WILEY FINANCE

SECOND EDITION

Financial Risk Manager handbook

- Learn the essentials of managing market, credit, and operational risk
- Identify regulatory, legal, and accounting issues
- Ideal for self-instruction and in-house training in risk management
- The official reference book for GARP's FRM certification program

PHILIPPE JORION

GARP
GLOBAL ASSOCIATION OF RISK PROFESSIONALS
RISK MANAGEMENT LIBRARY

Financial Risk Manager Handbook
Second Edition

Founded in 1807, John Wiley & Sons is the oldest independent publishing company in the United States. With offices in North America, Europe, Australia, and Asia, Wiley is globally committed to developing and marketing print and electronic products and services for our customers' professional and personal knowledge and understanding.

The Wiley Finance series contains books written specifically for finance and investment professionals, as well as sophisticated individual investors and their financial advisors. Book topics range from portfolio management to e-commerce, risk management, financial engineering, valuation, and financial instrument analysis, as well as much more.

For a list of available titles, please visit our Web site at www.WileyFinance.com.

Financial Risk Manager Handbook
Second Edition

Philippe Jorion

GARP

Wiley
John Wiley & Sons, Inc.

Published by John Wiley & Sons, Inc., Hoboken, New Jersey
Published simultaneously in Canada

Library of Congress Cataloging-in-Publication Data:

ISBN 0-471-43003-X

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

About the Author

Philippe Jorion is Professor of Finance at the Graduate School of Management at the University of California at Irvine. He has also taught at Columbia University, Northwestern University, the University of Chicago, and the University of British Columbia. He holds an M.B.A. and a Ph.D. from the University of Chicago and a degree in engineering from the University of Brussels.

Dr. Jorion has authored more than seventy publications directed to academics and practitioners on the topics of risk management and international finance. Dr. Jorion has written a number of books, including *Big Bets Gone Bad: Derivatives and Bankruptcy in Orange County,* the first account of the largest municipal failure in U.S. history, and *Value at Risk: The New Benchmark for Managing Financial Risk,* which is aimed at finance practitioners and has become an "industry standard."

Philippe Jorion is a frequent speaker at academic and professional conferences. He is on the editorial board of a number of finance journals and is editor in chief of the *Journal of Risk*.

About GARP

The **Global Association of Risk Professionals** (GARP), established in 1996, is a not-for-profit independent association of risk management practitioners and researchers. Its members represent banks, investment management firms, governmental bodies, academic institutions, corporations, and other financial organizations from all over the world.

GARP's mission, as adopted by its Board of Trustees in a statement issued in February 2003, is to be the leading professional association for risk managers, managed by and for its members dedicated to the advancement of the risk profession through education, training and the promotion of best practices globally.

In just seven years the Association's membership has grown to over 27,000 individuals from around the world. In the just six years since its inception in 1997, the FRM program has become the world's most prestigious financial risk management certification program. Professional risk managers having earned the FRM credential are globally recognized as having achieved a minimum level of professional competency along with a demonstrated ability to dynamically measure and manage financial risk in a real-world setting in accord with global standards. Further information about GARP, the FRM Exam, and FRM readings are available at *www.garp.com*.

Contents

Part V: Operational and Integrated Risk Management 531

Preface

The FRM Handbook provides the core body of knowledge for financial risk managers. Risk management has rapidly evolved over the last decade and has become an indispensable function in many institutions.

This Handbook was originally written to provide support for candidates taking the FRM examination administered by GARP. As such, it reviews a wide variety of practical topics in a consistent and systematic fashion. It covers quantitative methods, capital markets, as well as market, credit, operational, and integrated risk management. It also discusses the latest regulatory, legal, and accounting issues essential to risk professionals.

Modern risk management systems cut across the entire organization. This breadth is reflected in the subjects covered in this Handbook. This Handbook was designed to be self-contained, but only for readers who already have some exposure to financial markets. To reap maximum benefit from this book, readers should have taken the equivalent of an MBA-level class on investments.

Finally, I wanted to acknowledge the help received in the writing of this second edition. In particular, I would like to thank the numerous readers who shared comments on the previous edition. Any comment and suggestion for improvement will be welcome. This feedback will help us to maintain the high quality of the FRM designation.

Philippe Jorion
April 2003

Introduction

The *Financial Risk Manager Handbook* was first created in 2000 as a study support manual for candidates preparing for GARP's annual FRM exam and as a general guide to assessing and controlling financial risk in today's rapidly changing environment.

But the growth in the number of risk professionals, the now commonly held view that risk management is an integral and indispensable part of any organization's management culture, and the ever increasing complexity of the field of risk management have changed our goal for the Handbook.

This dramatically enhanced second edition of the Handbook reflects our belief that a dynamically changing business environment requires a comprehensive text that provides an in-depth overview of the various disciplines associated with financial risk management. The Handbook has now evolved into the essential reference text for any risk professional, whether they are seeking FRM Certification or whether they simply have a desire to remain current on the subject of financial risk.

For those using the FRM Handbook as a guide for the FRM Exam, each chapter includes questions from previous FRM exams. The questions are selected to provide systematic coverage of advanced FRM topics. The answers to the questions are explained by comprehensive tutorials.

The FRM examination is designed to test risk professionals on a combination of basic analytical skills, general knowledge, and intuitive capability acquired through experience in capital markets. Its focus is on the core body of knowledge required for independent risk management analysis and decision-making. The exam has been administered every autumn since 1997 and has now expanded to 43 international testing sites.

The FRM exam is recognized at the world's most prestigious global certification program for risk management professionals. As of 2002, 3,265 risk management professionals have earned the FRM designation. They represent over 1,450 different companies, financial institutions, regulatory bodies, brokerages, asset management firms, banks, exchanges, universities, and other firms from all over the world.

GARP is very proud, through its alliance with John Wiley & Sons, to make this flagship book available not only to FRM candidates, but to risk professionals, professors, and their students everywhere. Philippe Jorion, preeminent in his field, has once again prepared and updated the Handbook so that it remains an essential reference for risk professionals.

Any queries, comments or suggestions about the Handbook may be directed to frmhandbook@garp.com. Corrections to this edition, if any, will be posted on GARP's Web site.

Whether preparing for the FRM examination, furthering your knowledge of risk management, or just wanting a comprehensive reference manual to refer to in a time of need, any financial services professional will find the FRM Handbook an indispensable asset.

Global Association of Risk Professionals
April 2003

Financial Risk Manager
Handbook
Second Edition

PART one

Quantitative Analysis

Chapter 1

Bond Fundamentals

Risk management starts with the pricing of assets. The simplest assets to study are fixed-coupon bonds, for which cash flows are predetermined. As a result, we can translate the stream of cash flows into a present value by discounting at a fixed yield. Thus the valuation of bonds involves understanding compounded interest, discounting, as well as the relationship between present values and interest rates.

Risk management goes one step further than pricing, however. It examines potential changes in the value of assets as the interest rate changes. In this chapter, we assume that there is a single interest rate that is used to discount to all bonds. This will be our fundamental risk factor.

Even for as simple an instrument as a bond, the relationship between the price and the risk factor can be complex. This is why the industry has developed a number of tools that summarize the risk profile of fixed-income portfolios.

This chapter starts our coverage of quantitative analysis by discussing bond fundamentals. Section 1.1 reviews the concepts of discounting, present values, and future values. Section 1.2 then plunges into the price-yield relationship. It shows how the Taylor expansion rule can be used to measure price movements. These concepts are presented first because they are so central to the measurement of financial risk. The section then discusses the economic interpretation of duration and convexity.

1.1 Discounting, Present, and Future Value

An investor considers a zero-coupon bond that pays $100 in 10 years. Say that the investment is guaranteed by the U.S. government and has no default risk. Because the payment occurs at a future date, the investment is surely less valuable than an up-front payment of $100.

To value the payment, we need a **discounting factor**. This is also the **interest rate**, or more simply the **yield**. Define C_t as the cash flow at time $t = T$ and the discounting factor as y. Here, T is the number of periods until maturity, e.g. number of years, also known as **tenor**. The **present value** (PV) of the bond can be computed as

$$PV = \frac{C_T}{(1+y)^T} \tag{1.1}$$

For instance, a payment of $C_T = \$100$ in 10 years discounted at 6 percent is only worth \$55.84. This explains why the market value of zero-coupon bonds decreases with longer maturities. Also, keeping T fixed, the value of the bond decreases as the yield increases.

Conversely, we can compute the **future value** of the bond as

$$FV = PV \times (1+y)^T \tag{1.2}$$

For instance, an investment now worth $PV = \$100$ growing at 6 percent will have a future value of $FV = \$179.08$ in 10 years.

Here, the yield has a useful interpretation, which is that of an **internal rate of return** on the bond, or annual growth rate. It is easier to deal with rates of returns than with dollar values. Rates of return, when expressed in percentage terms and on an annual basis, are directly comparable across assets. An annualized yield is sometimes defined as the **effective annual rate (EAR)**.

It is important to note that the interest rate should be stated along with the method used for compounding. Equation (1.1) uses annual compounding, which is frequently the norm. Other conventions exist, however. For instance, the U.S. Treasury market uses semiannual compounding. If so, the interest rate y^S is derived from

$$PV = \frac{C_T}{(1+y^S/2)^{2T}} \tag{1.3}$$

where T is the number of periods, or semesters in this case. Continuous compounding is often used when modeling derivatives. If so, the interest rate y^C is derived from

$$PV = C_T \times e^{-y^C T} \tag{1.4}$$

where $e^{(\cdot)}$, sometimes noted as $\exp(\cdot)$, represents the exponential function. These are merely definitions and are all consistent with the same initial and final values. One has to be careful, however, about using each in the appropriate formula.

Example: Using different discounting methods

Consider a bond that pays \$100 in 10 years and has a present value of \$55.8395. This corresponds to an annually compounded rate of 6.00% using $PV = C_T/(1+y)^{10}$, or $(1+y) = C_T/PV^{1/10}$.

This rate can be easily transformed into a semiannual compounded rate, using $(1 + y^S/2)^2 = (1+y)$, or $y^S = ((1+0.06)^{(1/2)} - 1) \times 2 = 0.0591$. It can be also transformed into a continuously compounded rate, using $\exp(y^C) = (1+y)$, or $y^C = \ln(1+0.06) = 0.0583$.

Note that as we increase the frequency of the compounding, the resulting rate decreases. Intuitively, because our money works harder with more frequent compounding, a lower investment rate will achieve the same payoff.

Key concept:
For fixed present and final values, increasing the frequency of the compounding will decrease the associated yield.

Example 1-1: FRM Exam 1999—Question 17/Quant. Analysis

1-1. Assume a semiannual compounded rate of 8% per annum. What is the equivalent annually compounded rate?

a) 9.20%
b) 8.16%
c) 7.45%
d) 8.00%

Example 1-2: FRM Exam 1998—Question 28/Quant. Analysis

1-2. Assume a continuously compounded interest rate is 10% per annum. The equivalent semiannual compounded rate is

a) 10.25% per annum
b) 9.88% per annum
c) 9.76% per annum
d) 10.52% per annum

1.2 Price-Yield Relationship

1.2.1 Valuation

The fundamental discounting relationship from Equation (1.1) can be extended to any bond with a fixed cash-flow pattern. We can write the present value of a bond P as the discounted value of future cash flows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \tag{1.5}$$

where:

C_t = the cash flow (coupon or principal) in period t
t = the number of periods (e.g. half-years) to each payment
T = the number of periods to final maturity
y = the discounting factor

A typical cash-flow pattern consists of a regular coupon payment plus the repayment of the principal, or **face value** at expiration. Define c as the coupon *rate* and F as the face value. We have $C_t = cF$ prior to expiration, and at expiration, we have $C_T = cF + F$. The appendix reviews useful formulas that provide closed-form solutions for such bonds.

When the coupon rate c precisely matches the yield y, using the same compounding frequency, the present value of the bond must be equal to the face value. The bond is said to be a **par bond**.

Equation (1.5) describes the relationship between the yield y and the value of the bond P, given its cash-flow characteristics. In other words, the value P can also be written as a nonlinear function of the yield y:

$$P = f(y) \tag{1.6}$$

Conversely, we can define P as the current market price of the bond, including any accrued interest. From this, we can compute the "implied" yield that will solve Equation (1.6).

There is a particularly simple relationship for **consols**, or **perpetual bonds**, which are bonds making regular coupon payments but with no redemption date. For a consol, the maturity is infinite and the cash flows are all equal to a fixed percentage of the face value, $C_t = C = cF$. As a result, the price can be simplified from Equation (1.5) to

$$P = cF\left[\frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots\right] = \frac{c}{y}F \tag{1.7}$$

as shown in the appendix. In this case, the price is simply proportional to the inverse of the yield. Higher yields lead to lower bond prices, and vice versa.

Example: Valuing a bond

Consider a bond that pays $100 in 10 years and a 6% annual coupon. Assume that the next coupon payment is in exactly one year. What is the market value if the yield is 6%? If it falls to 5%?

The bond cash flows are $C_1 = \$6, C_2 = \$6, \ldots, C_{10} = \$106$. Using Equation (1.5) and discounting at 6%, this gives the present value of cash flows of $5.66, $10.68, ..., $59.19, for a total of $100.00. The bond is selling at par. This is logical because the coupon is equal to the yield, which is also annually compounded. Alternatively, discounting at 5% leads to a price appreciation to $107.72.

Example 1-3: FRM Exam 1998—Question 12/Quant. Analysis

1-3. A fixed-rate bond, currently priced at 102.9, has one year remaining to maturity and is paying an 8% coupon. Assuming the coupon is paid semiannually, what is the yield of the bond?

a) 8%
b) 7%
c) 6%
d) 5%

1.2.2 Taylor Expansion

Let us say that we want to see what happens to the price if the yield changes from its initial value, called y_0, to a new value, $y_1 = y_0 + \Delta y$. Risk management is all about assessing the effect of changes in risk factors such as yields on asset values. Are there shortcuts to help us with this?

We could recompute the new value of the bond as $P_1 = f(y_1)$. If the change is not too large, however, we can apply a very useful shortcut. The nonlinear relationship can be approximated by a **Taylor expansion** around its initial value[1]

$$P_1 = P_0 + f'(y_0)\Delta y + \frac{1}{2}f''(y_0)(\Delta y)^2 + \cdots \tag{1.8}$$

where $f'(\cdot) = \frac{dP}{dy}$ is the first derivative and $f''(\cdot) = \frac{d^2P}{dy^2}$ is the second derivative of the function $f(\cdot)$ valued at the starting point.[2] This expansion can be generalized to situations where the function depends on two or more variables.

Equation (1.8) represents an infinite expansion with increasing powers of Δy. Only the first two terms (linear and quadratic) are ever used by finance practitioners. This is because they provide a good approximation to changes in prices relative to other assumptions we have to make about pricing assets. If the increment is very small, even the quadratic term will be negligible.

Equation (1.8) is fundamental for risk management. It is used, sometimes in different guises, across a variety of financial markets. We will see later that this Taylor expansion is also used to approximate the movement in the value of a derivatives contract, such as an option on a stock. In this case, Equation (1.8) is

$$\Delta P = f'(S)\Delta S + \frac{1}{2}f''(S)(\Delta S)^2 + \ldots \tag{1.9}$$

where S is now the price of the underlying asset, such as the stock. Here, the first derivative $f'(S)$ is called *delta*, and the second $f''(S)$, *gamma*.

