.10	−0.09	6.83
GU12	8.43	3.65	−5.13	−6.92	−1.95	−0.15	0.15	−0.13	8.59
CNU12	7.73	2.63	−5.98	−6.21	0.94	−0.13	0.12	−0.11	7.88
YMU12	2.30	−0.26	−2.25	0.91	1.70	−0.08	0.08	−0.07	2.38
XMU12	7.10	2.48	−5.13	−5.36	0.13	−0.13	0.12	−0.11	7.23
ZYU12	2.74	−0.03	−2.77	0.24	−0.04	−0.07	0.07	−0.06	2.81
ZTU12	6.95	2.60	−4.72	−5.45	−0.06	−0.12	0.12	−0.10	7.07

In Table 14.2 we provide three calculated prices for the futures markets. The fair price is the calculated price from the basket using the methodology described in previous sections. This should be most closely related to the market price and provides a direct arbitrage opportunity if significantly different from the market price. The US futures market closes at 3:00pm Eastern time but the cash market is open usually up to 5:00pm. Cash prices are supposed to have been captured at 3:00pm, but sometimes there might be timing differences that could lead to discrepancies. The model price is based on using term structure prices for bonds in the basket and then calculating the CTD and the futures price. The reconstituted price is based on constructing the price of bonds in the basket from zero coupon bonds and using those prices for futures calculations. There is a closer relationship between the model or reconstituted price and the market price of the futures than between the fair price and the market price. It appears that most futures in the tables are cheap relative to fair value.

14.9 AUSTRALIAN BOND FUTURES

Australian and New Zealand bond futures are structured differently from those of other countries. The contracts are cash settled and thus there is no delivery in the contracts. The price of the bond futures is calculated from the average yield of all the bonds in the basket as follows:

$$p_f = 100(1 - y_{\text{av}} \tag{14.51}$$

However, for margin calculation, the price of a notional 10-year bond with a coupon of 6% is calculated using the average yield of the basket. Thus the contract price is calculated as

$$p = \frac{3(1 - x^{20})}{y/2} + 100x^{20}, \quad x = \frac{1}{1 + y/2} \tag{14.52}$$

The margin movement is based on the contract price.

14.10 REPLICATION OF BOND FUTURES

We can use linear optimization to replicate the risk profile of bond futures. The replication is useful for either arbitraging bond futures or replicating the performance of futures for funds that cannot invest in derivatives. The universe of securities that can be used to replicate bond futures can be one of the following three groups:

- Bonds in the basket.
- The universe of coupon bonds.
- The universe of all bonds including Strips.

Depending on the universe of securities that we use, the tracking error can vary. Under normal circumstances all strategies replicate the bond futures very closely. However, in times of stress where bond futures typically demand a liquidity premium, there will be some divergence between the replicating portfolio and the bond futures. Each one of the above universes can be optimized for either buying or selling bond futures. Due to the option characteristics of bond futures, it is not always possible to calculate all the risks without shorting some securities. Instead of shorting securities, we allow a small variance in the bend duration relative to the bond futures. Assuming a fully financed portfolio, we have to include the durations of financing in the aggregate durations. We use the following constraints to calculate the replicating portfolio:

$$\sum_i N_i D_{0,i} p_i - N_l D_{0,l} p_l = D_{0,x} p_x \tag{14.53}$$

$$\sum_i N_i D_{1,i} p_i - N_l D_{1,l} p_l = D_{1,x} p_x \tag{14.54}$$

$$\sum_i N_i D_{2,i} p_i - N_l D_{2,l}\, p_l = D_{2,x} p_x \pm 0.02 D_{0,x} p_x \tag{14.55}$$

$$\sum_i N_i\, p_i = N_l\, p_l \tag{14.56}$$

$$0.9\, p_x < \sum_i N_i\, p_i < 1.1 p_x \tag{14.57}$$

where N_i and N_l are the number of units (face amount) of bond i and of Libor, respectively; $D_{k,i}$, $D_{k,l}$, and $D_{k,x}$ represent the kth component of interest duration i, of the Libor duration through the delivery date, and of the duration of the futures, respectively; and p_i, p_l, and p_x are the price of bond i, the price of Libor for maturity on the delivery date, and the futures price, respectively.

Constraints (14.53) and (14.54) match the level and slope durations of the optimized portfolio with the futures, and (14.55) matches the bend duration with a 2% variance of the level duration. Constraint (14.56) requires equal market value for bonds and Libor financing, and (14.57) is a limit on the variance of market value. The objective function will be

$$\text{Obj} = \sum_i N_i D_{0,i} p_i y_i \tag{14.58}$$

For optimizing a portfolio that is designed to outperform bond futures, for example, for buying the portfolio and shorting the bond futures, we maximize the objective function. For an optimized portfolio that is constructed to underperform the bond futures, we minimize the objective function.

Table 14.4 shows a sample of replicating portfolios for TYU12. There are four replicating strategies: using coupon bonds only, using Strips as well as coupons, and long and short for either scenario. When the universe of bonds consists of coupons as well as Strips, the yield difference between the optimized portfolio and the futures yield is larger than when only coupon bonds are permitted.

We use the difference between the market price of a futures and its fair price as an indicator of cheapness/richness of the contract. For example, in Table 14.2, TYU12 is cheap by 134.95 − 134.66 = 0.29.

TABLE 14.4 Replicating futures risks, July 31, 2012

Ticker	Cusip	Cpn	Maturity	B/S	Univ	Number	Mv	Invoice	Sprd	Yield
TSY	912828SN	1.5	3/31/19	B	Cpn	34,504,563	36,007,464	104.356	−0.035%	0.901%
TSY	912810EE	8.5	2/15/20	B	Cpn	41,141,553	65,174,500	158.415	−0.047%	0.979%
TSY	912828TC	1	6/30/19	B	Cpn	33,963,089	34,101,410	100.407	−0.044%	0.950%
LBR			9/19/12	S	Cpn	−135,392,438	−135,283,375	99.919	0.003%	0.601%
Hedge				B	Cpn				−0.044%	0.958%
TYU 12			9/19/12	S	Cpn	−1,000			−0.048%	0.905%
TSY	912833KV	0	5/15/19	B	All	80,043,139	74,898,767	93.573	−0.001%	0.979%
TSY	912810EH	7.875	2/15/21	B	All	14,408,071	22,780,771	158.111	−0.056%	1.152%
TSY	912803A M	0	2/15/17	B	All	14,812,201	14,350,060	96.880	0.157%	0.698%
TSY	912810EE	8.5	2/15/20	B	All	12,583,085	19,933,527	158.415	−0.047%	0.979%
LBR			9/19/12	S	All	−132,069,511	−131,963,125	99.919	0.003%	0.601%
Hedge				B	All				−0.006%	0.997%
TYU 12			9/19/12	S	All	−1,000			−0.048%	0.905%
TSY	912828PV	0.5	11/30/12	S	Cpn	−8,141,208	−8,157,962	100.206	0.251%	0.133%
TSY	912828TH	0.875	7/31/19	S	Cpn	−86,285,992	−85,742,015	99.370	−0.046%	0.967%
TSY	912810QW	3	5/15/42	S	Cpn	−887,995	−977,238	110.050	0.051%	2.531%
TSY	912828TG	0.5	7/31/17	S	Cpn	−48,866,919	−48,670,970	99.599	−0.036%	0.581%
LBR			9/19/12	B	Cpn	143,663,912	143,548,186	99.919	0.003%	0.601%
Hedge				S	Cpn				−0.041%	0.897%
TYU 12			9/19/12	B	Cpn	1,000			−0.048%	0.905%
TSY	912803DX	0	5/15/42	S	All	−1,661,060	−738,457	44.457	0.073%	2.722%
TSY	912820RE	0	8/15/18	S	All	−98,973,980	−94,491,449	95.471	−0.055%	0.768%
TSY	912803AR	0	8/15/19	S	All	−36,790,354	−34,368,077	93.416	−0.067%	0.968%
TSY	912833KA	0	5/15/13	S	All	−13,946,437	−13,950,203	100.027	−0.087%	−0.034%
LBR			9/19/12	B	All	143,663,912	143,548,186	99.919	0.003%	0.601%
Hedge				S	All				−0.057%	0.872%
TYU 12			9/19/12	B	All	1,000			−0.048%	0.905%

14.11 BACKTESTING OF BOND FUTURES

To use the above information in a backtest of a strategy to buy bond futures when they are cheap and to short them when rich, we have to make sure that the difference between the market price and fair price is not a consequence of timing the price captures. We analyzed historical cheapness/richness of bond, note, and 5-year futures in the US. When there was a price difference of 0.35 or more for three consecutive days, we assumed that the price difference was real and not an artifact of different times for capturing price of futures and cash bonds. We then used the minimum difference as our value signal. If bond futures were rich, which they were in about 75% of cases, we sold bond futures and bought cash bonds to hedge them, similarly to what we did in Table 14.4. When bond futures were cheap, we bought bond futures and shorted cash bonds from the basket. The trades were held until 1 day before the first delivery date or if the opposite trade was triggered. There was no hedging of the trades during the holding period. It is possible that the shorted bonds had a lower repo rate than what we used and therefore our gain was exaggerated. However, even if we execute the strategy only when bond futures were rich, the gain per trade stayed about the same and we lost in only about 25% of cases. Table 14.5 shows the result of backtest from 2001 through mid-2012. Table 14.6 shows the list of trades that either lost money or had the least gain when we bought the futures and sold cash bonds.

What is common to all these trades is that the market yield moved by 100 bps or more in each case. Since we did not change the hedge during the trade period and bond futures have very low to negative convexity, the trades underperformed. Had we hedged after a market move of about 25–30 bps, the trades would all have contributed positively. Consequently, the performance of the backtest results is significantly underestimated.

TABLE 14.5 Bond futures backtest results, July 31, 2012

Avg Gain	Stdev	Number	IR
812	998	71	1.98

TABLE 14.6 Bond futures backtest underperformers, July 31, 2012

Ticker	Start	End	Market	Trade	G/L
USZ08	7/22/08	11/28/08	−16,375	15,391	−985
USH08	10/3/07	2/29/08	−8,304	7,906	−398
USZ07	7/5/07	11/30/07	−10,954	10,813	−141
TYU02	5/14/02	8/30/02	−10,177	10,094	−83
TYZ07	5/16/12	11/30/07	−8,495	8,578	83
TYZ08	7/3/07	11/28/08	−10,632	10,719	87
FVH08	7/22/08	2/29/08	−7,858	7,953	95

CHAPTER 15

Bond Options

Analysis of bond options requires the term structure of volatility as well as the term structure of rates. While the shape of the yield curve can change significantly, our term structure of volatility does not include explicit correlation parameters. Instead, we match the volatility at every point on the volatility surface with market volatility, using the adjustment table (as explained in Section 9.3), and therefore the implied market correlations are included implicitly in the volatility model. Bond options can be of three general categories:

- *European bond options* – The option can be exercised only at expiry.
- *American continuous options* – The option can be exercised at any time after the vesting time. For example, callable bonds cannot be called prior to a specific date, called the first call date.
- *American discrete options* – The option can be exercised only on specific dates, usually coupon payment dates, between the first call date and the expiration date.

The widely traded interest rate volatility is based on options on swaps (swaptions). A call swaption is an option to receive a fixed rate underlying bond on or before a certain expiration date and a put swaption is the right to pay fixed rate before expiration date. Swaptions trade with many different expiration dates for the options and maturity dates for the underlying swap.

An at-the-money call swaption is based on the implied forward coupon of the underlying swap. For example, if short rates are at 1% and 10-year rates are at 3%, the implied 10-year swap 1 year forward might be at 3.25%. The buyer of an American call swaption can exercise the options at any time in the first year and receive a coupon of 3.25% through the final maturity of the underlying which would be 11 years from the original trade date.

In an upward sloping yield curve, the buyer of a call option has a very high incentive to exercise early and start collecting the forward coupon, instead of earning short rates through the life of the option. The same is true for the exercise of a put option in an inverted yield curve.

The Advanced Fixed Income and Derivatives Management Guide. Saied Simozar

15.1 EUROPEAN BOND OPTIONS

We first analyze a European Libor bond option. The price of a European bond option is the discounted value of the expected exercised price of the option on the expiration date. We use (14.19) as the arbitrage-free forward price of the bond. The probability distribution of a bond yield at a future time t is given by

$$d\rho_{f,t} = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{(\ln(y)-\ln(y_0)-\mu)^2}{2\sigma_y{}^2}} d\ln(y) \tag{15.1}$$

where the standard deviation of the forward log of yield is

$$\sigma^2 = v_y^2 t \tag{15.2}$$

By definition, the sum of all probabilities of forward yields is equal to 1:

$$\int_0^\infty d\rho_{f,t} = \frac{1}{\sqrt{2\pi}\sigma_y} \int_0^\infty e^{-(\ln(y)-\ln(y_0)-\mu)^2/2\sigma_y^2} d\ln(y) = 1 \tag{15.3}$$

The drift μ is calculated in such a way that the expected value of the forward price of the bond is equal to its forward price. The resulting equation (15.4) is called arbitrage-free requirement:

$$p_{f,t} = \frac{1}{\sqrt{2\pi t}v_y} \int p_f(y) e^{-(\ln(y)-\ln(y_0)-\mu^2)/2v_y^2 t} d\ln(y) = \langle p_f(y) \rangle \tag{15.4}$$

The forward price of a call option with exercise yield y_x and time to expiration t_x will be

$$C(t_x) = \frac{1}{\sqrt{2\pi}\sigma_y} \int_0^{y_x} \left(p(y) - p(y_x)\right) e^{(\ln(y)-\ln(y_0)-\mu)^2/2\sigma_y^2} d\ln(y) \tag{15.5}$$

If price and yield had a linear relationship, we could calculate the distribution drift analytically, just as is done for equities. However, due to convexity, the drift needs to be calculated numerically. For computation purposes, it is much more convenient to write equation (15.5) in terms of the drifted unit normal distribution:

$$C(t_x) = \frac{1}{\sqrt{2\pi}} \int_0^{y_x} \left(p(y) - p(y_x)\right) e^{-(x-x_0)^2/2} dx \tag{15.6}$$

with

$$y = y_0 e^{\sigma_y x} \tag{15.7}$$

The present value of the forward call price, for most options, is calculated by discounting the future price by the risk-free yield at the time of option expiration:

$$C(0) = C(t_x) e^{-t_x r_x} \tag{15.8}$$

where r_x is the risk-free rate between the trade date and the expiration date of the option. For bond options, the above equation is not accurate, because the paths of interest rates that result in a call option being exercised imply lower forward rates and, in general, lower rates before the option expiration. We will estimate the discount function later in Section 15.3.

Using equity like options for bonds and then making a convexity adjustment to the option's price will violate the arbitrage-free requirement. The arbitrage-free requirement must be part of the drift calculation that includes convexity.

The most widely used model to price European bond options and swaptions is the Black-76 model which uses a log-normal distribution of rates. The option is calculated the same as equity options by replacing the price with forward yield. Thus, in the Black-76 model the expected value of the forward rate is equal to the forward rate. This, however, does not mean the expected value of the forward price will be equal to the forward price and as such it is not arbitrage-free. The Black-76 equation can be derived by using (14.14), replacing price by rate and noting that in a swaption the exercise of the option will lead to a stream of cash flows that is equal to the difference between the strike rate and the forward rate through the maturity of the bond. Using the Black–Scholes call price (14.16) and replacing the price with interest rate, the call rate can be calculated as

$$r_{\text{call}} = \frac{1}{\sqrt{2\pi t}v_y} \int_0^{r_x} (r - r_x) e^{(\ln(r/r_f) + v_y^2 t/2)^2 / 2v_y^2 t} d\ln(r) \tag{15.9}$$

The exercise of the call swaption will entitle the receiver to a stream of cash flows through the maturity of the underlying bond equal to the call rate. If r_f is the implied forward rate, the value of the stream of cash flows will be

$$p_{f,\text{call}} = 100 \sum_i^N \frac{r_{\text{call}}/m}{\left(1 + r_f/m\right)^{im}} = 100 \frac{r_{\text{call}}}{r_f} \left[1 - \left(1 + \frac{r_f}{m}\right)^{-Nm}\right] \tag{15.10}$$

Equation (15.10) is very widely used to price swaptions in the market and is known as the Black-76 model. Black-76 has three shortcomings:

- It is not strictly arbitrage-free. The arbitrage-free requirement must be based on the expected forward price; the arbitrage-free condition of the Black-76 model is based on rates. If the underlying bond has a low convexity or the option tenor is short, the price–yield relationship will be almost linear and Black-76 model will be accurate.
- When an at-the-money call is exercised, the implied forward rates are lower, implying that the discounting of the forward price must be done with slightly lower interest rates. Black-76 has no mechanism to make that adjustment.
- Black-76 cannot be easily adapted to American bond options.

The present value of a call price using Black-76 on July 8, 2011 for a 10-year swaption 1 year forward was 3.59, and the above model is very close at 3.55. Figure 15.1 shows a Bloomberg screenshot, using the same parameters, which prices the option at 3.71.

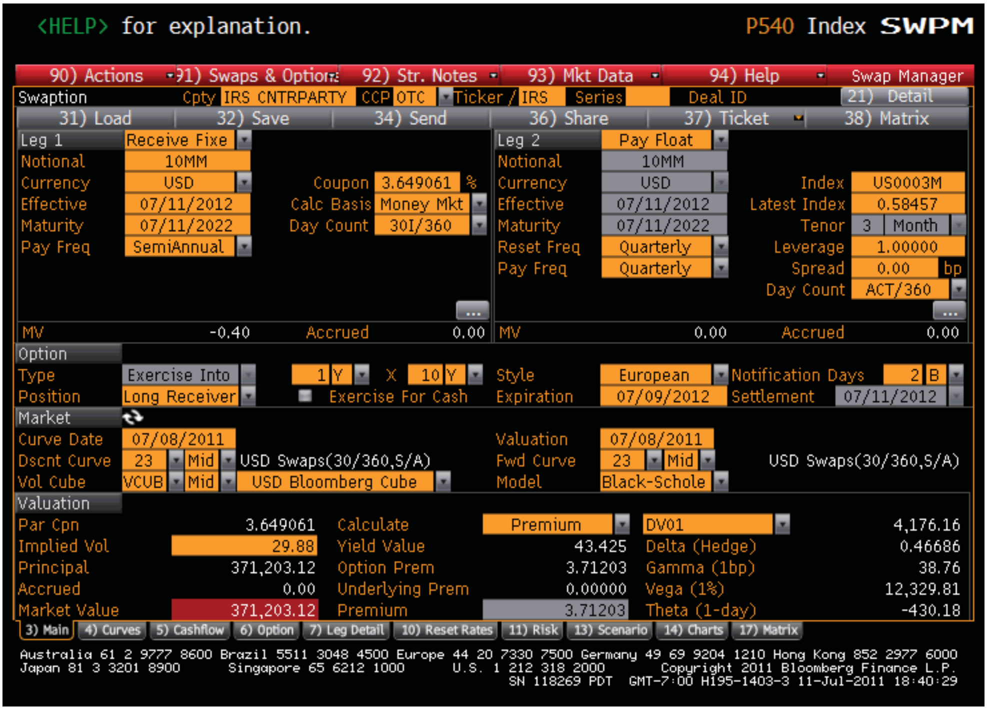


FIGURE 15.1 European at-the-money call swaption, July 8, 2011

For a longer dated option and longer maturity where the convexity can be more important, we price a 5-year option on a 20-year bond. The following table shows the inputs and calculated prices:

r_f	0.04809
m	2
N	20
t_f	5
v_r	20.1%
5-year discount rate	1.84%
Black-76 price	9.95
Our model	9.53

The Black-76 model is similar to an equity option, using a log-normal distribution for rates instead of prices. The range of bond prices is much more limited than the range of equity prices. In a log-normal distribution, the price of a stock or interest rates can range from zero to infinity. However, the price of a bond at a yield (rate) of zero is equal to the sum of all the cash flows and at a yield of infinity the price is zero. Thus, at extreme rates the Black-76 model fails.

15.2 EXERCISE BOUNDARY OF AMERICAN OPTIONS

Consider a forward distribution of rates where an American call option is exercisable. In Figure 15.2, one of the paths from the current level of rates, time t_0 and point A to point C is through point B. Assume that point C is the last exercise date of an American option for a bond that matures at point M and, as such, an option that is in-the-money will be exercised. We assume that there is no cost associated with the exercise of the option. Usually, callable bonds might have an exercise cost, since the call has to be funded and the raised funds require a brokerage fee. We evaluate the forward price of the bond at all forward points at time C that meet arbitrage-free requirement (15.4).

We then calculate the forward price distribution of the bond price at time B that meets the arbitrage-free requirement stated in DUND as

$$dp = \frac{1}{\sqrt{2\pi}} p(y) e^{-(x-x_B)^2/2} dx \tag{15.11}$$

At point B, another possible exercise date for the American option, we calculate the call price of the option if it is exercised and converted to a bond that matures at time M. At point B we then have to decide whether it is more economical to exercise the option or to leave it for possible exercise at point C. The distribution of rates at point B is governed by the volatility and drift of a forward bond at point B that matures at point M.

At each point B, we calculate the probability of all the paths leading to the final exercise time C. The volatility of the paths from B to C is governed by the volatility σ_{BC} and drift μ_{AB}. The volatility can be calculated by aggregating the volatility of cash flows using (9.37). We note the additive property of the drift as follows:

$$\mu_{AC} = \mu_{AB} + \mu_{BC} \tag{15.12}$$

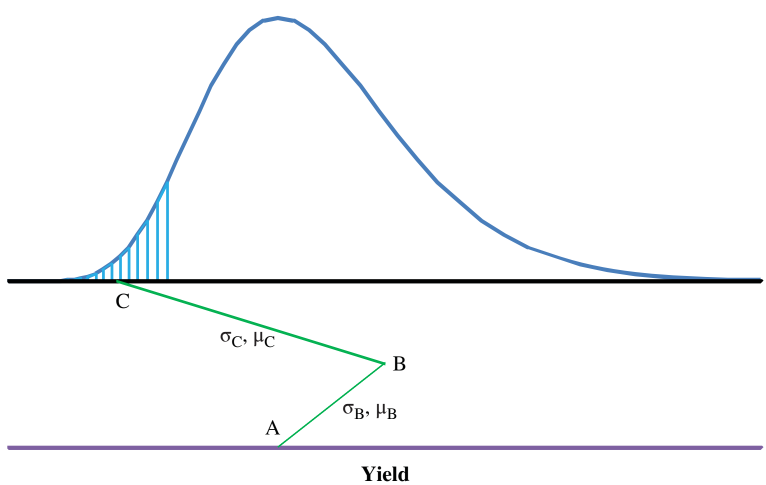


FIGURE 15.2 Log-normal probability distribution

Using unitized drifts (14.22),

$$\zeta_{BC} = \frac{\zeta_{AC}\sigma_{AC} - \zeta_{AB}\sigma_{AB}}{\sigma_{bc}} \tag{15.13}$$

Since we have already calculated the forward price of the option at every point at time C, we calculate the probability of all the paths from point B to C and calculate the forward price of an unexercised option at time C.

After discounting the forward prices of unexercised options at time C to time B, we can compare the price of an exercised option and an unexercised American option and calculate the yield at which exercised and unexercised prices will be identical. We call a collection of such points where the value of exercised and unexercised options is equal to the exercise boundary. The exercise boundary has to be calculated from the last exercise date to the earliest date sequentially. For call options, the yields below the exercise boundary imply early exercise. At higher yields, unexercised options are more valuable than exercised ones, even if they are in-the-money. For put options, the yields above the exercise boundary will result in early exercise.

15.3 PRESENT VALUE OF A FUTURE BOND OPTION

If a call option is exercised in the future, the paths that will lead to the exercise of the option are more likely to involve lower interest rates than is implied in the present. Therefore, the discount function for calculating the present value of a forward exercised option should also imply lower rates. In this section we will derive the approximate discount function for calculating the present value of a future exercise. For notational convenience, we replace μ_{AB} and σ_{AB} by μ_B and σ_B, respectively. Likewise, we replace μ_{BC} and σ_{BC} by μ_C and σ_C, respectively. The probability density of all yields at point B in Figure 15.2 for a bond that matures at M can be written as

$$d\rho_B = \frac{1}{\sqrt{2\pi}\sigma_B} e^{-(\ln(y_B)-\ln(y_A)-\mu_B)^2/2\sigma_B^2} d\ln\left(y_B\right) \tag{15.14}$$

where y_A is the forward yield of the bond for settlement at time B calculated at time A and y_B is the distribution of that yield at time B. Now let us consider a rate process that will take us from A to B to C. The combined probability density to go from A to B to C can be written as

$$d\rho_B d\rho_C = \frac{1}{2\pi\sigma_A\sigma_B} e^{-\left(\ln(y_B/y_A)-\mu_B\right)^2/2\sigma_B^2-\left(\ln(y_C/y_B)-\mu_C\right)^2/2\sigma_C^2} d\ln\left(y_B\right) d\ln\left(y_C\right) \tag{15.15}$$

where σ_B (σ_C) is the volatility of the forward bond at B (C), in the interval between A and B (B and C), and μ_B (μ_C) is the drift of the forward bond at B (C). Substituting

$$u = \ln\left(\frac{y}{y_0}\right) \tag{15.16}$$

for some constant y_0 and rearranging the parameters leads to

$$d\rho_B d\rho_C = \frac{1}{2\pi\sigma_B\sigma_C}\exp\left[-\frac{(u_C-\mu_C-u_A-\mu_B)^2}{2(\sigma_C^2+\sigma_B^2)}\right]du_C \\ \times\exp\left[-\frac{\sigma_C^2+\sigma_B^2}{2\sigma_B^2\sigma_C^2}\left(u_B-\frac{(u_C-\mu_C)\sigma_B^2+(u_A+\mu_B)\sigma_C^2}{\sigma_B^2+\sigma_C^2}\right)\right]du_B \tag{15.17}$$

To calculate the average $u_B(\overline{u_B})$ for all paths that start at point A and end at C, we can integrate the above equation multiplied by u_B over its range. $u_B - u_A$ is a measure of how many standard deviations the distribution has shifted. We can thus write

$$\overline{u_B}d\rho_C = \frac{1}{2\pi\sigma_B\sigma_C}\exp\left[-\frac{(u_C-\mu_C-u_A-\mu_B)^2}{2(\sigma_B^2+\sigma_C^2)}\right]du_C \\ \times\int\exp\left[-\frac{\sigma_B^2+\sigma_C^2}{2\sigma_B^2\sigma_C^2}\left(u_B-\frac{(u_C-\mu_C)\sigma_B^2+(u_A+\mu_B)\sigma_C^2}{\sigma_B^2+\sigma_C^2}\right)^2\right]u_B du_B \tag{15.18}$$

or

$$\overline{u_B}d\rho_C = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_B^2+\sigma_C^2}}\exp\left[-\frac{(u_C-\mu_C-u_A-\mu_B)^2}{2(\sigma_B^2+\sigma_C^2)}\right] \\ \times du_C\,\frac{(u_C-\mu_C)\sigma_B^2+(u_A+\mu_B)\sigma_C^2}{\sigma_B^2+\sigma_C^2} \tag{15.19}$$

In this equation

$$d\rho_C = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_B^2+\sigma_C^2}}\exp\left[-\frac{(u_C-\mu_C-u_A-\mu_B)^2}{2(\sigma_B^2+\sigma_C^2)}\right]du_C \tag{15.20}$$

is simply the density function of a random process that goes from point A to point C, with

$$\mu = \mu_B + \mu_C \\ \sigma^2 = \sigma_B^2 + \sigma_C^2 \tag{15.21}$$

Thus,

$$\overline{u_B} = \frac{(u_C-\mu_C)\sigma_B^2+(u_A+\mu_B)\sigma_C^2}{\sigma_B^2+\sigma_C^2} \tag{15.22}$$

μ_B and μ_C are the drift parameters. Assuming constant volatility in the interval between A and C, and the additive property of the drift, we can write

$$\begin{aligned} \sigma_B^2 &= \upsilon^2 t_B \\ \sigma_C^2 &= \upsilon^2 t_C \\ T &= t_B + t_C \\ \mu_B &= \lambda t_B \\ \mu_C &= \lambda t_C \end{aligned} \tag{15.23}$$

Substituting these values in (15.22) we find the shift in u of a bond forward as

$$\overline{u_B} - u_A = \frac{t_B(u_C - u_A)}{T} \tag{15.24}$$

This states that a bond forward having a shift in the log of yield equal to $u_C - u_A$, at time C (T), has an average shift at an intermediate point B, proportional to time at point B (t_B).

Our analysis up to this point has been for the forward security. However, we are interested in calculating the impact of changes in the forward rates on the discount function in the interval from A to C. At point B, the yield of the forward security has shifted by

$$u_B - u_A = \ln\left(\frac{y_B}{y_A}\right) \tag{15.25}$$

We can assume that the yield of all the points between B and C has also shifted proportionally. Likewise, the yield of the discount function has shifted proportionally. For the discount function, we add the subscript d to (15.24). Thus,

$$\frac{d\overline{u}}{dt_B} = \frac{u_{C,d} - u_{A,d}}{T} \tag{15.26}$$

This is the rate of change of the shift in $\overline{u}$ in the interval between A and C. When analyzing the discount function between A and C, we note that at time t_B any shift will impact the discount function from time t_B to t_C and not the whole range from t_A to t_C. To calculate the average shift in the discount function, we multiply the above equation by $T - t$, the area that is impacted by the shift, and integrate over its range:

$$\overline{u_d} - u_{A,d} = \frac{1}{T}\int_0^T \frac{d\overline{u}}{dt}(T-t)dt = \frac{u_{C,d} - u_{A,d}}{2} \tag{15.27}$$

Compare this with (15.24); the average shift for the end point of the discount function is half that of the forward bond. This is intuitively reasonable, since as we approach exercise time C, only the remaining portion of discount function is subject to volatility, while the forward bond has full volatility.

Next, we need to calculate the average volatility of the discount function. We first note that the discount function is an evolving function and, at any time t_B, only the segment between B and C can change. Additionally, the volatility at any forward point will propagate to the remaining part of the forward discount function. Thus, we need to calculate the average instantaneous volatility over the life of the discount function.