The Taylor expansion allows easy aggregation across financial instruments. If we have x_i units (numbers) of bond i and a total of N different bonds in the portfolio, the portfolio derivatives are given by

$$f'(y) = \sum_{i=1}^{N} x_i f_i'(y) \tag{1.10}$$

We will illustrate this point later for a 3-bond portfolio.

[1]This is named after the English mathematician Brook Taylor (1685–1731), who published this result in 1715. The full recognition of the importance of this result only came in 1755 when Euler applied it to differential calculus.

[2] This first assumes that the function can be written in polynomial form as $P(y + \Delta y) = a_0 + a_1\Delta y + a_2(\Delta y)^2 + \cdots$, with unknown coefficients a_0, a_1, a_2. To solve for the first, we set $\Delta y = 0$. This gives $a_0 = P_0$. Next, we take the derivative of both sides and set $\Delta y = 0$. This gives $a_1 = f'(y_0)$. The next step gives $2a_2 = f''(y_0)$. Note that these are the conventional mathematical derivatives and have nothing to do with derivatives products such as options.

1.2.3 Bond Price Derivatives

For fixed-income instruments, the derivatives are so important that they have been given a special name.[3] The negative of the first derivative is the **dollar duration (DD)**:

$$f'(y_0) = \frac{dP}{dy} = -D^* \times P_0 \tag{1.11}$$

where D^* is called the **modified duration.** Thus, dollar duration is

$$\text{DD} = D^* \times P_0 \tag{1.12}$$

where the price P_0 represent the *market* price, including any accrued interest. Sometimes, risk is measured as the **dollar value of a basis point (DVBP),**

$$\text{DVBP} = [D^* \times P_0] \times 0.0001 \tag{1.13}$$

with 0.0001 representing one hundredth of a percent. The **DVBP**, sometimes called the **DV01**, measures can be more easily added up across the portfolio.

The second derivative is the **dollar convexity (DC)**:

$$f''(y_0) = \frac{d^2P}{dy^2} = C \times P_0 \tag{1.14}$$

where C is called the **convexity**.

For fixed-income instruments with known cash flows, the price-yield function is known, and we can compute analytical first and second derivatives. Consider, for example, our simple zero-coupon bond in Equation (1.1) where the only payment is the face value, $C_T = F$. We take the first derivative, which is

$$\frac{dP}{dy} = (-T)\frac{F}{(1+y)^{T+1}} = -\frac{T}{(1+y)}P \tag{1.15}$$

Comparing with Equation (1.11), we see that the modified duration must be given by $D^* = T/(1+y)$. The conventional measure of **duration** is $D = T$, which does not

[3]Note that this chapter does not present duration in the traditional textbook order. In line with the advanced focus on risk management, we first analyze the properties of duration as a sensitivity measure. This applies to any type of fixed-income instrument. Later, we will illustrate the usual definition of duration as a weighted average maturity, which applies for fixed-coupon bonds only.

include division by $(1 + y)$ in the denominator. This is also called **Macaulay duration**. Note that duration is expressed in periods, like T. With annual compounding, duration is in years. With semiannual compounding, duration is in semesters and has to be divided by two for conversion to years.

Modified duration is the appropriate measure of interest-rate exposure. The quantity $(1 + y)$ appears in the denominator because we took the derivative of the present value term with discrete compounding. If we use continuous compounding, modified duration is identical to the conventional duration measure. In practice, the difference between Macaulay and modified duration is often small. With a 6% yield and semiannual compounding, for instance, the adjustment is only a factor of 3%.

Let us now go back to Equation (1.15) and consider the second derivative, which is

$$\frac{d^2P}{dy^2} = -(T+1)(-T)\frac{F}{(1+y)^{T+2}} = \frac{(T+1)T}{(1+y)^2} \times P \tag{1.16}$$

Comparing with Equation (1.14), we see that the convexity is $C = (T + 1)T/(1 + y)^2$. Note that its dimension is expressed in period squared. With semiannual compounding, convexity is measured in semesters squared and has to be divided by four for conversion to years squared.[4]

Putting together all these equations, we get the Taylor expansion for the change in the price of a bond, which is

$$\Delta P = -[D^* \times P](\Delta y) + \frac{1}{2}[C \times P](\Delta y)^2 \ldots \tag{1.17}$$

Therefore duration measures the first-order (linear) effect of changes in yield and convexity the second-order (quadratic) term.

Example: Computing the price approximation

Consider a 10-year zero-coupon bond with a yield of 6 percent and present value of \$55.368. This is obtained as $P = 100/(1 + 6/200)^{20} = 55.368$. As is the practice in the Treasury market, yields are semiannually compounded. Thus all computations should be carried out using semesters, after which final results can be converted into annual units.

[4] This is because the conversion to annual terms is obtained by multiplying the semiannual yield Δy by two. As a result, the duration term must be divided by 2 and the convexity term by 2^2, or 4, for conversion to annual units.

Here, Macaulay duration is exactly 10 years, as $D = T$ for a zero-coupon bond. Its modified duration is $D^* = 20/(1+6/200) = 19.42$ semesters, which is 9.71 years. Its convexity is $C = 21 \times 20/(1+6/200)^2 = 395.89$ semesters squared, which is 98.97 in years squared. Dollar duration is $\text{DD} = D^* \times P = 9.71 \times \$55.37 = \$537.55$. The DVBP is $\text{DVBP} = \text{DD} \times 0.0001 = \$0.0538$.

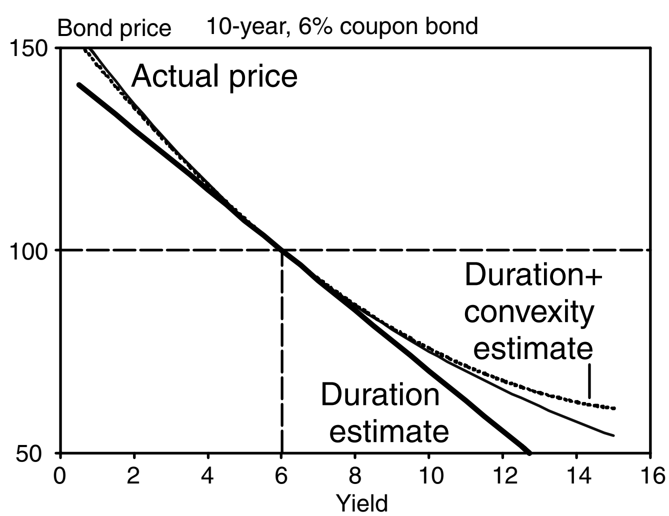
We want to approximate the change in the value of the bond if the yield goes to 7%. Using Equation (1.17), we have $\Delta P = -[9.71\times\$55.37](0.01)+0.5[98.97\times\$55.37](0.01)^2 = -\$5.375 + \$0.274 = -\$5.101$. Using the first term only, the new price is $\$55.368 - \$5.375 = \$49.992$. Using the two terms in the expansion, the predicted price is slightly different, at $\$55.368 - \$5.101 = \$50.266$.

These numbers can be compared with the exact value, which is \$50.257. Thus the linear approximation has a pricing error of −0.53%, which is not bad given the large change in yield. Adding the second term reduces this to an error of 0.02% only, which is minuscule given typical bid-ask spreads.

More generally, Figure 1-1 compares the quality of the Taylor series approximation. We consider a 10-year bond paying a 6 percent coupon semiannually. Initially, the yield is also at 6 percent and, as a result the price of the bond is at par, at \$100. The graph compares, for various values of the yield y:

1. The actual, exact price $P = f(y_0 + \Delta y)$
2. The duration estimate $P = P_0 - D^* P_0 \Delta y$
3. The duration and convexity estimate $P = P_0 - D^* P_0 \Delta y + (1/2) C P_0 (\Delta y)^2$

FIGURE 1-1 Price Approximation



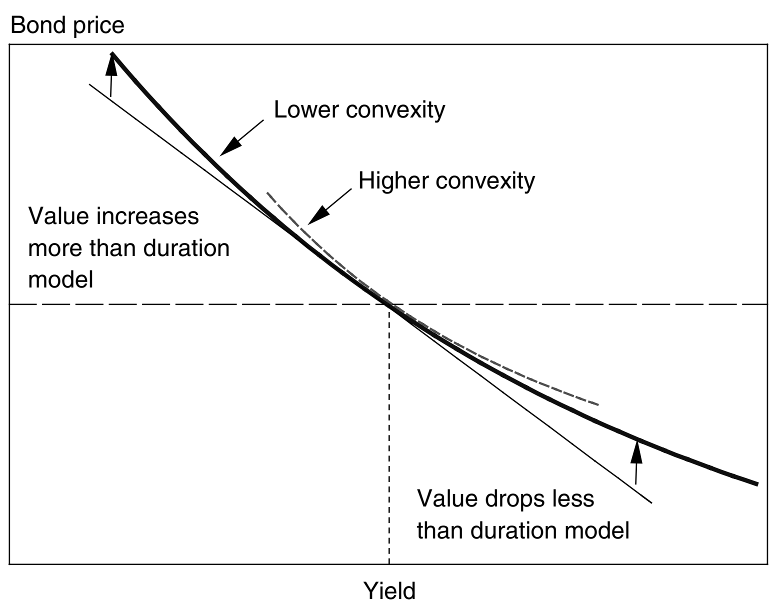
The actual price curve shows an increase in the bond price if the yield falls and, conversely, a depreciation if the yield increases. This effect is captured by the tangent to the true price curve, which represents the linear approximation based on duration. For small movements in the yield, this linear approximation provides a reasonable fit to the exact price.

> **Key concept:**
> Dollar duration measures the (negative) slope of the tangent to the price-yield curve at the starting point.

For large movements in price, however, the price-yield function becomes more curved and the linear fit deteriorates. Under these conditions, the quadratic approximation is noticeably better.

We should also note that the curvature is away from the origin, which explains the term convexity (as opposed to concavity). Figure 1-2 compares curves with different values for convexity. This curvature is beneficial since the second-order effect $0.5[C \times P](\Delta y)^2$ *must* be positive when convexity is positive.

FIGURE 1-2 Effect of Convexity



As Figure 1-2 shows, when the yield rises, the price drops but less than predicted by the tangent. Conversely, if the yield falls, the price increases faster than the duration model. In other words, the quadratic term is always beneficial.

> **Key concept:**
> Convexity is always positive for coupon-paying bonds. Greater convexity is beneficial both for falling and rising yields.

The bond's modified duration and convexity can also be computed directly from numerical derivatives. Duration and convexity cannot be computed directly for some bonds, such as mortgage-backed securities, because their cash flows are uncertain. Instead, the portfolio manager has access to pricing models that can be used to reprice the securities under various yield environments.

We choose a change in the yield, Δy, and reprice the bond under an upmove scenario, $P_+ = P(y_0 + \Delta y)$, and downmove scenario, $P_- = P(y_0 - \Delta y)$. **Effective duration** is measured by the numerical derivative. Using $D^* = -(1/P)dP/dy$, it is estimated as

$$D^E = \frac{[P_- - P_+]}{(2P_0\Delta y)} = \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{(2\Delta y)P_0} \tag{1.18}$$

Using $C = (1/P)d^2P/dy^2$, **effective convexity** is estimated as

$$C^E = [D_- - D_+]/\Delta y = \left[\frac{P(y_0 - \Delta y) - P_0}{(P_0\Delta y)} - \frac{P_0 - P(y_0 + \Delta y)}{(P_0\Delta y)}\right]/\Delta y \tag{1.19}$$

These computations are illustrated in Table 1-1 and in Figure 1-3.

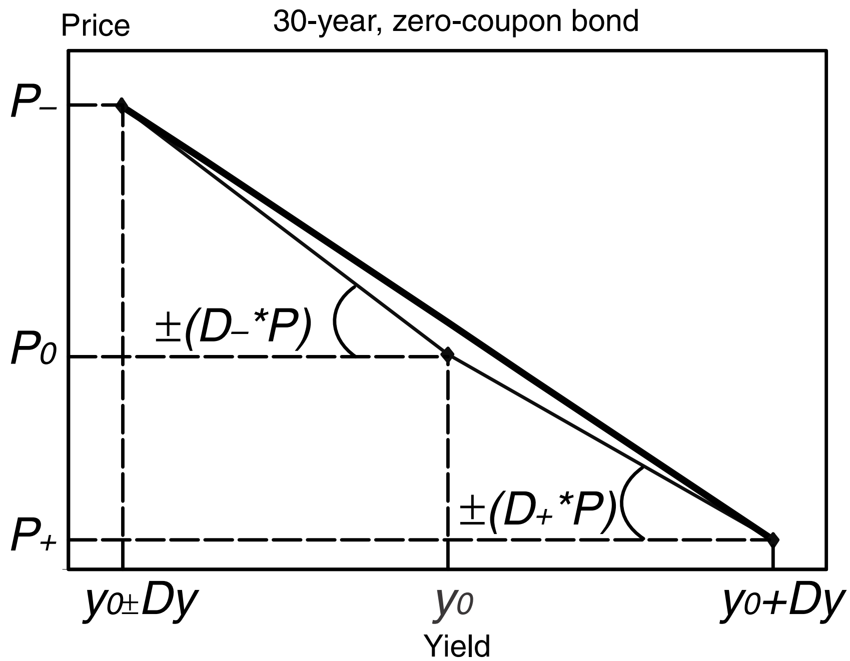
TABLE 1-1 Effective Duration and Convexity

State	Yield (%)	Bond Value	Duration Computation	Convexity Computation
Initial y_0	6.00	16.9733		
Up $y_0 + \Delta y$	7.00	12.6934		Duration up: 25.22
Down $y_0 - \Delta y$	5.00	22.7284		Duration down: 33.91
Difference in values			−10.0349	8.69
Difference in yields			0.02	0.01
Effective measure			29.56	869.11
Exact measure			29.13	862.48

As a benchmark case, consider a 30-year zero-coupon bond with a yield of 6 percent. With semiannual compounding, the initial price is $16.9733. We then revalue the bond at 5 percent and 7 percent. The effective duration in Equation (1.18) uses the two extreme points. The effective convexity in Equation (1.19) uses the difference between the dollar durations for the upmove and downmove. Note that convexity is positive if duration increases as yields fall, or if $D_- > D_+$.

The computations are detailed in Table 1-1, where the effective duration is measured at 29.56. This is very close to the true value of 29.13, and would be even closer if the step Δy was smaller. Similarly, the effective convexity is 869.11, which is close

FIGURE 1-3 Effective Duration and Convexity



to the true value of 862.48. In general, however, effective duration is a by-product of the pricing model. Inaccuracies in the model will distort the duration estimate.

Finally, this numerical approach can be applied to get an estimate of the duration of a bond by considering bonds with the same maturity but different coupons. If interest rates decrease by 100 basis points (bp), the market price of a 6% 30-year bond should go up, close to that of a 7% 30-year bond. Thus we replace a drop in yield of Δy by an increase in coupon Δc and use the effective duration method to find the **coupon curve duration**

$$D^{CC} = \frac{[P_+ - P_-]}{(2P_0\Delta c)} = \frac{P(y_0; c + \Delta c) - P(y_0; c - \Delta c)}{(2\Delta c)P_0} \tag{1.20}$$

This approach is useful for securities that are difficult to price under various yield scenarios. Instead, it only requires the market prices of securities with different coupons.