If $w(0, t_x, t_{y1}, t_{y2})$ is the volatility of a forward line segment (t_{y1}, t_{y2}) between time zero and t_x, then the incremental absolute volatility between t_x and $t_x + dt_x$ is calculated as

$$\begin{aligned} w(t_x, t_x + dt_x, t_{y1}, t_{y2})^2 \, dt_x = {} & w(0, t_x + dt_x, t_{y1}, t_{y2})^2 (t_x + dt_x) \\ & - w(0, t_x, t_{y1}, t_{y2})^2 t_x \end{aligned} \tag{15.28}$$

Thus,

$$w(t_x, t_x, t_{y1}, t_{y2})^2 = \frac{d\left(w(0, t_x, t_{y1}, t_{y2})^2 t_x\right)}{dt_x} \tag{15.29}$$

Equation (15.29) is general and applies to any forward yield (line segment). Integrating it leads to

$$\int_0^{t_x} w(0, t_x, t_{y1}, t_{y2})^2 \, dt_x = \int_0^{t_x} d\left(w(0, t_x, t_{y1}, t_{y2})^2 t_x\right) = w(0, t_x, t_{y1}, t_{y2})^2 t_x \tag{15.30}$$

To calculate the average volatility of the instantaneous forward rate, we evaluate (15.30) at $t_{y1} = t_{y2} = t_x = t_{\mathrm{C}}$. Making the necessary substitutions, average absolute volatility squared can be calculated:

$$\overline{W^2 t_{\mathrm{C}}} = w(0, t_{\mathrm{C}}, t_{\mathrm{C}}, t_{\mathrm{C}})^2 t_{\mathrm{C}} \tag{15.31}$$

Using a UND as in (14.25), Δx and Δu are related by

$$\Delta x = \frac{\ln(y_{\mathrm{C}}/y_{\mathrm{A}})}{\sigma} = \frac{\Delta u}{\sigma} \tag{15.32}$$

with

$$\sigma_d^2 = \overline{W^2(t_{\mathrm{C}} - t_{\mathrm{A}}} \tag{15.33}$$

where σ_d is the volatility of the discount function. Defining σ_f as the volatility of the forward security and using the UND from (14.25) and combining with (15.25), we find the number of standard deviations that the logarithm of yield has shifted at point C:

$$\ln\left(\frac{y_{\mathrm{C}}}{y_{\mathrm{A}}}\right) = u_{\mathrm{C}} - u_{\mathrm{A}} = \sigma_f \Delta x \tag{15.34}$$

Likewise, the number of standard deviations in yield shift for the discount function from (15.27) is

$$\ln\left(\frac{y_d}{y_{0,d}}\right) = \overline{u_d} - u_{\mathrm{A},d} = \frac{u_{\mathrm{C},d} - u_{\mathrm{A},d}}{2} = \frac{\sigma_d \Delta x}{2} \tag{15.35}$$

By equating the standard deviations, Δx in equations (15.34) and (15.35), we find

$$\frac{2}{\sigma_d} \ln\left(\frac{y_d}{y_{d0}}\right) = \frac{1}{\sigma_f} \ln\left(\frac{y_{\mathrm{C}}}{y_{\mathrm{A}}}\right) \tag{15.36}$$

Thus,

$$y_d = y_{d0}\left(\frac{y_C}{y_A}\right)^{\sigma_d/2\sigma_f} \tag{15.37}$$

As we can see, for $y_C = y_A$, the discount function will stay unchanged. The yield of the discount function will be lower if $y_C < y_A$ and vice versa. Using the UND function (14.24), this can be simplified as

$$y_d = y_{d0}e^{\sigma_d x/2} \tag{15.38}$$

For small values of σ_d we can simplify this by means of Taylor series to first order in x:

$$y_d \approx y_{d0}\left(1 + \frac{\sigma_d x}{2}\right) \tag{15.39}$$

We define the discount function d as

$$d = e^{-y_d t} = e^{-(1+\sigma_d x/2)y_{d0}t} \tag{15.40}$$

From (15.31), we can calculate σ_d as

$$\sigma_d = \sqrt{W^2(t - t_0)} \tag{15.41}$$

The adjustment to the discount factor is very small. For example, if the absolute yield volatility is 2% (200 bps) per year, for a 10-year period, $\sigma_d = 0.065$. For a shift of two standard deviations in yield (x = 2), assuming a yield of 5%, the change in the discount function is only about 3%, that is,

$$d = e^{-y_{d0}t}e^{-y_{d0}t\sigma_d x/2} \approx e^{-y_{d0}t} \times 0.968$$

This is the change in the discount function for the price of the option. The adjustment has to be made for all options when calculating the exercise boundary and discounting the value of exercised options.

15.4 FEEDFORWARD PRICING

Once we calculate the exercise boundary, we start at the earliest exercise date and calculate the price at all exercise yields. We then create a process at all forward yields where the option was not exercised and calculate forward paths. The exercise price times the weight of forward paths that land in the exercise zone is added to the option price and the weights of paths that end up in the non-exercise zone will be added to non-exercised weights.

For numerical calculation, if we slice a UND into N pieces and use Q sigma as the range of the UND (6 sigma gives sufficient accuracy), then x in (14.21) is given by

$$x_i = \frac{Q(2i - N)}{N} \tag{15.42}$$

The area of an even numbered slice i, between i – 1 and i + 1, using Simpson's rule is given by

$$\text{area}_i = \frac{2Q}{3N}\left(e^{-x_{i-1}^2/2} + 4e^{-x_i^2/2} + e^{-x_{i+1}^2/2}\right) \tag{15.43}$$

For every layer, we create a UND consisting of N intervals. The range of yields for exercise points that are farther in the future will be wider than range of yields for early exercise times. The number of intervals at every layer is, however, constant. Therefore, computation time increases linearly with the number of layers.

Every layer is divided into N buckets, with equal number of buckets at both sides of the drifted middle of the distribution. If $w_{n-1,j}$ is the weight of an unexercised option in the jth bucket of layer $n - 1$, it will progress to layer n as a log-normal distribution with drift and volatility calculated from (15.21). Assume that one of those paths that is a fraction η of $w_{n-1,j}$ will result in a value x_k that is in bucket i of the next layer to the right of the middle of the bucket. We calculate the contributed weight of that path to buckets i and $i + 1$ as follows:

$$\begin{aligned} w_{n,i} &= w_{n-1,j}\eta\frac{x_{n,i+1} - x_k}{x_{n,i+1} - x_{n,i}} \\ w_{n,i+1} &= w_{n-1,j}\eta\frac{x_k - x_{n,i}}{x_{n,i+1} - x_{n,i}} \end{aligned} \tag{15.44}$$

so that if a path leads to the boundary of two consecutive buckets, each will get half the contribution.

For each layer, the number of points that need to be analyzed is about N^2, since every point progresses to N points in the next layer. Even though this computation method is much more time-consuming than simple binomial tree method, the accuracy is significantly better, comparable to the accuracy of closed form solutions if they existed. The binomial tree method converges much less slowly and the number of nodes that need to be analyzed increases quadratically. Additionally, early exercise points are analyzed with far fewer steps than late exercise points. For most options, the earliest exercise times are the most valuable since the discounting is less. However, in binomial trees, the least emphasis is on the earliest times.

The spacing between layers does not need to be constant. The shorter dated layers can have closer spacing than long dated layers. This is in contrast to the binomial trees where shorter dated layers have fewer points and thus are less densely populated. For long dated swaptions and options, having more than 40 layers does not increase the accuracy in any meaningful way.

Table 15.1 shows the calculated premiums for a few European and American swaptions. There is a relatively large pricing difference between our calculation and Bloomberg's for American options, particularly put options.

Figures 15.3 and 15.4 show the screen images of American call and put options with 1 year to expiry and 10-year maturity. We can estimate the value of American versus European options by noting that an at-the-money European call is about 50% likely to be exercised and the likelihood is higher for American options. Given an immediate exercise, the underlying bond has a coupon of 3.65%, which is about 3.2% higher than the implied risk-free rate through the expiration of the option. If we assume 50% exercise probability spread over the year, on average we will receive 50% of the bond premium for 6 months, resulting in $0.5 \times 0.5 \times 3.2 = 0.80$ extra premium for the American call. Our model calculation of the American call appears more reasonable than the calculation in Figure 15.3.

TABLE 15.1 Bond option premiums, July 8, 2011

Parent			Option				Premium		
Cpn	Maturity	Fwd	C/P	A/E	Expiry	Strike	Fair	Model	Bloomberg
3.65	7/11/22	100	C	E	7/9/12	100	3.550	3.550	3.712
3.65	7/11/22	100	C	A	7/9/12	100	4.291	4.291	5.238
3.65	7/11/22	100	P	E	7/9/12	100	3.551	3.551	3.714
3.65	7/11/22	100	P	A	7/9/12	100	3.579	3.579	9.912

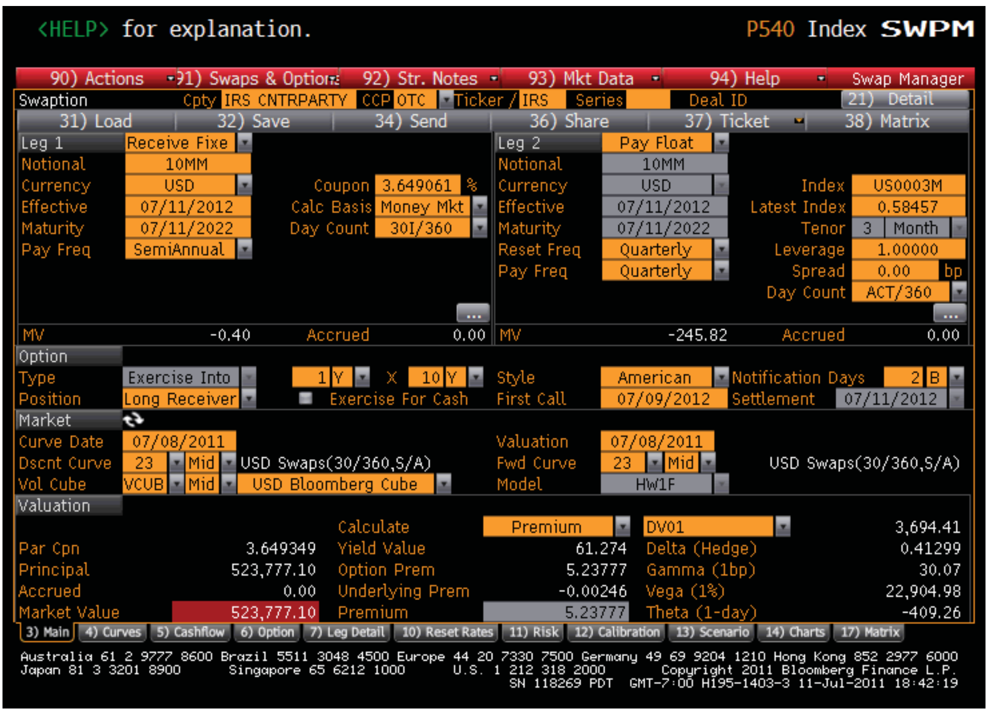


FIGURE 15.3 American at-the-money call swaption, July 8, 2011

In a steep yield curve environment, such as for the above options, an American put option is rarely exercised early since the put holder has to pay forward coupon early while the cash rate is much lower. It makes sense to exercise an American put in a steep yield curve only when the likelihood of exercise at expiration time is so high that the interest earned on the premium will be more than the time value of the option. The premium price of 9.912 in Figure 15.4 is highly suspect based on the shape of the curve and knowing the value of European put option.

The underlying price of an American call option increases approximately linearly with time and the difference between short rates and long rates. For example, if short rates are at 1% and long rates at 4%, the present value of a bond with a 1-year forward price of par will be about 103. If the option is exercised early, over the next year the

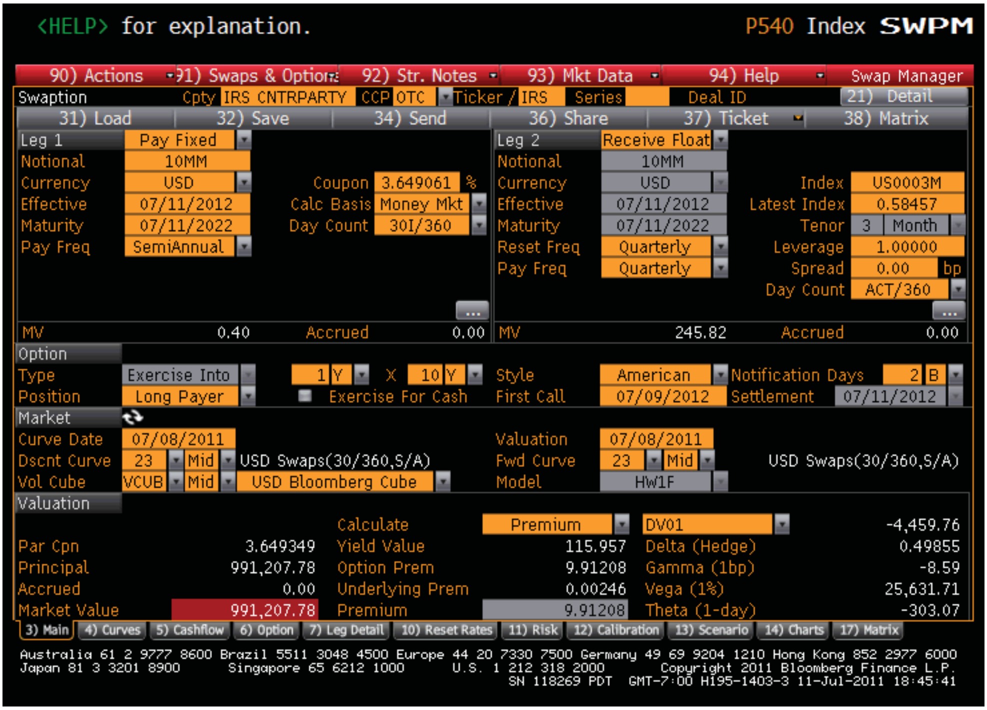


FIGURE 15.4 American at-the-money put swaption, July 8, 2011

bond will have a coupon of 4% instead of 1% for short rates. Likewise, for a 2-year forward, the added benefit will be approximately 6%. The price premium of an at-the-money option increases approximately with the square root of time. Therefore, in very steep yield curves, there is usually an expiration time at which the premium for early exercise will be more than the option premium and the American call option will be exercised immediately.

Table 15.2 is an example of four American options that are at the cusp of exercise. The price of the underlying securities if the options are exercised immediately is shown in the rightmost column. The parent price of the 10-year forward, maturing in 2025, of an option expiring in 4 years is 112.797. The optimal exercise for this option is immediate

TABLE 15.2 Early exercise of American call option, July 8, 2011

Parent			Option				Price		
Cpn	Maturity	Fwd	C/P	A/E	Expiry	Strike	Fair	Model	Parent
4.73	7/10/25	100	C	A	7/8/15	100	12.797	12.797	112.797
4.75	7/10/30	100	C	A	7/8/15	100	13.287	13.287	112.881
3.38	7/10/18	100	C	A	7/8/13	100	5.441	5.441	105.441
3.59	7/10/19	100	C	A	7/8/13	100	5.900	5.900	105.861

and the premium earned will be 112.797 − 100 = 12.797. However, an option with the same expiration date but 15-year maturity underlying, due to the higher volatility of the longer duration underlying, has more value if not exercised immediately. With a duration of 12.7 years, if long rates fall by 8 bps, it will be economical for this option to be exercised if the volatility does not change.

15.5 BOND OPTION GREEKS

Option Greeks refer to characteristics of options that influence their pricing. For a review of option Greeks see any standard options textbook or Haug [13]. For bond options, especially using TSIR and TSYV, the Greeks can be significantly more complicated than equity Greeks. The most commonly used option Greek is delta, the ratio of the change in the price of the option to the underlying security (parent). Vega, theta, and gamma are the sensitivity of an option's price to volatility, time, and second order change in the parent price, respectively. Thus,

$$\begin{aligned} O_d &= \frac{\partial p_o}{\partial p_u} \\ O_v &= \frac{\partial p_o}{\partial v_u} \\ O_t &= \frac{\partial p_o}{\partial t} \\ O_g &= \frac{\partial^2 p_o}{\partial p_u^2} \end{aligned} \tag{15.45}$$

where O_d, O_v, O_t, and O_g are respectively the option delta, vega, theta, and gamma, p_o is the option price, p_u is the underlying (parent) price, and v_u is volatility. Other option Greeks such as zomma, vomma, and vanna can also be calculated in a similar way. Refer to [13] for information on other option Greeks.

Since bond options are for an underlying security that is subject to maturity and thus a function of time, the most important task before one calculates the Greeks is to identify the underlying security. It is common practice for European options to use the forward security as the underlying security. However, this can cause consistency issues when we include American options and callable bonds, since the option delta can became larger than 1. While a delta larger than 1 is possible in some exotic options, for simple options it is never larger than 1. Alternatively, we can use the forward security based on the first exercisable date of an option. For options with discrete call dates, the underlying security changes when a call date expires, resulting in discontinuity in the risks of the underlying security. For consistency and simplicity, we use the spot underlying security for calculating Greeks.

Using the spot underlying security as the parent security for Greeks calculations will result in call and put at-the-money deltas that can be significantly different from 0.5 and –0.5, respectively. In nearly all options, buying a call and selling a put has the same risk as buying the underlying security for forward settlement. However, for American options this is not the case, since one leg of the option might be economical to exercise early.

TABLE 15.3 Bond option Greeks, July 8, 2011

Parent				Option				Greeks			
Desc.	**Cpn**	**Maturity**	**Price**	**C/P**	**A/E**	**Expiry**	**Strike**	**Delta**	**Gamma**	**Vega**	**Theta**
Option	3.65	7/11/22	100	C	E	7/9/12	100.00	0.426	0.019	0.000	–1.946
Option	3.65	7/11/22	100	C	A	7/9/12	100.00	0.564	0.018	0.000	–3.042
Option	3.65	7/11/22	100	P	E	7/9/12	100.00	–0.462	0.016	0.000	–1.948
Option	3.65	7/11/22	100	P	A	7/9/12	100.00	–0.471	0.015	0.000	–1.987
Bond	3.65	7/11/22	103.257								
Frwd	3.65	7/11/22	100								

Delta can be calculated by changing the yield of the underlying security and then calculating the impact of that change on the price of the security and the option. Using the TSIR, the price of the security can change due to changes in the level, slope, bend, etc. of the TSIR. Each one of those changes will result in a different delta. The calculated delta from the changes in the yield of the parent security is very close to the delta calculated from the changes in the level of TSIR. A European call swaption expiring in 3 years and with a forward maturity of 5 years has calculated deltas of 0.262, 0.161, and –0.235 from level, slope, and bend components, respectively. Thus, if the price change is from the slope of the yield curve, the delta hedging will not work as expected.

Vega, like delta, can be calculated by changing the volatility of the underlying security. However, using the TSYV, we can calculate the sensitivity of the option to the components of the term structure of volatility.

For bond options, theta and delta are related. If interest rates change, not only will the forward price of the option change, but also the discount function for calculating the present value of the option. This is one key ingredient that the Black-76 model is missing for correct evaluation of options. Using binomial trees addresses this issue, but the accuracy is not very high.

Table 15.3 shows common Greeks analytics for options listed in Table 15.1. The American call option has a higher than 50% probability of being exercised and therefore has a relatively high delta. This is reflected in the premium of the option as well in Table 15.1.

Ordinary options can only be hedged using other options that have the same expiration date but different strike prices. For American options that can be exercised continuously, there may not be any simple way of hedging them. The volatility model that was discussed in Chapter 9 can be used to hedge bond options similarly to the way TSIR exposures can be used for hedging interest rate risks.

15.6 RISK MEASUREMENT OF BOND OPTIONS

The risks of most options are measured by the delta of the option multiplied by the risks of the underlying security. For example, if an option has a delta of 0.4, the risk metrics of every unit of the option are equivalent to 0.4 units of the underlying security. Delta hedging for bonds does not work as well as for equities, currencies, and commodities.

By changing the components of the TSIR and measuring their impact on the price of the option and the underlying security, we can calculate a different delta for each component of the TSIR. Alternatively, the calculated level, slope, etc. durations of an option are equal to their specific delta times the durations of the parent security.

The price of callable bonds is equal to the underlying bond (no call bond) minus the call price. The price of a puttable bond is equal to the price of the underlying bond plus the put price. Writing subscript u for the underlying, c for call and p for put options, and b for a callable or puttable bond, we have the identities

$$\begin{aligned} p_b &= p_u - p_c + p_p, \\ D_{c,k} &= \frac{p_c(a_k - \Delta) - p_c(a_k + \Delta)}{2\Delta p_u} \\ D_{b,k} &= \frac{p_b(a_k - \Delta) - p_b(a_k + \Delta)}{2\Delta p_b} \end{aligned} \qquad (15.46)$$

where $D_{c,k}$ and $D_{b,k}$ are the kth duration component of a call option and callable bond respectively, and Δ is a small shift in the kth component of the TSIR a_k.

Table 15.4 lists the TSIR duration components of the options that were in Table 15.3. The American call is more likely to be exercised early, resulting in a much higher sensitivity to the level of interest rates. On the other hand, the American put has a low probability of early exercise in the steep yield curve environment, leading to risks that are very similar to those of the European put option. The duration risks of the underlying cash bond and the forward bond are also presented for comparison.

The sensitivity of the option price to changes in the volatility surface can also be measured by changing the components of the term structure of Libor volatility (TSLV) in (9.24). These changes are in units of absolute interest rate, implying that the measured sensitivities will have the unit of time or duration. Thus, we call the TSLV sensitivity "the duration of volatility". The kth duration of volatility is calculated as

$$D_{v,c,k} = \frac{p_c(b_k - \Delta) - p_c(b_k + \Delta)}{2\Delta p_u} \qquad (15.47)$$

Table 15.5 lists the TSLR duration components of the options in Table 15.3. The first or level component is similar to the level of interest rates sensitivity and is usually the largest contributor to the TSLV risk of an option. The sensitivities of options prices

TABLE 15.4 Bond option durations, July 8, 2011

C/P	A/E	Level	Slope	Bend	3rd	4th
C	E	3.87	2.22	−2.25	−3.96	−1.43
C	A	5.13	2.62	−2.86	−4.75	−2.15
P	E	−4.21	−2.61	2.56	4.54	1.46
P	A	−4.29	−2.65	2.60	4.62	1.49
Bond		9.10	4.09	−4.68	−8.00	−3.84
Frwd		8.41	5.02	−5.00	−8.84	−3.00

TABLE 15.5 Bond option TSLV sensitivities, July 8, 2011

C/P	A/E	Level	2nd	3rd	4th	5th	6th	7th	8th
C	E	−3.136	−0.125	3.030	1.743	1.199	−1.171	−0.628	−0.705
C	A	−2.915	−0.068	2.808	1.905	0.315	−1.183	−0.967	−1.008
P	E	−3.139	−0.125	3.032	1.744	1.200	−1.172	−0.629	−0.705
P	A	−3.142	−0.123	3.035	1.756	1.171	−1.177	−0.643	−0.719

to the components of the TSLV for American and European options are much closer than interest rate sensitivities.

15.7 TREASURY AND REAL BONDS OPTIONS

The term structure of yield volatility can only be calculated for Libor, where there is a liquid market in swaptions for a few currencies. For other currencies and treasury or real rates, where the volatility is not known, one has to estimate it in relation to either the volatility of known currencies or Libor volatility and the market relationships. For example, in the US, Libor and treasury rates have a correlation of more than 98%, implying that we can use Libor volatility to price treasury options.

Unlike yield which can be observed, volatility cannot be observed. Only historically volatility can be measured; future volatility can be traded and hedged, but cannot be realized. For example, an investment in a risk-free zero coupon bond with a yield of 5%, will result in annualized return of 5% at maturity. The return of a transaction (not an investment) in a risk-free asset with a volatility of 15% will only be known at expiration or exercise time.

There is an implicit assumption in all volatility calculations that volatility is independent of yield. For equities, it is assumed that volatility is independent of stock price. In general, out-of-the-money options tend to have a higher implied volatility than at-the-money options. This is called the volatility smile (as opposed to the volatility frown if the opposite is true). It is possible to estimate the volatility smile for some highly traded options, but the data for calculating the term structure of volatility smile are currently not very reliable.

There is significant evidence that the yield volatility is a function of yield. Prior to the Lehman bankruptcy, when short rates were about 4–5%, the short term volatility of 1-year bonds was about 20%. When Fed Funds rates were below 0.5%, the same rates had a yield volatility of more than 80% which stayed at elevated levels years after the crisis had ended. This can be explained by noting that the Federal Reserve usually moves rates in increments of 0.25% when they are in a tightening or easing cycle and the increments are not significantly different whether rates are at 5% or 1%. This is also one reason why volatility can be modeled much more accurately as absolute volatility instead of relative volatility in our TSYV.

Aside from the yield dependence of volatility, for most securities, there is the uncertainty of the relationship between the security volatility and that of Libor. Of the fundamental rates which include treasury, real, and Libor, the most uncertainty is in

TABLE 15.6 Bond option beta sensitivities, July 8, 2011

Beta	Premium
0.4	1.43
0.5	1.79
0.6	2.15
0.7	2.51
0.8	2.86
0.9	3.22
1	3.57
1.1	3.92
1.2	4.27
1.3	4.62
1.4	4.97
1.5	5.31

the long term volatility structure of real rates, since they have the lowest correlation with Libor. One way to estimate the volatility of real rates is to use beta adjustment.

Beta is a measure of the relative yield volatility of a security relative to the volatility of Libor. If y_r is the yield of real bonds and y_l is Libor yield, then

$$\beta = \sqrt{\frac{\sum_i \left(\Delta y_{r,i} / y_{r,i}\right)^2}{\sum_i \left(\Delta y_{l,i} / y_{l,i}\right)^2}} \tag{15.48}$$

For example, if the yield volatility of real rates is 40% of the yield volatility of Libor rates, we can multiply all forward volatilities by 0.4 to estimate the forward real rate volatility. Option premiums increase almost linearly with beta. Table 15.6 shows a sample of at-the-money, 1-year expiration, 10-year maturity European call premiums for treasuries with different betas.

15.8 BOND OPTIONS WITH CREDIT RISK

Up until now, our analysis of bond options assumed that there was no credit risk in the option and arbitrage-free requirements applied to Libor quality bonds. Since the term structure of volatility is derived from the Libor market, in the absence of a liquid market for other securities, their volatility has to be derived from Libor volatility. Beta adjustment is a reasonable approach for treasuries, since they have a very high correlation with Libor and both are related to the general state of the economy. For corporate bonds, the spread of the security is a function of the economy as well as the credit fundamentals of the company. When the economy slows or recession is forecasted, nominal rates fall and spreads widen. During recoveries inflation falls and both nominal rates and spreads fall. During periods of non-inflationary growth, spreads fall and rates tend to be stable. One way to model the volatility of non-Libor bonds is to assume that the bond spread relative to Libor varies according to its beta and correlation coefficient.

Correlation is a measure of how the spread and Libor move relative to each other. When inflation is not a threat, if the economy slows, rates fall while spreads widen, since a slowing economy is risky for corporations. If the economy accelerates, rates rise while spreads fall, since companies have stronger revenue to pay down debt. In this scenario, spreads and Libor are negatively correlated. If inflation is on the rise, the Fed raises rates to combat inflation, which usually results in higher rates as well as higher spreads. In this case, Libor and spreads will be positively correlated.

The correlation coefficient of change between two processes is defined as

$$\rho = \frac{\sum_i \left(\ln\left(y_{l,i}/y_{l,i-1}\right) - \overline{\mu_l}\right)\left(\ln\left(s_{c,i}/s_{c,i-1}\right) - \overline{\mu_c}\right)}{\sqrt{\sum_i \left(\ln\left(y_{l,i}/y_{l,i-1}\right) - \overline{\mu_l}\right)^2 \sum_i \left(\ln\left(s_{c,i}/s_{0,i-1}\right) - \overline{\mu_c}\right)^2}} \tag{15.49}$$

where $y_{l,i}$ is Libor yield at time t_i, $s_{c,i}$ is the security spread at time t_i, $\overline{\mu_l}$ is the mean of changes in Libor over the sample period, and $\overline{\mu_c}$ is the mean of changes in the spread over the sample period.

Mathematically speaking, correlation is equal to the cosine of the angle between two vectors. Assuming that the changes in Libor yield and credit spread are respectively Δy_l and Δs_c, the change in the yield of the bond is

$$\Delta y = \sqrt{(\Delta y_l)^2 + (\Delta s_c)^2 + 2\rho \Delta y_l \Delta s_c} \tag{15.50}$$

Historically, treasury rates and Libor or swap rates have been very highly correlated, and therefore one can use the TSLV for pricing callable treasury bonds or options on treasury bonds. During times of crisis, such as the Lehman bankruptcy, the correlation between Libor and treasury rates falls slightly, but it is still more than 90%, and the correlation is higher for longer maturity bonds. Overall, treasury rates have a slightly lower volatility than Libor and we can assume that treasury bonds have a beta less than 1.

Real rate bonds have a much lower correlation with Libor than nominal bonds, and their volatility is significantly lower. Options on real rate bonds can be priced using an estimated beta for real rates. For example, the ratio of 3-month historical market volatility for a specific real rate bond and a comparable Libor bond can be used to estimate the beta of real rate bonds. The beta can then be used in conjunction with the TSLV to estimate the market implied yield volatility of a real rate bond.

Analysis of options on credit securities is significantly more complicated than treasuries or interest rate swaps. For a given change in forward Libor rate, we cannot use the mean change in the spread based on correlation, because if the correlation is zero, the mean change in the spread will always be zero. Therefore, one needs to calculate a distribution of spread change scenarios that will result in the correct correlation and volatility of the spread, a process that can be computationally expensive. For practical purposes, one can use a weighted sum of about 10–20 scenarios that produces the expected correlation and beta.

First, let us consider two correlated normal distributions, a and b, with means of zero and standard deviations σ_a and σ_b and correlation ρ_{ab}, defined as

$$a = \ln\left(\frac{y_l}{y_{l,0}}\right) - \mu_l$$
$$b = \ln\left(\frac{s_c}{s_{c,0}}\right) - \mu_c \tag{15.51}$$

The joint distribution probability can be written as

$$g(a,b)da\,db = \frac{1}{2\pi\sigma_a\sigma_b}\exp\left[-\frac{a^2}{2\sigma_a^2} - \frac{(b-\lambda a)^2}{2\sigma_b^2}\right]da\,db$$
$$\lambda = \rho_{ab}\frac{\sigma_b}{\sigma_a} = \rho_{ab}\beta \tag{15.52}$$

where β is the ratio of the standard deviations. It is a trivial exercise to calculate the average of b, $\bar{b}$, for a given value of a, and its standard deviation $\overline{b^2}$ as follows:

$$\bar{b} = \lambda a = \rho_{ab}\frac{\sigma_b}{\sigma_a}a = \rho_{ab}\beta a$$
$$\overline{b^2} = \lambda^2 a^2 + \sigma_b^2 = \rho_{ab}^2\beta^2 a^2 + \sigma_b^2 \tag{15.53}$$

Thus, the distribution of b is normal with mean $\rho_{ab}\beta a$ and standard deviation σ_b^2. By definition, a small change in the relative spread is proportional to the change in the relative Libor yield times the correlation. The proportionality factor is β. Thus, the spread can be written as

$$\frac{ds_c}{s_c} = \frac{dy_l}{y_l}\rho\beta$$
$$s_c = s_{c,0}\left(\frac{y_l}{y_{l,0}}\right)^{\rho\beta} \tag{15.54}$$

Substituting for a and b from (15.51) into the above equation, the yield of the security is derived as

$$y(x_c) = y_l + s_c = y_l + s_{c,0}\left(\frac{y_l}{y_{l,0}}\right)^{\rho\beta} \tag{15.55}$$

In correlated distributions, there are two arbitrage-free requirements that need to be met: one for Libor and one for the security. The arbitrage-free requirements for corporate bond options will be discussed in more detail in the next few sections.