Example: Computation of coupon curve duration

Consider a 10-year bond that pays a 7% coupon semiannually. In a 7% yield environment, the bond is selling at par and has modified duration of 7.11 years. The prices of 6% and 8% coupon bonds are \$92.89 and \$107.11, respectively. This gives a coupon curve duration of $(107.11 - 92.89)/(0.02 \times 100) = 7.11$, which in this case is the same as modified duration.

Example 1-4: FRM Exam 1999—Question 9/Quant. Analysis

1-4. A number of terms in finance are related to the (calculus!) derivative of the price of a security with respect to some other variable.
Which pair of terms is defined using second derivatives?
a) Modified duration and volatility
b) Vega and delta
c) Convexity and gamma
d) PV01 and yield to maturity

Example 1-5: FRM Exam 1998—Question 17/Quant. Analysis

1-5. A bond is trading at a price of 100 with a yield of 8%. If the yield increases by 1 basis point, the price of the bond will decrease to 99.95. If the yield decreases by 1 basis point, the price of the bond will increase to 100.04. What is the modified duration of the bond?
a) 5.0
b) 5.0
c) 4.5
d) −4.5

Example 1-6: FRM Exam 1998—Question 22/Quant. Analysis

1-6. What is the price impact of a 10-basis-point increase in yield on a 10-year par bond with a modified duration of 7 and convexity of 50?
a) −0.705
b) −0.700
c) −0.698
d) −0.690

Example 1-7: FRM Exam 1998—Question 20/Quant. Analysis

1-7. Coupon curve duration is a useful method to estimate duration from market prices of a mortgage-backed security (MBS). Assume the coupon curve of prices for Ginnie Maes in June 2001 is as follows: 6% at 92, 7% at 94, and 8% at 96.5. What is the estimated duration of the 7s?
a) 2.45
b) 2.40
c) 2.33
d) 2.25

Example 1-8: FRM Exam 1998—Question 21/Quant. Analysis

1-8. Coupon curve duration is a useful method to estimate convexity from market prices of an MBS. Assume the coupon curve of prices for Ginnie Maes in June 2001 is as follows: 6% at 92, 7% at 94, and 8% at 96.5. What is the estimated convexity of the 7s?

a) 53
b) 26
c) 13
d) −53

1.2.4 Interpreting Duration and Convexity

The preceding section has shown how to compute analytical formulas for duration and convexity in the case of a simple zero-coupon bond. We can use the same approach for coupon-paying bonds. Going back to Equation (1.5), we have

$$\frac{dP}{dy} = \sum_{t=1}^{T} \frac{-tC_t}{(1+y)^{t+1}} = -[\sum_{t=1}^{T} \frac{tC_t}{(1+y)^{t+1}}]/P \times P = -\frac{D}{(1+y)}P \tag{1.21}$$

which defines duration as

$$D = \sum_{t=1}^{T} \frac{tC_t}{(1+y)^t}/P \tag{1.22}$$

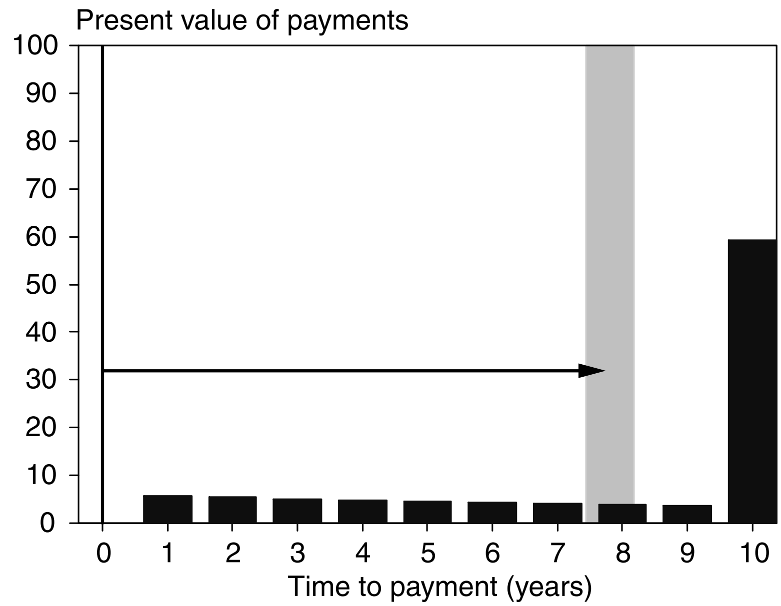
The economic interpretation of duration is that it represents the average time to wait for each payment, weighted by the present value of the associated cash flow. Indeed, we can write

$$D = \sum_{t=1}^{T} t \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} = \sum_{t=1}^{T} t \times w_t \tag{1.23}$$

where the weights w represent the ratio of the present value of cash flow C_t relative to the total, and sum to unity. This explains why the duration of a zero-coupon bond is equal to the maturity. There is only one cash flow, and its weight is one.

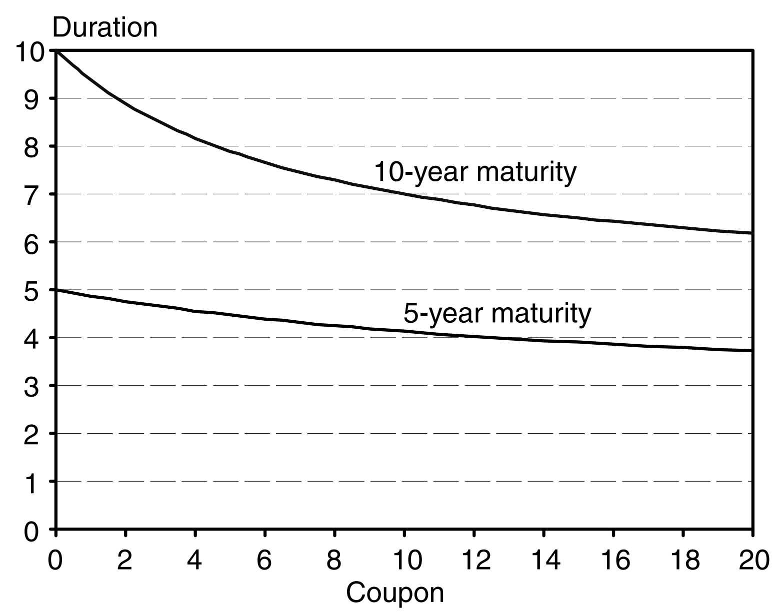
Figure 1-4 lays out the present value of the cash flows of a 6% coupon, 10-year bond. Given a duration of 7.80 years, this coupon-paying bond is equivalent to a zero-coupon bond maturing in exactly 7.80 years.

FIGURE 1-4 Duration as the Maturity of a Zero-Coupon Bond



For coupon-paying bonds, duration lies between zero and the maturity of the bond. For instance, Figure 1-5 shows how the duration of a 10-year bond varies with its coupon. With a zero coupon, Macaulay duration is equal to maturity. Higher coupons place more weight on prior payments and therefore reduce duration.

FIGURE 1-5 Duration and Coupon



Duration can be expressed in a simple form for **consols**. From Equation (1.7), we have $P = (c/y)F$. Taking the derivative, we find

$$\frac{dP}{dy} = cF\frac{(-1)}{y^2} = (-1)\frac{1}{y}[\frac{c}{y}F] = (-1)\frac{1}{y}P = -\frac{D_C}{(1+y)}P \tag{1.24}$$

Hence the Macaulay duration for the consol D_C is

$$D_C = \frac{(1+y)}{y} \tag{1.25}$$

This shows that the duration of a consol is finite even if its maturity is infinite. Also, it does not depend on the coupon.

This formula provides a useful rule of thumb. For a long-term coupon-paying bond, duration must be lower than $(1+y)/y$. For instance, when $y = 6\%$, the upper limit on duration is $D_C = 1.06/0.06$, or approximately 17.5 years. In this environment, the duration of a par 30-year bond is 14.25, which is indeed lower than 17.5 years.

Key concept:
The duration of a long-term bond can be approximated by an upper bound, which is that of a consol with the same yield, $D_C = (1+y)/y$.

Figure 1-6 describes the relationship between duration, maturity, and coupon for regular bonds in a 6% yield environment. For the zero-coupon bond, $D = T$, which is a straight line going through the origin. For the par 6% bond, duration increases monotonically with maturity until it reaches the asymptote of D_C. The 8% bond has lower duration than the 6% bond for fixed T. Greater coupons, for a fixed maturity, decrease duration, as more of the payments come early.

Finally, the 2% bond displays a pattern intermediate between the zero-coupon and 6% bonds. It initially behaves like the zero, exceeding D_C initially then falling back to the asymptote, which is common for all coupon-paying bonds.

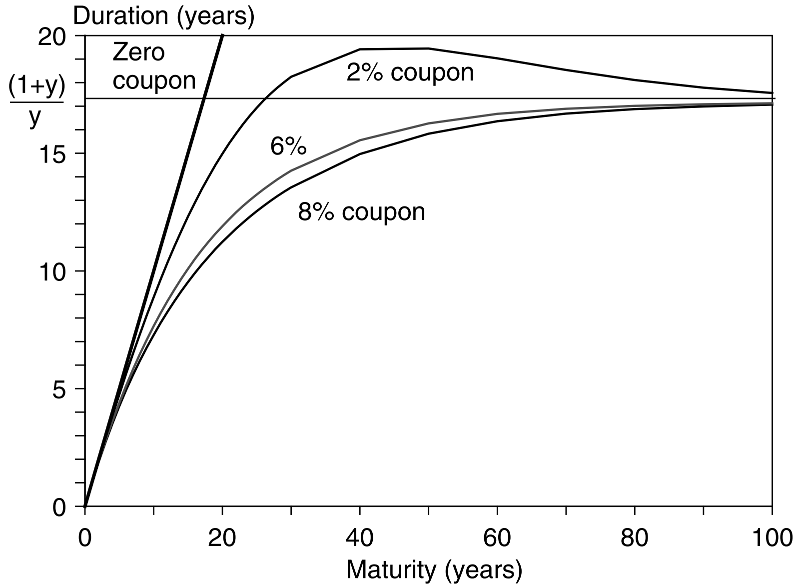
Taking now the second derivative in Equation (1.5), we have

$$\frac{d^2P}{dy^2} = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} = \left[\sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}}\right] \Big/ P \times P \tag{1.26}$$

which defines convexity as

$$C = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+y)^{t+2}} \Big/ P \tag{1.27}$$

FIGURE 1-6 Duration and Maturity



Convexity can also be written as

$$C = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^2} \times \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t} = \sum_{t=1}^{T} \frac{t(t+1)}{(1+y)^2} \times w_t \qquad (1.28)$$

which basically involves a weighted average of the square of time. Therefore, convexity is much greater for long-maturity bonds because they have payoffs associated with large values of t. The formula also shows that convexity is always positive for such bonds, implying that the curvature effect is beneficial. As we will see later, convexity can be negative for bonds that have uncertain cash flows, such as **mortgage-backed securities** (MBSs) or callable bonds.

Figure 1-7 displays the behavior of convexity, comparing a zero-coupon bond with a 6 percent coupon bond with identical maturities. The zero-coupon bond always has greater convexity, because there is only one cash flow at maturity. Its convexity is roughly the square of maturity, for example about 900 for the 30-year zero. In contrast, the 30-year coupon bond has a convexity of about 300 only.

As an illustration, Table 1-2 details the steps of the computation of duration and convexity for a two-year, 6 percent semiannual coupon-paying bond. We first convert

FIGURE 1-7 Convexity and Maturity

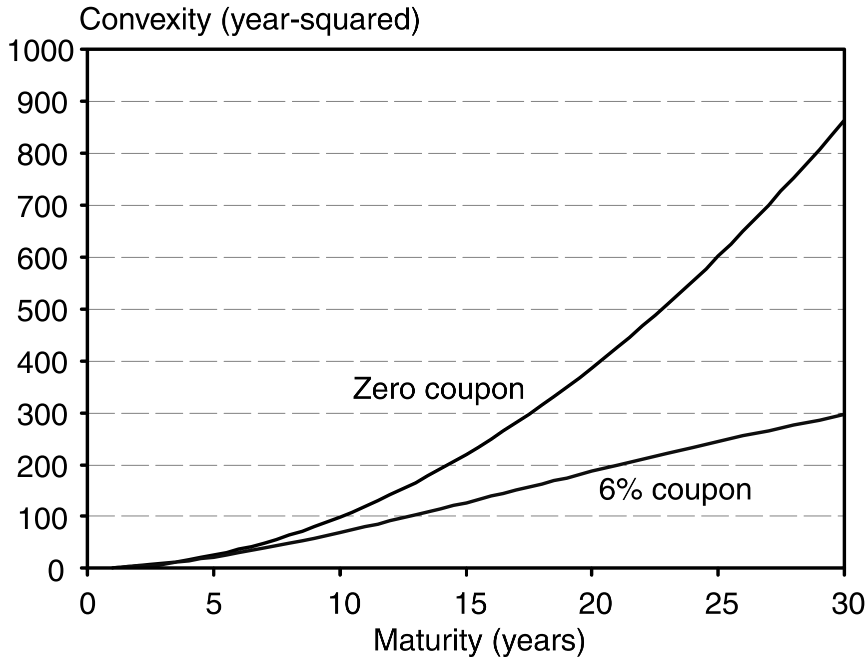


TABLE 1-2 Computing Duration and Convexity

Period (half-year) t	Payment C_t	Yield (%) (6 mo)	PV of Payment $C_t/(1+y)^t$	Duration Term tPV_t	Convexity Term $t(t+1)PV_t/(1+y)^2$
1	3	3.00	2.913	2.913	5.491
2	3	3.00	2.828	5.656	15.993
3	3	3.00	2.745	8.236	31.054
4	103	3.00	91.514	366.057	1725.218
Sum:			100.00	382.861	1777.755
(half-years)				3.83	17.78
(years)				1.91	
Modified duration				1.86	
Convexity					4.44

the annual coupon and yield into semiannual equivalent, $3 and 3 percent each. The *PV* column then reports the present value of each cash flow. We verify that these add up to $100, since the bond must be selling at par.

Next, the duration term column multiplies each *PV* term by time, or more precisely the number of half years until payment. This adds up to $382.86, which divided by the price gives $D = 3.83$. This number is measured in half years, and we need to divide by two to convert to years. Macaulay duration is 1.91 years, and modified duration $D^* = 1.91/1.03 = 1.86$ years. Note that, to be consistent, the adjustment in the denominator involves the semiannual yield of 3%.

Finally, the right-most column shows how to compute the bond's convexity. Each term involves PV_t times $t(t+1)/(1+y)^2$. These terms sum to 1,777.755, or divided by the price, 17.78. This number is expressed in units of time squared and must be divided by 4 to be converted in annual terms. We find a convexity of $C = 4.44$, in year-squared.

Example 1-9: FRM Exam 2001—Question 71

1-9. Calculate the modified duration of a bond with a Macauley duration of 13.083 years. Assume market interest rates are 11.5% and the coupon on the bond is paid semiannually.

a) 13.083
b) 12.732
c) 12.459
d) 12.371

Example 1-10: FRM Exam 2001—Question 66

1-10. Calculate the duration of a two-year bond paying a annual coupon of 6% with yield to maturity of 8%. Assume par value of the bond to be $1,000.

a) 2.00 years
b) 1.94 years
c) 1.87 years
d) 1.76 years

Example 1-11: FRM Exam 1998—Question 29/Quant. Analysis

1-11. A and B are two perpetual bonds, that is, their maturities are infinite. A has a coupon of 4% and B has a coupon of 8%. Assuming that both are trading at the same yield, what can be said about the duration of these bonds?

a) The duration of A is greater than the duration of B.
b) The duration of A is less than the duration of B.
c) A and B both have the same duration.
d) None of the above.

Example 1-12: FRM Exam 1997—Question 24/Market Risk

1-12. Which of the following is *not* a property of bond duration?

a) For zero-coupon bonds, Macaulay duration of the bond equals its years to maturity.

b) Duration is usually inversely related to the coupon of a bond.

c) Duration is usually higher for higher yields to maturity.

d) Duration is higher as the number of years to maturity for a bond. selling at par or above increases.

Example 1-13: FRM Exam 1999—Question 75/Market Risk

1-13. Suppose that your book has an unusually large short position in two investment grade bonds with similar credit risk. Bond A is priced at par yielding 6.0% with 20 years to maturity. Bond B also matures in 20 years with a coupon of 6.5% and yield of 6%. If risk is defined as a sudden and large drop in interest rate, which bond contributes greater market risk to the portfolio?

a) Bond A.

b) Bond B.

c) Both bond A and bond B will have similar market risk.

d) None of the above.

Example 1-14: FRM Exam 2000—Question 106/Quant. Analysis

1-14. Consider these five bonds:

Bond Number	Maturity (yrs)	Coupon Rate	Frequency	Yield (ABB)
1	10	6%	1	6%
2	10	6%	2	6%
3	10	0%	1	6%
4	10	6%	1	5%
5	9	6%	1	6%

How would you rank the bonds from the shortest to longest duration?

a) 5-2-1-4-3

b) 1-2-3-4-5

c) 5-4-3-1-2

d) 2-4-5-1-3

Example 1-15: FRM Exam 2001—Question 104
1-15. When the maturity of a plain coupon bond increases, its duration increases
a) Indefinitely and regularly
b) Up to a certain level
c) Indefinitely and progressively
d) In a way dependent on the bond being priced above or below par

1.2.5 Portfolio Duration and Convexity

Fixed-income portfolios often involve very large numbers of securities. It would be impractical to consider the movements of each security individually. Instead, portfolio managers aggregate the duration and convexity across the portfolio. A manager with a view that rates will increase, for instance, should shorten the portfolio duration relative to that of the benchmark. Say for instance that the benchmark has a duration of 5 years. The manager shortens the portfolio duration to 1 year only. If rates increase by 2 percent, the benchmark will lose approximately $5 \times 2\% = 10\%$. The portfolio, however, will only lose $1 \times 2\% = 2\%$, hence "beating" the benchmark by 8%.

Because the Taylor expansion involves a summation, the portfolio duration is easily obtained from the individual components. Say we have N components indexed by i. Defining D_p and P_p as the portfolio modified duration and value, the portfolio dollar duration (DD) is

$$D_p^* P_p = \sum_{i=1}^{N} D_i^* x_i P_i \tag{1.29}$$

where x_i is the number of units of bond i in the portfolio. A similar relationship holds for the portfolio dollar convexity (DC). If yields are the same for all components, this equation also holds for the Macaulay duration.

Because the portfolio total market value is simply the summation of the component market values,

$$P_p = \sum_{i=1}^{N} x_i P_i \tag{1.30}$$

we can define the **portfolio weight** w_i as $w_i = x_i P_i / P_p$, provided that the portfolio market value is nonzero. We can then write the portfolio duration as a weighted average of individual durations

$$D_p^* = \sum_{i=1}^{N} D_i^* w_i \tag{1.31}$$

Similarly, the portfolio convexity is a weighted average of individual convexity numbers

$$C_p = \sum_{i=1}^{N} C_i w_i \tag{1.32}$$

As an example, consider a portfolio invested in three bonds, described in Table 1-3. The portfolio is long a 10-year and 1-year bond, and short a 30-year zero-coupon bond. Its market value is \$1,301,600. Summing the duration for each component, the portfolio dollar duration is \$2,953,800, which translates into 2.27 years. The portfolio convexity is −76,918,323/1,301,600=−59.10, which is negative due to the short position in the 30-year zero, which has very high convexity.

Alternatively, assume the portfolio manager is given a benchmark that is the first bond. He or she wants to invest in bonds 1 and 2, keeping the portfolio duration equal to that of the target, or 7.44 years. To achieve the target value and dollar duration, the manager needs to solve a system of two equations in the amounts x_1 and x_2:

$$\text{Value: } \$100 = x_1 \$94.26 + x_2 \$16.97$$

$$\text{Dol. Duration: } 7.44 \times \$100 = 0.97 \times x_1 \$94.26 + 29.13 \times x_2 \$16.97$$

TABLE 1-3 Portfolio Duration and Convexity

	Bond 0	Bond 1	Bond 2	Portfolio
Maturity (years)	10	1	30	
Coupon	6%	0%	0%	
Yield	6%	6%	6%	
Price P_i	\$100.00	\$94.26	\$16.97	
Mod. duration D_i^*	7.44	0.97	29.13	
Convexity C_i	68.78	1.41	862.48	
Number of bonds x_i	10,000	5,000	−10,000	
Dollar amounts $x_i P_i$	\$1,000,000	\$471,300	−\$169,700	\$1,301,600
Weight w_i	76.83%	36.21%	−13.04%	100.00%
Dollar duration $D_i^* P_i$	\$744.00	\$91.43	\$494.34	
Portfolio DD: $x_i D_i^* P_i$	\$7,440,000	\$457,161	−\$4,943,361	\$2,953,800
Portfolio DC: $x_i C_i P_i$	68,780,000	664,533	−146,362,856	−76,918,323

The solution is $x_1 = 0.817$ and $x_2 = 1.354$, which gives a portfolio value of \$100 and modified duration of 7.44 years.[5] The portfolio convexity is 199.25, higher than the index. Such a portfolio consisting of very short and very long maturities is called a **barbell portfolio**. In contrast, a portfolio with maturities in the same range is called a **bullet portfolio**. Note that the barbell portfolio has much greater convexity than the bullet bond because of the payment in 30 years. Such a portfolio would be expected to outperform the bullet portfolio if yields move by a large amount.

In sum, duration and convexity are key measures of fixed-income portfolios. They summarize the linear and quadratic exposure to movements in yields. As such, they are routinely used by portfolio managers.

Example 1-16: FRM Exam 1998—Question 18/Quant. Analysis

1-16. A portfolio consists of two positions: One position is long \$100,000 par value of a two-year bond priced at 101 with a duration of 1.7; the other position is short \$50,000 of a five-year bond priced at 99 with a duration of 4.1. What is the duration of the portfolio?

a) 0.68
b) 0.61
c) −0.68
d) −0.61

Example 1-17: FRM Exam 2000—Question 110/Quant. Analysis

1-17. Which of the following statements are *true*?

I. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 10-year, 6% bond.
II. The convexity of a 10-year zero-coupon bond is higher than the convexity of a 6% bond with a duration of 10 years.
III. Convexity grows proportionately with the maturity of the bond.
IV. Convexity is always positive for all types of bonds.
V. Convexity is always positive for "straight" bonds.

a) I only
b) I and II only
c) I and V only
d) II, III, and V only

[5]This can be obtained by first expressing x_2 in the first equation as a function of x_1 and then substituting back into the second equation. This gives $x_2 = (100 - 94.26x_1)/16.97$, and $744 = 91.43x_1 + 494.34x_2 = 91.43x_1 + 494.34(100 - 94.26x_1)/16.97 = 91.43x_1 + 2913.00 - 2745.79x_1$. Solving, we find $x_1 = (-2169.00)/(-2654.36) = 0.817$ and $x_2 = (100 - 94.26 \times 0.817)/16.97 = 1.354$.

1.3 Answers to Chapter Examples

Example 1-1: FRM Exam 1999—Question 17/Quant. Analysis

b) This is derived from $(1 + y^S/2)^2 = (1 + y)$, or $(1 + 0.08/2)^2 = 1.0816$, which gives 8.16%. This makes sense because the annual rate must be higher due to the less frequent compounding.

Example 1-2: FRM Exam 1998—Question 28/Quant. Analysis

a) This is derived from $(1 + y^S/2)^2 = \exp(y)$, or $(1 + y^S/2)^2 = 1.105$, which gives 10.25%. This makes sense because the semiannual rate must be higher due to the less frequent compounding.

Example 1-3: FRM Exam 1998—Question 12/Quant. Analysis

d) We need to find y such that $\$4/(1 + y/2) + \$104/(1 + y/2)^2 = \$102.9$. Solving, we find $y = 5\%$. (This can be computed on a HP-12C calculator, for example.) There is another method for finding y. This bond has a duration of about one year, implying that, approximately, $\Delta P = -1 \times \$100 \times \Delta y$. If the yield was 8%, the price would be at \$100. Instead, the change in price is $\Delta P = \$102.9 - \$100 = \$2.9$. Solving for Δy, the change in yield must be approximately -3%, leading to $8 - 3 = 5\%$.

Example 1-4: FRM Exam 1999—Question 9/Quant. Analysis

c) First derivatives involve modified duration and delta. Second derivatives involve convexity (for bonds) and gamma (for options).

Example 1-5: FRM Exam 1998—Question 17/Quant. Analysis

c) This question deals with effective duration, which is obtained from full repricing of the bond with an increase and a decrease in yield. This gives a modified duration of $D^* = -(\Delta P/\Delta y)/P = -((99.95 - 100.04)/0.0002)/100 = 4.5$.

Example 1-6: FRM Exam 1998—Question 22/Quant. Analysis

c) Since this is a par bond, the initial price is $P = \$100$. The price impact is $\Delta P = -D^* P \Delta y + (1/2) C P (\Delta y)^2 = -7\$100(0.001) + (1/2)50\$100(0.001)^2 = -0.70 + 0.0025 = -0.6975$. The price falls slightly less than predicted by duration alone.

Example 1-7: FRM Exam 1998-Question 20/Quant. Analysis

b) The initial price of the 7s is 94. The price of the 6s is 92; this lower coupon is roughly equivalent to an upmove of $\Delta y = 0.01$. Similarly, the price of the 8s is 96.5; this higher coupon is roughly equivalent to a downmove of $\Delta y = 0.01$. The effective modified duration is then $D^E = (P_- - P_+)/(2\Delta y P_0) = (96.5 - 92)/(2 \times 0.01 \times 94) = 2.394$.

Example 1-8: FRM Exam 1998—Question 21/Quant. Analysis

a) We compute the modified duration for an equivalent downmove in y as $D_- = (P_- - P_0)/(\Delta y P_0) = (96.5 - 94)/(0.01 \times 94) = 2.6596$. Similarly, the modified duration for an upmove is $D_+ = (P_0 - P_+)/(\Delta y P_0) = (94 - 92)/(0.01 \times 94) = 2.1277$. Convexity is $C^E = (D_- - D_+)/(\Delta y) = (2.6596 - 2.1277)/0.01 = 53.19$. This is positive because modified duration is higher for a downmove than for an upmove in yields.

Example 1-9: FRM Exam 2001-Question 71

d) Modified duration is $D^* = D/(1 + y/200)$ when yields are semiannually compounded. This gives $D^* = 13.083/(1 + 11.5/200) = 12.3716$.

Example 1-10: FRM Exam 2001—Question 66

b) Using an 8% annual discount factor, we compute the present value of cash flows and duration as

Year	C_t	PV	$t\ PV$
1	60	55.56	55.55
2	1,060	908.78	1,817.56
Sum		964.33	1,873.11

Duration is 1,873.11/964.33 = 1.942 years. Note that the par value is irrelevant for the computation of duration.

Example 1-11: FRM Exam 1998—Question 29/Quant. Analysis

c) Going back to the duration equation for the consol, Equation (1.25), we see that it does not depend on the coupon but only on the yield. Hence, the durations must be the same. The price of bond A, however, must be half that of bond B.

Example 1-12: FRM Exam 1997—Question 24/Market Risk

c) Duration usually increases as the time to maturity increases (Figure 1-4), so (d) is correct. Macaulay duration is also equal to maturity for zero-coupon bonds, so (a) is correct. Figure 1-5 shows that duration decreases with the coupon, so (b) is correct. As the yield increases, the weight of the payments further into the future decreases, which *decreases* (not increases) the duration. So, (c) is false.

Example 1-13: FRM Exam 1999—Question 75/Market Risk

a) Bond B has a higher coupon and hence a slightly lower duration than for bond A. Therefore, it will react less strongly than bond A to a given change in yields.

Example 1-14: FRM Exam 2000—Question 106/Quant. Analysis

a) The nine-year bond (number 5) has shorter duration because the maturity is shortest, at nine years, among comparable bonds. Next, we have to decide between bonds 1 and 2, which only differ in the payment frequency. The semiannual bond (number 2) has a first payment in six months and has shorter duration than the annual bond. Next, we have to decide between bonds 1 and 4, which only differ in the yield. With lower yield, the cash flows further in the future have a higher weight, so that bond 4 has greater duration. Finally, the zero-coupon bond has the longest duration. So, the order is 5-2-1-4-3.

Example 1-15: FRM Exam 2001—Question 104

b) With a fixed coupon, the duration goes up to the level of a consol with the same coupon. See Figure 1-6.

Example 1-16: FRM Exam 1998—Question 18/Quant. Analysis

d) The dollar duration of the portfolio must equal the sum of the dollar durations for the individual positions, as in Equation (1.29). First, we need to compute the market value of the bonds by multiplying the notional by the ratio of the market price to the face value. This gives for the first bond \$100,000 (101/100) = \$101,000 and for the second \$50,000 (99/100) = \$49,500. The value of the portfolio is $P = \$101,000 - \$49,500 = \$51,500$.

Next, we compute the dollar duration as $\$101,000 \times 1.7 = \$171,700$ and $-\$49,500 \times 4.1 = -\$202,950$, respectively. The total dollar duration is $-\$31,250$. Dividing by \$51.500, we find a duration of $DD/P = -0.61$ year. Note that duration is negative due to the short position. We also ignored the denominator $(1 + y)$, which cancels out from the computation anyway if the yield is the same for the two bonds.

Example 1-17: FRM Exam 2000—Question 110/Quant. Analysis

c) Because convexity is proportional to the square of time to payment, the convexity of a bond will be driven by the cash flows far into the future. Answer I is correct because the 10-year zero has only one cash flow, whereas the coupon bond has several others that reduce convexity. Answer II is false because the 6% bond with 10-year duration must have cash flows much further into the future, say in 30 years, which will create greater convexity. Answer III is false because convexity grows with the square of time. Answer IV is false because some bonds, for example MBSs or callable bonds, can have negative convexity. Answer V is correct because convexity must be positive for coupon-paying bonds.

Appendix: Applications of Infinite Series

When bonds have fixed coupons, the bond valuation problem often can be interpreted in terms of combinations of infinite series. The most important infinite series result is for a sum of terms that increase at a geometric rate:

$$1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a} \tag{1.33}$$

This can be proved, for instance, by multiplying both sides by $(1 - a)$ and canceling out terms.

Equally important, consider a geometric series with a finite number of terms, say N. We can write this as the difference between two infinite series:

$$1 + a + a^2 + a^3 + \cdots + a^{N-1} = (1 + a + a^2 + a^3 + \cdots) - a^N(1 + a + a^2 + a^3 + \cdots) \tag{1.34}$$

such that all terms with order N or higher will cancel each other.