15.9 THEOREM: CREDIT PRICES ARE NOT ARBITRAGE-FREE

At any given time, the values of ρ, β and y_l in (15.55) are constant. $y(x_c)$ is dominated by y_l at very large values of y_l in the forward distribution of rates, if the correlation is negative. Likewise, at very small values of y_l, $y(x_c)$ will be dominated by $s_{c,0}\left(y_l/y_{l,0}\right)^{\rho\beta}$, which becomes arbitrarily large. Thus, for a negative correlation, the yield of a credit security can be arbitrarily large for either very high values of rates or low values of rates, and the resulting forward price can be arbitrarily close to zero at both extremes. Given the continuous nature of the price–yield relationship, the price will attain the maximum value somewhere between high and low rates. At the implied forward maximum price, there can be no distribution of forward rates that can result in the expected value of the price, and the forward yield volatility has to be zero. A call option struck at the yield corresponding to the maximum price can never be in-the-money and its price and volatility have to be zero. Since the forward yield volatility is not zero, a constant negative correlation is mathematically inconsistent with arbitrage-free pricing. We conclude that the correlation must be a variable or changing with rates to ensure the stability and arbitrage-free status of forward pricing.

Historically, the correlations of high grade, high yield, and emerging markets bonds with interest rates have been negative. Using weekly treasury rates from Federal Reserve H.15 tables and Moody's BAA yields that are also available on the Fed's website, we can estimate the historical correlation of treasury rates and BAA spreads. By subtracting the 10-year constant maturity treasury rate from the BAA yield, we can estimate the spread of the high grade bonds relative to the treasury rate. The calculated correlation from 1995 to 2013 is equal to −0.50 with a beta of 0.33.

Given the long term persistence of the negative correlation between treasury rates and credit spreads, especially when interest rates are falling, one has to question the market efficiency of credit spread pricing. In all crises, including the early 1990s mortgage crisis, the 1998 Russian/LTCM crisis, the 2001 technology bubble, and the 2008 Lehman bankruptcy, when interest rates fell sharply, credit spreads widened, resulting in negative correlation between rates and credit spreads.

It is intuitively reasonable to assume that during a crisis, when investors seek principal protection by selling credit securities, the Fed or central banks in general lower rates to mitigate the effects of risk aversion resulting in negative correlation between rates and spreads. However, this phenomenon is inconsistent with arbitrage-free pricing and efficient market hypothesis. Estimating the correlations and betas of the market, one can estimate the maximum price that a security can attain and construct a portfolio of short call options to take advantage of the market behavior. The negative correlation at the time that interest rates fall is part of the structure of the market and is not likely to change. It is thus logical to conclude that credit rates do indeed have a maximum price/minimum spread that can be mathematically calculated. From (15.55), we can write

$$\begin{aligned} y(x_c) &= y_{l,0}f + s_{c,0}f^{\rho\beta} \\ f &= \frac{y_l}{y_{l,0}} \end{aligned} \tag{15.56}$$

Minimizing yield by differentiating with respect to *f* results in

$$f = \left(-\frac{y_{l,0}}{s_{c,0}\rho\beta}\right)^{1/(\rho\beta-1)} \tag{15.57}$$

For a negative correlation, the above equation has a positive definite root that defines the minimum value of the yield that a credit security can assume. Once the minimum value of the spread is attained, further fall in interest rates will result in spread widening.

Based on this argument, one has to conclude that the spread market is not efficient and there are long term arbitrage opportunities in it.

15.10 CORRELATION MODEL

Zero or very small positive correlations are most often unstable and the arbitrage-free condition cannot be met. We know that a negative correlation is not stable at all rates and has to become positive at some point. Given the stability of positive correlation, should one assume that a positive correlation can be constant? How do we differentiate between a positive correlation and the positive component of a negative correlation? For consistency reasons, and considering that positive correlation could indeed be a manifestation of the variable nature of the negative correlation at different forward rates, we will assume that correlation is a function of yield such that the correlation is equal to the mean value of the correlation at all forward rates.

Correlation is always between –1 and +1; one way to model it is to use the tangent hyperbolic function as follows:

$$\begin{aligned} \rho &= \frac{e^h - e^{-h}}{e^h + e^{-h}} \\ h &= \frac{\alpha_\rho \ln\left(y/y_0\right)}{\sigma_b} + \eta \end{aligned} \tag{15.58}$$

The values of α_ρ and η need to be calculated in such a way as to ensure that the average weighted correlation is equal to the expected correlation. Using the probability distribution of forward rates, we require the equality

$$\rho_0 = \frac{1}{\sqrt{2\pi}\sigma_y} \int \rho e^{-\left(\ln(y)-\ln(y_0)-\mu\right)^2 / 2\sigma_y^2} d\ln\left(y\right) \tag{15.59}$$

We can only solve for α_ρ or η using the above equation. To satisfy the arbitrage-free requirement, α_ρ has to be negative, so that at low rates correlation is positive. We can select the value of α_ρ and calculate η, which is the decay coefficient for the correlation coefficient.

We can also use a correlation model that requires only one parameter to be calculated and positive correlation at low rates is guaranteed. An exponential function similar to the time decay function (2.16) that has a range of $(-1,1)$ can be used as follows:

$$\rho = 2e^{-\alpha y_l / y_{l,0}} - 1, \qquad \alpha = \ln\left(\frac{1+\rho}{2}\right) \tag{15.60}$$

The drawback of this model is that it is not symmetric at low and high rates. Figure 15.5 shows the shapes of the hyperbolic and exponential correlation models.

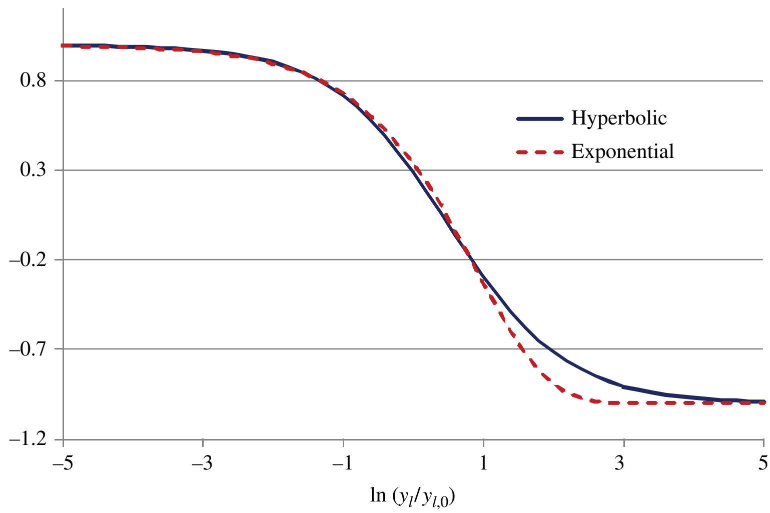


FIGURE 15.5 Correlation functions

15.11 CREDIT BOND OPTIONS EXAMPLES

For non-floating corporate bonds, a parallel shift in the credit curve or interest rate (treasury curve) results in the same price sensitivity, that is, duration components. Arbitrage-free requirements guarantee that a parallel shift in interest rates or credit curve results in the same price for the underlying security. For options, when the Libor curve is shifted, the drift needs to be recalculated to ensure arbitrage-free pricing of Libor and the security. However, shifting the credit curve, the Libor drift does not change but the credit drift needs to be recalculated and therefore the credit distribution will be slightly different from the case where Libor is shifted. This will result in different duration components for Libor and credit for a credit option, even if both correlation and beta are equal to unity.

In deciding to exercise an early call option, the issuer needs to consider its financing option and therefore discount the option value and premium by its credit curve.

Once the choice of option exercise is made by the issuer, the value of the option has to be discounted back by Libor for the bond holder.

The present value of an option for a bond holder can be significantly higher than for the issuer. Therefore, callable bonds must trade more cheaply than non-callable bonds and issuers of such bonds are penalized through the overestimation of the call value.

Table 15.7 shows the call value and 3-month at-the-money call for two custom callable securities. Security A has a fixed coupon of 4% that steps up to 10% in 2016 and is priced at 99.5. Security B has a fixed coupon of 5% that becomes floating based on 6-month Libor with a spread of 5% after 2012 and is also priced at 99.5. The "A beta" and "B beta" calculations are based on a constant correlation of 1 and using the beta adjustment explained in Section 15.7. The calculated call prices are based on a beta of 2.5 for both securities using three different correlation coefficients.

TABLE 15.7 Call values of credit bonds, July 8, 2011

Security	Corr	Beta	Cpn	Maturity	Price	Call Value	3 Mo. Call
A	0.7	2.5	4	1/15/26	99.50	8.20	5.38
A	0	2.5	4	1/15/26	99.50	5.16	2.00
A	−0.7	2.5	4	1/15/26	99.50	1.95	1.43
B	0.7	2.5	5	1/15/26	99.50	4.06	3.90
B	0	2.5	5	1/15/26	99.50	2.10	0.72
B	−0.7	2.5	5	1/15/26	99.50	2.79	2.79
A Beta		2.5	4	1/15/26	99.50	14.04	10.07
B Beta		2.5	5	1/15/26	99.50	6.12	6.27

There is a very large variation in the call value of a callable bond or the price of an option on a bond based on correlation assumptions. Securities A and B could simply be two bonds issued by the same issuer with fixed and floating coupons. Other than using a correlation model, there is no way to calculate the price of options on different securities of an issuer in a consistent way. Estimating the market volatility based on historical data is incorrect because it is not clear if the source of volatility was due to interest rates or the spread. If an issuer has both floating and fixed rate bonds, the decomposition of interest rate volatility, spread volatility and correlation is the only approach that can result in a consistent and arbitrage-free pricing of their options.

Correlation, beta, forward yield, spread, and arbitrage-free requirements all contribute to the price of an option. Without option calculation, it is not always easy or intuitive to estimate the option's price. For example, floating coupon security B has a lower premium value with zero correlation than with a negative correlation of –0.7. The correlation function will force positive correlation at low rates and negative correlation at high rates. The negative correlation at higher rates will result in more stable prices, while at lower rates the price increases very rapidly. The arbitrage-free requirement shifts the distribution of yields and will result in lower probability of exercise. On the other hand, starting with a negative correlation, the price of the security is dominatevd by the spread that has a higher duration and beta, and the resulting distribution will be closer to the center.

Next, we used security A with a correlation of 0.5 and calculated the call and put option premiums as well as the call value using different values of α_ρ in equation (15.58). Table 15.8 compares the calculated prices of options using the correlation models. The call values are very close for a wide range of α_ρ and have a significantly tighter range than the typical bid–ask spread for the price of such options.

Bond options can be calculated similarly to callable/puttable bonds. The spread correlation and spread beta can be estimated much more reliably from historical data for short dated bond options. For such bonds, the change in spread correlation and beta is likely to be small, leading to more accurate pricing and hedging of the option.

TABLE 15.8 Option values for varying correlation parameters, July 8, 2011

Model	Slope	Call Value	3 Mo. Call	3 Mo. Put
Hyperbolic	0.20	6.84	4.17	4.18
Hyperbolic	0.40	6.60	4.10	4.12
Hyperbolic	0.60	6.59	4.12	4.14
Hyperbolic	0.80	6.68	4.17	4.20
Hyperbolic	1.00	6.72	4.22	4.25
Hyperbolic	1.20	6.72	4.26	4.28
Hyperbolic	1.40	6.73	4.27	4.29
Hyperbolic	1.60	6.70	4.27	4.29
Hyperbolic	1.80	6.67	4.26	4.28.
Hyperbolic	2.00	6.65	4.26	4.28
Decay		7.30	4.24	4.24
Average Hyper		6.69	4.21	4.23
Stdev		0.07	0.06	0.06

15.12 RISK MEASUREMENT OF COMPLEX BOND OPTIONS

Options on most credit bonds can be calculated by using the correlation and beta as outlined in the previous sections. The pricing of complex securities or bonds can depend on all four curves – treasury, real, Libor, and credit – along with their associated correlations and betas. We can, therefore, calculate all four groups of durations and convexities plus the sensitivity of a security to the TSYV as well as options Greeks for an option on a credit security.

Most of the risk measures of bond options cannot be calculated analytically and numerical methods have to be used as outlined in (15.46). For example, to calculate the credit duration of a call option, we use the formula

$$D_{p,c,k} = \frac{p_p(a_{c,k} - \Delta) - p_p(a_{c,k} + \Delta)}{2\Delta p_u} \tag{15.61}$$

where $D_{p,c,k}$ is the kth credit duration of the put, p_p the put price (premium), p_u the price of the underlying (parent) security, $a_{c,k}$ the kth component of the TSCR, and Δ a small change in the kth component of the TSCR.

Fixed coupon credit bonds have equal credit and treasury duration. However, options on the same bonds have different credit and treasury durations, even if we use a correlation and beta of unity. This is an artifact of the arbitrage-free requirement model that is used for option pricing. For example, in order to calculate the credit and interest rate durations of a security, we need to take the following steps:

- *Shift the treasury curve* – Calculate the drift to ensure arbitrage-free pricing of Libor. The calculated drift is slightly different from the unshifted treasury curve. The credit curve is also shifted to maintain the constant credit spread. The drift for the credit curve needs to be recalculated to ensure arbitrage-free pricing.

- *Shift the credit curve* – The Libor curve along with its associated drift will not change, but the drift of the credit curve needs to be recalculated.
- *Calculate the durations* – The two above cases will result in slightly different distributions for the credit curve and the resulting options prices and calculated durations will be slightly different.

Table 15.9 shows three components of the credit and treasury durations of the securities that were listed in Table 15.7.

TABLE 15.9 Call risks of credit bonds, July 8, 2011

Security	Corr	Credit			Treasury		
		Level	Slope	Bend	Level	Slope	Bend
A	0.7	2.32	1.60	−0.30	2.46	1.73	−0.25
A	0	2.07	1.70	0.08	1.67	1.25	−0.06
A	−0.7	3.42	3.45	1.07	2.54	2.50	0.73
B	0.7	1.21	0.93	−0.21	0.58	0.18	−0.39
B	0	1.14	1.01	0.17	0.33	0.12	−0.20
B	−0.7	1.40	1.16	0.10	0.46	0.14	−0.32
A Beta		3.23	1.86	−0.86	3.23	1.87	−0.85
B Beta		1.82	1.29	−0.13	0.96	0.34	−0.54

15.13 REMARKS ON BOND OPTIONS

Bond options trade very heavily in fixed income markets. Exchange traded options on bond and note futures are among the most liquid instruments of any kind in the market. Pricing these short dated options can easily be done using Black–Scholes formulas. For longer dated European options which are heavily traded as swaptions, Black-76 is the most widely used model for pricing. It is very fast for calculations and reasonably accurate for short dated swaptions. However, as mentioned in Section 15.1, it is not completely free of defects and the power of today's computers has all but eliminated its speed advantage compared to more complex algorithms.

Pricing American call options is usually done by simulation or using a binomial tree structure. Both of these models converge relatively slowly and accuracy is usually an issue. The convergence of both methods is proportional to the square root of the number of simulation paths or steps in the binomial tree. For example, if the pricing error is \$0.50, you need to increase the number of steps by a factor of 25 to get the error to about \$0.10. By using Simpson's rule for integration, the model presented in this chapter converges almost quadratically.

Most high yield bond companies issue callable bonds. Considering the size of the market, there are not many successful models to price such long dated bond options. Additionally, some of those bonds can have either floating or fixed coupon rates. It appears that, without a correlation and beta model, it is not possible to price fixed and floating bond options of an issuer on a consistent basis.

We showed that using the historical market dynamics implies that corporate securities pricing is subject to arbitrage; however, the pricing model must be arbitrage-free. There clearly is an issue with the correlation of spreads and treasuries at times of crisis, implying that corporate yields peak at both ends of the interest rate spectrum which will not be addressed by the pricing models. If the pricing model is subject to arbitrage, one can immediately construct an arbitrage portfolio and take advantage of the opportunity.

Given the variability of correlations and betas, it is not clear if one can calculate the maximum price of corporate securities knowing the market dynamics. If one is able to estimate the maximum achievable price of a corporate security, he can sell call options at that strike price and realize free money, since the options will never be in-the-money.

We know that during times of easy credit, the excesses will build in the economy until they are no longer sustainable. At some point investors decide that the marginal gains from investing in credit securities are not worth the required level of risk and they start pulling their money out of risky ventures. Once the tide turns, more and more investors follow suit until a critical stage is reached where the central bank steps in and eases monetary policy. This dynamic is not likely to change in the future, hence credit prices will continue to be subject to arbitrage.

Identification of the peaks and the willingness to step out of credit markets, when the spreads are too tight, makes the case that active investment management can add value compared to indexation.

CHAPTER 16

Currencies

Currencies are very important components of any global portfolio. There is significant academic and non-academic debate as to whether currency is an investment asset class or is simply the attribute of a transaction. A corporation can issue its debt in multiple currencies and they can all have the same credit risk and should all trade at the same default probability relative to the local currency's risk-free rate. Likewise, countries can issue debt in foreign currencies which carries a credit risk.

Currency is one of the tools that many central banks use as a monetary policy vehicle. A depreciating currency increases import prices of goods in an open economy and can result in inflation. On the other hand, an appreciating currency can put domestic producers at a competitive disadvantage relative to imported goods. For these and other political reasons, many countries attempt to manage or manipulate their currencies. Currencies management can fall into one of the following regimes.

- *Pegged* – A pegged currency is one that has a fixed exchange rate compared to a major currency, usually the US dollar or sometimes sterling, the euro, etc. For a peg to be maintained, the country has to have a strong economy, a relatively large current account surplus and very large foreign currency reserves. During times of higher inflation than the host country or at times of crisis, the central bank needs to intervene in the markets by selling dollars or other currencies and buying their own currency to maintain the peg. The Hong Kong dollar is an example of a successful peg to the US dollar. The Argentine peso is an example that failed after about 10 years in 2001 – the peso lost about 75% of its value in a short period of time.
- *Strongly managed* – A strongly managed currency is similar to a pegged currency, except that the currency has a tight range within which it can move relative to a major currency or basket of currencies. Usually, the country institutes strong capital restrictions to enforce the currency range. For currencies with a large current account surplus and growth rate such as Chinese renminbi, the country may limit the flow of foreign capital. Some countries such as Brazil have instituted tax for capital inflows and outflows. The central bank often buys foreign currency as a way to prevent rapid appreciation of the currency. The former communist countries

had strongly managed currencies and prevented foreign capital from flowing in and domestic capital from flowing out. Most countries that manage their currency strongly tend to develop two separate interest rates or even exchange rates, known as on-shore and off-shore rates.

- *Weakly managed* – A weakly managed currency is one that has a relatively wide range relative to a reference or a basket of major currencies. Weakly managed currencies are often supported by monetary policy such as raising interest rates, to attract capital. Many eastern European currencies and the Danish krone fall in this category.
- *Free floating* – Nearly all major currencies are free floating and the market forces drive their valuations. In rare instances, if a major currency becomes very rich or cheap, central banks may intervene. However, due to the enormous liquidity of major currencies, the capital needed to move them can be very large and may require a coordinated effort by central banks. In the early 2000s there was an intervention to stop the slide of the euro relative to the US dollar.

16.1 CURRENCY FORWARDS

Most currency transactions take place using forwards. A forward is an agreement to purchase a currency for forward or future settlement. For example, if 3-month forward EUR is \$1.30, in 3 months, if the EUR spot price is \$1.32, the buyer of €1.0 million will be entitled to pay \$1.3 million and receive €1.0 million at settlement. For cash settled transactions, the buyer will be entitled to receive the difference between the spot price and the transaction price in cash instead of receiving the currency. The date at which the final price is established is called the *value date* and is usually two or three business days before the settlement date, at which time the exchange takes place.

The forward price of a free floating currency is calculated in such a way that the forward value of a short term investment in both currencies will be the same (arbitrage-free). If r_1 and r_2 are the short rates of currencies 1 and 2, and x_s and x_f are the number of units of currency 2 for one unit of currency 1 in the spot and forward markets respectively, then

$$x_f(1 + r_1 t) = x_s(1 + r_2 t) \tag{16.1}$$

The left hand side of equation (16.1) represents investment in one unit of the first currency for a period of t and then converting the result to the second currency in the future. The right hand side is converting the first currency to the second in the spot market and then investing in the short term deposit rate of the second currency. The forward price can be simplified as

$$\begin{aligned} x_f &= x_s \frac{1 + r_2 t}{1 + r_1 t} \\ \frac{x_f - x_s}{x_s} &\approx (r_2 - r_1)t \end{aligned} \tag{16.2}$$

For small interest rate differentials, the forward price relative shift is equal to the interest rate difference times the time to the forward time. For example, if the spot price of BRL (Brazil Real) is 2, i.e., 2 BRL's is equal to one dollar, and short rates in the US and

Brazil are 1% and 9% respectively, the three month forward price of BRL will be 2.04, implying that BRL is expected to devalue by about 2%.

16.2 CURRENCY AS AN ASSET CLASS

Nearly all investment asset classes are based on the issuance of a security or asset that is supported by a physical property, asset or good faith of the issuing institution. As such, there is a limited supply of the asset that cannot be increased by anyone other than the issuer. To short the asset, one has to borrow the asset from an asset holder and receive the repo rate in exchange for the cash that is the collateral for the borrowed asset.

Most currency transactions are like swaps and are governed by ISDA agreements. A currency transaction is symmetric and a zero sum game, and its expected return is zero. If one side of the transaction makes money, the other side will lose the same amount (except for transaction costs). However, when purchasing a traditional asset class, such as a government or a corporate bond, if rates go down, the investor profits, but the issuer does not lose. Since the expected returns of currencies are zero, assuming the efficient market hypothesis, currency investment should not be an asset class.

Fama [14] found that there was a higher correlation between the current spot exchange rates and future spot rates than between forward rates and future spot rates. In the above example, BRL will be more likely to be closer to 2.0 than to 2.04 in 3 months' time. This is particularly true if inflation rates are about the same in both countries. The exchange rate patterns in recent history also support the notion that higher yielding currencies outperform lower yielding currencies.

When productivity is high, the return of capital is also high. The demand for money for investment results in high real rates. Likewise, foreign capital seeking a higher rate of return flows into the host country, resulting in appreciation of the exchange rate. This is true even if the trade balance is in deficit, since capital flows much faster and more freely than goods. Likewise, when the demand for capital is low or the savings rate is high, the excess capital results in lower interest rates and capital flows to higher return currencies. A high yield currency is also known as a carry currency and a low yield currency is known as a funding currency. During times of crisis, capital repatriation results in sharp declines in carry currencies in a short period of time.

Historical data are abundant with long term currency appreciations as developing countries enact market friendly policies and encourage foreign investments resulting in very high productivity rates as they narrow the standard of living gap with developed countries. The Japanese yen stood at more than 300 to the dollar in the 1960s and early 1970. The German Mark stood at 4 to the dollar in the same period. These are examples of countries closing the technological gap with the US. Many Asian and South American countries are on a similar trajectory, and African countries will surely follow.

Table 16.1 is a simple analysis of 35 currencies (AUD, BRL, CAD, CHF, CLP, CNY, COP, CZK, DKK, EUR, GBP, HKD, HUF, IDR, ILS, INR, ISK, JPY, KRW, MXN, MYR, NOK, NZD, PEN, PHP, PLN, RUB, SEK, SGD, THB, TRY, TWD, UAH, USD, and ZAR) in the period 1998–2011. On the last business day of every month, the seven highest yielding currencies were purchased and seven lowest yielding currencies were sold short 1 month forward. At the end of each month, the forwards would become equal to the future spot exchange rates and the gains (G) and losses (L) were tabulated using equal

TABLE 16.1 Long/short currency trades

Year	G/L
1998	14.3%
1999	28.0%
2000	16.5%
2001	16.8%
2002	11.6%
2003	24.5%
2004	17.0%
2005	16.0%
2006	4.8%
2007	10.4%
2008	–2.8%
2009	19.5%
2010	1.7%
2011	5.1%
Average	13.7%
Stdev	8.7%

Source: Bloomberg

weights for all currencies. For a portfolio of $1 million there were a net amount of $1 million long and short positions every month.

The overwhelming evidence is that currency investing can have net positive return, even if it is against the efficient market hypothesis (EMH) principle and therefore is an asset class. For most global or multi-sector portfolios currency transactions are used for hedging the currency risk of bonds or stocks or for generating alpha. Currency transactions are ideal for generating excess return in portfolios in a consistent way since they do not require any upfront capital at initiation.

16.3 CURRENCY TRADING AND HEDGING

Currencies are always traded in pairs. The first currency or the home currency is called the base currency and the second or foreign currency is called the quote or counter currency. The price of a currency is the number of the quote currency units for each unit of the base currency. For example, a quoted price of JPY in USD of 88.5 implies 88.5 yen per US dollar, and is shown by the International Organization for Standardization (ISO) codes of the respective currencies as USD/JPY or USDJPY.

If the base is explicitly stated, the market convention for the base is used. The euro is always assumed to be the base unless another currency is explicitly stated. For example, when EUR is quoted to be 1.32, it implies that EUR 1 is equal to USD 1.32. The order of precedence of currencies as base is as follows: euro, pound sterling, Australian dollar, New Zealand dollar, US dollar, Canadian dollar, Swiss franc, and Japanese yen.

Currency forwards are often priced based on the spot price plus points or pips. A pip is typically the fifth significant figure of the price spread, but it can change if the digits of the currency price changes. For example, for a USD based account the euro may be quoted as 1.2432/1.2435 as bid/ask prices for EUR 1. The spread between bid and ask of 0.0003 is called 3 pips in trading. For Japanese yen, when it trades at an exchange rate of more than 100, the pip may be the fifth significant figure (e.g. 108.23/108.20), but at lower than 100 it could be the fourth significant figure (e.g. 92.56/92.54).

The quoted prices of the base and quote currencies are the opposite of each other and can be confusing to inexperienced traders. For the euro, since it is the base currency, the offer price is higher, since it represents the number of US dollars per euro. However, for the Japanese yen the offer is lower, since it is the number of yen per US dollar.

Forward currency prices are usually also quoted in terms of pips plus spot price. For example, if BRL spot is at 1.92, a 3-month forward may be quoted at +155, implying a price of 1.9355.

Global fixed income portfolios should be managed on a currency hedged basis, and currency should be used as an overlay to take active currency bets. In a typical fixed income portfolio with a duration of 5 years and absolute interest rate volatility of 100 bps, the volatility due to interest rate movements is about 5%, while the currency volatility is usually about 10%. Therefore, an unhedged global portfolio has a higher volatility from currency and becomes more of a currency investment than bond investment. In a currency hedged portfolio the rate and currency decisions can be made independently, while in unhedged portfolios they have to be made together. A potentially good rate opportunity cannot be implemented if the currency fundamentals are not supportive in an unhedged portfolio.

A country with a steep yield curve offers very attractive carry for hedged currency return, since the cost of currency hedging is short rates while the yield earned is from long rates. On the other hand, an inverted yield curve is a much better candidate to buy the currency and short the bond, since inverted yield curves usually imply high real short rates which supports the currency. Historical analysis shows that steep yield curves have the highest future returns especially if combined with high real rates. As we saw in Table 5.1, in the long run the biggest contribution to return is from yield or carry.

When purchasing a foreign currency bond, the hedge can be instituted at the same time or usually 1 day later. Most currencies are settled T+2, implying two business days from the trade date, while bonds are usually T+3. The currency hedge is simply a currency spread where short term currency is purchased for settlement on the bond settlement day and the forward (typically 1 month or 3 months) is sold short at the same time. This is exactly like a short currency roll, where the short end is covered and a new short at the longer end is opened. The transaction cost of the currency spread is significantly lower than a naked currency position, since there is no exposure to the currency. The only exposure is to the short term interest rate, which is usually an order of magnitude less volatile than the exposure to the currency.

16.4 VALUATION AND RISKS OF CURRENCY POSITIONS

The forward price of currency positions can be calculated from (16.2) provided that the short term rate and spot currency are known. The volatility of unhedged portfolios can be significantly higher than hedged currency portfolios due to currency fluctuations.

Pricing inconsistency can be another source of relative volatility. For example, a global index may capture the bond and currency prices at 1:00pm London time, while a US based portfolio may capture currency prices at 2:00pm New York time. If currency prices have moved between New York and London closing times, the portfolio will show a tracking error purely based on the timing of price captures. Calculating currency prices for a portfolio and its benchmark on a consistent basis provides much more reliable risk metrics and tracking errors.

Pricing and risk management can be easily achieved by calculating the term structure of rates for each currency. Knowing the short term rates of the host currency, we can calculate the continuously compounded short rates. For example, for the US, where the deposit rates follow the Actual/360 convention, the future value of a time deposit with n_1 days to maturity is

$$fv = p_0 \left(1 + \frac{n_1}{360} r\right) = p_0 e^{y_1 t_1}, \quad t_1 = \frac{n_1}{365.2425} \tag{16.3}$$

Given the mid-point of the bid–ask forward prices of a currency using the base currency (USD in this example), we can calculate the respective implied short rate of the foreign currency from (16.1) as

$$r_2 = \frac{x_f \left(1 + r_1 \, n_1/360\right) - x_s}{x_s \, n_2 / M_2} \tag{16.4}$$

where n_2 and M_2 are the number of days in the period and year for the quote currency for a time deposit. Given two liquid forward prices of the quote currency, we can calculate the implied short term rates at two points and then calculate the term structure of rates using two parameters and assuming that the remaining three parameters are zero. Having calculated the term structure of short rates in different currencies, all forward rates can be calculated from the spot prices as

$$x_f = x_s e^{(y_q - y_b)t} \tag{16.5}$$

where y_b and y_q are the continuously compounded short rates of the base and quote currencies, respectively.

Most accounting or record keeping systems break down a currency transaction into two forward settled cash transactions. For example, a 3-month BRL forward at a price of 1.92 per USD and a face value of 19.2 million will be broken down into a BRL 19.2 million credit and a \$10.0 million debit with a transaction date of 3 months into the future. The present value of the BRL credit can be calculated from its term structure, and the duration components can be calculated as

$$D_k = \psi_k t \tag{16.6}$$

where ψ_k is the kth component of the term structure of rates. Since deposit rates are of Libor quality, the duration contribution of currencies is Libor based and the durations have to be aggregated with Libor durations. The contribution of currencies to interest rate risk is small, but for levered portfolios or if currency positions are longer than 3 months, the contribution of durations to the portfolio's host currency needs to be calculated and properly aggregated. There is a much larger universe of tradable currencies than interest rate swaps and the short term structure of interest rates can be estimated for all currencies that have active forward markets.

16.5 CURRENCY FUTURES

Currency (FX) futures, like bond futures, are exchange traded instruments with standardized contract size and expiration dates. Most contracts are deliverable and there is daily margin movement, so the gain and losses are current on a daily basis. Unlike over-the-counter currency transactions that are broken into two positions for the respective currency pairs, FX futures are maintained by accounting and record keeping systems as one position, with the number of shares representing the number of contracts.

The price quotation of FX futures is the number of units of the domestic currency where the contracts trade per unit of the foreign currency regardless of the market convention for the base and quote currency. For example, the market convention for EUR/USD is base EUR and for JPY/USD base USD. Thus EUR=1.25 means that EUR 1 buys USD 1.25, but JPY=125 means that USD 1 buys JPY 125. In exchange traded futures, EUR would be quoted as 1.25 and JPY would be quoted as 0.0080.

Analysis of currency futures requires breaking a futures contract into the respective positions for the currency pair adjusted for the contract size of each contract and then performing the necessary risk analysis. Due to daily margin movement, the value of the quote currency changes every day depending on the price of the base currency. For example, for one EUR futures contract with a face value of €125,000, if the price of EUR changes from 1.20 to 1.19, there will be a margin movement of 125,000 × 0.01 = $1250 debit from the long position or credit to the short position. The USD face value changes from 125,000 × 1.2 = $150,000 to $148,750.

In over-the-counter trading, a fixed amount of quote currency is exchanged for a fixed amount of base currency on the value date. However, in FX futures trading, due to daily margin movement, a fixed amount of the base currency is exchanged for a variable amount of the quote currency. The amount of quote currency changes similarly to all other futures such as bond futures.

16.6 CURRENCY OPTIONS

Currency options are the most traded of all options. Most exotic options, such as single and double barrier, digital, lookback, and basket, are primarily used for currency trading. Most record keeping systems report currency options as a single position. However, for proper risk measurement, the currency option has to be broken up into its components and the face amounts have to be adjusted by the option delta. For example, a call option on 10 million EUR/USD at a strike price of 1.30 in 3 months, when the forward EUR is at 1.27 with a delta of 0.3 and premium of $0.01 for a USD based investor, should be broken up into:

- 0.3 × 10,000,000 long EUR forward position (delta adjusted).
- 0.3 × 10,000,000 × 1.3 short USD forward position (delta adjusted).

- $0.01 \times 10{,}000{,}000$ in USD for the value of the option.
- The present value of 3 months forward $0.3 \times 10{,}000{,}000 \times (1.3 - 1.27)$ USD as cash adjustment.

The last item is necessary to maintain the overall market value of the portfolio. The forward EUR and USD positions do not cancel each other, since they are based on the strike price of the option which is different from the forward price of the currency.

The duration risks of the synthetic forwards must be calculated using the TSIR for each respective currency.

CHAPTER 17
Prepayment Model

Mortgage bonds are bonds whose principal is collateralized by an underlying property or real estate. The borrower for most residential single family mortgages has the option to prepay any or all of the principal back at any time, and therefore it is like a bond with continuous call option. The borrower usually does not always have the flexibility or the sophistication of an institutional borrower to call the bond at optimum time and yield. Therefore, before analyzing mortgage bonds, we need to develop a model, based on historical data, that explains the behavior of home owners under different interest rate scenarios. Such a model is called a prepayment model.

Mortgage bonds with prepayment restrictions are like regular bonds without call option through the restriction period. The principal of a mortgage loan is usually paid back in one of the following ways:

- Regular amortization of the loan.
- Accelerated payments – for example, some borrowers make 13 payments per year to shorten the life of the loan.
- Selling the house – if the home owner sells his house, the new buyer has to get a new loan and the old loan is paid back in full, unless it is assumed by the new buyer, which is very rare.
- The home owner refinances the loan at a lower rate – the remaining principal is paid back in full.
- Default – the home owner defaults at which point the insurance takes effect and the insurance company pays back the principal.
- Payback – after a number of years the home owner decides to repay the loan in full.

There are significant costs for refinancing a mortgage. The borrower has to pay for title search and insurance as well as points for new loans. The cost to the borrower is typically about 0.5–0.75% of additional interest for the loan.

We will analyze each of the above components in the following sections.

The Advanced Fixed Income and Derivatives Management Guide. Saied Simozar

17.1 HOME SALE

When a home is sold, the principal of the loan is paid back in full. In rare cases where the loan is assumable and the buyer assumes the loan, the principal is not paid, but such loans are extremely uncommon for single family home loans.

When an individual buys a home or refinances the loan of an existing home, he/she is not likely to sell or refinance the home immediately due to the associated costs of a new loan. Therefore, the prepayment probability at the beginning of a mortgage is low. In the prepayment speed assumption (PSA) model, it is assumed that it takes 30 months for a mortgage loan to become seasoned. A seasoned home has a turnover probability of about 6% per year, and during the seasoning period the turnover probability increases linearly with time. In the mortgage bond trading business, a prepayment rate of 6% per year is called a prepayment rate of 100 PSA.

We assume that the change in the relative factor due to home sale is an exponentially decaying function of time,

$$\frac{df_s}{f_s} = -a_s dt + b_s e^{-c_s t} dt \tag{17.1}$$

where f_s is the mortgage factor due to home sale, df_s/f_s is the relative change in the mortgage factor due to home sale in time dt, a_s is the factor for home sale after a very long seasoning period, b_s is the initial factor for home sale, and c_s is the decay coefficient for home sale.

At the beginning of the mortgage, the change in mortgage factor due to home sale is zero. Also, after a very long time, at 100 PSA, the annual home sale rate is 6%, thus we can write

$$a_s = b_s = 0.06 \tag{17.2}$$

Upon integration, we have

$$\ln\left(\frac{f_s}{f_{s0}}\right) = -a_s t + \frac{a_s}{c_s}(1 - e^{-c_s t}) \tag{17.3}$$

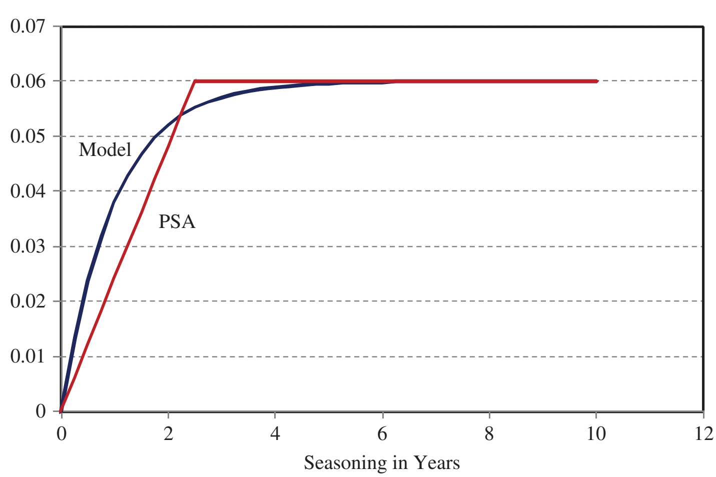


FIGURE 17.1 Fraction of homes sold per year

Our model is somewhat like the PSA model, except that it is a continuous function of time, as shown in Figure 17.1 for $c_s = 1$. The PSA average of 6% is a long term average. For calculation purposes, one can scale the model depending on the expected housing turnover.

Home sales are very seasonal. For example, the average number of houses sold in August is more than double that in February. We can incorporate the seasonality component into the data.

17.2 REFINANCING

Home owners start to refinance their home when it is economically beneficial. The refinancing is usually done at a time and rate that is not optimal. The cost of refinancing including title search, broker fees, and other costs is about 0.50% in rate of the new mortgage. A home owner usually has the option of refinancing at a rate that is about 0.50% higher and paying no closing costs and fees. When the interest rate differential is more than 0.50%, home owners start to refinance their mortgages. We define the incentive to refinance as the difference between the existing mortgage rate on a home and the market rate plus 0.50%. For example, if the mortgage rate is 6% and the current market rate is 4.9%, the incentive to refinance is $6 - 4.9 - 0.5 = 0.6\%$, the amount in rate that the call option is in-the-money.

After rates have been low enough to make economic sense for home owners to refinance for a while, the eligible candidates refinance their mortgages. The remainder, for economic reasons, or due to credit score or other restrictions, are less likely to refinance. This phenomenon is called "burnout".

We first define the refinancing incentive of a mortgage in terms of the difference between the existing mortgage rate, the current mortgage, and the cost of refinancing as

$$r_i(t) = r_0 - r_t - s \tag{17.4}$$

where $r_i(t)$ is the refinancing incentive at time t, r_0 is the mortgage rate, r_t is the rate at a time t after initiation of the mortgage, and s is the spread to account for the cost (of mortgage or refinancing) mortgage (i.e. 0.50%).

We model the prepayment rate related to refinancing as

$$\frac{df_r}{f_r} = -a_r r_i dt - b_r e^{-c_r R} r_i(t) dt \tag{17.5}$$

where f_r is the mortgage factor due to refinancing, df_r/f_r is the relative change in the mortgage factor due to refinancing in time dt, a_r is the factor for refinancing after very high burnouts, b_r is the initial factor for burnouts, c_r is the decay coefficient for burnout, and R is the cumulative historical incentive. Note that

$$R = \int_{t_0}^{0} r_i(t)dt, \quad r_i(t) > 0 \tag{17.6}$$

Where t_0 is the starting time of the mortgage. At the beginning, where the burnout is zero, the prepayment rate is proportional to $a+b$. After very high burnouts, the prepayment rate falls proportional to a.

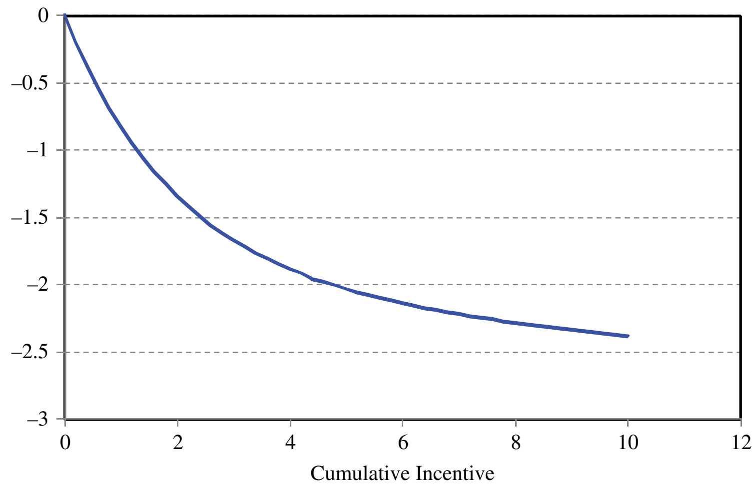


FIGURE 17.2 Natural log of mortgage factor due to incentive

Integrating (17.5) results in

$$\ln\left(\frac{f_r}{f_{r0}}\right) = -a_r R - \frac{b_r}{c_r}(1 - e^{-c_r R}) \tag{17.7}$$

Figure 17.2 shows the natural log of the mortgage factor due to refinancing. In the absence of burnouts, the curve would be linear.

17.3 ACCELERATED PAYMENTS

The monthly mortgage payment is the minimum payment due. Some home owners pay extra amounts on a regular basis or, when their balance is small, pay it off in full. Our model for accelerated payments is similar to the home sale model. We assume that the change in the relative factor due to accelerated payments is an exponentially decaying function of time:

$$\frac{df_a}{f_a} = -a_a dt + b_a e^{-c_a t} dt \tag{17.8}$$

where f_a is the accelerated payment factor, df_a/f_a is the relative change in the mortgage factor due to accelerated payments, a_a is the factor for accelerated payments after a very long seasoning period, b_a is the initial factor for accelerated payments, and c_a is the decay coefficient for accelerated payments.

At the beginning of the mortgage loan, the change in mortgage factor due to accelerated payments is zero. Thus,

$$a_a = b_a \tag{17.9}$$

Integration leads to

$$\ln\left(\frac{f_a}{f_{a0}}\right) = -a_a t + \frac{a_a}{c_a}(1 - e^{-c_a t}) \tag{17.10}$$

17.4 PREPAYMENT FACTOR

To calculate the prepayment factor, assuming that the factors due to home sale, refinancing, and accelerated payments are unrelated, we need to multiply the three respective factors. Note that at the initiation of a mortgage all factors are equal to unity:

$$f_{s0} = f_{r0} = f_{a0} = 1 \tag{17.11}$$

The prepayment factor is

$$\ln(f_p) = \ln(f_s f_r f_a) = \ln(f_s) + \ln(f_r) + \ln(f_a) \tag{17.12}$$

For a conventional 30-year mortgage the following factors can be found empirically:

$$a_s = b_s = 0.06, \quad c_s = 1 \tag{17.13}$$

$$a_r = 0.04, \quad b_r = 1.50, \quad c_r = 1.00 \tag{17.14}$$

$$a_a = b_a = 0.015, \quad c_a = 0.0142 \tag{17.15}$$

CHAPTER 18

Mortgage Bonds

Mortgage bonds are bonds whose principal is collateralized by an underlying property or real estate. When a home owner borrows a loan to purchase a house, the term of the loan, the interest rate, and the amount of loan are used to calculate the monthly mortgage payment. For conventional loans, the constant payment rate consists of both interest payment and principal payment.

At the beginning of a loan most of the payment is interest payment, and towards the end of the loan most of the payment will be principal payment. Most mortgage loans have a 30-year amortization schedule. Fifteen-year loans have a higher principal payment at the beginning than 30-year loans.

Not all mortgages have a fixed coupon with a term of 30 or 15 years. Some other mortgage structures include:

- *Interest-only* – The home owner pays only interest on the mortgage and the principal is due after a predetermined period, e.g., 5 years. Some interest-only mortgages can become amortizing at the end of the interest-only period with a predetermined interest rate.
- *Adjustable rate mortgage* – The interest rate of the mortgage is adjustable, based on a benchmark such as the Fed Funds rate or Libor plus a spread. Most adjustable rate mortgages have caps and floors. The cap is the maximum rate that the mortgage rate can go to and the floor is the minimum rate.
- *Balloons* – A balloon mortgage is one where the principal is due after a specific period of time, usually 5 or 7 years. Most balloons amortize like a 30-year mortgage; however, at the end of the balloon period, the home owner has to refinance the mortgage and pay back the principal. A balloon can be an interest-only mortgage.

Mortgages that meet certain size and quality requirements are generally securitized by the US agencies, Fannie Mae, Freddie Mac or Ginnie Mae. In this book we only cover conventional 15- and 30-year mortgages, which are by far the largest segments of the market. The analysis for other mortgages is similar and requires the development of a prepayment model for them.

The Advanced Fixed Income and Derivatives Management Guide. Saied Simozar

Most mortgages are pooled together by lenders based on a coupon rate, the original and remaining term, geographic area, and credit scores of the borrowers. The pools are securitized and sold in the secondary market. Mortgage bonds have the following characteristics:

- Lenders, such as banks and mortgage companies pool together many mortgage loans and securitize them as one security. Pooled loans typically have comparable coupon and original and remaining terms.
- The borrower pays a gross coupon to the lender. The lender or servicer takes some of the proceeds of the coupon for servicing and insurance and pays the balance, called net coupon, to the buyers of the securitized mortgage pool. All the principal payments are passed through to the bond holders.
- Most mortgage loans amortize over the life of the loan. The borrower pays a fixed monthly payment which covers the interest and part of the principal of the loan.
- There is a significant cost for refinancing a mortgage, as observed at the beginning of Chapter 17.

The weighted average coupon (WAC) of a pool is the outstanding face value weighted gross coupon that borrowers pay the mortgage servicer. The servicer pays the net coupon of the pool to the investors and uses the difference between the WAC and net as servicing costs. For most mortgage bonds the servicing cost is about 0.5–0.9%. The net coupon is often in increments of 0.5%. For example, for a WAC of 6.62%, the net coupon can be 6%. The weighted average maturity (WAM) of a mortgage pool is the average maturity in months and is an integer. The WAM is different from the maturity of the pool, which is equal to the maturity of the longest loan in the pool. The servicers provide the current WAC and WAM of a pool as well as the original WAC and WAM. The WAC of a pool does not change significantly over its life, but the WAM falls by roughly one unit every month.

Mortgage trading is usually done as TBA (to be announced); the exact securities that are to be delivered are not known at the trade time. Only the face amount, net coupon and price are agreed upon. For seasoned pools, additional specifications such as the WAM, geographic area of the loans or even the pool number may be agreed upon as well. Generally, 2 days before settlement, the pool numbers of the securities to be delivered are communicated with the buyer. Trades that are larger than \$1 million face, have to be delivered in lots of 1 million with a small variance. In the early to mid-1990s, the variance was 3%, that is, the seller had the option to deliver 1 million face ± 30,000. However, the variance has been reduced to only 0.01%. Up to three pools can be combined to make a whole lot of 1 million.

18.1 MORTGAGE VALUATION

Consider a mortgage loan with a principal amount of q, annual coupon rate (WAC) of c, payable monthly with 30/360 convention, and term of n years. The constant monthly payment rate m can be calculated from

$$q = \sum_{i=1}^{12n} \frac{m}{\left(1 + c/1200\right)^{i}} \tag{18.1}$$

Alternatively, we can write

$$q = \sum_{i=1}^{12n} \frac{m}{\left((1 + c/1200)^{12} \right)^{t_i}} = \sum_{i=1}^{12n} \frac{m}{e^{rt_i}} = \sum_{i=1}^{12n} m e^{-rt_i}, \quad t_i = \frac{i}{12} \tag{18.2}$$

where t is time in years and

$$r = 12 \ln\left(1 + \frac{c}{1200}\right) \tag{18.3}$$

It is considerably easier to work in the continuous compounding of payments by converting the summation in (18.2):

$$q = \sum_{i=1}^{12n} m e^{-rt_i} \Rightarrow \int_0^{t_n} G e^{-rt} dt \tag{18.4}$$

In differential equation form, if q is the principal, G is the constant payment amount, and r is the continuously compounded coupon rate, then

$$-dq = -G dt + rq dt \tag{18.5}$$

In this equation, $rqdt$ is the amount of interest payment in the period dt. The remainder of the payment will be a reduction in principal. Solving the above equation leads to

$$\frac{G - q_0 r}{G - qr} = e^{-rt} \tag{18.6}$$

where q_0 is the original principal of the mortgage bond. At maturity t_n the remaining principal q is zero. We can therefore calculate the continuous payment amount G as

$$G = \frac{q_0 r}{1 - e^{-rt_n}} \tag{18.7}$$

By substitution, the remaining principal at any time t is given by

$$q = q_0 \frac{1 - e^{-rt_n + rt}}{1 - e^{-rt_n}} \tag{18.8}$$

Now consider a pool of mortgages consisting of many original mortgages, some of them have already paid out in full. Equation (18.8) will apply to the pool at any time if the WAM of the pool in years is t_n, and the current outstanding balance is q_0.

Due to refinancing, home sale, and faster paydown of the principal, the principal payment is usually higher than the regular principal payment rate calculated by taking the derivative of (18.8),

$$dq = q_0 r e^{-rt_n} e^{rt} dt \tag{18.9}$$

We denote the additional principal payment rate, called the prepayment rate, as $h(t)$; the continuous payment amount will also be a function of time and the amount of outstanding principal. The calculation of the principal payment is a function of the gross WAC, but the pass-through to the investor is only net coupon plus all principal payments. The coupon contribution will simply be the ratio of the net coupon and the WAC of the pool. Thus, if $g(t)$ is the contribution from the coupon payment rate plus amortization to the bond holder, the price function of a mortgage pool can be written as

$$p = \frac{100}{q_0}\int_0^{t_n}\left[h(t) + g(t)\right]e^{-yt}dt \tag{18.10}$$

Since the borrower has the option to repay the mortgage at any time, when interest rates drop, the expected remaining life of the mortgage bond drops. Most mortgage pools have an original term of 30 years that pay on a monthly basis, resulting in 360 payments.

To analyze a mortgage bond, we need to construct a probability tree at different time horizons. Since building such a tree could be computationally expensive for 360 payments, we need to find approximate methods using fewer data points.

We first convert (18.10) to a summation form as follows:

$$p = \frac{100}{q_0}\sum_{i=1}^{n}\left(h_i + g_i\right)\, e^{-y_i t_i} \tag{18.11}$$

where h and g are the monthly cash flow contributions to the price function. We can now write the contribution of the cash flows in the interval from j to k as

$$p_{jk} = \frac{100}{q_0}\sum_{i=j}^{k}\left(h_i + g_i\right)e^{-y_i t_i} = \frac{100}{q_0}e^{-y_k t_k}\sum_{i=j}^{k}\left(h_i + g_i\right)f_{ik} \tag{18.12}$$

where f_{ik} is the factor to calculate the future value at time k of a cash flow at time i.

Since home owners do not exercise their options optimally, we do not need to construct the exercise boundary; instead, we use the prepayment model to capture the behavior of borrowers at different times and interest rates. Unlike bonds, we cannot calculate the forward coupon of mortgages by discounting future cash flows, since mortgages have a spread over Libor and are continuously callable. The prevailing mortgage rate at any given time is called the *current coupon*. We will first try to find a model to calculate the current coupon at future times to calculate the refinancing incentive for home owners.

18.2 CURRENT COUPON

The current coupon of a mortgage depends on several parameters, including interest rates, the volatility surface (option price), the spread demanded by investors, initiation costs, and the credit score of the borrower. Analysis of the historical current coupon of mortgages shows a very strong dependence on the level and slope of interest rates and weaker but meaningful dependence on the level of volatility.

We analyzed the average current coupon of 30-year and 15-year conventional mortgages initiated each month versus the average level and slope of the term structure of interest rates for that month and obtained the following fitted parameters.

The current coupon was assumed to be proportional to the level of interest rates and it also depended on the slope of treasury rates as well as volatility. One-year volatility for a forward 10-year term was used. Other volatility measures, including 3yr × 10yr and 2yr × 10yr, also have similar accuracies.

For conventional 30-year mortgages, we found the following parameters from 1997 through 2012:

$$r = 100a_{l,0} + 42.4a_{t,1} + 0.023v_{1,10} + 0.700 \tag{18.13}$$

where $a_{l,0}$ is the level of the Libor term structure, $a_{t,1}$ is the slope of the treasury term structure, $v_{1,10}$ is the swaption volatility of a 1-year option for the 10-year forward rate, and r is the continuously compounded mortgage rate from equation (18.3). The standard error of the fit or the standard deviation of the calculated coupon rate versus the market rate is 0.13%. If we remove the 12 months after the Lehman bankruptcy from the data, the error is only 0.10%.

Figure 18.1 shows the graph of the conventional 30-year mortgage rates versus the fitted data. The fitted parameters were calculated from 1997 to 2012 when the volatility was available. The model data from 1991 to 1997 assumed a volatility of 14% for 1-year options in a 10-year forward rate.

Figure 18.2 shows the difference between the market mortgage rate and the calculated rate. A positive value indicates cheap mortgage bonds and a negative value indicates expensive rates.

For conventional 15-year mortgages we found the following parameters from 1997 through 2012:

$$r = 100a_{l,0} + 32.8a_{t,1} + 0.020v_{1,10} + 0.43 \tag{18.14}$$

Figure 18.3 shows the graph of the 15 year mortgage market and calculated rates. The error of the fit is 0.16%.

The strong relationship between the current coupon and the components of the term structure of rates can be used to estimate future current coupons. Considering the number of factors that can affect the current coupon, including the credit score of the

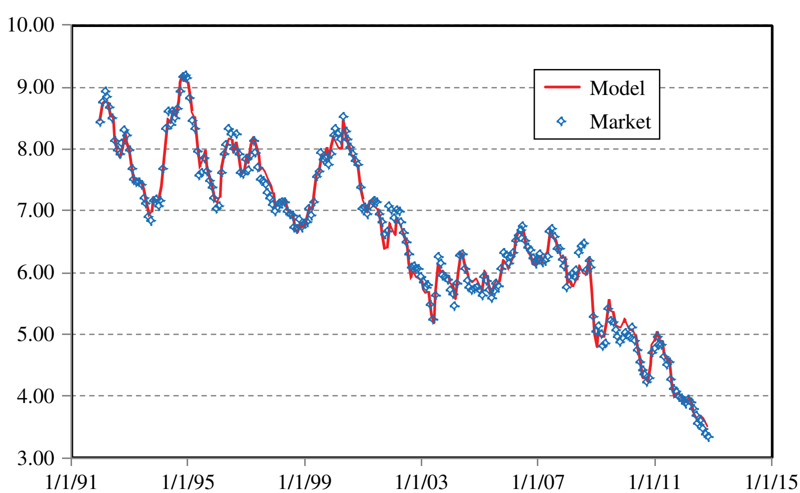


FIGURE 18.1 Conventional 30-year mortgage rates

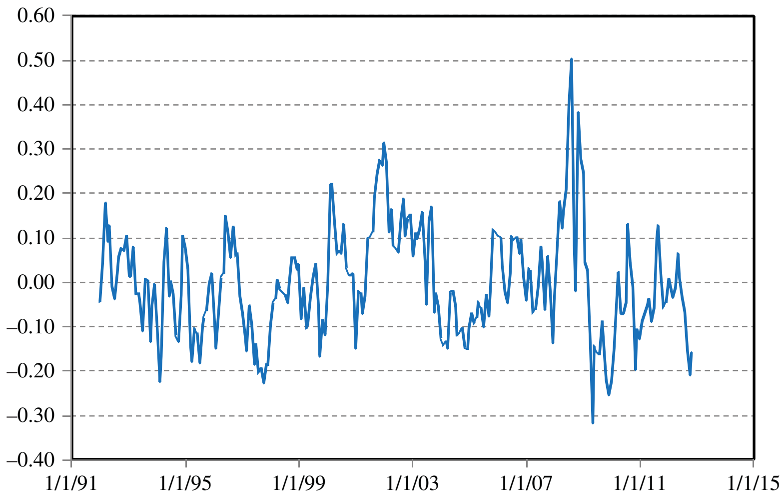


FIGURE 18.2 Calculation error for 30-year conventional mortgages

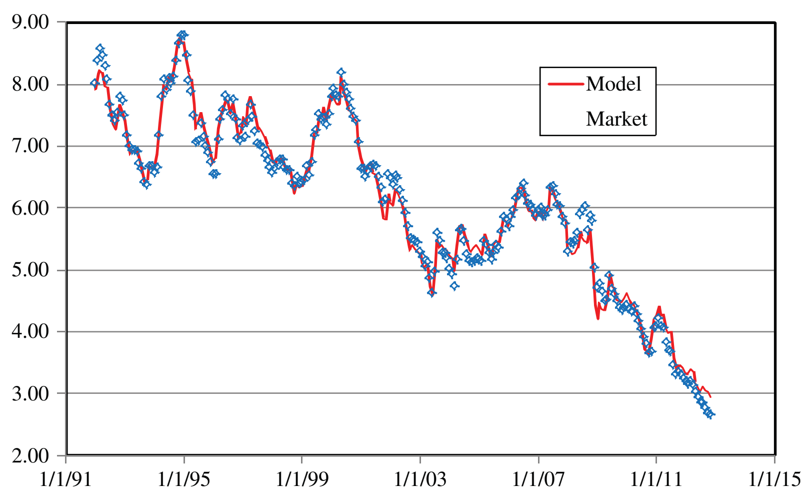


FIGURE 18.3 Conventional 15-year mortgage rates

borrower and the points received by the issuer, the accuracy is remarkable. Most mortgages have an issuance cost of about 2 points or 2% of the principal of the mortgage. Zero point mortgages typically have a higher interest rate by about 0.5–0.625%.

18.3 MORTGAGE ANALYTICS

We are now ready to calculate mortgage analytics based on the future path of interest rates and our prepayment model.

Consider a point in the future with a cumulative refinancing incentive (RI) of $R(t)$, the then incentive of $r_i(t)$ and the mortgage factor of $f(t)$, similar to point B in Figure

15.2. We will now consider a distribution that will transform point B to point C after a forward time Δt. The marginal contribution to RI will be

$$\Delta R = \frac{r_i(t) + r_i(t + \Delta t)}{2} \Delta t \tag{18.15}$$

In this equation, we use the average of RI at points B and C times the time difference between those two points. Likewise, we can calculate the marginal change in the factor due to refinancing as

$$\ln\left(\frac{f_{\mathrm{C}}}{f_{\mathrm{B}}}\right) = \ln\left(\frac{f(R_{\mathrm{B}} + \Delta R)}{f(R_{\mathrm{B}})}\right) \tag{18.16}$$

For small changes in RI, the change in factor will be

$$\frac{\Delta f}{f} = -a_r \Delta R - b_r \Delta R e^{-c_r R} \tag{18.17}$$

(see (17.7)). Since the RI is non-linear, prepayment rates are path dependent. For example, if future interest rates go down first and then up, future prepayment rates will be different than if interest rates go up first and then down. Thus, we need to find out how to aggregate the paths that end up in the same level of rates at some point in the forward space.

If the average factor is not accurately maintained, there will be leakage in the principal payments and the sum of all principal payments will not add up to 100. For example, if the factor from one path is 0.8 and from another path with equal probability is 0.9, the average factor for the end point should be 0.85. Based on this factor, we need to calculate the RI that would correspond to a factor of 0.85.