We can then write

$$1 + a + a^2 + a^3 + \cdots + a^{N-1} = \frac{1}{1-a} - a^N \frac{1}{1-a} \tag{1.35}$$

These formulas are essential to value bonds. Consider first a consol with an infinite number of coupon payments with a fixed coupon rate c. If the yield is y and the face value F, the value of the bond is

$$\begin{aligned}
P &= cF\left[\frac{1}{(1+y)} + \frac{1}{(1+y)^2} + \frac{1}{(1+y)^3} + \cdots\right] \\
&= cF\frac{1}{(1+y)}[1 + a^2 + a^3 + \cdots] \\
&= cF\frac{1}{(1+y)}\left[\frac{1}{1-a}\right] \\
&= cF\frac{1}{(1+y)}\left[\frac{1}{1-(1/(1+y))}\right] \\
&= cF\frac{1}{(1+y)}\left[\frac{(1+y)}{y}\right] \\
&= \frac{c}{y}F
\end{aligned}$$

Similarly, we can value a bond with a *finite* number of coupons over T periods at which time the principal is repaid. This is really a portfolio with three parts:

(1) A long position in a consol with coupon rate c

(2) A short position in a consol with coupon rate c that starts in T periods

(3) A long position in a zero-coupon bond that pays F in T periods

Note that the combination of (1) and (2) ensures that we have a finite number of coupons. Hence, the bond price should be

$$P = \frac{c}{y}F - \frac{1}{(1+y)^T}\frac{c}{y}F + \frac{1}{(1+y)^T}F = \frac{c}{y}F\left[1 - \frac{1}{(1+y)^T}\right] + \frac{1}{(1+y)^T}F \qquad (1.36)$$

where again the formula can be adjusted for different compounding methods.

This is useful for a number of purposes. For instance, when $c = y$, it is immediately obvious that the price must be at par, $P = F$. This formula also can be used to find closed-form solutions for duration and convexity.

Chapter 2

Fundamentals of Probability

The preceding chapter has laid out the foundations for understanding how bond prices move in relation to yields. Next, we have to characterize movements in bond yields or, more generally, any relevant risk factor in financial markets.

This is done with the tools of probability, a mathematical abstraction that describes the distribution of risk factors. Each risk factor is viewed as a random variable whose properties are described by a probability distribution function. These distributions can be processed with the price-yield relationship to create a distribution of the profit and loss profile for the trading portfolio.

This chapter reviews the fundamental tools of probability theory for risk managers. Section 2.1 lays out the foundations, characterizing random variables by their probability density and distribution functions. These functions can be described by their principal moments, mean, variance, skewness, and kurtosis. Distributions with multiple variables are described in Section 2.2. Section 2.3 then turns to functions of random variables. Finally, Section 2.4 presents some examples of important distribution functions for risk management, including the uniform, normal, lognormal, Student's, and binomial.

2.1 Characterizing Random Variables

The classical approach to probability is based on the concept of the **random variable**. This can be viewed as the outcome from throwing a die, for example. Each realization is generated from a fixed process. If the die is perfectly symmetric, we could say that the probability of observing a face with a six in one throw is $p = 1/6$. Although the event itself is random, we can still make a number of useful statements from a fixed data-generating process.

The same approach can be taken to financial markets, where stock prices, exchange rates, yields, and commodity prices can be viewed as random variables. The assumption of a fixed data-generating process for these variables, however, is more tenuous than for the preceding experiment.

2.1.1 Univariate Distribution Functions

A random variable X is characterized by a **distribution function**,

$$F(x) = P(X \leq x) \tag{2.1}$$

which is the probability that the realization of the random variable X ends up less than or equal to the given number x. This is also called a **cumulative distribution function**.

When the variable X takes discrete values, this distribution is obtained by summing the step values less than or equal to x. That is,

$$F(x) = \sum_{x_j \leq x} f(x_j) \tag{2.2}$$

where the function $f(x)$ is called the **frequency function** or the **probability density function** (p.d.f.). This is the probability of observing x.

When the variable is continuous, the distribution is given by

$$F(x) = \int_{-\infty}^{x} f(u)du \tag{2.3}$$

The density can be obtained from the distribution using

$$f(x) = \frac{dF(x)}{dx} \tag{2.4}$$

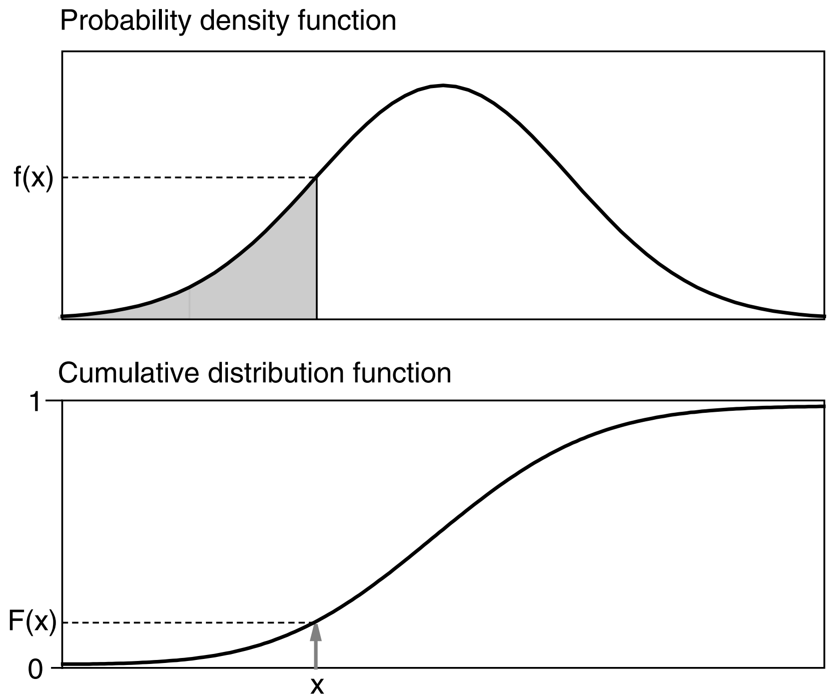
Often, the random variable will be described interchangeably by its distribution or its density.

These functions have notable properties. The density $f(u)$ must be positive for all u. As x tends to infinity, the distribution tends to unity as it represents the total probability of any draw for x:

$$\int_{-\infty}^{\infty} f(u)du = 1 \tag{2.5}$$

Figure 2-1 gives an example of a density function $f(x)$, on the top panel, and of a cumulative distribution function $F(x)$ on the bottom panel. $F(x)$ measures the area under the $f(x)$ curve to the left of x, which is represented by the shaded area. Here, this area is 0.24. For small values of x, $F(x)$ is close to zero. Conversely, for large values of x, $F(x)$ is close to unity.

FIGURE 2-1 Density and Distribution Functions



Example: Density functions

A gambler wants to characterize the probability density function of the outcomes from a pair of dice. Out of 36 possible throws, we can have one occurrence of an outcome of two (each die showing one). We can have two occurrences of a three (a one and a two and vice versa), and so on.

The gambler builds the frequency table for each value, from 2 to 12.

- From this, he or she can compute the probability of each outcome. For instance, the probability of observing three is equal to 2, the frequency $n(x)$, divided by the total number of outcomes, of 36, which gives 0.0556. We can verify that all the probabilities indeed add up to one, since all occurrences must be accounted for.
- From the table, we see that the probability of an outcome of 3 or less is 8.33%.

2.1.2 Moments

A random variable is characterized by its distribution function. Instead of having to report the whole function, it is convenient to focus on a few parameters of interest.

TABLE 2-1 Probability Density Function

Outcome x_i	Frequency $n(x)$	Probability $f(x)$	Cumulative Probability $F(x)$
2	1	0.0278	0.0278
3	2	0.0556	0.0833
4	3	0.0833	0.1667
5	4	0.1111	0.2778
6	5	0.1389	0.4167
7	6	0.1667	0.5833
8	5	0.1389	0.7222
9	4	0.1111	0.8333
10	3	0.0833	0.9167
11	2	0.0556	0.9722
12	1	0.0278	1.0000
Sum	36	1.0000	

It is useful to describe the distribution by its **moments.** For instance, the expected value for x, or **mean**, is given by the integral

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) dx \tag{2.6}$$

which measures the *central tendency*, or *center of gravity* of the population.

The distribution can also be described by its **quantile**, which is the cutoff point x with an associated probability c:

$$F(x) = \int_{-\infty}^{x} f(u) du = c \tag{2.7}$$

So, there is a probability of c that the random variable will fall *below* x. Because the total probability adds up to one, there is a probability of $p = 1 - c$ that the random variable will fall *above* x. Define this quantile as $Q(X, c)$. The 50% quantile is known as the **median**.

In fact, value at risk (VAR) can be interpreted as the cutoff point such that a loss will not happen with probability greater than $p = 95\%$ percent, say. If $f(u)$ is the distribution of profit and losses on the portfolio, VAR is defined from

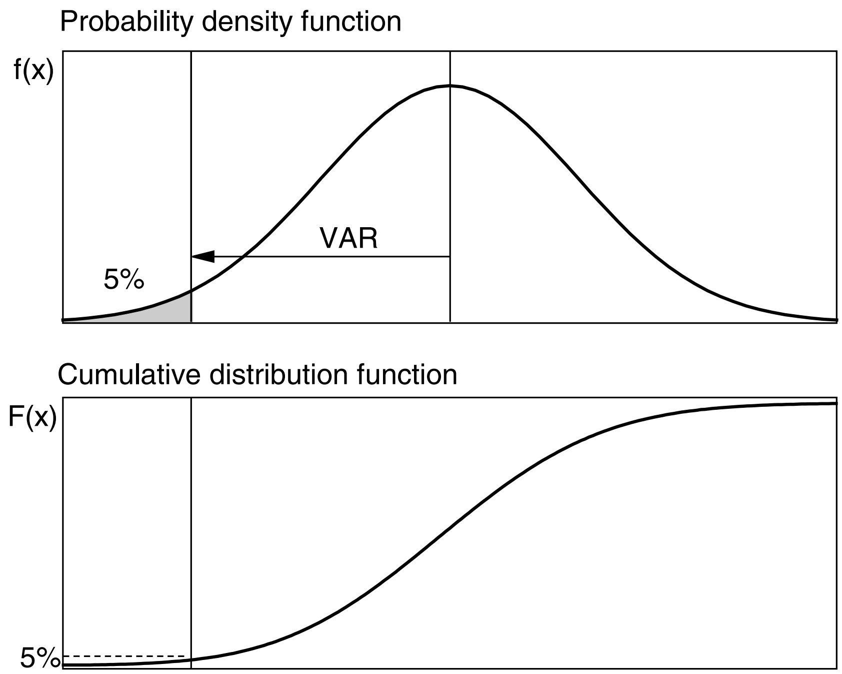
$$F(x) = \int_{-\infty}^{x} f(u) du = (1 - p) \tag{2.8}$$

where p is the right-tail probability, and c the usual left-tail probability. VAR can then be defined as the deviation between the expected value and the quantile,

$$\text{VAR}(c) = E(X) - Q(X, c) \tag{2.9}$$

Figure 2-2 shows an example with $c = 5\%$.

FIGURE 2-2 VAR as a Quantile



Another useful moment is the squared dispersion around the mean, or **variance**, which is

$$\sigma^2 = V(X) = \int_{-\infty}^{+\infty} [x - E(X)]^2 f(x)dx \tag{2.10}$$

The **standard deviation** is more convenient to use as it has the same units as the original variable X

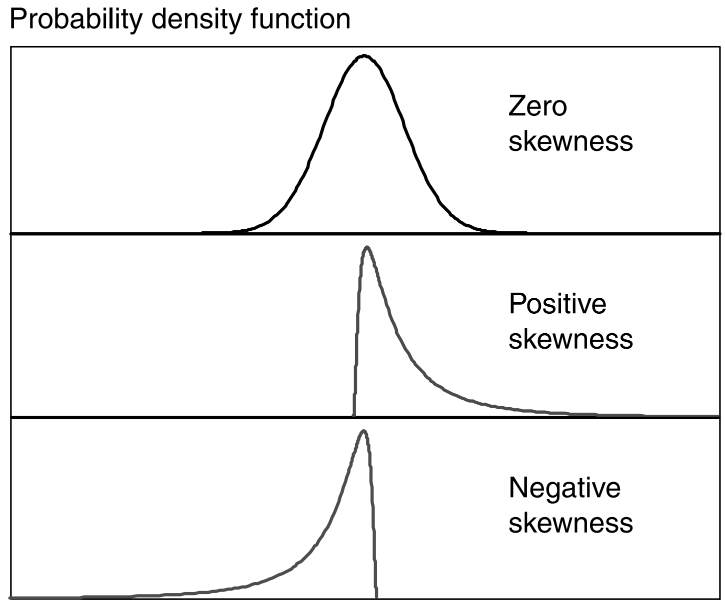
$$\mathrm{SD}(X) = \sigma = \sqrt{V(X)} \tag{2.11}$$

Next, the scaled third moment is the **skewness**, which describes departures from symmetry. It is defined as

$$\gamma = \left(\int_{-\infty}^{+\infty} [x - E(X)]^3 f(x)dx \right) \Big/ \sigma^3 \tag{2.12}$$

Negative skewness indicates that the distribution has a long left tail, which indicates a high probability of observing large negative values. If this represents the distribution of profits and losses for a portfolio, this is a dangerous situation. Figure 2-3 displays distributions with various signs for the skewness.

FIGURE 2-3 Effect of Skewness



The scaled fourth moment is the **kurtosis**, which describes the degree of "flatness" of a distribution, or width of its tails. It is defined as

$$\delta = \left(\int_{-\infty}^{+\infty} [x - E(X)]^4 f(x) dx \right) \Big/ \sigma^4 \qquad (2.13)$$

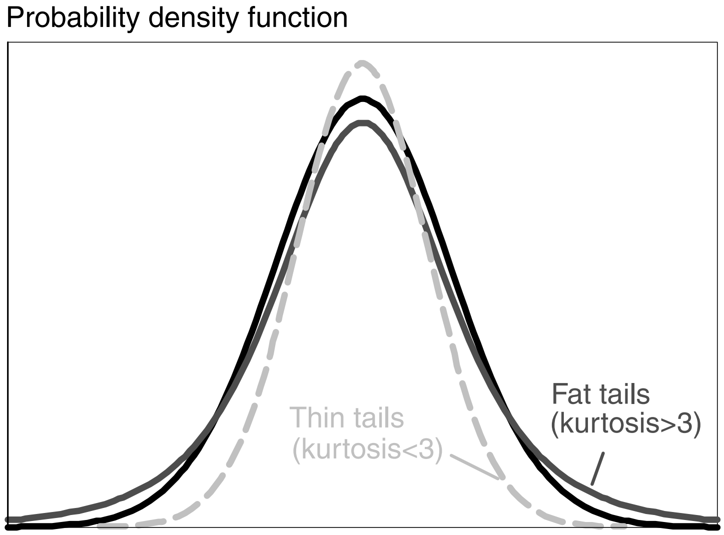
Because of the fourth power, large observations in the tail will have a large weight and hence create large kurtosis. Such a distribution is called **leptokurtic**, or **fat-tailed**. This parameter is very important for risk measurement. A kurtosis of 3 is considered average. High kurtosis indicates a higher probability of extreme movements. Figure 2-4 displays distributions with various values for the kurtosis.