To simplify the iterative process of calculating the factor, first assume that point B is an aggregate of points with factors f_i and weights w_i. We can write the net factor as:

$$F = \sum_i w_i f_i = \sum_i w_i e^{-aR_i - \frac{b}{c}(1 - e^{-cR_i})} \tag{18.18}$$

If R is the implied RI that corresponds to F, it can be calculated from

$$F = e^{-aR - \frac{b}{c}(1 - e^{-cR})} \tag{18.19}$$

Substituting

$$R_i = R + r_i \tag{18.20}$$

we can write (18.18) as

$$F = \sum_i w_i e^{-a(R + r_i) - \frac{b}{c}(1 - e^{-c(R + r_i)})} \tag{18.21}$$

For a small value of r, i.e., for relatively tight distributions or small changes in time, we can expand the exponent using Taylor series up to first order, resulting in

$$F \approx \sum_i w_i F\left[1 - r_i(a + be^{-cR})\right] \tag{18.22}$$

By definition,

$$\sum_i w_i = 1 \tag{18.23}$$

Thus,

$$\sum_i w_i r_i (a + be^{-cR}) = (a + be^{-cR}) \sum_i w_i r_i \approx 0 \tag{18.24}$$

To calculate the change in the factor from a change in RI, we replace r_i with $r_i + \Delta R$ in equation (18.22), leading to

$$F + \Delta F \approx \sum_i w_i F \left[1 - (r_i + \Delta R)(a + be^{-cR}) \right] \tag{18.25}$$

After simplification and applying (18.24), we have

$$\frac{\Delta F}{F} \approx -\Delta R(a + be^{-cR}) \tag{18.26}$$

This result for changes in the aggregate factor is similar to (18.17) that we found for a specific factor. The only difference is that we have to use the implied RI of the aggregate. The implied RI is calculated from (18.19). We can use this equation to estimate the change in the RI that corresponds to a given factor at a forward point.

Before we start constructing the forward tree for the path of interest rates, we need to construct forward current coupons. The forward coupon is calculated from the term structure of forward rates. The term structure of forward rates can be calculated by calculating five forward yields for Libor and treasury curves. For example, to calculate the current coupon 5 years forward, we can calculate the forward yield at maturities that correspond to $-0.8\tau, -0.4\tau, 0.0\tau, 0.4\tau$, and 0.8τ. Knowing the forward yields, we can use equation (3.24) to calculate the polynomial representation of the yield curve from these five points. Chebyshev basis functions can then be calculated by simple transformation and the forward level and slope of Libor and treasury curves can be calculated for use in (18.13) or (18.14).

Next, we need to calculate the 5-year forward, 1-year volatility of a 10-year Libor bond. This requires estimating the coupon of a 10-year Libor bond that starts 1 year after the forward date (6 years forward in this example). The forward coupon can be calculated from the forward partial yield of the security, converting it to the estimated coupon. The forward partial yield can be calculated analytically and, unlike yield calculation, does not require iteration. The forward 1-year volatility of a 10-year Libor can be calculated using equation (9.37) for each cash flow of the forward security. In this equation $x_2 = y_1$, thus

$$w(t_{x1}, t_{x2}, t_{x2}, t_{y2}) = \sqrt{\frac{\left(w(0, t_{x2}, t_{x2}, t_{y2})\right)^2 t_{x2} - \left(w(0, t_{x1}, t_{x2}, t_{y2})\right)^2 t_{x1}}{t_{x2} - t_{x1}}} \tag{18.27}$$

We also need the historical term structure of rates to estimate the historical current coupons. The difference between the WAC of a mortgage pool and the estimated current coupon from (18.13) or (18.14) is the premium or discount of the coupon of the pool.

Our process of building the paths of interest rates and refinancing is summarized as follows:

- We build forward Libor distribution and calculate the drifts at forward points.
- The forward current coupon is calculated and its difference with the forward 10-year Libor rate will be the mortgage–Libor forward spread s_{ml}.
- Given a point in the forward interest rate tree, we know the factor and RI.
- We evolve this point on the tree to all future paths and calculate the factor at each forward point, using the known RI for that point.
- The distribution of rates will be based on the distribution of 10-year Libor adjusted by s_{ml} to calculate the forward mortgage rate for a given level of Libor.
- We take the average factor at each point and calculate the implied RI for that point and continue to evolve the tree.

The average factor at point C from all points B will be calculated as

$$f(\mathrm{C}) = \frac{\sum_j f_j(t) w_j + \sum_j \Delta f w_j}{\sum_j w_j} \tag{18.28}$$

where $f_j(t)$ is the factor at time t (or point B in time space), Δf is the amount of principal payment due to amortization and prepayment, and w_j is the probability weight of interest rates going from point j at time B to C.

The historical TSIR needs to be provided to calculate the original premium or discount of the mortgage pool. The difference between the calculated coupon and the market coupon in Figure 18.1 is usually due to the borrowing habits of the borrowers or their risk premium. For example, a borrower who borrows with no point is likely to do the same in future refinancing and will thus pay a higher coupon rate. Likewise, a borrower who has a lower credit rating is likely to pay a higher coupon rate in a future refinancing as well.

The premium or discount of a pool's coupon relative to the estimated coupon rate from (18.18) is likely to persist and needs to be added to the current coupon to estimate the future coupon rate of the borrower or to calculate his RI. Thus, if the calculated coupon rate for a pool is 5.5%, but the gross coupon rate of the pool is 5.63%, then the additional premium of the pool for refinancing is likely to stay at 0.13%.

Next, we need to choose the spacing between the layers, similar to the layers in American options. For the first 6–12 months we can space the layers on a monthly basis. The spacing can then be increased to 2, 3, 6, and 12 months. If the spacing is longer than 1 month, we can assume that the accumulated cash flows take place in the middle of the range. For example, to aggregate cash flows for April, May, and June, we can assume that all cash flows happen in May for valuation and risk measurement.

For a given time period, we can use (18.8) to calculate the expected change in the principal at the beginning and end of the period due to amortization. We also calculate the prepayment amount by interpolating between the interest rate at the beginning and end of the period along with the expected coupon payment and assume that all the cash flows take place in the middle of the period. Thus, for cash flows that are 6 or more years into the future, annual spacing between layers will provide the necessary accuracy for price calculation.

Knowing the historical RI of a given pool, we start by propagating interest rates to the first layer. At each point, we recalculate the prepayment amount, RI and calculate the factor for each point on the lattice. We progress from one layer to the next, keeping track of the factor and RI until the maturity of the pool.

The most widely used measure of value for mortgage bonds is the option adjusted spread (OAS). Once we construct the tree of forward paths and payments, we have to find the spread that, when added to Libor or treasury curve and used to discount the future expected cash flows, will result in the market price of the security.

18.4 MORTGAGE RISK MEASUREMENT AND VALUATION

Like American call options, mortgage bonds are sensitive to interest rates as well as to volatilities. For a given term structure of Libor and volatility surface, we construct the path of all forward rates. The forward distribution of rates is based on the calculated forward volatility surface which can be calculated from our term structure of volatility very accurately by using the adjustment table. From the spot term structure of Libor rates, term structure of interest rates and term structure of Libor volatility, the forward level of the TSLR, slope of TSIR, and forward volatility of 1-year by 10-year Libor can be calculated for estimating the forward current coupon using (18.13) or (18.14). At every node of the distribution, the burnout that is consistent with the factor at that point is calculated.

The forward paths of interest rates for mortgages cannot be recombined without effectively accounting for the burnout, since a rise in interest rates followed by a fall will result in different burnout than a fall in rates followed by a rise. Once the interest rate tree is constructed, the OAS can be calculated by finding the spread that will match the market value of the cash flows with the price of the bond.

The OAS of a mortgage bond has no equivalence in any other area of fixed income. Conventional mortgage bonds are virtually risk-free and thus their forward distribution of coupons and prepayments can be estimated from Libor volatility that is actively traded in the market. The calculated spread can thus be realized if one has an accurate prepayment model with minimal risk. It is common practice in the marketplace to label the spread of corporate bonds as OAS. This is misleading since the volatility surface of corporate bonds is not actively traded and the default probability and credit risks are not traded on an active basis.

The best measure of value for a mortgage bond is its OAS. The value of an option for mortgages can be misleading, particularly in upward sloping yield curves. In order to calculate the option value, we will first have to calculate the price of the parent security. Discounting future cash flows based on scheduled amortization and coupon payment of mortgage bonds is not logical, since home owners do indeed move and sometimes prepay their principal even if it does not make economic sense. When interest rates rise and property values decline, home sales fall but defaults rise, leading to principal paydowns. We can estimate the option price of a mortgage bond by calculating the difference between the discounted future cash flows with and without refinancing. Paydowns due to home sales and aging will still influence prepayments regardless of interest rates.

In an upward sloping yield curve a mortgage bond that prepays modestly can be worth more than a mortgage that does not prepay at all. For example, if short rates are

TABLE 18.1 Valuation of mortgage bonds, settlement August 3, 2012

			Price							Value		
Cpn	**Wac**	**Wam**	**Market**	**Invoice**	**Parent**	**Fair**	**IO**	**PO**	**Option**	**OAS**	**Yield**	**Theta**
5	5.7	310	113.5	113.53	117.94	115.77	26.07	87.46	2.18	0.34%	1.83%	2.03%
4.25	5.03	324	109.5	109.52	113.22	112.45	23.03	86.49	0.77	0.44%	1.96%	2.15%
4	4.62	335	108.5	108.52	112.00	111.73	22.70	85.83	0.27	0.47%	2.02%	2.23%
4	4.51	346	108.5	108.52	112.79	112.34	22.72	85.80	0.44	0.56%	2.10%	2.30%
3.5	3.92	351	105	105.02	109.38	105.79	10.22	94.80	3.59	0.23%	1.55%	1.09%
3	3.45	360	104	104.02	105.65	105.54	11.49	92.52	0.11	0.35%	1.69%	2.16%

close to zero as they were in 2012 in the US, a mortgage bond with a coupon of 3% earns its investor an additional 3% per year over short rates. If most of the principal is returned to the investor within a few years, the present value of those cash flows could be higher than if the principal is returned after many years where the discounting yield is significantly higher. Table 18.1 shows a sample of mortgage bonds along with the valuation parameters.

An interest-only (IO) bond receives only the interest rate cash flows of a mortgage bond. The principal-only (PO) bond is complementary to the IO and receives only the principal payments.

The average life of a mortgage bond is the weighted average time of all the principal payments of the bond,

$$\text{average life} = \frac{\sum_i c_{p,i} t_i}{100} \tag{18.29}$$

where $c_{p,i}$ is the principal payment at time t_i. The duration components of mortgage bonds can be calculated similarly to callable bonds by shifting the components of the TSIR. For mortgages, the shifts in the components of the curve are significantly more complicated, since the forward current coupons also depend on the components of the curve. For example, to shift the slope of the treasury curve, we need to calculate the slope of forward curves at all forward points as well. Likewise, for shifting the components of the volatility curve, we need to calculate the future expected forward volatilities and then calculate the expected future current coupons from (18.13) or (18.14). If the forward slope of the curve is not adjusted, the calculated slope duration can be off by several years.

Table 18.2 lists the risk measures of the securities in Table 18.1 as well as the IO and PO of the last mortgage bond. In general, the average life and level duration of mortgage bonds with a premium price increase with seasoning. The level convexity of all pass through mortgages is negative; however, with seasoning, the convexity becomes less negative.

The negative convexity of a current coupon bond can be very large. For example, in Table 18.2, the bond with a coupon of 3.0 and WAM of 360 has a convexity of –557 which requires a zero coupon bond with a maturity of 24 years just to hedge the convexity.

TABLE 18.2 Risk measures of mortgage bonds, July 31, 2012

					Duration					Convexity			
	Cpn	Wac	Wam	Avg Life	Level	Slope	Bend	4th	5th	X00	X01	X11	X02
Mtg	5	5.7	310	6.80	5.34	0.75	−2.51	−2.07	−0.74	−86	453	−3888	−1524
Mtg	4.25	5.03	324	7.07	5.91	240	−2.63	−2.05	−0.61	−90	1554	1986	−1704
Mtg	4	4.62	335	740	6.21	2.83	−2.65	−2.11	−0.62	−101	1908	2981	−1784
Mtg	4	4.51	346	7.29	6.02	2.31	−2.64	−2.00	−0.51	−105	1509	1384	−1721
Mtg	3.5	3.92	351	346	0.79	−6.73	−1.17	−0.64	−0.47	−87	−560	−26009	−98
Mtg	3	3.45	360	4.65	2.37	−6.59	−1.68	−0.55	−0.27	−557	−1627	−27345	−416
IO	3	3.45	360	4.65	−50.39	−256.3	−249	−12.99	−4.03	1082	134025	−964522	1302
PO	3	3.45	360	4.65	8.13	20.63	−1.59	0.85	0.17	−774	15766	74767	−1214

TABLE 18.3 Principal components of mortgage volatility, July 31, 2012

			Principal Components			
Cpn	**Wac**	**Wam**	**1st**	**2nd**	**3rd**	**4th**
5	5.7	310	–3.9	–0.9	–0.9	–0.2
4.25	5.03	324	–2.2	–0.5	–0.6	–0.2
4	4.62	335	–1.4	–0.4	–0.5	–0.2
4	4.51	346	–1.9	–0.5	–0.6	–0.2
3.5	3.92	351	–20.0	–4.5	–4.8	–1.1
3	3.45	360	–3.3	–0.6	–1.0	–0.4

IOs have very large negative duration. If interest rates rise, refinancing falls and IO will receive coupons for a longer period of time, resulting in higher prices, hence negative duration. The slope and bend durations and cross-convexities of IO and PO bonds can be extremely large and highly unstable, and therefore it is very difficult to hedge them.

Mortgages are similar to American call options with continuous incremental exercise points. However, since the option is not exercised optimally, they are unlike any tradable option. The duration of volatility of a mortgage provides a window into how it can be best hedged. Table 18.3 lists the first four principal components of the duration of volatility that is calculated by multiplying the term structure duration of volatilities by the matrix of principal components provided in Table 9.2. We can see that in nearly all cases the ratio of the absolute value of the first component to the second component is about 4.0 .

We next calculate the principal components of the duration of volatility of American and European swaptions to find a swaption that could be best used to hedge the volatility exposure of mortgages. Many market participants use 3-year expiration by 10-year maturity (3×10) European call swaption to hedge the volatility of mortgages. Table 18.4 lists a number of American and European call and put swaptions for comparison that could be best used for hedging the mortgage volatilities. It is clear from Table 18.4 that none of the swaptions have a risk profile that can be used as a hedge for mortgages. The ratio of the first principal component to the second for mortgages is about 4, but swaptions do not follow the same pattern.

We used linear programming to find the optimal weight of swaptions to hedge the 4% 346 mortgage in Table 18.3 by minimizing the minimum face value of swaptions for hedging. Table 18.5 summarizes the results. The most effective replication of the volatility is a combination of two long calls and a short put with a premium of about 0.74. Thus, to hedge the volatility of the mortgage, we have to perform the opposite of the replication trade which will cost a premium of 0.74 points as well.

The biggest risk in hedging a mortgage is in the prepayment model. The volatility and interest rate risks are very well understood and the current coupon can be estimated very accurately, given all the changes in the market, as Figures 18.1 and 18.3 suggest. Adding seasonal factors to home sales can improve monthly estimation of

TABLE 18.4 Principal components of swaption volatility, July 31, 2012

				Principal Components			
Issue	C/P	A/E	Premium	1st	2nd	3rd	4th
l×20	C	E	2.75	−4.8	−1.4	−2.0	−0.5
3×10	C	E	4.30	−9.3	−4.4	−3.0	−0.3
4×9	C	A	5.30	−11.2	−7.0	−4.2	−0.4
5×20	C	E	5.09	−0.8	−4.6	−2.7	−0.6
2×10	P	E	5.57	−1.8	−5.8	−3.4	−0.7
5×15	P	A	7.70	−2.3	−7.5	−4.3	−0.7
5×20	P	A	10.63	−6.0	−9.4	−5.0	−0.4
5×5	P	E	9.07	4.9	−5.8	−3.4	−0.9
5×10	P	E	7.52	4.4	−5.2	−3.1	−0.9

TABLE 18.5 Hedging volatility of a mortgage

Issue	C/P	A/E	Premium	1st	2nd	3rd	Weight
1×20	C	E	0.67	−1.2	−0.4	−0.5	0.24
3×10	C	E	0.27	−0.6	−0.3	−0.2	0.06
5×6	P	E	−0.20	−0.1	0.1	0.1	−0.02
Total			0.74	−1.9	−0.5	−0.6	

prepayments, but has small effect on the valuation of mortgage bonds. Unlike inflation linked securities where seasonal patterns can be important for bonds with a short maturity, mortgage bonds have a steady paydown of the principal which tends to average out the first order seasonal effects. Mortgage bonds with low maturities have very small outstanding principals and are rarely traded.

It is well known that as home prices appreciate, people are more likely to sell their home and trade up to a more expensive one. Thus, a prepayment model should be dependent on home prices. However, home prices are known only after the fact and modeling future home price appreciations will add too much complexity to a prepayment model. Home price appreciation due to falling interest rates is captured by the prepayment model to a large extent, lumped together with the refinancing incentive.

CHAPTER 19

Product Design and Portfolio Construction

There is significant overlap between product design and portfolio construction, in the sense that they both attempt to achieve the highest return for a given level of risk. In product design, we are concerned about the viability of the product for a long term, usually 10 years or more. In portfolio construction, a typical investment horizon is 3–12 months. In both cases, we need to use historical data as a guide to evaluate correlations and volatilities, similarly to equity portfolio construction.

Most institutional fixed income products are based on indexes. An index is a collection of bonds or assets that meet certain credit quality, maturity, liquidity or size criteria. For example, the Citi Treasury Index is a capitalized weighted index of all US treasuries with a maturity of 1 year or more. The Short Treasury Index is a capitalized weighted index of all US treasuries with a maturity of 1–3 years. Indexes can be combined to construct broader indexes or aggregates. For example, the JP Morgan Emerging Markets Index consists of the liquid bonds of more than ten developing countries. The Barclays Aggregate Bond Index is an aggregate of most US dollar denominated bonds with maturities longer than 1 year, including treasuries, corporate bonds, and mortgage bonds.

Indexes are supply driven and the largest weights belong to the largest issuers. For example, in the late 1990s, during the treasury surplus years in the US, the weight of US treasuries fell relative to mortgage bonds as real estate prices soared and borrowing increased. Therefore, an index is not necessarily an optimal portfolio. Nevertheless, an index is an unbiased measure of the investable market and its performance is a reasonable gauge of the expected performance of a bond manager. All market participants can own a capitalization weighted index, and therefore it represents the best indicator of the market.

In product design, one has to understand the needs of the client and the duration of the liabilities, if any, and create a benchmark whose performance can be independently verified and meets the client's needs as closely as possible. For a pension plan that has liabilities with a duration of 15 years, using the treasury index does not provide enough duration. A better benchmark might be treasuries with minimum maturities of 5 or 10 years, which when combined with an equity allocation provide the necessary duration and return expectation. Index providers often publish the return data for such customized indexes as well.

A key ingredient in designing a product or constructing a portfolio is to understand the interactions between different sectors under normal as well as stress situations. As mentioned in Section 15.9, the long term correlation of spreads and interest rates tends to be negative. This implies that if credit is overweighed, the treasury duration needs to be longer to mitigate the contribution of credit spread to risk. Consider a correlation between spreads and treasury rates of -0.5 and beta of spread of 0.33 for a portfolio with a duration of 5 years and yield and spread of 3.0% and 3.6%, respectively. Denoting the spread of the security by s, the treasury yield by y, the correlation coefficient by ρ, and the beta of spread relative to yield by β, we can calculate the effect of a change in rates on the spread from (15.54) as follows:

$$ds = s\frac{dy}{y}\rho\beta = 3.6\% \times \frac{dy}{3.0\%} \times (-0.5) \times 0.33 \approx -0.2dy$$

Thus, for a change of 1 basis point in treasury rates, the average expected change in spread will be –0.2 bps. If we are long the spread duration by 0.9 years, we need to be long the treasury duration by 0.18 years as well. If treasury rates rise by 30 bps, the negative impact on the portfolio will be $-0.30\% \times 0.18 = -0.054\%$. Likewise the expected tightening in spread is 20% of 30 bps with a positive contribution to the portfolio of $0.30\% \times 0.2 \times 0.9 = 0.054\%$. The two positions negate the risks of each other on average. However, in an upward sloping yield curve, both positions are likely to contribute positively to excess performance. For example, if short rates are 1%, the extra duration of 0.18 for a portfolio of 5-year duration will require an additional equivalent of 0.18/5 = 3.6% of the 5-year treasury. The likely contribution of financed treasury over a year is $3.6\%(3.0\% - 1\%) = 0.072\%$. Likewise, the expected excess contribution of the financed credit component with a weight of 18% (duration 0.9 years) is $18\%(3.0\% + 3.6\% - 1\%) = 1.008\%$. The two strategies add about 108 bps of excess return without increasing the risk significantly.

This example illustrates the principal behind Markowitz portfolio theory which is widely covered in books on finance. In practice, things are not always this straightforward; correlations and volatilities are not stable, and during times of crisis the correlation of spread and rates will approach –1 and betas will increase significantly. In general, the application of portfolio theory to fixed income is not as straightforward as it is for equities for the following reasons:

- Fixed income is more about liability management than simple risk and return. For a pension fund with a vested duration of liabilities of 15 years, the risk-free rate is a bond with duration of 15 years, while for a corporate treasury that intends to make an acquisition the risk-free rate is probably 3-month Libor.
- Fixed income sectors are highly correlated. Nearly all fixed income products are dependent on interest rates and correlations become useless at times of crisis, when they are needed the most as they approach 1 or –1.
- Principal preservation is much more important in fixed income since the upside in any investment is limited to the principal plus interest while the downside is unlimited.
- In fixed income, the portfolio structure can be independent of strategies to add alpha or excess return through the use of derivatives. Careful structuring of a short duration and a long duration portfolio can be achieved in such a way that their relative performance will be similar compared to their respective benchmarks.

Our approach to product design, is based on portfolio theory. However, we implement it in the context of "what-if" analysis which is much more conducive to team management. In an equity portfolio, one may have to consider more than one thousand different tickers and calculate the correlations among them. The core of fixed income can be broken down into a handful of sectors including treasuries, agencies, corporates, mortgages, high yield, and emerging markets. Other sectors, including commercial mortgage backed securities and convertible notes, are a much smaller part of core fixed income in most institutional accounts. The bonds in each of these sectors are highly correlated and, as noted in the previous sections, for treasuries, only three parameters can account for 98% of the movements of the entire market.

We break each of the fixed income sectors into subsectors based on maturity or credit quality buckets to build a product analyzer that can be used for designing products in a variety of sectors and maturity buckets.

19.1 PRODUCT ANALYZER

The product analyzer can be used to analyze the historical performance and risks of a portfolio and to design a product or to construct a portfolio that meets certain risk and return requirements. We provide the steps that are necessary to build a generic product analyzer for nearly all types of fixed income product designs. The product analyzer can be built using a popular spreadsheet using monthly historical return data for sectors of fixed income. The historical data can generally be obtained from index providers and the tables need to be maintained on a monthly basis.

Fixed income products are typically broken down by maturity buckets of short (1–3 years), intermediate (1–10 years) and full index. For a US based fund manager, to have the maximum flexibility, we need to have data for the maturity buckets 1–3, 1–10, 1–30, 3–10, 5+, and 10+ years. Each one of these data series will occupy a column in a spreadsheet. We also need to store the durations of each of these sectors in the columns of a separate sheet of the worksheet. Similar historical data need to be obtained and stored in the worksheet for high grade corporates with ratings of BBB, A, and AA and high yield with ratings of BB, B, and CCC. There are usually only two maturity buckets of 15 and 30 years available for mortgages. We add data series for emerging markets, convertible securities, carry currencies (high yield), funding currencies, and return attribution of level, slope, and bend components of the TSIR. For euro or other currency based products, we can add the time series of the respective fixed income sectors that are available locally.

First, let us show how the risks of two correlated securities are calculated. Suppose that securities A and B have risks r_A and r_B and correlation ρ_{AB}. The combined risk of A and B is calculated as

$$\begin{aligned} r &= \sqrt{r_A^2 + r_B^2 + 2\rho_{AB} r_A r_B} \\ &= \sqrt{\begin{vmatrix} r_A & r_B \end{vmatrix} \times \begin{vmatrix} 1 & \rho_{AB} \\ \rho_{AB} & 1 \end{vmatrix} \times \begin{vmatrix} r_A \\ r_B \end{vmatrix}} \end{aligned} \tag{19.1}$$

For a portfolio of several securities or sectors with respective risks of r_i shown as a vector **R**, and correlation matrix, **ρ**, the overall risk can be expressed as

$$r = \sqrt{|\mathbf{R}|^T \times |\boldsymbol{\rho}| \times |\mathbf{R}|} = \sqrt{\sum_i \sum_j r_i r_j \rho_{ij}} \tag{19.2}$$

We can rewrite the above equation to calculate the contribution to tracking error. Define

$$w_i = \frac{\sum_j r_j \rho_{ij}}{\sqrt{\sum_i \sum_j r_i r_j \rho_{ij}}} \tag{19.3}$$

The risks can be written as

$$r = \sum_i w_i r_i \tag{19.4}$$

where w_i is the weight of the risk of security i in the portfolio and $w_i r_i$ is the contribution of the risk of security i to the total risk. For example, if securities A, B, and C have all risks of one unit and AB has a correlation of 0.6 and AC and BC have correlations of –0.5, then the contribution of each security to risk of the portfolio will be $w_i = (0.742, 0.742, 0.0)$ and the overall risk of the portfolio will be 1.483 units. The risk weight w_i has a non-linear relationship with the risks of other securities in a portfolio; however, for small changes in the risks of each security, the relationship will be almost linear. For example, if the risk of security C increased to 1.1, then $w_i = (0.706, 0.706, 0.067)$ and overall risk will be 1.486. This change is very small, considering that the risk of C in the original portfolio was zero. If we calculate the risks of a portfolio of just A and B, then the contribution to risks will be $w_i = (0.894, 0.894)$ and the overall risk will be 1.788. If security C has a positive expected return, its addition to the portfolio will not only increase the return, it will lower the risk as well.

Constructing the correlation matrix of different sectors in a spreadsheet enables us to perform what-if analysis and review the hypothetical performance of the portfolio in historical times of crises. If the monthly durations of each sector are also available, we can see how different durations at different times contributed to risk or return.

Table 19.1 is a sample spreadsheet for the analysis of annualized risk, performance simulation, and what-if analysis. By changing the date ranges for correlation and the performance range, one can see how the portfolio would respond to different sectors. By increasing the weight of sectors that have a negative contribution to risk, we lower the overall annualized risk. Note that the sum of weights can be more or less than 100%. Currency positions do not require upfront cash and treasury duration can be added by derivatives such as futures depending on investment policy.

The risks for all sectors and asset classes have to be stated in the same units. For example, if we use annualized return volatility in basis points as a measure of risk, we have to use the same units for currency, bond futures, swaps etc. as well. Monthly volatilities can be converted to annual volatility by multiplying them by $\sqrt{12}$.

TABLE 19.1 Sample portfolio analyzer output

	Start	End
Correlation	**1/1/2002**	**7/31/2008**
Performance	**8/31/2008**	**3/31/2009**
Duration	4.80	
Yield	5.74%	
Spread	2.52%	
Total Weight	111.3%	
Annualized Risk, bps	311.0	
Performance	−3.4%	
Historical Volatility, bps	365	

Sector	Weight	Contribution to Risk
Agy 1–3	5.0%	−0.7
Agy 3–7	3.8%	−2.1
Agy 7–10	2.5%	−5.6
Agy 10+	1.7%	−1.5
BBB 1–3	0.9%	3.4
BBB 3–7	2.3%	16.3
BBB 7–10	4.0%	44.3
BBB 10+	2.0%	35.8
B 1–7	4.9%	44.7
CCC	6.3%	59.8
Tsy 1–3	9.3%	−2.3
Tsy 3–7	6.6%	−4.2
Tsy 7–10	2.8%	−8.4
Tsy 10+	10.0%	−11.1
Mtg 30	33.2%	62.6
Mtg 15	6.1%	7.9
Carry Fx	10%	72.0

Once the historical risk profile of a product is analyzed and found to be desirable, we can expect the future risk profile to be similar. However, the expected return of the product will be approximately equal to the yield adjusted for default loss. Since yields have been on a secular decline from 1980 to 2012, the same pattern cannot be expected to continue, and therefore expected future returns are likely to be less.

19.2 PORTFOLIO ANALYZER

Our portfolio analyzer is similar to the product analyzer, except that it is used to construct a portfolio and analyze its short term (3–12 months) return expectations. Additionally, the portfolio analyzer needs to be constructed in concert with the investment process. For example, if allocation is made to different sectors by duration, then the weights will be based on contribution to duration.

Given that most portfolios are managed against benchmarks, the risks are generally measured as the difference in spread duration between the portfolio and the benchmark. Likewise, return expectations are measured relative to benchmark. We need to make adjustments to the product analyzer to use it as a portfolio analyzer.

High grade corporate, agency, and global government bonds have a high correlation with treasuries and the spreads are generally not very volatile. Thus, allocation to these sectors can be made on the basis of spread duration. On the other hand, high yield corporate bonds and currencies have a much lower correlation with treasuries and it is more logical to allocate the funds to these sectors by market value.

To allocate by spread duration, the risk is measured by the volatility of the spread or OAS, whichever is available. Theoretically, we have to measure the relative volatility, that is, changes in spread divided by the spread, and then multiply the realized volatility by the spread to calculate the absolute spread volatility. However, in practice it is easier to use the absolute change in spread as input into the correlation matrix. For example, if the historical monthly spread of the jth sector is denoted by $s_{j,i}$, then the square of its risk will be calculated as

$$v_{s,j}^2 = \frac{12}{n}\sum_{i=1}^{n}\left(s_{i+1} - s_i - \overline{\Delta s}\right)^2 = \frac{12}{n}\sum_{i=1}^{n}\left(\Delta s_i - \overline{\Delta s}\right)^2 \tag{19.5}$$

where $\overline{\Delta s}$ is the drift or the average change in the spread in one period. The risk of sector j in basis points using spread duration weights will then be

$$r_j = 10{,}000 D_{s,j} v_{s,j} \tag{19.6}$$

For currencies, the risk is calculated as

$$v_{x,j}^2 = \frac{12}{n}\sum_{i=1}^{n}\left(\ln\left(\frac{p_{i+1}}{p_i} - \overline{\Delta u}\right)\right)^2 = \frac{12}{n}\sum_{i=1}^{n}\left(\Delta u_i - \overline{\Delta u}\right)^2 \tag{19.7}$$

where p_i is the price of the currency at interval i, Δu_i is its performance, and $\overline{\Delta u}$ is the drift or the average periodic performance (the subscript j having been dropped for convenience). Instead of the correlation matrix, we need to calculate the covariance matrix which includes the risks. The covariance matrix for sectors j and k, representing duration weighted sectors and market value weighted sectors respectively, is calculated as

$$c_{jk} = \sum_i\left(\Delta s_{j,i} - \overline{\Delta s_{j,i}}\right)\left(\Delta u_{k,i} - \overline{\Delta u_{k,i}}\right) = \rho_{jk} v_j v_k \tag{19.8}$$

To complete the portfolio analyzer, we need to load in the risks of different sectors of the benchmark. For a more useful portfolio analyzer, the risk factors of multiple benchmarks such as 1 to 3 year, 1 to 10 year, etc., can be provided; these can be accessed using dropdown menus. Instead of using absolute allocations, the net allocation should be used and the resulting risk measure will be the total tracking error of the portfolio relative to the benchmark. The net contribution of each sector to risk is called marginal contribution to tracking error, and the spread and durations will be measured relative to the benchmark.

One has to remember that the estimated tacking error is just that, and the market environment can result in a significantly different tracking error than anticipated. A useful feature to add to the portfolio analyzer is the volatility scale. During times of crisis the interest rate volatility spikes up, and to account for it one can use a scale factor to scale the tracking factor by. For example, if the historical absolute volatility of a 10-year swap has been 80 bps (absolute volatility is equal to relative volatility times the level of rates) and the current absolute volatility is 100 bps, we can multiply the tracking error by 1.25 to account for the increased volatility.

The portfolio analyzer is a useful tool to review the positions and bets in the portfolio and to carry out what-if analysis in a team environment to see how different ideas impact the risk and yield of the portfolio. It can also be used for regulatory or reporting requirements. The analyzer has many limitations and is not a product to use on a daily basis. The markets change on a regular basis and correlations are not often stable. Default rates tend to be cyclical and correlations tend to go to extremes at times of crisis, when they are needed the most. However, the portfolio analyzer provides a portfolio manager a great tool to see the approximate tracking error that is being taken and the expected spread. Additionally, it provides insights to increase the yield without increasing risk or lowering risk while maintaining the yield.

We can also calculate the mean reversion of the bets in the portfolio if they are indeed mean reverting and estimate the projected excess performance over the investment time horizon. Since spreads tend to be mean reverting, we can estimate the mean reversion half-life and calculate the projected spread change over a 3-month horizon for net positions. See Section 3.7 for details.

The portfolio analyzer is a relatively high level product for portfolio structuring and is useful for macro bets. Security selection and active management should account for similar contributions to the performance of a fixed income portfolio.

19.3 COMPETITIVE UNIVERSE

Sometimes a product that is very well structured may not be competing in the right universe of funds or it is not clear what benchmarks the competitive universe uses. Many funds that include “income” in their name fall into this category. The definition of income can differ considerably among portfolio managers. Technically, all fixed income products are income funds, since there is no growth in fixed income through maturity. The most that you can get is the principal and interest at maturity. Funds such as “strategic income” and “high yield income” are typically funds that provide the portfolio manager additional tools to add income to the portfolio and may include high yield, emerging markets, sovereign bonds, and currencies which are usually not a part of core bond funds.

Depending on market environment, many of these funds tend to have relatively large changes in their structure; however, as a group they do not change significantly over a long period of time. One can use style analysis to estimate the composition of the market. Then a fund can be structured with a benchmark that mimics that market more closely.

First, we need to access historical pricings of the competitive universe on a daily or weekly basis and calculate the performance on a daily or weekly basis. Most funds that have a price of about \$10 will round the price to \$0.01, which is 10 bps of performance. If there are 50 funds in the competitive universe, the error becomes $10/\sqrt{50} = 1.4$ bps. Since nearly all funds have distributions, their net asset value changes on a daily basis. To calculate the performance of the fund on a daily basis, we use the formula

$$\text{perf} = \frac{\text{nav}_{i+1} + \text{dist}}{\text{nav}_i} \tag{19.9}$$

where perf is performance, nav is net asset value, and dist is distribution. We then use the historical performance of different sectors along with a few major currencies, emerging markets, and global bonds for the same dates that fund pricings are available and perform linear regression to find the weights of significant components of the competitive universe. The calculated weights may not add to 100%, but in most cases will come very close to 100%. This method provides a very good picture of the competitive market and allows the bond manager to design a product or restructure the benchmark to be able to compete in the competitive universe more effectively.

19.4 PORTFOLIO CONSTRUCTION

Portfolio construction is the process of designing a portfolio based on the inputs of analysts, economists, and traders in such a way as to satisfy the product's policies, guidelines, and constraints. The portfolio analyzer is a very good first step in sector allocation of a portfolio. It can incorporate the portfolio manager's views on the direction of spreads, currencies, rates, curve position and provide an indication of the expected return, risk, and tracking error of the portfolio relative to the benchmark.

We can distinguish three kinds of portfolio management style. The top-down approach is based on analysis of the economy, inflation or disinflation pressures, growth, unemployment, etc. Based on economic fundamentals, one decides the direction of interest rates, the path of the shape of the curve, and spreads. The biggest risk to the top-down approach is that the market already has a view about economic fundamentals and many of those pressures are already priced in. For example, when the market anticipates Fed rate cuts, the front end of the curve outperforms the remainder of the market and the curve steepens. Thus, a top-down approach has to be more accurate than the collective wisdom of the market participants to work, and most top-down fixed income managers tend to have volatile portfolios and often underperform the market.

The bottom-up approach is based on the analysis of the issuers and works best for corporate and mortgage bonds. It involves analysis of corporate filings as well as analysis of their competitors, suppliers, and products. Since there are a limited number of analysts that cover a company, a good analyst can often select issuers that are most likely to pay their debt. Unlike equity analysis where growth is a significant component of valuation, a stable business is more desirable than growth prospects for bond holders. For mortgage bonds, sometimes off-the-run coupons become very slightly cheap or rich depending on market environment, and a mortgage analyst can increase overweight to premium or discount coupons to take advantage of the cheaper securities.

The quantitative approach is similar to the methods that have been discussed in this book, such as linear optimization, using inflation swaps plus interest rate swaps instead of using inflation linked bonds, etc. The quantitative approach can be used in conjunction with top-down or bottom-up analysis. For example, for constructing a corporate bond portfolio, an analyst can select the names of issuers from the universe of investable issuers that he likes and these names can be used as input into the optimization process to structure the cheapest portfolio. We will discuss this method later in this chapter.

The tracking error calculated by the portfolio analyzer is often understated, since only a fraction of the bonds or issuers from the corporate sector will be included in the portfolio. For funds that can trade swaps, exposure to the corporate sector can often be achieved by buying a total return swap, where the investment manager receives the returns of the corporate sector and pays short term Libor. The transaction cost is comparable to structuring the corporate portfolio by buying individual bonds; however, the portfolio manager cannot add value by security selection or sector allocation. This approach is useful for fund managers who do not have the required number of analysts to cover corporate bonds or for top-down managers.

The most consistent contributor to the performance of a portfolio of risk-free assets is the carry or yield. Maximizing the yield in a steep yield curve environment without constraint will result in a portfolio of very long duration. Likewise maximizing the carry with a steep yield curve will concentrate the portfolio usually in the 5–10-year maturity range depending on the steepness of the curve. Every fixed income portfolio has to have a major emphasis on maximizing yield or carry.

Historical analysis of global bonds suggests that buying 10-year interest rate swaps of countries with steep yield curves generates positive alpha with information ratio greater than unity. Steep yield curves are usually a result of central banks lowering short rates, when inflationary pressures are very low, resulting in relatively high real long rates. High real rates usually result in further falls in inflation and eventual bull-flattening of the yield curve.

In Chapter 5 we discussed how linear optimization can add value even for a treasury portfolio. For corporate and emerging markets bonds, linear optimization can add significantly more value without changing the portfolio structure or the weights of issuers in the portfolio. Since default is a real possibility in corporate or emerging markets bonds, we need to provide constraints to limit the maximum market value allocation to each issuer. Likewise, we can limit the maximum spread duration for each issuer relative to the benchmark or in absolute terms. Transaction costs are relatively significant for illiquid bonds and we need to take into consideration such transactions by buying a bond at the offer price and selling it at the bid price. For corporate bonds, the bid–ask can be about 5–10 bps of yield, implying a price bid–ask of about 0.50% for a bond with a maturity of 10 years.

Analysts can provide the list of tickers that they like to buy or hold in the portfolio, and the tickers can be used to find all bonds for input into the linear optimization. For example, for a corporate portfolio we may require that no issuer can have a market value of more than 1% in excess of the benchmark and a maximum contribution to duration of 0.1 years over the benchmark.

Table driven constraints for the optimization can be very useful for team portfolio management, where the analysts as well as the portfolio managers can more easily visualize the structure of the portfolio and bets in specific sectors.

Table 19.2 is a sample list of constraints for a table driven linear optimization. Each group limits the sum of the contributions of all sectors in the group to the specified value. For example, the sum of spread duration for AR + EC + UA + VE has to be less than or equal to 0.4 years. To optimize a high yield portfolio, specific sectors or issuers can be grouped to limit their maximum contribution to the portfolio. Table 19.3 shows a sample output of trading recommendations made by the optimizer rounded to $5000 face values.

TABLE 19.2 Sample linear optimization constraints

Field	Country	Absolute	Relative to Bench	Sign	Group
Market Value	AR		2%	<=	
Market Value	BG		5%	<=	
Market Value	BR		5%	<=	
Market Value	CO		3%	<=	
Market Value	EC		2%	<=	
Market Value	MX		10%	<=	
Market Value	ID		3%	<=	
Market Value	PA		3%	<=	
Market Value	PE		3%	<=	
Spread Duration	PH		0.4	<=	
Spread Duration	HU		0.2	<=	
Spread Duration	HR		0.2	<=	
Spread Duration	TR		0.4	<=	
Spread Duration	ZA		0.4	<=	
Spread Duration	AR		0.4	<=	1
Spread Duration	EC		0.4	<=	1
Spread Duration	UA		0.4	<=	1
Spread Duration	VE		0.4	<=	1
Market Value	AR	6.0%		<=	2
Market Value	EC	6.0%		<=	2

TABLE 19.3 Sample linear optimization trades, July 31, 2012

Description	Cpn	Maturity	Current	B/S	B/A	Trade	Bid	Offer
CO Republic 6 1/8% due 41	6.125	1/18/41	200,000	Cut	Bid	−25,000	137.75	138.75
MX UMS 5 3/4% due 2110	5.75	10/12/10	6,180,000	Cut	Bid	−50,000	125.00	126.00
RU Ministry Fin 12 3/4% due 28	12.75	6/24/28	43,610,000	Cut	Bid	−1,455,000	190.25	191.25
UA Republic 7.65% due 13	7.65	6/11/13	1,000,000	Cut	Bid	−60,000	99.50	99.50
VE Republic 12 3/4% due 22	12.75	8/23/22	23,535,000	Cut	Bid	−100,000	99.25	100.00
CO Republic 7 3/8% due 37	7.375	9/18/37	590,000	HId		0	155.50	156.50
ID Republic 11 5/8% due 19	11.625	3/4/19	805,000	HId		0	151.25	152.25
ID Republic 7 3/4% due 38	7.75	1/17/38	805,000	HId		0	147.00	148.00
TR Republic 7% due 19	7	3/11/19	260,000	HId		0	120.13	120.88
AR Discount 8.28% due 33 Exch	8.28	12/31/33	10,505,000	Add	Ask	50,000	64.25	65.32
BR Republic 12 1/4% due 30	12.25	3/6/30	5,870,000	Add	Ask	130,000	204.50	205.50
PE Republic 5 5/8% due 50	5.625	11/18/50	550,000	Add	Ask	70,000	131.00	132.00
PH Republic 9 1/2% due 30	9.5	2/2/30	3,815,000	Add	Ask	610,000	170.50	171.00
TR Republic 7 1/2% due 19	7.5	11/7/19	2,860,000	Add	Ask	25,000	124.38	125.13
TR Republic 7 1/2% due 17	7.5	7/14/17	7,135,000	Add	Ask	45,000	118.75	119.50
TR Republic 6 3/4% due 18	6.75	4/3/18	12,775,000	Add	Ask	80,000	117.13	117.88
ID Republic 8 1/2% due 35	8.5	10/12/35	0	Buy	Ask	25,000	155.13	156.13
PH Republic 10 5/8% due 25	10.625	3/16/25	0	Buy	Ask	245,000	171.13	171.88
TR Republic 7 1/4% due 15	7.25	3/15/15	0	Buy	Ask	1,035,000	110.88	111.38

Source: Prices from J. P. Morgan Securities

The advantage of table driven linear optimization is that each analyst or portfolio manager can update their recommendations in the table and the overnight processing will produce the optimum trades to achieve the portfolio restructuring or rebalancing. The trader can see the estimated bid–ask spreads and the expected prices for each security before a trade is initiated. If the market price of a security is significantly different from the price used in the optimization, the trader can postpone the transaction.

For corporate portfolios, liquidity may be a constraint for buying securities. Often, the optimizer picks securities that have higher spreads but are not available for trade. A table of exclusions can be updated to remove illiquid securities from the universe of securities that is used for optimization. Optimization of corporate bond portfolios often requires trial and error and a few iterations to find desirable securities. Many traders prefer to trade benchmark securities or very liquid securities which tend to have a premium price and a lower yield. In the long run lower yields will cost the portfolio excess returns.

Without linear optimization, there is no other way to efficiently structure the portfolio by making the necessary trade-offs of bid–ask and yield or spread advantage. The optimization makes the entire portfolio available for sale at the bid price to raise cash and the entire universe in excess of the portfolio can be purchased at the offer price. A security that an analyst has a sell rating on is taken off the buy list and is automatically added to the sell list.

In general a decision to hold is the same as a decision to buy; thus, many fixed income mangers require analysts to have only buy or sell recommendations. Traders can often ignore small size trades that are produced by the optimizer. For example, a trade for $25,000 will not make a significant difference in the risk or return profile of a portfolio with a market value of $100 million.

Linear optimization is also very useful for end-of-month rebalancing when the index is also rebalanced and it can provide a peek into the restructured portfolio in advance. More importantly, a comparison of the positions in the current portfolio, the index, and the future portfolio, will make it easier to visualize the final product before implementation. Table 19.4 is a sample of such a report.

A monthly rebalancing of emerging markets or high yield portfolios using linear programming generates about 100–200 additional basis points on an annual basis while maintaining the same level of risk.

TABLE 19.4 Sample portfolio preview

		Contribution to MV			Contribution to SD			Contribution to Carry		
Country	S & P	Index	Port	Optim	Index	Port	Optim	Index	Port	Optim
AR	B	2.1%	4.1%	4.1%	0.178	0.334	0.333	0.22%	0.47%	0.47%
BG	BBB	0.5%	0.0%	0.0%	0.010	0.000	0.000	0.01%	0.00%	0.00%
BR	BBB–	12.4%	6.9%	7.1%	1.089	0.674	0.694	0.15%	0.13%	0.14%
CO	BBB–	5.6%	0.7%	0.7%	0.492	0.094	0.091	0.07%	0.01%	0.01%
EC	B–	0.2%	0.0%	0.0%	0.006	0.000	0.000	0.02%	0.00%	0.00%
HR	BBB–	1.4%	0.0%	0.0%	0.088	0.000	0.000	0.07%	0.00%	0.00%
HU	BBB–	2.1%	0.0%	0.0%	0.154	0.000	0.000	0.10%	0.00%	0.00%
ID	BB+	6.6%	1.4%	1.4%	0.524	0.122	0.125	0.14%	0.03%	0.03%
MX	BBB	13.8%	4.4%	4.4%	1.221	0.834	0.826	0.17%	0.09%	0.09%
PA	BBB–	3.1%	0.0%	0.0%	0.297	0.000	0.000	0.05%	0.00%	0.00%
PE	BBB–	4.2%	0.4%	0.5%	0.481	0.072	0.081	0.05%	0.01%	0.01%
PH	BB	9.6%	3.6%	4.5%	0.865	0.384	0.466	0.15%	0.07%	0.08%
RU	BBB	11.7%	46.9%	45.4%	0.662	4.253	4.114	0.24%	1.27%	1.22%
TR	BB	13.2%	15.5%	16.3%	1.026	0.742	0.762	0.34%	0.41%	0.43%
UA	B+	1.3%	0.6%	0.5%	0.059	0.005	0.004	0.11%	0.05%	0.04%
VE	B+	9.7%	13.9%	13.8%	0.579	0.716	0.713	0.99%	1.57%	1.57%
ZA	BBB+	2.7%	0.0%	0.0%	0.179	0.000	0.000	0.04%	0.00%	0.00%
Cash	AA	0.0%	1.6%	1.4%	0.000	0.000	0.000	0.00%	0.00%	0.00%
Total		100.0%	100.0%	100.0%	7.912	8.230	8.209	2.94%	4.10%	4.09%

CHAPTER 20

Calculating Parameters of the TSIR

The components of the term structure of interest rates can be calculated by finding the set of parameters that would best approximate the market price of treasuries. The most widely used method for such an exercise is least squares error fitting. We can use one of the following two optimization functions to calculate the components of the TSIR:

$$Z = \sum_{i}^{N} w_i \; p_{m,i} \left(1 - \frac{p_{t,i}}{p_{m,i}}\right)^2 \tag{20.1}$$

$$Z = \sum_{i}^{N} \frac{w_i \; p_{m,i}}{D_{m,i}^2} \left(1 - \frac{p_{t,i}}{p_{m,i}}\right)^2 \tag{20.2}$$

where w_i is the weight or outstanding face amount of a given issue, $p_{m,i}$ is the market price (plus accrued interest) of a bond, $p_{t,i}$ is the calculated price of a bond based on the TSIR, $D_{m,i}$ is the Macaulay duration (which can be replaced by level duration), and N is the number of bonds to be used for the term structure calculation.

By minimizing Z, the term structure parameters can be calculated. Using outstanding amounts as a weighting function for bonds has the advantage that small issues that are not liquid and tend to have pricing errors will have lower weights in the optimization process. Since some bonds can be on-special in the repo market or have bad prices, one should run the optimization process at least twice, screening out securities with a standard error of more than 4σ after the first pass. Callable bonds should not be included in the optimization.

By replacing the calculated price $p_{t,i}$ using (5.11), we can write (20.2) as

$$Z = \sum_{i} w_i p_{m,i} s_{b,i}^2 \tag{20.3}$$

where $s_{b,i}$ is the implied yield spread of the security relative to the TSIR. Equation (20.1) leads to the TSIR on the basis of price optimization for all treasuries weighted by the

The Advanced Fixed Income and Derivatives Management Guide. Saied Simozar

respective outstanding amount of each issue. Equation (20.2) uses yield optimization, which results in a relatively uniform yield fit throughout the curve. In the US treasury market, both models result in almost identical yield curves. In less developed markets, price optimization results in a more accurate yield calculation at long maturities and therefore provides a better valuation and risk assessment for a portfolio. Yield optimization results in a more accurate fit at the short end of the curve and therefore is better suitable for yield sensitive calculations such as short dated mortgages or options.

Another alternative is a compromise between price and yield optimization by replacing the square of duration with duration in (20.2):

$$Z = \sum_i \frac{w_i\, p_{m,i}}{D_{m,i}} \left(1 - \frac{p_{t,i}}{p_{m,i}}\right)^2 \tag{20.4}$$

This method improves the fitting for the short end and long end of the curve compared to price optimization and yield optimization respectively.

If price optimization is not used, it is a good exercise to require as a constraint that the sum of calculated and observed market values of all treasuries be identical, i.e.,

$$\sum_i w_i (p_{m,i} - p_{t,i}) = 0 \tag{20.5}$$

If the TSIR is fitted with (20.5) as a constraint for an index, it will price the index accurately at any given time and therefore the contribution of security selection to the performance of the index will be zero as long as the index does not change. This implies that the optimal TSIR coefficients are index dependent. For example, consider portfolios A and B, which are managed against treasury benchmarks with minimum maturities of one and five years respectively. If we apply (20.5) to benchmark A, then a small performance of benchmark B cannot be accounted for by the term structure durations and convexity contributions and has to be attributed to security selection.

For convenience, we use the same optimization form for (20.1), (20.2) and (20.4) as follows:

$$Z = \sum_i \mu_i w_i p_{m,i} \left(1 - \frac{p_{t,i}}{p_{m,i}}\right)^2 \tag{20.6}$$

with μ_i equal to 1, $1/D^2_{m,i}$ or $1\,/\,D_{m,i}$ for (20.1), (20.2) or (20.4), respectively. To incorporate (20.5) as an optimization constraint, we need to use the Lagrange multiplier. Differentiating (20.6) with respect to the coefficients of the TSIR yields

$$\sum_i \left(\mu_i \left(1 - \frac{p_{t,i}}{p_{m,i}}\right) - \lambda\right) w_i \frac{\partial pt_{t,i}}{\partial a_k} = 0 \qquad k = 0, 1, 2\ldots \tag{20.7}$$

where λ is the Lagrange multiplier, $\frac{\partial p_{t,i}}{\partial a_k}$ is the price derivative relative to the kth component of the TSIR of the ith security defined in (4.3) or (10.38), depending on whether we use the standard TSIR or the convexity adjusted TSIR (10.28). We can use (20.7) and (20.5) to simultaneously solve for the coefficients of the TSIR and λ.

For convexity adjusted TSIR, we will also need to include vega optimization defined in (10.40), that is,

$$\sum_i \left(\mu_i \left(1 - \frac{p_{t,i}}{p_{m,i}} \right) - \lambda \right) w_i \frac{\partial p_{t,i}}{\partial \sigma_y} = 0 \tag{20.8}$$

We will now show that the optimization of (20.4) will also nearly satisfy (20.5) for the universe of bonds from which TSIR is being calculated. We first take the derivative of (20.4) relative to the first component of the TSIR:

$$\sum_i \frac{w_i}{D_{m,i}} \left(1 - \frac{p_{t,i}}{p_{m,i}} \right) \frac{\partial p_{t,i}}{\partial a_k} = 0, \qquad k = 0 \tag{20.9}$$

It follows that

$$\frac{\partial p_{t,i}}{\partial a_0} = p_{t,i} D_{t,i} \approx p_{m,i} D_{m,i} \tag{20.10}$$

Substituting (20.10) into (20.9) leads to (20.5). This implies that constraint (20.5) will not be necessary and the Lagrange multiplier can be set to zero if the optimization is based on (20.4).

It is mathematically and computationally easier to calculate the coefficients of the TSIR using the polynomial expansion (see Chapter 3) and then transform the result into the Chebyshev representation.

20.1 OPTIMIZING TSIR

The objective of the optimization is to find the set of term structure parameters that would minimize (20.1), (20.2) or (20.4). We rewrite the optimization functions in a slightly different form for convenience as one that would maximize

$$Z = -\frac{1}{2} \sum_i^N w_i \; p_{m,i} \mu_i \left(1 - \frac{p_{t,i}}{p_{m,i}} \right)^2 \tag{20.11}$$

The optimization can be subject to the following constraint if we choose to match the market values:

$$\sum_i u_i \left(p_{m,i} - p_{t,i} \right) = 0 \tag{20.12}$$

The weight of securities u_i in constraint (20.12) can be different from w_i in (20.11). By including constraint (20.12), the general optimization equation can be changed to one that would maximize

$$Z = -\frac{1}{2} \sum_i w_i \; p_{m,i} \mu_i \left(1 - \frac{p_{t,i}}{p_{m,i}} \right)^2 + \lambda \sum_i u_i (p_{m,i} - p_{t,i}) \tag{20.13}$$

We assume a generalized equation for the volatility as

$$\sigma = \sum_{i=0}^{L-1} b_i \psi_{v,i} \tag{20.14}$$

where $\psi_{v,i}$ is the ith basis function (i=0, 1,..., L – 1) for the term structure of volatility and b_i are its coefficients. For notational convenience, we will define the variable g_k as

$$g_k = \begin{cases} a_k & k = 0,1,\cdots,n-1, \\ b_{k-n} & k = n, n+1,\cdots, L+n-1 \end{cases} \tag{20.15}$$

After differentiation, the optimization function leads to solving the following sets of simultaneous equations:

$$Z_k = \frac{\partial Z}{\partial g_k} = \sum_j \left[w_j \mu_j \left(1 - \frac{p_{t,j}}{p_{m,j}}\right) - \lambda u_j \right] \frac{\partial p_{t,j}}{\partial g_k} = 0 \tag{20.16}$$

$$Z_\lambda = \sum_j u_j (p_{m,j} - p_{t,j}) = 0 \tag{20.17}$$

With the inclusion of λ, there are $K + 1 = n + L + 1$ equations and as many unknowns. The equations can be solved by using Newton's method in multiple dimensions. Figure 20.1 shows how Newton's optimization method works in one dimension. At starting point A, we calculate the slope of the tangent line, which is equal to the derivative of the function, and calculate the intercept of the line with the horizontal axis, point B. The value of the function at point B is calculated (point C) and its tangent is calculated, until the intercept of the function with the horizontal axis is found.

Using Newton's fastest descent method, from trial starting values of $\{a_0, a_1, \ldots, a_{n-1}, b_0, \ldots, b_{L-1}, \lambda_0\}$ we can calculate the shifts in the trial values by solving the set of simultaneous equations as follows:

$$\sum_{l=0}^{n+L-1} \frac{\partial Z_k}{\partial g_l} \delta g_l + \frac{\partial Z_k}{\partial \lambda} \delta\lambda + Z_k = 0 \qquad k = 0,1,\cdots,K-1 \tag{20.18}$$

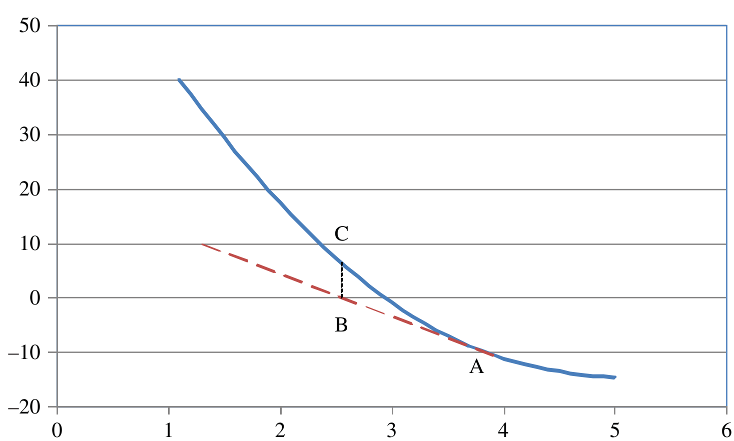


FIGURE 20.1 Newton's optimization method

$$\sum_{l=0}^{n+L-1} \frac{\partial Z_\lambda}{\partial g_l} \delta g_l + Z_\lambda = 0 \qquad k = 0,1,\ldots,K-1 \tag{20.19}$$

where δg_l is the shift in the trial value of the lth parameter and

$$\frac{\partial Z_k}{\partial g_l} = \frac{\partial Z_l}{\partial g_k} = \sum_j \left\{ \left[w_j \mu_j \left(1 - \frac{p_{t,j}}{p_{m,j}} \right) - \lambda u_j \right] \frac{\partial p_{t,j}^2}{\partial g_k \partial g_l} - \frac{w_j \mu_j}{p_{m,j}} \frac{\partial p_{t,j}}{\partial g_k} \frac{\partial p_{t,j}}{\partial g_l} \right\} \tag{20.20}$$

$$\frac{\partial Z_\lambda}{\partial g_l} = \frac{\partial Z_l}{\partial \lambda} = -\sum_j u_j \frac{\partial p_{t,j}}{\partial g_l} \tag{20.21}$$

The second order derivatives are given by

$$\frac{\partial p_t^2}{\partial a_k \partial a_l} = \sum_i \frac{c_i t_i^2 \psi_k \psi_l}{\left(1 + y_s v_y^2 t_i^2\right)^2} \left[1 + \frac{v_y^2 t_i}{1 + y_s v_y^2 t_i^2} \right] e^{-y_s t_i} \tag{20.22}$$

$$\frac{\partial p_t^2}{\partial a_k \partial b_l} = \sum_i \frac{c_i y_s v_y \; t_i^3 \psi_k \psi_{v,l}}{\left(1 + y_s v_y^2 t_i^2\right)^2} \left[\frac{1}{1 + y_s v_y^2 t_i^2} + 1 - y_s t_i \right] e^{-y_s t_{i_i}} \tag{20.23}$$

$$\frac{\partial p_t^2}{\partial b_k \partial b_l} = \sum_i \frac{c_i y_s^2 \; t_i^3 \psi_{v,k} \psi_{v,l}}{\left(1 + y_s v_y^2 t_i^2\right)^2} \left[\frac{-1}{1 + y_s v_y^2 t_i^2} + 2 + y_s^2 v_y^2 t_i^3 \right] e^{-y_s t_i} \tag{20.24}$$

The first order derivatives are

$$\frac{\partial p_t}{\partial a_k} = -\sum_i \frac{c_i t_i \psi_k}{1 + y_s v_y^2 t_i^2} e^{-y_s t_i} \tag{20.25}$$

$$\frac{\partial p_t}{\partial b_k} = \sum_i \frac{c_i y_s^2 v_y \; t_i^3 \psi_{v,k}}{1 + y_s v_y^2 t_i^2} e^{-y_s t_i} \tag{20.26}$$

The following is the list of equations that are needed to calculate the derivatives:

$$\frac{\partial p_t}{\partial x} = -\sum_i c_i t_i \frac{\partial y_s}{\partial x} e^{-y_s t_i} \tag{20.27}$$

$$\frac{\partial y_s}{\partial a_k} = \frac{\partial y_s}{\partial y_t} \frac{\partial y_t}{\partial a_k} \tag{20.28}$$

$$\frac{\partial y_s}{\partial y_t} = \frac{1}{1 + y_s v_y^2 t^2} \tag{20.29}$$

$$\frac{\partial y_t}{\partial a_k} = \psi_k \tag{20.30}$$

$$\frac{\partial y_s}{\partial a_k} = \frac{\psi_k}{1 + y_s v_y^2 t^2} \tag{20.31}$$

$$\frac{\partial y_s}{\partial b_k} = -\frac{{y_s}^2 v_y \, t^2 \psi_{v,k}}{1 + y_s v_y^2 t^2} \tag{20.32}$$

Since the contribution of yield volatility v to the price function is second order, its first order derivative is zero at $v = 0$. This leads to a trivial solution of (20.16) for $v = 0$

which is not optimal. Therefore Newton's fastest descent method would converge to accurate yield curve parameters only if a reasonably close estimate of the volatility can be made. Alternatively, one can use optimization techniques that operate on the function such as the simplex method to calculate the yield curve parameters.

The simplex method for function optimization is different from the simplex method for linear programming. It is based on bracketing the minimum values of the function by varying each variable and finding the minimum by iteration. It is not as efficient as Newton's method, but it is usually much more stable. For a review of different methods of finding optimum values of functions, see Press *et al.* [15].