Example: Computing moments

Our gambler wants to know the expected value of the outcome of throwing two dice. He or she computes the product of the probability and outcome. For instance, the first entry is $xf(x) = 2 \times 0.0278 = 0.0556$, and so on. Summing across all events, this gives the mean as $\mu = 7.000$. This is also the median, since the distribution is perfectly symmetric.

Next, the variance terms sum to 5.8333, for a standard deviation of $\sigma = 2.4152$. The skewness terms sum to zero, because for each entry with a positive deviation $(x - \mu)^3$, there is an identical one with a negative sign and with the same probability.

FIGURE 2-4 Effect of Kurtosis



Finally, the kurtosis terms $(x-\mu)^4 f(x)$ sum to 80.5. Dividing by σ^4, this gives a kurtosis of $\delta = 2.3657$.

2.2 Multivariate Distribution Functions

In practice, portfolio payoffs depend on numerous random variables. To simplify, start with two random variables. This could represent two currencies, or two interest rate factors, or default and credit exposure, to give just a few examples.

We can extend Equation (2.1) to

$$F_{12}(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2) \tag{2.14}$$

which defines a joint bivariate distribution function. In the continuous case, this is also

$$F_{12}(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{12}(u_1, u_2) du_1 du_2 \tag{2.15}$$

where $f(u_1, u_2)$ is now the **joint density**. In general, adding random variables considerably complicates the characterization of the density or distribution functions.

The analysis simplifies considerably if the variables are **independent**. In this case, the joint density separates out into the product of the densities:

$$f_{12}(u_1 u_2) = f_1(u_1) \times f_2(u_2) \tag{2.16}$$

and the integral reduces to

$$F_{12}(x_1, x_2) = F_1(x_1) \times F_2(x_2) \tag{2.17}$$

TABLE 2-2 Computing Moments of a Distribution

Outcome x_i	Prob. $f(x)$	Mean $xf(x)$	Variance $(x-\mu)^2 f(x)$	Skewness $(x-\mu)^3 f(x)$	Kurtosis $(x-\mu)^4 f(x)$
2	0.0278	0.0556	0.6944	-3.4722	17.3611
3	0.0556	0.1667	0.8889	-3.5556	14.2222
4	0.0833	0.3333	0.7500	-2.2500	6.7500
5	0.1111	0.5556	0.4444	-0.8889	1.7778
6	0.1389	0.8333	0.1389	-0.1389	0.1389
7	0.1667	1.1667	0.0000	0.0000	0.0000
8	0.1389	1.1111	0.1389	0.1389	0.1389
9	0.1111	1.0000	0.4444	0.8889	1.7778
10	0.0833	0.8333	0.7500	2.2500	6.7500
11	0.0556	0.6111	0.8889	3.5556	14.2222
12	0.0278	0.3333	0.6944	3.4722	17.3611
Sum	1.0000	7.0000	5.8333	0.0000	80.5000
Denominator				14.0888	34.0278
		Mean	StdDev	Skewness	Kurtosis
		7.0000	2.4152	0.0000	2.3657

In other words, the joint probability reduces to the product of the probabilities.

This is very convenient because we only need to know the individual densities to reconstruct the joint density. For example, a credit loss can be viewed as a combination of (1) default, which is a random variable with a value of one for default and zero otherwise, and (2) the exposure, which is a random variable representing the amount at risk, for instance the positive market value of a swap. If the two variables are independent, we can construct the distribution of the credit loss easily. In the case of the two dice, the probability of a joint event is simply the product of probabilities. For instance, the probability of throwing two ones is equal to $1/6 \times 1/6 = 1/36$.

It is also useful to characterize the distribution of x_1 abstracting from x_2. By integrating over all values of x_2, we obtain the **marginal density**

$$f_1(x_1) = \int_{-\infty}^{\infty} f_{12}(x_1, u_2) du_2 \tag{2.18}$$

and similarly for x_2. We can then define the **conditional density** as

$$f_{1\cdot 2}(x_1 \mid x_2) = \frac{f_{12}(x_1, x_2)}{f_2(x_2)} \tag{2.19}$$

Here, we keep x_2 fixed and divide the joint density by the marginal probability of observing x_2. This normalization is necessary to ensure that the conditional density is a proper density function that integrates to one. This relationship is also known as **Bayes' rule**.

When dealing with two random variables, the comovement can be described by the **covariance**

$$\mathrm{Cov}(X_1, X_2) = \sigma_{12} = \int_1 \int_2 [x_1 - E(X_1)][x_2 - E(X_2)] f_{12}(x_1, x_2) dx_1 dx_2 \qquad (2.20)$$

It is often useful to scale the covariance into a unitless number, called the **correlation coefficient**, obtained as

$$\rho(X_1, X_2) = \frac{\mathrm{Cov}(X_1, X_2)}{\sigma_1 \sigma_2} \qquad (2.21)$$

The correlation coefficient is a measure of linear dependence. One can show that the correlation coefficient always lies in the $[-1, +1]$ interval. A correlation of one means that the two variables always move in the same direction. A correlation of minus one means that the two variables always move in opposite direction.

If the variables are independent, the joint density separates out and this becomes

$$\mathrm{Cov}(X_1, X_2) = \left\{ \int_1 [x_1 - E(X_1)] f_1(x_1) dx_1 \right\} \left\{ \int_2 [x_2 - E(X_2)] f_2(x_2) dx_2 \right\} = 0,$$

by Equation (2.6), since the average deviation from the mean is zero. In this case, the two variables are said to be **uncorrelated**. Hence independence implies zero correlation (the reverse is not true, however).

Example: Multivariate functions

Consider two variables, such as the Canadian dollar and the euro. Table 2-3a describes the joint density function $f_{12}(x_1, x_2)$, assuming two payoffs only for each variable.

TABLE 2-3a Joint Density Function

x_1 / x_2	−5	+5
−10	0.30	0.15
+10	0.20	0.35

- From this, we can compute the marginal densities, the mean and standard deviation of each variable. For instance, the marginal probability of $x_1 = -5$ is given by $f_1(x_1) = f_{12}(x_1, x_2 = -10) + f_{12}(x_1, x_2 = +10) = 0.30 + 0.20 = 0.50$. Table 2-3b shows that the mean and standard deviations are $\overline{x}_1 = 0.0, \sigma_1 = 5.0$, $\overline{x}_1 = 1.0$, $\sigma_2 = 9.95$.

Finally, Table 2-3c details the computation of the covariance, which gives Cov = 15.00. Dividing by the product of the standard deviations, we get $\rho = \mathrm{Cov}/(\sigma_1 \sigma_2) = 15.00/(5.00 \times 9.95) = 0.30$. The positive correlation indicates that when one variable goes up, the other is more likely to go up than down.

TABLE 2-3b Marginal Density Functions

	Variable 1				Variable 2		
x_1	Prob. $f_1(x_1)$	Mean $x_1 f_1(x_1)$	Variance $(x_1 - \bar{x}_1)^2 f_1(x_1)$	x_2	Prob. $f_2(x_2)$	Mean $x_2 f_2(x_2)$	Variance $(x_2 - \bar{x}_2)^2 f_2(x_2)$
−5	0.50	−2.5	12.5	−10	0.45	−4.5	54.45
+5	0.50	+2.5	12.5	+10	0.55	+5.5	44.55
Sum	1.00	0.0	25.0		1.00	1.0	99.0
		$\bar{x}_1 = 0.0$	$\sigma_1 = 5.0$			$\bar{x}_2 = 1.0$	$\sigma_2 = 9.95$

TABLE 2-3c Covariance and Correlation

	$(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)f_{12}(x_1, x_2)$	
	x_1 =-5	x_1 =+5
x_2 =-10	(-5-0)(-10-1)0.30=16.50	(+5-0)(-10-1)0.15=-8.25
x_2 =+10	(-5-0)(+10-1)0.20=-9.00	(+5-0)(+10-1)0.35=15.75
Sum	Cov=15.00	

Example 2-1: FRM Exam 1999—Question 21/Quant. Analysis

2-1. The covariance between variable *A* and variable *B* is 5. The correlation between *A* and *B* is 0.5. If the variance of *A* is 12, what is the variance of B?

a) 10.00
b) 2.89
c) 8.33
d) 14.40

Example 2-2: FRM Exam 2000—Question 81/Market Risk

2-2. Which one of the following statements about the correlation coefficient is *false*?

a) It always ranges from −1 to +1.
b) A correlation coefficient of zero means that two random variables are independent.
c) It is a measure of linear relationship between two random variables.
d) It can be calculated by scaling the covariance between two random variables.

2.3 Functions of Random Variables

Risk management is about uncovering the distribution of portfolio values. Consider a security that depends on a unique source of risk, such as a bond. The risk manager could model the change in the bond price as a random variable directly. The problem with this choice is that the distribution of the bond price is not stationary, because the price converges to the face value at expiration. Instead, the practice is to model changes in yields as random variables because their distribution is better behaved.

The next step is to characterize the distribution of the bond price, which is a nonlinear function of the yield. A similar issue occurs for an option-trading desk, which contains many different positions all dependent on the value of the underlying asset, in a highly nonlinear fashion.

More generally, the portfolio contains assets that depend on many sources of risk. The risk manager would like to describe the distribution of portfolio values from information about the instruments and the joint density of all the random variables. Generally, the approach consists of integrating the joint density function over the appropriate space. This is no easy matter, unfortunately. We first focus on simple transformations, for which we provide expressions for the mean and variance.

2.3.1 Linear Transformation of Random Variables

Consider a transformation that multiplies the original random variable by a constant and add a fixed amount, $Y = a + bX$. The expectation of Y is

$$E(a + bX) = a + bE(X) \tag{2.22}$$

and its variance is

$$V(a + bX) = b^2 V(X) \tag{2.23}$$

Note that adding a constant never affects the variance since the computation involves the *difference* between the variable and its mean. The standard deviation is

$$\text{SD}(a + bX) = b\text{SD}(X) \tag{2.24}$$

Example: Currency position plus cash

Consider the distribution of the dollar/yen exchange rate X, which is the price of one Japanese yen. We wish to find the distribution of a portfolio with \$1 million in cash plus 1,000 million worth of Japanese yen. The portfolio value can be written as $Y = a + bX$, with fixed parameters (in millions) $a = \$1$ and $b = Y1,000$.

Therefore, if the expectation of the exchange rate is $E(X) = 1/100$, with a standard deviation of $\text{SD}(X) = 0.10/100 = 0.001$, the portfolio expected value is $E(Y) = \$1 + Y1,000 \times 1/100 = \$11$ million, and the standard deviation is $\text{SD}(Y) = Y1,000 \times 0.001 = \$1$ million.

2.3.2 Sum of Random Variables

Another useful transformation is the summation of two random variables. A portfolio, for instance, could contain one share of Intel plus one share of Microsoft. Each stock price behaves as a random variable.

The expectation of the sum $Y = X_1 + X_2$ can be written as

$$E(X_1 + X_2) = E(X_1) + E(X_2) \tag{2.25}$$

and its variance is

$$V(X_1 + X_2) = V(X_1) + V(X_2) + 2\text{Cov}(X_1, X_2) \tag{2.26}$$

When the variables are uncorrelated, the variance of the sum reduces to the sum of variances. Otherwise, we have to account for the cross-product term.

> **Key concept:**
> The expectation of a sum is the sum of expectations. The variance of a sum, however, is only the sum of variances if the variables are uncorrelated.

2.3.3 Portfolios of Random Variables

More generally, consider a linear combination of a number of random variables. This could be a portfolio with fixed weights, for which the rate of return is

$$Y = \sum_{i=1}^{N} w_i X_i \tag{2.27}$$

where N is the number of assets, X_i is the rate of return on asset i, and w_i its weight.

To shorten notation, this can be written in matrix notation, replacing a string of numbers by a single vector:

$$Y = w_1X_1 + w_2X_2 + \cdots + w_NX_N = [w_1 w_2 \ldots w_N]\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} = w'X \tag{2.28}$$

where w' represents the transposed vector (i.e., horizontal) of weights and X is the vertical vector containing individual asset returns. The appendix for this chapter provides a brief review of matrix multiplication.

The portfolio expected return is now

$$\text{E}(Y) = \mu_p = \sum_{i=1}^{N} w_i \mu_i \tag{2.29}$$

which is a weighted average of the expected returns $\mu_i = E(X_i)$. The variance is

$$V(Y) = \sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} w_i w_j \sigma_{ij} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \sigma_{ij} \quad (2.30)$$

Using matrix notation, the variance can be written as

$$\sigma_p^2 = [w_1 \ldots w_N] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \ldots & \sigma_{1N} \\ \vdots & & & & \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \ldots & \sigma_N \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

Defining Σ as the covariance matrix, the variance of the portfolio rate of return can be written more compactly as

$$\sigma_p^2 = w' \Sigma w \quad (2.31)$$

This is a useful expression to describe the risk of the total portfolio.

Example: Computing the risk of a portfolio

Consider a portfolio invested in Canadian dollars and euros. The joint density function is given by Table 2-3a. Here, x_1 describes the payoff on the Canadian dollar, with $\mu_1 = 0.00$ and $\sigma_1 = 5.00$. For the euro, $\mu_2 = 1.00$ and $\sigma_1 = 9.95$. The covariance was computed as $\sigma_{12} = 15.00$, with the correlation $\rho = 0.30$. If we have 60% invested in Canadian dollar and 40% in euros, what is the portfolio volatility?

Following Equation (2.31), we write

$$\sigma_p^2 = [0.60\ 0.40] \begin{bmatrix} 25.00 & 15.00 \\ 15.00 & 99.00 \end{bmatrix} \begin{bmatrix} 0.60 \\ 0.40 \end{bmatrix} = [0.60\ 0.40] \begin{bmatrix} 21.00 \\ 48.60 \end{bmatrix} = 32.04$$

Therefore, the portfolio volatility is $\sigma_p = 5.66$. Note that this is hardly higher than the volatility of the Canadian dollar alone, even though the risk of the euro is much higher. The portfolio risk has been kept low due to diversification effects. Keeping the same data but reducing ρ to -0.5 reduces the portfolio volatility even further, to $\sigma_p = 3.59$.

2.3.4 Product of Random Variables

Some risks result from the product of two random variables. A credit loss, for instance, arises from the product of the occurrence of default and the loss given default.

Using Equation (2.20), the expectation of the product $Y = X_1 X_2$ can be written as

$$E(X_1\ X_2) = E(X_1)E(X_2) + \text{Cov}(X_1, X_2) \quad (2.32)$$

When the variables are independent, this reduces to the product of the means.

The variance is more complex to evaluate. With independence, it reduces to

$$V(X_1 \; X_2) = E(X_1)^2 V(X_2) + V(X_1)E(X_2)^2 + V(X_1)V(X_2) \tag{2.33}$$

2.3.5 Distributions of Transformations of Random Variables

The preceding results focus on the mean and variance of simple transformations only. They say nothing about the distribution of the transformed variable $Y = g(X)$ itself. The derivation of the density function of Y, unfortunately, is usually complicated for all but the simplest transformations $g(\cdot)$ and densities $f(X)$.