20.2 OPTIMIZING TSCR

To calculate the term structure of spreads for credit securities, we use (20.18) and (20.19) as the starting point for the optimization equation. Assuming no cash flow guarantee, the price of a credit security with recovery can be calculated from (13.17) as

$$p_t = \sum_i \left(r_i c_{f,c,i} + c_i\right) e^{-(s_{s,i}+y_{s,i})t_i} \tag{20.33}$$

We redefine the recovery adjusted effective cash flow as

$$c_{e,i} = r_i c_{f,c,i} + c_i \tag{20.34}$$

The spread adjusted discount function can thus be written as

$$d_{s,i} = e^{-(s_{s,i}+y_{s,i})t_i} \tag{20.35}$$

Assume

$$s_t = \sum_{i=0}^{n-1} a_{c,i}\psi_i \tag{20.36}$$

$$\sigma_c = \sum_{i=0}^{L-1} b_i\psi_i \tag{20.37}$$

$$x_{s,i} = \frac{1}{1 + s_s v_s^2 t_i^2} \tag{20.38}$$

where v_s is the spread volatility. We note that for a unit principal outstanding, the implied forward spread coupon from (13.13) is given by

$$c_{f,c,i} = (e^{s_{s,i}t_i - s_{s,i-1}t_{i-1}} - \tag{20.39}$$

Equations (20.22)–(20.24) for credit securities take the following forms:

$$\begin{aligned}\frac{\partial p_{t,c}^2}{\partial a_k \partial a_l} &= \sum_i c_{e,i} t_i^2 x_{s,i}^2 (1 + v_s^2 t_i x_{s,i})\psi_k \psi_l d_{s,i} \\ &\quad - \sum_i r_i \left[t_i x_{s,i} \left(\frac{\partial c_f}{\partial a_l}\psi_k + \frac{\partial c_f}{\partial a_k}\psi_l \right) - \frac{\partial^2 c_f}{\partial a_k \partial a_l} \right] d_{s,i}\end{aligned} \tag{20.40}$$

$$\begin{aligned}\frac{\partial p_{t,c}^2}{\partial a_k \partial b_l} &= \sum_i c_{e,i} s_s v_s t_i^3 x_{s,i}^2 \psi_k \psi_l (1 + x_{si} - s_s t_i) d_{s,i} \\ &\quad + \sum_i r_i \left[t_i x_{s,i} \left(s_s^2 v_s t_i^2 \psi_l \frac{\partial c_f}{\partial a_k} - \psi_k \frac{\partial c_f}{\partial b_l} \right) + \frac{\partial^2 c_f}{\partial a_k \partial b_l} \right] d_{s,i}\end{aligned} \tag{20.41}$$

$$\begin{aligned}\frac{\partial p_{t,c}^2}{\partial b_k \partial b_l} &= \sum_i c_{e,i} s_s^2 t_i^3 x_{s,i}^2 (2 - x_{s,i} + s_s^2 v_s^2 t_i^3) \psi_k \psi_l d_{s,i} \\ &\quad + \sum_i r_i \left[s_s^2 v_s t_i^3 x_{s,i} \left(\frac{\partial c_f}{\partial b_l} \psi_k + \frac{\partial c_f}{\partial b_k} \psi_l \right) + \frac{\partial^2 c_f}{\partial b_k \partial b_l} \right] d_{s,i}\end{aligned} \tag{20.42}$$

Equations (20.40)–(20.42) are the primary equations that are needed to calculate the term structure of credit rates. The following are all the equations that are necessary for the calculation of the TSCR:

$$\frac{\partial c_f}{\partial a_{c,k}} = \left[\frac{t_i \psi_{k,i}}{1 + s_{s,i} v_{s,i}^2 t_i^2} - \frac{t_{i-1} \psi_{k,i-1}}{1 + s_{s,i-1} v_{s,i-1}^2 t_{i-1}^2} \right] (1 + c_f) \tag{20.43}$$

$$\begin{aligned}\frac{\partial c_f}{\partial b_k} &= \left[t_i \frac{\partial s_{s,i}}{\partial b_k} - t_{i-1} \frac{\partial s_{s,i-1}}{\partial b_k} \right] \left(1 + c_{fi} \right) \\ &= \left(-s_{s,i}^2 v_{s,i} t_i^3 x_{s,i} \psi_{ki} + s_{s,i-1}^2 v_{s,i-1} t_{i-1}^3 x_{s,i-1} \psi_{k,i-1} \right) \left(1 + c_{f,i} \right)\end{aligned} \tag{20.44}$$

$$\begin{aligned}\frac{\partial^2 c_f}{\partial a_k \partial a_l} &= \left(-v_{s,i}^2 t_i^3 x_{s,i}^3 \psi_{ki} \psi_{li} + v_{s,i-1}^2 t_{i-1}^3 x_{s,i-1}^3 \psi_{k,i-1} \psi_{l,i-1} \right) \left(1 + c_{f,i} \right) \\ &\quad + \frac{\partial c_f}{\partial a_k} \times \frac{\partial c_f}{\partial a_l} \times \frac{1}{1 + c_{f,i}}\end{aligned} \tag{20.45}$$

$$\begin{aligned}\frac{\partial^2 c_f}{\partial a_k \partial b_l} &= \left(-s_{s,i} v_{s,i} t_i^3 x_{s,i}^2 (1 + x_{s,i}) \psi_{ki} \psi_{li} + RPT_{i-1} \right) \left(1 + c_{f,i} \right) \\ &\quad + \frac{\partial c_f}{\partial a_k} (-s_{s,i}^2 v_{s,i} t_i^3 x_{s,i} \psi_{li} + RPT_{i-1})\end{aligned} \tag{20.46}$$

$$\begin{aligned}\frac{\partial^2 c_f}{\partial b_k \partial b_l} &= \left[s_{s,i}^2 t_i^3 x_{s,i} (2 - 2x_{s,i} - x_{s,i}^2) \psi_{k,i} \psi_{l,i} - RPT_{i-1} \right] \left(1 + c_{f,i} \right) \\ &\quad + \frac{\partial c_f}{\partial b_k} \times \frac{\partial c_f}{\partial b_l} \times \frac{1}{1 + c_{f,i}}\end{aligned} \tag{20.47}$$

(where RPT_{i-1} means calculating the previous term at time t_{i-1}),

$$\frac{\partial s_{s,i}}{\partial a_k} = \frac{\psi_{k,i}}{1 + s_s v_{s,i}^2 t_i^2} = x_{s,i} \psi_{k,i} \tag{20.48}$$

$$\frac{\partial s_{s,i}}{\partial b_k} = -s_{s,i}^2 v_{s,i} t_i^2 x_{s,i} \psi_{k,i} \tag{20.49}$$

$$\frac{\partial x_{s,i}}{\partial a_k} = -v_{s,i}^2 t_i^2 x_{s,i}^3 \psi_{k,i} \tag{20.50}$$

$$\frac{\partial x_{si}}{\partial b_k} = -s_{s,i} v_{s,i} t_i^2 x_{s,i}^2 (1 + x_{s,i}) \psi_{ki} \tag{20.51}$$

$$\frac{\partial p_{t,c}}{\partial a_k} = -\sum_i \left(c_{e,i} t_i x_{s,i} \psi_k - r_i \frac{\partial c_{f,i}}{\partial a_k} \right) d_{s,i} \tag{20.52}$$

$$\frac{\partial p_{t,c}}{\partial b_k} = \sum_i \left(c_{e,i} s_{s,i}^2 v_{s,i} t_i^3 x_{s,i} \psi_k + r_i \frac{\partial c_{f,i}}{\partial b_k} \right) d_{s,i} \tag{20.53}$$

$$\frac{\partial s_{s,i}}{\partial a_k} = \frac{\partial s_{s,i}}{\partial s_{t,i}} \frac{\partial s_{t,i}}{\partial a_k} \tag{20.54}$$

$$\frac{\partial s_{s,i}}{\partial s_{t,i}} = \frac{1}{1 + s_s v_{s,i}^2 t_i^2} \tag{20.55}$$

$$\frac{\partial s_t}{\partial a_k} = \psi_k \tag{20.56}$$

$$\frac{\partial s_t}{\partial b_k} = \psi_k \tag{20.57}$$

In our formulation, we implied that the contribution of convexity to credit yield can be modeled by the term structure of volatility as in (20.37). In practice, at most one component of volatility can be estimated from market data. For US treasury rates the volatility can be calculated relatively accurately due to the efficiency of the market. However, for credit securities, there is not enough data or there is too much noise in prices to use the data for estimating volatility.

20.3 OPTIMIZING TSCR WITH NO CONVEXITY

Even though the contribution of volatility and convexity is more important for most credit securities than treasuries, at the present time the market is not efficient enough to measure the contribution of volatility to the TSCR. Assuming that the contribution of volatility is zero, our equations simplify significantly as follows:

$$\frac{\partial p_{t,c}^2}{\partial a_k \partial a_l} = \sum_i c_e t_i^2 \psi_k \psi_l d_{s,i} - \sum_i r_i \left[t_i \left(\frac{\partial c_f}{\partial a_l} \psi_k + \frac{\partial c_f}{\partial a_k} \psi_l \right) - \frac{\partial^2 c_f}{\partial a_k \partial a_l} \right] d_{s,i} \tag{20.58}$$

$$\frac{\partial^2 p_{t,c}}{\partial a_k \partial b_l} = \frac{\partial^2 p_{t,c}}{\partial b_k \partial b_l} = 0 \tag{20.59}$$

$$\frac{\partial c_f}{\partial a_k} = \left[t_i \psi_{k,i} - t_{i-1} \psi_{k,i-1} \right] (1 + c_f) \tag{20.60}$$

$$\frac{\partial^2 c_f}{\partial a_k \partial a_l} = \frac{\partial c_f}{\partial a_k} \times \frac{\partial c_f}{\partial a_l} \times \frac{1}{1 + c_{f,i}} \tag{20.61}$$

20.4 ESTIMATING RECOVERY VALUE

The formulation for the calculation of the parameters of the TSCR was derived for securities with recovery value in Section 20.2. We can use the coefficient of the recovery value as one of the adjustable parameters that can be optimized to derive the TSCR and market implied recovery value. For example, to calculate a TSCR with five components where the fourth and fifth components are matched with the TSIR, we will be optimizing the TSCR on only three components. We can include the recovery value as an additional component to calculate.

The recovery value can only lie in the range (0%, 100%), and this can cause potential problems with the optimization of the data. Since the optimal solution is found by Newton's fastest descent or simplex algorithms, the trial intermediate solutions can have a recovery value that is negative or larger than 100%, leading to instability in the solution. Such instabilities can easily be overcome if smaller steps can be taken. If the data quality is very bad or the maturity range of data is very tight, unstable components of the TSCR or recovery value will be calculated that can be significantly different from one day to the next.

It is advisable to estimate the market implied recovery value by trial and error in the range (0%, 100%). We first assume a zero recovery value and calculate the TSCR, we then try a 10% recovery value and see if the yield error of the TSCR is lower, then we try 20%, … until we can bracket the optimum recovery value and then use the simplex method to estimate the optimum recovery value.

While this method is computationally more expensive than outright optimization of the recovery value, it provides a much more reliable and stable result.

20.5 ROBUSTNESS OF THE TERM STRUCTURE COMPONENTS

The stability and robustness of term structure components require many bonds with diverse maturities and efficient market prices. The transformation of time to τ as in (2.16) using an exponentially decaying time function results in a range of values in [–1, 1]. For nearly all applications five components of the TSIR are enough for valuation and risk management.

In order to fit five components of the TSIR or TSCR, there need to be at least five bonds at different maturities. However, having just five bonds will result in a fit that reproduces the price of all those bonds exactly no matter how bad the prices are. If an issuer has issued eight bonds in the last 2 years with original maturities of 10 years, all bonds will fall in a relatively tight range in the τ space that we use. Since they are all tightly bunched together, it is difficult to calculate five parameters from these eight bonds and have confidence in them. In fact, even the slope of the TSIR cannot be reliably calculated from such data.

Recall from equation 2.21 that the Chebyshev basis functions have a sinusoidal function of the form

$$\psi_n = \cos(n \arccos(\tau))$$

Using five parameters for the TSIR requires $n = 4$ which is a polynomial of degree 4 in τ space and intersects the τ axis at four points, dividing the range interval [–1,1] into five segments. A logical choice, in order to use five components, would be to require bonds with maturities in all five segments. While this approach is reasonable, it is not very practical since the shortest time intersection with the τ axis would be at about 0.30 years or 3.5 months. Most governments do not issue bonds with such short maturities. In the US there are treasury bills which are very liquid, but for most other countries such securities are relatively illiquid or not available. Older bonds that have short maturities tend to be very illiquid as well.

It is probably more practical to use a linear τ space and require minimum ranges for a given number of term structure parameters. For most countries that have a primitive swap or government curve, the range of data is typically 2–5 year maturities. Assuming a decay coefficient of 0.13, the difference between τ for the maturities of five and two year swaps will be 0.498=-0.044+0.542 and level and slope would be sufficient to calculate such curves. Thus, for practical purposes, a range of 0.45 in τ should be the minimum for using two components. Likewise for three, four, and five components 0.85, 1.15, and 1.35 respectively, are reasonable ranges for data to calculate robust and stable term structure components.

The range of τ for maturities of 2–20 years is (–0.54, 0.85) and this should be the minimum range for calculating five components. We can divide the range of data into maturity buckets in τ space and require the presence of data in most maturity buckets for calculating five parameters. For example, we can divide the $[-1, 1]$ range for τ into eight equally spaced maturity buckets and require that there be at least one bond in five non-adjacent buckets. Such a criterion works better than performing a uniformity test on the data. If there are 20 bonds in one bucket and most other buckets have one or two bonds, the calculated TSIR would be robust even though the data are not uniform.

In general, for bend calculation, there should be bonds with maturities near 2, 5, and 8 years or more. For most corporate or credit bonds, there is no need to calculate the fourth and fifth components and they should be set equal to the treasury rates if available.

20.6 CALCULATING THE COMPONENTS OF THE TSYV

The term structure of yield volatility can be calculated by minimizing the difference between the calculated and the market volatility price function (9.16). We rewrite (9.16) as

$$p_{m,j} D_j y_{x,j} v_{m,j} = \sum_i c_{j,i} y_{j,i} v_{j,i} t_i e^{-y_i t_i} \tag{20.62}$$

where $v_{m,j}$ is the market volatility of security j, $v_{j,i}$ is the volatility of cash flow at time t_i of security j, and $y_{x,j}$ is the continuous compounded (exponential) market yield of security j.

We write the optimization function so as to maximize the equation

$$Z = -\frac{1}{2} \sum_j w_j \left(\sum_i c_{i,j} y_{i,j} v_{i,j} t_{i,j} e^{-y_i t_i} - p_{m,j} y_{x,j} v_{m,j} D_j \right)^2 \tag{20.63}$$

where w_j is the weight of each bond in the optimization. We can write (9.19) as

$$u = y(t_x, t_f)v(t_x, t_f) = \sum_i a_i \psi_i \tag{20.64}$$

with

$$q_j = p_{m,j} y_{x,j} v_{m,j} D_j \tag{20.65}$$

$$d_i = c_i t_i e^{-y_i t_i} \tag{20.66}$$

$$\frac{\partial Z}{\partial a_k} = 0 = -\sum_j w_j \left[\left(\sum_i d_i u_i - q_j \right) \left(\sum_i d_i \frac{\partial u_i}{\partial a_k} \right) \right] \tag{20.67}$$

$$\frac{\partial u_{ij}}{\partial a_k} = \psi_k(i, j) \tag{20.68}$$

The coefficients of a_k can be solved using the set of simultaneous equations represented by (20.67). w_j can be equal to $1, \left(1/p_{m,j}D_j\right)^2$ or $1/p_{m,j}D_j$ to optimize for absolute volatility, price weighted volatility or a compromise between the two extremes.

CHAPTER 21

Implementation

Implementation of the methodologies in this book requires the development of modules that can handle all aspects of the infrastructure for an automated daily analysis and reporting. Calculation of the term structure of interest rates requires a module for generation of cash flows. However, cash flow generation of real rates or credit default swaps requires the term structure of nominal rates and Libor rates. These two projects will be heavily interrelated and must work together. In this chapter we discuss the development of modules that are necessary for the implementation of our methodology.

21.1 TERM STRUCTURE

The term structure module must be very flexible to handle all exceptions and calculate robust term structure of rates and credits. Using polynomial basis functions results in faster computation times and the result can then be converted to Chebyshev polynomials. For most developed countries, government bonds with varying maturities can be used as input for cash flow generation. Many emerging countries may not have an efficient or established government curve. In some cases better data are available for Libor (interest rate swaps) than for government debt and it can be used as the primary curve.

We first need to develop the primary curve of any country and use it as the basis for the development of other curves.

21.1.1 Primary Curve

To develop the primary nominal curve we need to get the list of cash flows along with the time to cash flow of non-callable government bonds or interest rate swaps as input. For each bond, the invoice price, number of shares, and duration should also be provided for weighting each bond in the optimization function. Refer to equation (20.3) and the relevant explanation for more details.

The first step in calculating the term structure components is to identify the maximum number of parameters, up to five, that can be used. As we discussed in Section 20.5, a broad distribution of maturities is required to calculate up to five parameters.

One of the keys to finding the optimal term structure, especially if the data are not very good, is to have a good starting point for the components. This can be achieved by calculating only the level of rates which should be close to the average yield of the input bonds. Then two components can be calculated, using the calculated level for the first component and initializing the second component at zero. This process can be continued until all the required components are calculated.

Once the term structure components are calculated, we need to calculate the adjustment table. First, the bonds must be sorted by maturity. If there are fewer bonds than the size of the adjustment table, the size of the adjustment table will be equal to the number of bonds. If there are more bonds than size of the adjustment table, we can divide the adjustment table into maturity buckets and choose one bond for each bucket beginning with the lowest maturity bucket, until the remaining number of bonds and buckets are equal, at which point all remaining bonds will have their own buckets.

The adjustment table can be calculated by starting with the lowest maturity bucket. The spread of the bond in this bucket and all prior cash flows will be equal to the spread that will match the market price of the bond. For the second bucket, we adjust the yield of all the cash flows that have a maturity less than the first maturity bucket in such a way that the interpolated adjustments to the yields of cash flows between the two buckets will reproduce the price of the second bond exactly. The process will continue until the adjustments for all buckets are calculated.

Generally, an adjustment table with 24 bonds provides enough accuracy for nearly all applications. Using the adjustment table, nearly all swaps can be calculated exactly.

For countries where there is a liquid treasury market, up to five components can be calculated for the treasury term structure of rates. For Libor, three components are sufficient and the last two components can be matched with the treasury components. For countries where the swap curve is more liquid, the opposite can be done.

At times of stress where liquidity premium is very high and the spread between on-the-run and off-the-run is very high, as in Figure 2.10, it is best to use off-the-run or coupon Strips for calculating the curve. In general, on-the-run bonds have a premium price due to liquidity and are not necessarily representative of the market.

Calculating the implied volatility of the yield curve can be very difficult using Newton's steepest convergence, since volatility is second order and its derivation at zero is zero. Using the triangular simplex method of optimization (not to be confused with linear optimization) usually works better for calculating implied volatility. In general, if the yields of long maturity bonds are not lower than the yields of intermediate to long bonds, the calculated implied volatility is zero.

21.1.2 Real Curve

The calculation of the term structure of real rates is very similar to the calculation of the TSIR. However, we must first calculate the TSIR to be able to strip the nominal portion of the cash flows of real bonds. Using (11.16), the real portion of a cash flow can be calculated as

$$c_r = c_n e^{-yt+y_n t_n} \tag{21.1}$$

where the subscript n refers to the inflation reference time for the cash flow at time t. The inflation lag is equal to $t - t_n$. For example, if the real coupon of a bond is 3% and the inflation lag is 2 months and forward nominal yield is 4%, the calculated real cash flow will be

$$c_r = 3e^{-4\%\times\frac{2}{12}} = 2.993$$

Next we need to calculate the nominal cash flows and accruals and subtract them from the market price of the bond. Consider an inflation report on March 15 and a real bond that pays coupon on April 15 with 2 months of inflation lag. On March 16 we know the nominal value of the coupon that will be received on April 15 and in fact we can calculate the market price of the bond up to the last day of April since the inflation factors are known through the end of April. If the factor for April coupon payment is 1.6 and the real coupon rate is 3%, the semi-annual coupon amount will be 2.4 per 100 of face value of the original bond. We have to subtract the present value of this coupon payment from the invoice purchase price to calculate the real purchase price.

We also know that inflation for the first 16 days of March has already happened but not reported, and we need to estimate it. This can be done by using the average of historical inflation values for the last 5 years for the month of March to estimate the resulting factor for the first 16 days of March and multiplying all cash flows by this factor.

Next, the cash flows need to be seasonally adjusted beginning with the next month based on day of the month reference. For example, if there is a coupon payment on July 15, and inflation has been reported through the end of February, then inflation for March, April, and May will impact the coupon on July 15. If seasonal factors for March, April, and May are 1.01, 1.005, and 0.99 respectively, then the contribution of seasonality for July 15 coupon payment will be

$$1.01\times1.005\times(0.99\times\tfrac{15}{31}+1\times\tfrac{16}{31}) = 1.010138$$

After this process, the cash flows have all been converted to real seasonally adjusted cash flows and the term structure of real rates can be calculated similarly to nominal rates.

21.1.3 Credit Curve and Recovery Value

Calculation of the credit curve is similar to the calculation of the primary nominal curve. Pricing of credit securities is much less transparent and inefficient than treasuries and swaps. However, with the increasing popularity of CDS, the spreads of many credit issuers are traded and can be used as additional data points to calculate the credit curves. In general, two or three parameters will offer a significant advantage compared to the simple spread calculation. The fourth and fifth components should be set equal to the components of the primary curve.

Even for issuers that have only one bond, a TSCS can be calculated by using identical components for the remaining four components and finding the parallel spread curve. Floating rate securities can also be included, as input for the TSCS, since the floating component will be based on Libor or treasury.

Calculation of the recovery value is somewhat more complicated since it cannot be done precisely. The recovery can be estimated by finding the credit curve that has the lowest fitting error and is therefore very sensitive to bond prices. Before finding the implied recovery value, the credit yield has to be higher than the Libor rate at all maturities, if discounting the recovery value by Libor. Since recovery is limited to (0, 100) range, using the fastest descent method may create instability in the calculated curve. The implied recovery value can be calculated after calculating the credit curve by trial and error. First, the recovery value can be estimated and the credit curve is recalculated. Using interpolation, ensuring that the recovery value is always in the range of (0, 100), the optimal implied recovery value can be calculated.

Since market practitioners usually use spreads based on a benchmark security to price other corporate securities, the relative pricing of all securities of an issuer remains unchanged to a large extent and the estimated recovery value does not change from one day to the next.

An adjustment table is not necessary or useful for credit curves since in most cases there are not that many bonds and the pricing may not be efficient. The spread of each security relative to its term structure is a measure of value for that security.

21.2 DISCOUNT FUNCTION AND RISK MEASUREMENT

In general, the discount function can be written as

$$d_i = e^{(\pm y_i \pm r_i \pm l_i \pm c_i)t_i} \tag{21.2}$$

where d_i is the discount function, y_i is the risk-free rate, r_i is the risk-free real rate, $l_i = y_i + l_{s,i}$ is the Libor or swap rate ($l_{s,i}$ being the Libor spread), and $c_i = y_i + c_{s,i}$ is the credit rate ($c_{s,i}$ being the credit spread over treasury). Theoretically, any combination of the above yields can be used for a discount function. However, there are only six practical discount yields, as shown in Table 21.1.

For implementation, each cash flow can be assigned a flag that encodes how it must be discounted. The flag can be used with bit patterns to identify the yield curve as well as the sign of the discount yield. For example, we can use 0x1 (hexagonal base number) for treasury yield, 0x2 for real, 0x4 for Libor, and 0x8 for credit. Likewise for the sign we use 0x10 for negative treasury, 0x20 for negative real, etc. Thus, 0x1B (0x01 + 0x02 + 0x08 + 0x10) is the flag for discounting real credit rates.

The discount function of cash flows that have a real yield dependency must be decomposed into real and nominal components. If the function `get_real_time`

TABLE 21.1 Practical discount yields

y_i	Risk-free rate
r_i	Risk-free real rate
l_i	Libor or swap rate
c_i	Credit rate
$l_i - y_i + r_i$	Real Libor yield
$c_i - y_i + r_i$	Real credit yield

returns the real time to a cash flow, then the following code can be used for the discount function of real bonds:

```
if((ti_r = get_real_time( ti, flag)) < ti ) then
   di_r = get_discount( ti_r, flag_r )
                            flag_r is 0x22 logical ANDED with flag
   di_n = get_discount( ti, flag_n )
                            flag_n is 0xdd logical ANDED with flag
   di_nr = get_discount( ti_r, flag_n )
   di = di_r * di_n / di_nr
else
   di = get_discount( ti, flag )
```

The risk parameters can also be calculated in a similar fashion:

```
If((ti_r = get_real_time( ti, flag)) < ti ) then
   dur_k_r[ k ] = di * ti_r * ψ(ti_r, k)
                             contribution to the kth real duration
   dur_k_n[ k ] = di *[ ti *ψ(ti, k)- ti_r *ψ(ti_r, k)]
                                          the kth nominal duration
```

The risks of floating rate coupons are also measured similar to the risks of credit securities with no real component in the float as in (12.27) and (12.29). If the floating coupon is based on the inflation or real rate, then we have to adjust for the inflation lag. This is the case for the floating leg of inflation swaps which are traditionally zero coupon instruments. Inflation swaps do not need to be zero coupon and we can calculate their risk using floating inflation. Unlike other floating rate bonds where the float is fixed just before the start of the coupon accrual period, an inflation swap coupon is truly floating and is not known until the inflation lag before the coupon payment. The risks of inflation swaps can be found in formulas (11.34)–(11.38).

21.3 CASH FLOW ENGINE

Most bonds have a fixed coupon rate and are very simple to analyze. However, many corporations, especially high yield issuers or emerging countries, have issued bonds with varying degrees of complexity of cash flows to satisfy their own capital requirements as well as investors' demands for higher yields. For example, a company that is expected to have a high growth rate may prefer to pay a lower coupon at the beginning and to pay a higher coupon as its business matures. It may issue a 10-year bond with a coupon rate of 3% for the first 3 years, then 5% for the following 3 years, followed by 7% for its remaining life. The cash flow structure of such bonds and more complex structures need to be stored in a database and made available to the module analyzing the instrument.

Sometimes a company issues a bond that pays in kind (PIK), that is, instead of paying a coupon rate, the implied coupon rate is added to the principal of the bond, like a

TABLE 21.2 Practical floating discount benchmarks

y_i	Risk-free rate, TSY
r_i	Risk-free real rate, RTS
l_i	Libor or swap rate, LBR
$y_i - r_i$	Implied treasury based inflation rate, ITS
$l_i - y_i + r_i$	Real Libor yield, RLB
$l_i - r_i$	Implied Libor based inflation rate, ILB

zero coupon bond. However, most bonds that pay in kind do so for a limited time and revert to coupon payment thereafter. A company or country can pay part of a coupon and PIK the balance. Such bonds are called capitalizing bonds.

A floating coupon can be based on one of the risk-free or quasi-risk-free benchmark rates listed in Table 21.2.

Corporate or emerging markets' floating rate bonds are usually issued with a spread over a benchmark to compensate for the additional risks of the issuer.

Table 21.3 lists different types of cash flows.

TABLE 21.3 Types of cash flow

Coupon	A fixed interest amount payment
Principal	The final payment of the principal
Float	A floating coupon based on a benchmark
Capitalized	A coupon or part of it that is added to the principal
PIK	Similar to capitalization. Often PIKs are optional
Sink	Part of a principal that is paid before maturity.
Cap-Float	A fixed amount of capitalization based on a floating coupon. The balance is paid as regular coupon
Float-Cap	A floating amount of capitalization after a fixed coupon based on a floating benchmark
PIK-Float	A fixed amount of PIK based on a floating coupon. The balance is regular coupon
Float-PIK	A floating amount of PIK after a fixed coupon based on a floating benchmark
RIG	Rolling interest guarantee provided as insurance if the regular coupon is not paid
Prin-Guarantee	Guaranteed part of a principal if it is not paid
Recovery	The recovery amount if there is default
Previous	The coupon is equal to the previous coupon. The previous coupon can be a floating rate coupon

Each associated cash flow can have its own discount rate. A principal guarantee may have to be discounted by treasury or Libor rate depending on the quality of the guarantee. Thus, for each cash flow type we need to know how to generate it and how to discount it, independently of each other. For example, a corporate bond that is based on 6-month floating Libor with a spread of 3% will have to be discounted by credit curve, while the coupon is generated by Libor curve.

There are bonds in the market whose coupons are fixed for 5 years and reset to the prevailing 5-year treasury rate plus a spread after 5 years. In order to analyze such bonds, we need to calculate the forward treasury rate and add the spread to calculate the coupon rate. The calculated coupon will be used for another 5 years and that is where the cash flow type "Previous" in Table 21.3 refers to.

The role of the cash flow engine (CFE) is to generate the forward cash flows based on the three primary curves of treasury, Libor, and real rates. For most bonds, the CFE needs to be used only once; however, for complex securities, such as a bond whose coupon is based on the forward 5-year treasury rate, it may be called multiple times to calculate the risks. For example, a company may issue a floating rate bond with semi-annual coupon that is equal to 6-month Libor plus a spread of 150 bps. The CFE uses the Libor curve to calculate the forward 6-month Libor at semi-annual intervals and adds 1.5% to calculate the forward coupon of the security. We assume that treasury curve is the fundamental driver of rates and Libor is a follower with a variable spread. In this example, if 6-month Libor is at 4% four years from now and the treasury rate for the same period is at 3.6%, then the Libor spread is 0.4%. The fixed component of the coupon of the security will thus be 1.5 + 0.4 = 1.9% and the remaining 3.6% will be floating treasury. If Libor spreads widen for such a security, the cash flows increase and the value of the security increases. Thus a floating rate bond has a negative Libor duration.

The CFE produces cash flows from the term structure of rates and using the adjustment table for Libor or treasury rates for accurate analysis. For example, for inflation swaps, the generated cash flows are based on implied forward inflation rates which are the differences between treasury rates and real rates lagged by 2 months. Once the cash flows are generated, they need to be scaled to the coupon period using market conventions. A corporate bond may pay quarterly coupons based on 6-month Libor rates. Corporate bonds use the 30/360 convention, but floating Libor uses the Actual/360 convention.

Many high yield companies issue bonds that can either be a fixed coupon or a PIK depending on their choice. Investors generally demand a higher rate for a PIK than for coupon payment, since if a company chooses to PIK the cash flow, it may be under financial pressure and hence a higher premium is required for the extra risks that investors bear. For example, a company may issue a bond with a coupon rate of 6% with the option to PIK at 8%. The CFE can often estimate the economical cash flow if the company has a few bonds outstanding and an estimate of the TSCS can be made. If the implied forward yield of the issuer is at 6% or below, the company can borrow at a rate that is below 6% and will likely pay the coupon. However, if the implied coupon is more than 8%, the company is likely to PIK the cash flow. At rates between 6% and 8%, it is not clear what the company will do, but it can be assumed that below 7% it pays cash coupon and above 7% it PIKs.

21.4 INVOICE PRICE

Most bonds in the market are traded as "clean price", implying that interest accrual is not added to the price. The invoice price of a bond is calculated by adding the accrued interest to the clean price of the bond. In some developing countries, bonds trade with an accrual known as "dirty price". Defaulted securities also trade with a flat price and there is no accrual for them.

Most bonds with sink and/or capitalization (PIK) as well as inflation linked securities trade with clean price and factor. Sink and capitalization or PIK are the opposites of each other, and their calculations are different. If $c_{c,i}$ and $c_{s,i}$ are respectively the capitalization and sink cash flows of a bond, the factor by which the principal of the bond changes is calculated as follows:

$$\begin{aligned} f_c &= \left(1+\frac{c_{c,1}}{100}\right)\times\left(1+\frac{c_{c,2}}{100}\right)\times\cdots = \prod_i\left(1+\frac{c_{c,i}}{100}\right) \\ f_s &= 1-\frac{c_{s,1}}{100}-\frac{c_{s,2}}{100}-\cdots = 1-\frac{1}{100}\sum_i c_{s,i} \end{aligned} \tag{21.3}$$

The outstanding principal of a bond with both sink and capitalization is the product of both factors.

The invoice price of a bond with capitalization is calculated differently from other bonds, and principal accrual needs to be calculated as well. If x is the accrual fraction period, for a bond with a coupon of c_i, of which c_c is capitalized, the capitalization accrual will be xc_c. However, if the bond is trading at a price of 60, the capitalization is not worth as much as coupon accrual and has to be scaled by the price of the bond. The invoice price of such a bond is calculated as

$$p_{\text{inv}} = f_s f_c\left(p + x(c_i - c_c) + xc_c\frac{p}{100}\right) \tag{21.4}$$

There have been bonds issued in the market that have step-up coupon and split accrual at the beginning and end of the period. For example, for a bond that pays semi-annual coupons with consecutive coupons of 4% and 6%, the cash flow at the time of 6% payment is the average of the two coupons and is equal to 5%. The invoice price of such a bond in the first half of the split period is accrued at the rate of 4% and in the second half at the rate of 6%.

Inflation linked securities in the US and Canada trade with clean factor. The inflation factor calculated based on the inflation reference point is multiplied by the clean price plus accrued to calculate the invoice price. Thus, the factor for a bond is the product of factors for sink, capitalization, and inflation. All cash flows have to be multiplied by the factor as well.

21.5 ANALYTICS

The first step in calculating risks and valuations is to calculate the spread of the security relative to its curve for non-callable securities. The spread is calculated by iteration, generally using Newton's method as outlined in Chapter 20.

The default probability of the security is then calculated from the difference between the credit and treasury curves. If there is zero recovery or guarantee for the security, risk measurements are straightforward and can be calculated analytically for fixed or simple floating coupons.

For securities with recovery or guarantee value, the interest rate, real, and Libor risks can be calculated analytically. Since the default probability is only a function of the spread of the security, the effective cash flows from default can be calculated from (13.15) and they will not change if interest rates or Libor rates change. We also need to keep track of the floating component of the cash flow to calculate its risks analytically using (13.26).

It is much more convenient to calculate the credit durations and convexities of a bond with recovery by shifting the credit curve, recalculating default probabilities and new prices. Using (13.27) and (13.28), the credit durations and analytics can be calculated from the shifted prices. This process requires the CFE to be called only once, since the effective cash flows can be readily calculated from the default probabilities.

The risks of most bonds can be calculated analytically; however, there are bonds in the market whose risks can be calculated only by shifting the curves, regenerating the cash flows, and calculating durations from changes in the prices.

Complex cash flow bonds such as those whose coupon depends on the forward 5-year treasury rate require additional effort to measure their risks. By shifting the treasury or Libor curve and generating cash flows and maintaining the spreads, one can calculate new prices and, from (13.27) and (13.28), calculate analytics.

The durations of capitalizing bonds based on floating coupons have to be calculated by shifting the curve and calculating the allocation to capitalization and payment for each level of interest rates. This requires multiple calls to the CFE to calculate the risks. If interest rates fall significantly, all the interest payment may have to be capitalized. The government of Ecuador had such a US dollar based bond that was the result of Brady restructuring where the interest rate was lower than the scheduled capitalization.

In general, we need to use one of the five methods to calculate the risks of any security, depending on its cash flow structure.

- *Simple* – The risks of fixed rate bonds, including step-ups, sinking, and capitalizing bonds whose cash flows are known in advance, can be calculated analytically.
- *Floaters* – Floating rate bonds whose coupon depend on the forward Libor or treasury rates, require an additional step in calculating their risks. We need to keep track of the prior coupon dates by using equation (12.25) for analytical calculation.
- *Recovery* – For measuring risks of bonds with recovery, but no guarantee, equation (13.26) can be used for analytical calculations. The interest rate, real, and Libor risks of credit securities do not depend on the credit curve or recovery and can be found analytically, unless the coupon is complex.
- *Guarantee* – The credit risks of bonds with RIG can be measured non-analytically by shifting the components of the credit curve and calculating its impact on the price using (13.27) and (13.28). If a bond has principal-only guarantee, its risk can be calculated analytically, similar to recovery rate. Other risks can be calculated analytically depending on their structure.
- *Complex cash flow* – Bonds whose cash flows depend on another bond, such as a bond whose coupon depends on the forward 5-year treasury, require a much

TABLE 21.4 Matrix of methods of risk calculation

Risk	Simple	Float	Recovery	Guarantee	Complex
Treasury	Treasury, corporate	Floating rate note	NA	NA	Complex credit
Real	Tips	Inflation swap	NA	NA	Complex credit
Libor	Fixed rate swap	Floating rate swap	NA	NA	Complex credit
Credit	Corporate	NA	Corporate	Brady bonds	NA

longer process for their risk measurement. By shifting the level of rates, the forward coupon of the underlying security is calculated and the cash flows of the bond are regenerated. The cash flows then need to be accrued based on the calendar convention. For example, if the accrual convention is Actual/360, each cash flow needs to be calculated on that basis and new prices can be calculated by maintaining the spread of the security relative to its curve. Equations (13.27) and (13.28) can then be used to calculate the risks. Only the risks of benchmark curves (treasury, real, and Libor) can be based on complex cash flows. Credit risk cannot be complex, unless a company issues a bond whose coupon depends on the coupon of another credit.

Table 21.4 shows examples of securities and methods of calculating their risks. Some of the methods are not applicable to some risks. For example, there is no special calculation for the treasury duration of a bond with a recovery value. Since default is only a function of spread, the treasury duration can be calculated as if the security had simple cash flows.

Complex credit securities are often issued by foreign corporations or quasi-sovereign foreign agencies denominated in a non-host country currency such as USD.

21.6 TRADE DATE VERSUS SETTLE DATE

It is customary to use the settle date for yield and invoice price calculation of cash bonds. The settle date convention varies by market; US treasuries are settled in one business day in the US, while international treasuries are settled in three business days. Swaps are usually settled in two business days and corporate bonds are settled in three business days.

The settlement date is usually stated as T+n, where n is the number of business days to be added to the trade date to calculate the settle date.

For the analysis of most securities the settle date should not be used, in particular if the settlement is for a forward date, even though the difference is usually very small. For example, if a buyer purchases a corporate bond for T+3 settlement that is over a long weekend (such as a Monday holiday), the actual settlement date will be six calendar days from the trade date. If the company files for bankruptcy before the settle date, the buyer is still exposed to its risk. Thus, the measurement of risk has to start at the trade date. The trade date is particularly important for short dated options. For example, an option that will expire 30 days from the trade date would expire in 25 days if it is based on settle date over a weekend.

The trade date/settle date convention has a small impact on duration measurements as well. For example, if a zero coupon bond of a credit security is purchased for settlement five calendar days from the trade date and maturity of 1 year from the trade date, then the credit duration of the security will be exactly 1 year. However, the interest rate duration of the security will be $\frac{365-5}{365} = 0.986$ years. To be more precise, there is a negative exposure to the repo rate due to the implied lending of the security between the trade date and settle date by the buyer.

21.7 AMERICAN OPTIONS

Analysis of American options requires the construction of a diffusion process that will propagate the yield of the forward bond through a normal process to the next layer. The accuracy of the calculation does not increase materially with an increase in the number of layers beyond 30–40 for a long maturity and exercise date bond. Thus, it can be assumed that at about 40 layers, the accuracy is comparable to a closed form solution which of course does not exist for bond options. Unlike binomial trees, in the methodology that was explained in Chapter 15, the spacing between layers need not be constant.

A very important consideration for building the layers is their spacing. For example, for a 10-year option, we can space the layers on a quarterly basis. Unlike binomial distributions, the spacing between layers need not be constant, and optimally it should not be. For deep out-of-the-money options the exercise is likely to take place in the latter half of the option, and for in-the-money options it is more likely in the first half if the carry is favorable. In high interest rate environments, the present value of an early exercise is much higher than that of a late exercise. In most cases, optimizing the number and spacing of layers can save significant computation time.

If we are only interested in the price of an option, we can use equally spaced layers for the calculation. However, in most cases we need to calculate the risks as well. To calculate the five treasury, three credit or Libor and eight volatility term structure risks plus convexities and Greeks, we need more than 40 price calculations. After a trial run of equally spaced layers, we can calculate the marginal contribution to the price of the option from each layer. We can then use about half as many layers, choosing them in such a way that the contribution from each layer, before expiration date, to the price of the option is about the same. For example, for an option whose contribution to price before expiration is \$2.00 and where 20 layers have been used, we create a new tree structure with 10 layers. Suppose the contribution to the price for the first four layers is 0.06, 0.08, 0.1, and 0.13; we choose the first layer of the new distribution at a time between the second and third layer at a distance that is 60% away from the second layer.

For bond options that have discrete exercise dates, the layers must be scheduled on the allowable exercise dates. In most cases, there is a record date for callable bonds before the call exercise. The calculation for discrete options needs to be on the basis of the record date for each exercise date.

Once the dates of layers are selected, the forward price, yield, and volatility of the bond are calculated for each layer. To calculate the distribution of all yields, we need to calculate the drift for each layer. Calculation of the drift can start by using the lognormal distribution of rates' drift (15.9), which is equal to $-v^2t/2$, and solve iteratively to calculate the drift so as to satisfy the arbitrage-free requirement. Since calculating the price requires a significantly higher computation time, only one set of prices needs to be calculated centered about the mean of the distribution, using the initial drift. The drift is then calculated by maintaining the prices and shifting the distribution, instead of maintaining the distribution and recalculating the prices.

To measure the Libor risks of an option, we need two copies of the TSLR stored in memory. One copy is used for shifting the components of the curve, and the other copy with unshifted components. Since the TSYV is based on nominal rates, we need to calculate the forward Libor yield to calculate the relative yield volatility at all forward dates. The forward Libor yield can be calculated using the unshifted copy of the TSLR.

Credit securities require calculation of the correlation parameters, depending on the correlation model used for their calculations. Even though in Section 15.9 we showed that, based on historical behavior, the pricing of credit markets is subject to arbitrage, for implementation we need to have arbitrage-free pricing, implying that at very low rates the correlation between treasury rates and credit spreads has to be positive.

Consider the yield distribution of a bond at a forward time A that propagates to another distribution at time B. The yield (price) distributions of the bond at times A and B can be written using (14.19) as

$$\begin{aligned} p_{f,A} &= \frac{1}{\sqrt{2\pi t v_A}} \int p_f(y) e^{-(\ln(y)-\ln(y_{0\,A})-\mu_A)^2/2v_A^2 t_A} d\ln(y) \\ p_{f,B} &= \frac{1}{\sqrt{2\pi t v_B}} \int p_f(y) e^{-(\ln(y)-\ln(y_{0\,B})-\mu_B)^2/2v_B^2 t_B} d\ln(y) \end{aligned} \tag{21.5}$$

where y_{0A} is the implied forward yield of the security at t_A. The implementation can be simplified by using the unit normal distribution function of the probability function as follows:

$$d\rho = \frac{1}{\sqrt{2\pi t v}} e^{-(\ln(y)-\ln(y_0)-\mu)^2/2v^2 t} d\ln(y) \equiv \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \tag{21.6}$$

Using an 81-point lattice (80 subintervals) that spans $\pm 6\sigma$, the yield at each point on the lattice is calculated as

$$y = y_0 e^{xv\sqrt{t}+\mu} = y_0 f_x, \quad x = 6\,\frac{i-40}{40}\bigg|_{i=0}^{80} \tag{21.7}$$

where f_x is the factor by which the forward yield is multiplied to calculate the yield at point i of the lattice. The distribution of rates and prices can be calculated for all the layers across the points of a normal distribution. Using 81 points for the normal distribution provides a very high level of accuracy for each layer. However, given the number of calculations and propagations that each point in the distribution goes through, it is not overkill. If we use 20 layers, we need to calculate the price of the underlying security at 81 × 20 = 1620 points for calculating the price of an option; the remainder of the calculations will be done by interpolation. If layers A and B are adjacent to each other, the volatility and drift parameters of diffusion from A to B are as follows:

$$\begin{aligned} v_{AB}^2(t_B - t_A) &= v_B^2 t_B - v_A^2 t_A \\ \mu_{AB} &= \mu_B - \mu_A + \ln\left(\frac{y_{0B}}{y_{0A}}\right) \end{aligned} \tag{21.8}$$

If the markets are inefficient or the calculation of the volatility surface is not accurate, there may be instances where the implied forward volatility is negative. If that happens, either layer A or B needs to be removed from the calculation.

The next step in calculating the price of an American option is to construct the exercise boundary. At the expiration date of an option or the last date that a callable bond can be exercised, if the option is in-the-money it will be exercised. We then analyze one layer before the last and calculate the present value of the option at all forward yields. At every yield, we create a diffusion process to the next layer using the volatility and drift from (21.8) and calculate the price of the underlying bond from its yield.

The forward price of the security is calculated by parabolic interpolation. The spacing between different points on the lattice may be too large for linear interpolation and the effect of convexity cannot be ignored for long dated options. If the forward yield falls between points 5 and 6 of the lattice, we can use points 5, 6, and 7 or 4, 5, and 6 for parabolic interpolation of the price of the underlying:

$$\begin{aligned} a &= \frac{1}{y_5 - y_7}\left(\frac{p_6 - p_5}{y_6 - y_5} - \frac{p_7 - p_6}{y_7 - y_6}\right) \\ b &= \frac{p_6 - p_5}{y_6 - y_5} - a(y_6 - y_5) \\ c &= p_5 - ap_5^2 - bp_5 \\ p &= ay^2 + by + c \end{aligned} \tag{21.9}$$

The weighted sum of the option price is calculated from the forward prices and is discounted by the appropriate discount function to the previous layer resulting in the unexercised price of the option. The discount function between layers is also a function of the forward yield. The same factor that scales the forward yield of the security in (21.7) is used to scale the yield of the discount function between layers. Thus, if $y_{f0,ji}$ is the forward discount yield of the jth layer at point i, then

$$y_{f,ji} = y_{f0,ji} f_x \tag{21.10}$$

By comparing the unexercised and exercised prices, we calculate the transition point or the exercise boundary at every layer. This process is performed for all the layers until the exercise boundary is calculated across the possible range of exercise times. On one side of the exercise boundary immediate exercise is economical, and on the other side the unexercised option is worth more.

Starting from the first exercise date, we build the probability distribution of all yields. We divide the yields into 80 buckets, such that bucket j in layer i is defined in the range $\left(j - \frac{1}{2}, j + \frac{1}{2}\right)$ and is denoted by (i, j). The buckets that are on the exercise side of the boundary are exercised and their net proceeds are discounted to the present time. Next, the unexercised buckets are propagated forward to the next layer where they are distributed based on their probability weights. Since the buckets of two consecutive layers are not perfectly aligned, the end point of each diffusion path is distributed to two buckets. Suppose that bucket j from layer i, with a weight of $w_{i,j}$, is propagated forward to the next layer to a point with a yield that corresponds to bucket k. The bucket index can be calculated from (21.7) as follows

$$\begin{aligned} x &= \frac{\ln(y/y_0) - \mu}{v\sqrt{t}} = 6\frac{k-40}{40} \\ k &= \frac{40x}{6} + 40 \end{aligned} \tag{21.11}$$

The probability weights of this transition are given by the interlayer volatility and drift (21.8) as follows:

$$u = \frac{\ln(y/y_0) - \mu_{AB}}{v_{AB}\sqrt{t_B - t_A}}, \quad \rho = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \tag{21.12}$$

If the landing point in the bucket (i+1, k) is higher than its half-way point, the weight is distributed between buckets k and k+1. Likewise, if the landing point is below the midpoint of the bucket, its impact is distributed between buckets k –1 and k. Figure 21.1 shows the propagation process between two layers.

For example, if $w_{i+1,k}$ is the weight of bucket k, then the contribution from bucket (i,j) to bucket k will be:

```
v1 = (40 * x) / 6 + 40.0 - 0.5
If( v1 > (double)k ) then
   w[i+1,k] = w[i+1,k] + ro * w[i,k] * (1. - v1 + k )
   w[i+1,k+1] = w[i+1,k+1] + ro * w[I,k] * (v1-k)
else
   w[i+1,k] = w[i+1,k] + ro * w[i,k] * (1. + v1 - k )
   w[i+1,k-1] = w[i+1,k-1] + ro * w[I,k] * (k-v1)
end
```

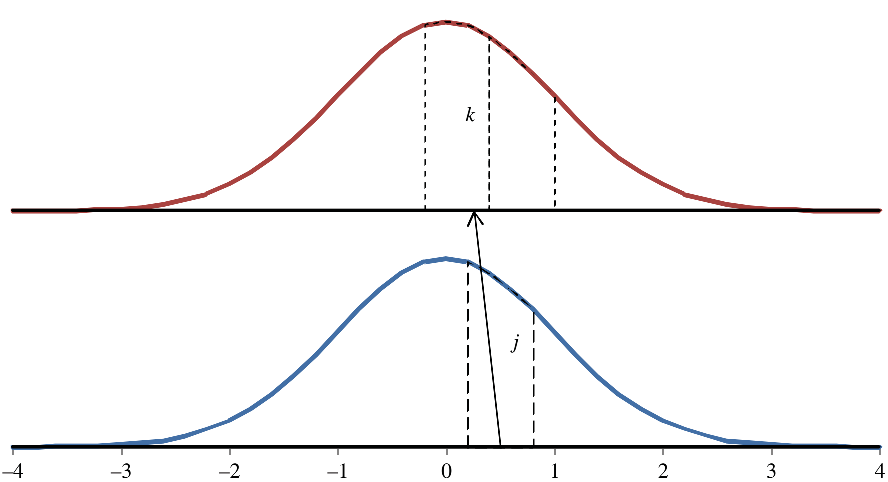


FIGURE 21.1 Propagation from bucket j to bucket k

This process is repeated for all buckets in all layers until the option expiration. The impact of the bucket that the exercise boundary runs through is very small, but it can be broken down into exercise and no-exercise portions. At every layer, the buckets that are in the exercise zone are discounted to the present time using the bucket's discount function and the sum of all contributions adds up to the option price.

The above analysis is for Libor based bonds only or for treasuries assuming that they have the same TSYV. For corporate bonds with correlation, we need to take additional steps into consideration to ensure that we match the specified correlation. For every level of Libor, the correlation and beta define the distribution of the credit rates. The distribution of the credit rates for a given level of Libor is given by (15.53). If we use a lattice of 81 points to evaluate the distribution of spread relative to Libor, the benefits of the additional computation time are very small. Using a lattice with 13–21 data points provides an accuracy that is well within the typical bid–ask spread for pricing such transactions.

21.8 LINEAR PROGRAMMING

As we have shown throughout this book, linear programming is a very useful tool for fixed income management. While there are many LP software systems commercially available, the key to the usefulness of the software is its integration and automation with daily portfolio analytics and processes. Instead of developing one from scratch, one can build the product on the foundations of the open software that is currently available. We will not provide details of how this can be accomplished; however, we provide a few guidelines that can be helpful. There is a very good review and analysis of LP and computer code in Press *et al.* [15] that can be used as a reference. For LP to be useful for portfolio management, the following features are desirable:

- *Scaling* – the range of values in a portfolio optimization can be very large. A portfolio with a market value of \$100 million can have a constraint on yield that can be eight orders of magnitude smaller than the market value of the portfolio. For example, to optimize a treasury portfolio with a duration of 4 years with the requirement that the duration weighted spread be larger than 1 basis point, we need the following constraints:

$$\sum_i m_i = 100{,}000{,}000, \quad \sum_i m_i v_i = 40{,}000, \quad \sum_i m_i v_i s_i \geq 4 \tag{21.13}$$

  where m_i and v_i are the market value and VBP of each bond in the universe. If the constraints are not scaled to comparable values, significant loss of accuracy will result.
- *Capacity* – A typical LP for optimization of high grade corporates can have three to five thousand constraints, if there are constraints on the security or issuer level for market value as well as durations. There has to be capacity for all the constraints and a way to add them in an automated way.
- *Flexibility* – The constraints can have many different attributes imposed by policy, investment committee, portfolio manager, and analyst. The constraints typically involve ratings (Moody's and S&P), duration, spread duration, and market value in absolute terms or in relation to the benchmark.

- *Transaction costs* – Transaction costs need to be properly accounted for in the optimization process to avoid unnecessary rebalancing. Usually, the portfolio or benchmark is priced at the bid. However, purchases are made at the offer price. Therefore, to sell securities from the portfolio the bid price has to be used, and for purchases the offer price.
- *Rounding* – LP will result in market values that are generally not traded. The result will need to be rounded to more tradable values depending on the market value of the portfolio. For example, for a treasury portfolio of $100 million, where only three to six bonds are needed, face values can be rounded to about $0.5 million. On the other hand, for a corporate portfolio of the same size with 100–200 securities, the face values may have to be rounded to $50,000 per security.
- *Clean-up* – Transactions smaller than the rounding size need to be removed or rounded up. These last two steps will result in a portfolio that does not meet the exact constraints and the resulting market value can be larger than the available cash. If this is the case, some of the face values have to be rounded down to make sure that the market value does not exceed the available cash.
- *Iteration* – Quite often in optimizing corporate portfolios, some of the selected securities are not available or there can be bad pricing for some issues. For such portfolios, the portfolio construction is usually an iterative process where securities that are not available are taken out of the universe of securities and the optimization process is repeated. It may take several days or weeks to construct a corporate portfolio using LP, since most of the time the cheapest securities are the least liquid.
- *Trading* – Due to market moves, the optimized prices are almost never available. It is best to trade in pairs if rebalancing a portfolio, buying a cheap security and selling a rich security making sure that the net transaction cost is not higher than the estimated bid–ask spread. For funding a portfolio, only securities that have moved less than the market should be traded on any given day.

21.9 MORTGAGE ANALYSIS

Analysis of mortgage bonds is somewhat similar to the analysis of American bond options, but requires a few more steps as follows:

- From the daily historical TSIR and TSLR the monthly averages need to be calculated as well as the average 1x10 swaption volatility. These values will be used to estimate the expected historical current coupon using (18.13) or (18.14). The difference between the expected historical current coupon and the actual WAC of the mortgage pool is the discount or premium of the pool at the time of issuance. It is a measure of the creditworthiness of the borrowers, and we assume that the spread will carry forward to the present time.
- Using the current TSIR and TSLR, we calculate the forward level of Libor and slope of the treasury curve by calculating forward TSIR and TSLR. The forward term structure of rates can be calculated by finding five points on the forward curve and then calculating a curve that will run through all five points. This can be done efficiently by using key rate basis functions and equation (3.24) to calculate the forward TSIR in polynomial basis functions and convert the result to Chebyshev

basis functions. This calculation needs to be done only once for the analysis of all mortgage pools. Likewise, the forward 1x10 volatility can also be calculated from the TSLV for all forward points. The resulting parameters will be used to calculate the expected forward current coupons before prepayments can be calculated.

- Next, we need to calculate the shifted forward curves for the measurement of the sensitivity of the mortgages to interest rates and volatilities. This calculation needs to be done only once. For example, we shift the first component of the TSLV by ±5 bps and calculate the shifts at all forward points (360). To calculate the duration of volatility of a mortgage, when we shift a component of the TSLV, all forward volatilities need to be adjusted to estimate the resulting forward current coupons. We can save significant computation time by performing this calculation only once and using the calculated shifts for the analysis of all bonds.
- At each forward point, we calculate the implied forward coupon of 10-year Libor as well as the current coupon of the mortgage. The spread between these two coupons is used to estimate the current coupon for the distribution of Libor rates. The distribution of forward Libor is calculated from its forward volatility, and at each node the forward coupon is calculated by adding the coupon spreads to the Libor rate.
- Once the interest rate tree is constructed and all the cash flows calculated, we find the OAS that is required to match the market price of the mortgage bond.
- At each node the expected factor of the bond is calculated by adding all the contributions from the previous layer that end in the bucket of the node under consideration. The factor is then used to calculate the corresponding burnout or prepayment incentive that will be used for further development of the tree.
- For the first 12 months we use monthly layers, for the following 12 months bimonthly then quarterly, then for the following 12 years semi-annually and the remainder annually. The cash flows for each layer are assumed to occur in the weighted average time for that layer. For example, for the third year when cash flows are analyzed on a quarterly basis, the cash flows for April, May, and June are assumed to occur at the end of May. There is an additional lag of 15 or 25 days for Freddie Mac and Fannie May cash flows, respectively.

References

[1] Boero, G., and Torricelli, C. A comparative evaluation of alternative models of the term structure of interest rates. *European Journal of Operational Research* 93 (1996), 205–223.
[2] Ferguson, R., and Raymar, S. A comparative analysis of several popular term structure estimation models. *Journal of Fixed Income* 7(4) (1998), 17–33.
[3] Willner, R. A new tool for portfolio managers: level, slope, and curvature durations. *Journal of Fixed Income* 6(1) (1996), 48–59.
[4] Nelson, C., and Siegel, A. Parsimonious modeling of yield curves. *Journal of Business* 60(4) (1987), 475–489.
[5] Ho, T.S.Y. Key rate durations: measures of interest rate risks. *Journal of Fixed Income* 2(2) (1992), 29–44.
[6] Litterman, R. and Scheinkman, J. Common factors affecting bond returns. *Journal of Fixed Income* 1(1) (1991), 54–61.
[7] Choudhry, M. *Advanced Fixed Income Analysis*. Elsevier Butterworth-Heinemann, Amsterdam, 2004.
[8] Vanderbei, R.J. *Linear Programming: Foundations and Extensions*, 3rd edn. Springer-Verlag, New York, 2008.
[9] Gass, S.I. *Linear Programming: Methods and Applications*, 5th edn. Dover, New York, 2003.
[10] Dybvig, P., Ingersoll Jr, J.E., and Ross, S.A. Long forward and zero-coupon rates can never fall. *Journal of Business* 59(1) (1996), 1–25.
[11] US Census Bureau. The X-13ARIMA-SEATS Seasonal Adjustment Program. http://www.census.gov/srd/www/x12a/ (accessed November 19, 2014).
[12] http://ec.europa.eu/eurostat
Search for Seasonal Adjustment.
[13] Haug, E.G. *The Complete Guide to Options Pricing Formulas*, 2nd edn. McGraw-Hill, New York, 2006.
[14] Fama, E.F. Forward and spot exchange rates. *Journal of Monetary Economics* 14 (1984). 319–338.
[15] Press, W.H., Teukolsky, S.A., Vetterling, W.T., and Flannery, B.P. (2007), *Numerical Recipes: The Art of Scientific Computing*, 3rd edn, Cambridge University Press, Cambridge.

Index

SAIED SIMOZAR, PhD, has spent almost 30 years in fixed income portfolio management, fixed income analytics, scientific software development and consulting. He is a principal at Fipmar, Inc., an investment management consulting firm in Beverly Hills, CA. Prior to that, Saied was a Managing Director at Nuveen Investments, with responsibilities for all global fixed income investments. He has also been a Managing Director at Bank of America Capital Management responsible for all global and emerging markets portfolios of the fixed income division. Prior to that, he was a senior portfolio manager at Putnam Investments and DuPont Pension Fund Investments.

For any queries about the book, please contact the author at: **fixedincomebook@fipmar.com**

SIMOZAR **The Advanced Fixed Income and Derivatives Management Guide**

WILEY

A New Framework for Analyzing and Managing Fixed Income Portfolios

Global traders implementing alpha transfer or complex fixed income strategies need a stable and accurate term structure of interest rates (TSIR) for all fundamental rates. However, theoretical models of TSIR often lack accuracy and practical models such as splines are not mathematically stable.

In *The Advanced Fixed Income and Derivatives Management Guide*, author Saied Simozar provides a unique and novel solution. In this detailed and practical guide, Simozar lays out a new framework, one that allows analysts and traders to evaluate all global fixed income investments by discount functions, based on a stable TSIR, and perform valuation, risk measurement and performance attribution across all asset classes and currencies in a consistent and accurate way.

Packed with over 700 useful equations and pages of explanation, this book offers investors the most detailed analysis of many fixed income sectors including inflation linked and corporate securities and their respective derivatives as well as American bond options currently available.

You'll learn to estimate recovery value from market data and assess the impact of recovery value on risks and valuations. The book gives you deeper insight into portfolio construction and optimization, performance attribution, security selection and uncovering arbitrage opportunities. Numerous market based examples of identifying alpha trades are included and many of the analytics are available on the book's dedicated website (www.wiley.com/go/simozar). You'll find worksheets, complete with macros, which you can download and use to measure risks and valuations based on the TSIR.

Analysts, portfolio managers and traders, keep this valuable guide at hand and learn to better manage your fixed income investments.

Cover Design: Wiley
Top Image: ©Shutterstock.com/bluebay
Bottom Image: ©Shutterstock.com/wongwean

WILEY

Also available as an e-book

ISBN 978-1-119-01414-0

9 781119 014140