Even if there is no closed-form solution for the density, we can describe the cumulative distribution function of Y when $g(X)$ is a one-to-one transformation from X into Y, that is can be inverted. We can then write

$$P[Y \le y] = P[g(X) \le y] = P[X \le g^{-1}(y)] = F_X(g^{-1}(y)) \tag{2.34}$$

where $F(\cdot)$ is the cumulative distribution function of X. Here, we assumed the relationship is positive. Otherwise, the right-hand term is changed to $1 - F_X(g^{-1}(y))$.

This allows us to derive the quantile of, say, the bond price from information about the distribution of the yield. Suppose we consider a zero-coupon bond, for which the market value V is

$$V = \frac{100}{(1+r)^T} \tag{2.35}$$

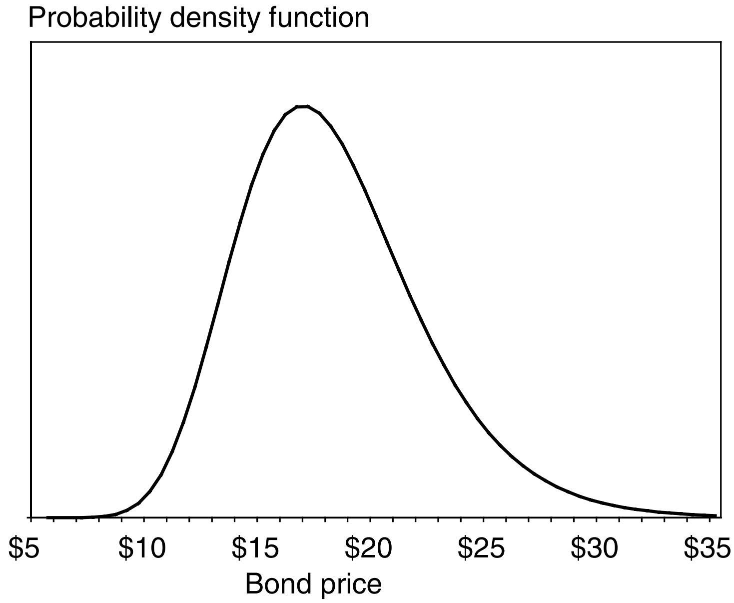
where r is the yield. This equation describes V as a function of r, or $Y = g(X)$. Using $r = 6\%$ and $T = 30$ years, this gives the current price $V = \$17.41$. The inverse function $X = g^{-1}(Y)$ is

$$r = (100/V)^{1/T} - 1 \tag{2.36}$$

We wish to estimate the probability that the bond price could fall below \$15. Using Equation (2.34), we first invert the transformation and compute the associated yield level, $g^{-1}(y) = (100/\$15)^{1/T} - 1 = 6.528\%$. The probability is given by

$$P[Y \le \$15] = F_X[r \ge 6.528\%] \tag{2.37}$$

FIGURE 2-5 Density Function for the Bond Price



Assuming the yield change is normal with volatility 0.8%, this gives a probability of 25.5 percent.[1] Even though we do not know the density of the bond price, this method allows us to trace out its cumulative distribution by changing the cutoff price of $15. Taking the derivative, we can recover the density function of the bond price. Figure 2-3 shows that this p.d.f. is skewed to the right.

Indeed the bond price can take large values if the yield falls to small values, yet cannot turn negative. On the extreme right, if the yield falls to zero, the bond price will go to $100. On the extreme left, if the yield goes to infinity, the bond price will fall to, but not go below, zero. Relative to the initial value of $15, there is a greater likelihood of large movements up than down.

This method, unfortunately, cannot be easily extended. For general densities, transformations, and numbers of random variables, risk managers need to turn to numerical methods. This is why credit risk models, for instance, all describe the distribution of credit losses through simulations.

---

[1] We shall see later that this is obtained from the standard normal variable $z = (6.528 - 6.000)/0.80 = 0.660$. Using standard normal tables, or the "=NORMSDIST(−0.660)" Excel function, this gives 25.5%.

2.4 Important Distribution Functions

2.4.1 Uniform Distribution

The simplest continuous distribution function is the **uniform distribution**. This is defined over a range of values for x, $a \leq x \leq b$. The density function is

$$f(x) = \frac{1}{(b-a)}, \quad a \leq x \leq b \tag{2.38}$$

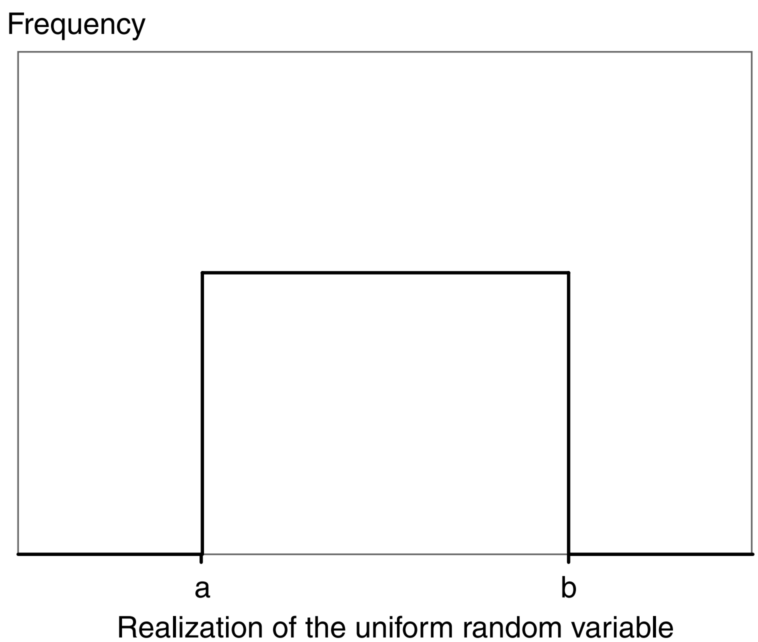
which is constant and indeed integrates to unity. This distribution puts the same weight on each observation within the allowable range, as shown in Figure 2-6. We denote this distribution as $U(a, b)$.

Its mean and variance are given by

$$E(X) = \frac{a+b}{2} \tag{2.39}$$

$$V(X) = \frac{(b-a)^2}{12} \tag{2.40}$$

FIGURE 2-6 Uniform Density Function



The uniform distribution $U(0, 1)$ is useful as a starting point for generating random numbers in simulations. We assume that the p.d.f. $f(Y)$ and cumulative distribution $F(Y)$ are known. As any cumulative distribution function ranges from zero to unity, we can draw X from $U(0, 1)$ and then compute $y = F^{-1}(x)$. As we have done in the previous section, the random variable Y will then have the desired distribution $f(Y)$.

2.4.2 Normal Distribution

Perhaps the most important continuous distribution is the **normal distribution**, which represents adequately many random processes. This has a bell-like shape with more weight in the center and tails tapering off to zero. The daily rate of return in a stock price, for instance, has a distribution similar to the normal p.d.f.

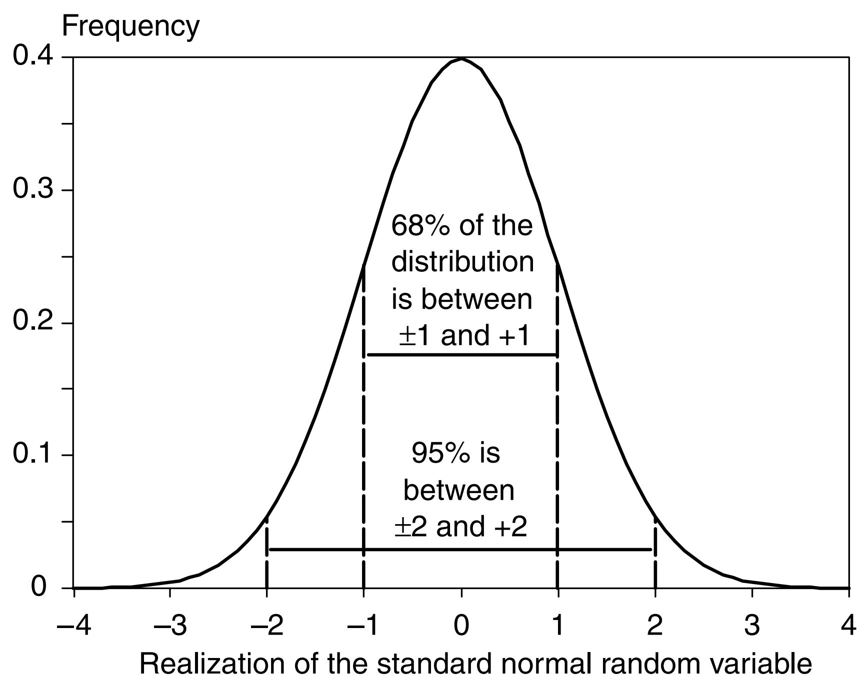
The normal distribution can be characterized by its first two moments only, the mean μ and variance σ^2. The first parameter represents the location; the second, the dispersion. The normal density function has the following expression

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-\frac{1}{2\sigma^2}(x-\mu)^2] \tag{2.41}$$

Its mean is $E[X] = \mu$ and variance $V[X] = \sigma^2$. We denote this distribution as $N(\mu, \sigma^2)$.

Instead of having to deal with different parameters, it is often more convenient to use a **standard normal variable** as ϵ, which has been standardized, or normalized, so that $E(\epsilon) = 0, V(\epsilon) = \sigma(\epsilon) = 1$. Define this as $f(\epsilon) = \Phi(x)$. Figure 2-7 plots the **standard normal distribution**.

FIGURE 2-7 Normal Density Function



First, note that the function is symmetrical around the mean. Its mean of zero is the same as its **mode** (most likely, or highest, point) and **median** (which has a 50 percent probability of occurrence). The skewness of a normal distribution is 0, which indicates that it is symmetric around the mean. The kurtosis of a normal distribution is 3. Distributions with fatter tails have a greater kurtosis coefficient.

About 95 percent of the distribution is contained between values of $\epsilon_1 = -2$ and $\epsilon_2 = +2$, and 68 percent of the distribution falls between values of $\epsilon_1 = -1$ and $\epsilon_2 = +1$. Table 2-4 gives the values that correspond to right-tail probabilities, such that

$$\int_{-\alpha}^{\infty} f(\epsilon)d\epsilon = c \tag{2.42}$$

For instance, the value of -1.645 is the quantile that corresponds to a 95% probability.[2]

TABLE 2-4 Lower Quantiles of the Standardized Normal Distribution

	Confidence Level (percent)								
c	99.99	99.9	99	97.72	97.5	95	90	84.13	50
Quantile ($-\alpha$)	−3.715	−3.090	−2.326	−2.000	−1.960	−1.645	−1.282	−1.000	−0.000

The distribution of any normal variable can then be recovered from that of the standard normal, by defining

$$X = \mu + \epsilon\sigma \tag{2.43}$$

Using Equations (2.22) and (2.23), we can show that X has indeed the desired moments, as $E(X) = \mu + E(\epsilon)\sigma = \mu$ and $V(X) = V(\epsilon)\sigma^2 = \sigma^2$.

Define, for instance, the random variable as the change in the dollar value of a portfolio. The expected value is $E(X) = \mu$. To find the quantile of X at the specified confidence level c, we replace ϵ by $-\alpha$ in Equation (2.43). This gives $Q(X, c) = \mu - \alpha\sigma$. Using Equation (2.9), we can compute VAR as

$$\text{VAR} = E(X) - Q(X, c) = \mu - (\mu - \alpha\sigma) = \alpha\sigma \tag{2.44}$$

For example, a portfolio with a standard deviation of $10 million would have a VAR, or potential downside loss, of $16.45 million at the 95% confidence level.

[2] More generally, the cumulative distribution can be found from the Excel function "=NORMDIST". For example, we can verify that "=NORMSDIST(−1.645)" yields 0.04999, or a 5% left-tail probability.

> **Key concept:**
> With normal distributions, the VAR of a portfolio is obtained from the product of the portfolio standard deviation and a standard normal deviate factor that reflects the confidence level, for instance 1.645 at the 95% level.

The normal distribution is extremely important because of the **central limit theorem** (CLT), which states that the mean of n independent and identically distributed variables converges to a normal distribution as the number of observations n increases. This very powerful result, valid for any distribution, relies heavily on the assumption of independence, however.

Defining $\bar{X}$ as the mean $\frac{1}{n}\sum_{i=1}^{n} X_i$, where each variable has mean μ and standard deviation σ, we have

$$\bar{X} \rightarrow N\left(\mu, \frac{\sigma^2}{n}\right) \tag{2.45}$$

It explains, for instance, how to diversify the credit risk of a portfolio exposed to many independent sources of risk. Thus, the normal distribution is the limiting distribution of the average, which explain why it has such a prominent place in statistics.[3]

Another important property of the normal distribution is that it is one of the few distributions that is stable under addition. In other words, a linear combination of jointly normally distributed random variables has a normal distribution.[4] This is extremely useful because we only need to know the mean and variance of the portfolio to reconstruct its whole distribution.

> **Key concept:**
> A linear combination of jointly normal variables has a normal distribution.

---

[3]Note that the CLT deals with the mean, or center of the distribution. For risk management purposes, it is also useful to examine the tails beyond VAR. A theorem from the **extreme value theory** (EVT) derives the generalized Pareto as a limit distribution for the tails.

[4]Strictly speaking, this is only true under either of the following conditions: (1) the univariate variables are independently distributed, or (2) the variables are multivariate normally distributed (this invariance property also holds for jointly elliptically distributed variables).

Example 2-3: FRM Exam 1999—Question 12/Quant. Analysis

2-3. For a standard normal distribution, what is the approximate area under the cumulative distribution function between the values −1 and 1?

a) 50%
b) 68%
c) 75%
d) 95%

Example 2-4: FRM Exam 1999—Question 11/Quant. Analysis

2-4. You are given that X and Y are random variables each of which follows a standard normal distribution with $\text{Cov}(X, Y) = 0.4$. What is the variance of $(5X + 2Y)$?

a) 11.0
b) 29.0
c) 29.4
d) 37.0

Example 2-5: FRM Exam 1999—Question 13/Quant. Analysis

2-5. What is the kurtosis of a normal distribution?

a) Zero
b) Cannot be determined, because it depends on the variance of the particular normal distribution considered
c) Two
d) Three

Example 2-6: FRM Exam 2000—Question 108/Quant. Analysis

2-6. The distribution of one-year returns for a portfolio of securities is normally distributed with an expected value of €45 million, and a standard deviation of €16 million. What is the probability that the value of the portfolio, one year hence, will be between €39 million and €43 million?

a) 8.6%
b) 9.6%
c) 10.6%
d) 11.6%

Example 2-7: FRM Exam 1999—Question 16/Quant. Analysis
2-7. If a distribution with the same variance as a normal distribution has kurtosis greater than 3, which of the following is *true*?
a) It has fatter tails than normal distribution.
b) It has thinner tails than normal distribution.
c) It has the same tail fatness as the normal distribution since variances are the same.
d) Cannot be determined from the information provided.

2.4.3 Lognormal Distribution

The normal distribution is a good approximation for many financial variables, such as the rate of return on a stock, $r = (P_1 - P_0)/P_0$, where P_0 and P_1 are the stock prices at time 0 and 1.

Strictly speaking, this is inconsistent with reality since a normal variable has infinite tails on both sides. Due to the limited liability of corporations, stock prices cannot turn negative. This rules out returns lower than minus unity and distributions with infinite left tails, such as the normal distribution. In many situations, however, this is an excellent approximation. For instance, with short horizons or small price moves, the probability of having a negative price is so small as to be negligible.

If this is not the case, we need to resort to other distributions that prevent prices from going negative. One such distribution is the lognormal.

A random variable X is said to have a **lognormal distribution** if its logarithm $Y = \ln(X)$ is normally distributed. This is often used for continuously compounded returns, defining $Y = \ln(P_1/P_0)$. Because the argument X in the logarithm function must be positive, the price P_1 can never go below zero. Large and negative large values of Y correspond to P_1 converging to, but staying above, zero.

The lognormal density function has the following expression

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(\ln(x) - \mu)^2\right], \quad x > 0 \tag{2.46}$$

Note that this is more complex than simply plugging $\ln(x)$ in Equation (2.41), because x also appears in the denominator. Its mean is

$$E[X] = \exp\left[\mu + \frac{1}{2}\sigma^2\right] \tag{2.47}$$

and variance $V[X] = \exp[2\mu + 2\sigma^2] - \exp[2\mu + \sigma^2]$. The parameters were chosen to correspond to those of the normal variable, $E[Y] = E[\ln(X)] = \mu$ and $V[Y] = V[\ln(X)] = \sigma^2$.

Conversely, if we set $E[X] = \exp[r]$, the mean of the associated normal variable is $E[Y] = E[\ln(X)] = (r - \sigma^2/2)$. This adjustment is also used in the Black-Scholes option valuation model, where the formula involves a trend in $(r - \sigma^2/2)$ for the log-price ratio.

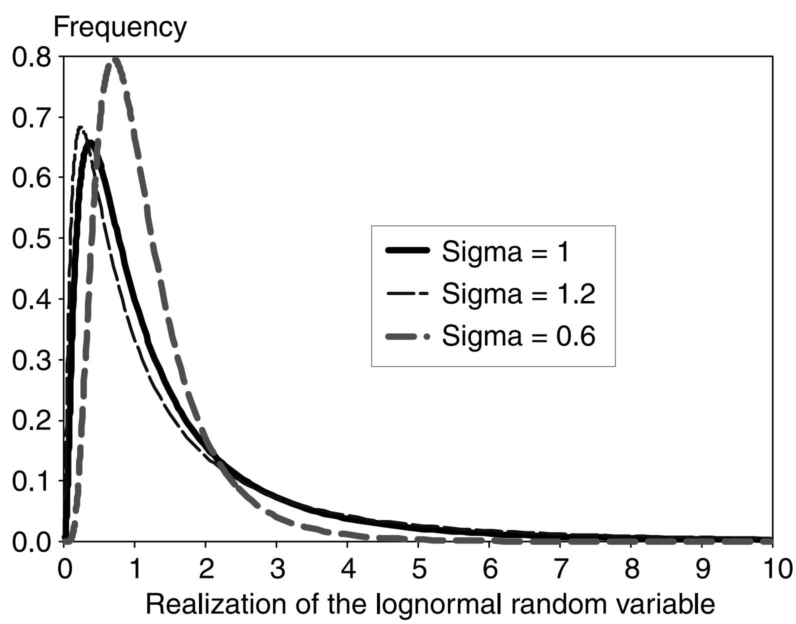
Figure 2-8 depicts the lognormal density function with $\mu = 0$, and various values $\sigma = 1.0, 1.2, 0.6$. Note that the distribution is skewed to the right. The tail increases for greater values of σ. This explains why as the variance increases, the mean is pulled up in Equation (2.47).

We also note that the distribution of the bond price in our previous example, Equation (2.35), resembles a lognormal distribution. Using continuous compounding instead of annual compounding, the price function is

$$V = 100 \exp(-rT) \tag{2.48}$$

which implies $\ln(V/100) = -rT$. Thus if r is normally distributed, V has a lognormal distribution.

FIGURE 2-8 Lognormal Density Function



Example 2-8: FRM Exam 2001—Question 72

2-8. The lognormal distribution is
a) Positively skewed
b) Negatively skewed
c) Not skewed, that is, its skew equals 2
d) Not skewed, that is, its skew equals 0

Example 2-9: FRM Exam 1999—Question 5/Quant. Analysis

2-9. Which of the following statements best characterizes the relationship between the normal and lognormal distributions?
a) The lognormal distribution is the logarithm of the normal distribution.
b) If the natural log of the random variable X is lognormally distributed, then X is normally distributed.
c) If X is lognormally distributed, then the natural log of X is normally distributed.
d) The two distributions have nothing to do with one another.

Example 2-10: FRM Exam 1998—Question 10/Quant. Analysis

2-10. For a lognormal variable X, we know that $\ln(X)$ has a normal distribution with a mean of zero and a standard deviation of 0.2. What is the expected value of X?
a) 0.98
b) 1.00
c) 1.02
d) 1.20

Example 2-11: FRM Exam 1998—Question 16/Quant. Analysis

2-11. Which of the following statements are *true*?
I. The sum of two random normal variables is also a random normal variable.
II. The product of two random normal variables is also a random normal variable.
III. The sum of two random lognormal variables is also a random lognormal variable.
IV. The product of two random lognormal variables is also a random lognormal variable.
a) I and II only
b) II and III only
c) III and IV only
d) I and IV only

Example 2-12: FRM Exam 2000—Question 128/Quant. Analysis
2-12. For a lognormal variable X, we know that $\ln(X)$ has a normal distribution with a mean of zero and a standard deviation of 0.5. What are the expected value and the variance of X?
a) 1.025 and 0.187
b) 1.126 and 0.217
c) 1.133 and 0.365
d) 1.203 and 0.399

2.4.4 Student's *t* Distribution

Another important distribution is the **Student's *t* distribution**. This arises in hypothesis testing, because it describes the distribution of the ratio of the estimated coefficient to its standard error.

This distribution is characterized by a parameter k known as the **degrees of freedom**. Its density is

$$f(x) = \frac{\Gamma[(k+1)/2]}{\Gamma(k/2)} \frac{1}{\sqrt{k\pi}} \frac{1}{(1+x^2/k)^{(k+1)/2}} \tag{2.49}$$

where Γ is the gamma function.[5] As k increases, this function converges to the normal p.d.f.

The distribution is symmetrical with mean zero and variance

$$V[X] = \frac{k}{k-2} \tag{2.50}$$

provided $k > 2$. Its kurtosis is

$$\delta = 3 + \frac{6}{k-4} \tag{2.51}$$

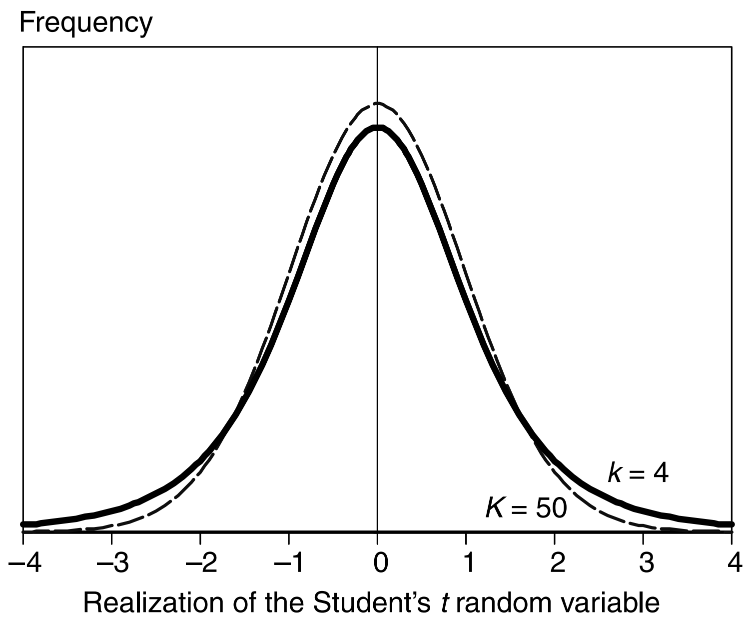
provided $k > 4$. Its has fatter tails than the normal which often provides a better representation of typical financial variables. Typical estimated values of k are around four to six. Figure 2-9 displays the density for $k = 4$ and $k = 50$. The latter is close to the normal. With $k = 4$, however, the p.d.f. has noticeably fatter tails.

Another distribution derived from the normal is the **chi-square distribution**, which can be viewed as the sum of independent squared standard normal variables

$$x = \sum_{j=1}^{k} z_j^2 \tag{2.52}$$

[5]The gamma function is defined as $\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx$.

FIGURE 2-9 Student's *t* Density Function



where k is also called the degrees of freedom. Its mean is $E[X] = k$ and variance $V[X] = 2k$. For k sufficiently large, $\chi^2(k)$ converges to a normal distribution $N(k, 2k)$. This distribution describes the sample variance.

Finally, another associated distribution is the F **distribution**, which can be viewed as the ratio of independent chi-square variables divided by their degrees of freedom

$$F(a, b) = \frac{\chi^2(a)/a}{\chi^2(b)/b} \tag{2.53}$$

This distribution appears in joint tests of regression coefficients.

Example 2-13: FRM Exam 1999—Question 3/Quant. Analysis

2-13. It is often said that distributions of returns from financial instruments are leptokurtotic. For such distributions, which of the following comparisons with a normal distribution of the same mean and variance *must* hold?

a) The skew of the leptokurtotic distribution is greater.
b) The kurtosis of the leptokurtotic distribution is greater.
c) The skew of the leptokurtotic distribution is smaller.
d) The kurtosis of the leptokurtotic distribution is smaller.

2.4.5 Binomial Distribution

Consider now a random variable that can take discrete values between zero and n. This could be, for instance, the number of times VAR is exceeded over the last year, also called the number of **exceptions**. Thus, the binomial distribution plays an important role for the backtesting of VAR models.

A binomial variable can be viewed as the result of n independent **Bernoulli trials**, where each trial results in an outcome of $y = 0$ or $y = 1$. This applies, for example, to credit risk. In case of default, we have $y = 1$, otherwise $y = 0$. Each Bernoulli variable has expected value of $E[Y] = p$ and variance $V[Y] = p(1 - p)$.

A random variable is defined to have a **binomial distribution** if the discrete density function is given by

$$f(x) = \binom{n}{x} p^x (1 - p)^{n-x}, \quad x = 0, 1, \ldots, n \tag{2.54}$$

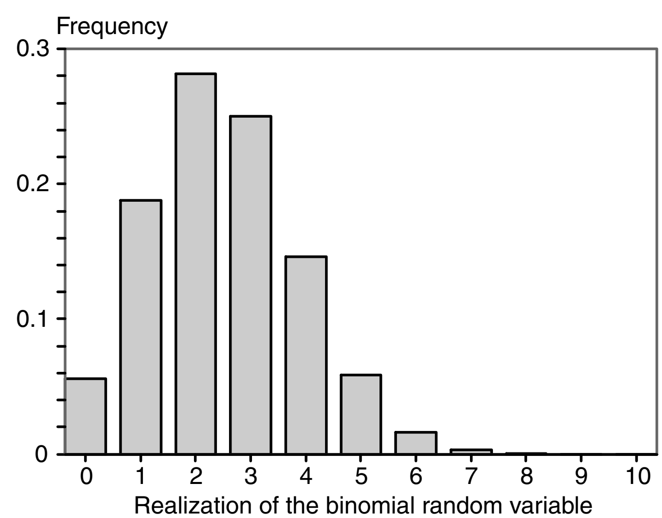
where $\binom{n}{x}$ is the number of combinations of n things taken x at a time, or

$$\binom{n}{x} = \frac{n!}{x!(n - x)!} \tag{2.55}$$

and the parameter p is between zero and one. This distribution also represents the total number of successes in n repeated experiments where each success has a probability of p.

The binomial variable has expected value of $E[X] = pn$ and variance $V[X] = p(1 - p)n$. It is described in Figure 2-10 in the case where $p = 0.25$ and $n = 10$. The probability of observing $X = 0, 1, 2 \ldots$ is 5.6%, 18.8%, 28.1% and so on.

FIGURE 2-10 Binomial Density Function with $p = 0.25$, $n = 10$



For instance, we want to know what is the probability of observing $x = 0$ exceptions out of a sample of $n = 250$ observations when the true probability is 1%. We should expect to observe about 2.5 exceptions in such a sample. We have

$$f(X = 0) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x} = \frac{250!}{1 \times 250!}0.01^0 0.99^{250} = 0.081$$

So, we would expect to observe 8.1% of samples with zero exceptions, under the null hypothesis. Alternatively, the probability of observing 10 exception is $f(X = 8) = 0.02\%$ only. Because this probability is so low, observing 8 exceptions would make us question whether the true probability is 1%.

When n is large, we can use the CLT and approximate the binomial distribution by the normal distribution

$$z = \frac{x - pn}{\sqrt{p(1-p)n}} \sim N(0, 1) \tag{2.56}$$

which provides a convenient shortcut. For our example, $E[X] = 0.01 \times 250 = 2.5$ and $V[X] = 0.01(1-0.01) \times 250 = 2.475$. The value of the normal variable is $z = (8-2.5)/\sqrt{2.475} = 3.50$, which is very high, leading us to reject the hypothesis that the true probability of observing an exception is 1% only.

Example 2-14: FRM Exam 2001—Question 68

2-14. EVT, Extreme Value Theory, helps quantify two key measures of risk:

a) The magnitude of an X year return in the loss in excess of VAR

b) The magnitude of VAR and the level of risk obtained from scenario analysis

c) The magnitude of market risk and the magnitude of operational risk

d) The magnitude of market risk and the magnitude of credit risk

2.5 Answers to Chapter Examples

Example 2-1: FRM Exam 1999—Question 21/Quant. Analysis

c) From Equation (2.21), we have $\sigma_B = \text{Cov}(A, B)/(\rho\sigma_A) = 5/(0.5\sqrt{12}) = 2.89$, for a variance of $\sigma_B^2 = 8.33$.

Example 2-2: FRM Exam 2000—Question 81/Market Risk

b) Correlation is a measure of linear association. Independence implies zero correlation, but the reverse is not always true.

Example 2-3: FRM Exam 1999—Question 12/Quant. Analysis

b) See Figure 2-7.

Example 2-4: FRM Exam 1999—Question 11/Quant. Analysis

d) Each variable is standardized, so that its variance is unity. Using Equation (2.26), we have $V(5X + 2Y) = 25V(X) + 4V(Y) + 2 * 5 * 2 * \text{Cov}(X, Y) = 25 + 4 + 8 = 37$.

Example 2-5: FRM Exam 1999—Question 13/Quant. Analysis

d) Note that (b) is not correct because the kurtosis involves σ^4 in the denominator and is hence scale-free.

Example 2-6: FRM Exam 2000—Question 108/Quant. Analysis

b) First, we compute the standard variate for each cutoff point $\epsilon_1 = (43 - 45)/16 = -0.125$ and $\epsilon_2 = (39 - 45)/16 = -0.375$. Next, we compute the cumulative distribution function for each $F(\epsilon_1) = 0.450$ and $F(\epsilon_2) = 0.354$. Hence, the difference is a probability of $0.450 - 0.354 = 0.096$.

Example 2-7: FRM Exam 1999—Question 16/Quant. Analysis

a) As in Equation (2.13), the kurtosis adjusts for σ. Greater kurtosis than for the normal implies fatter tails.

Example 2-8: FRM Exam 2001—Question 72

a) The lognormal distribution has a long left tail, as in Figure 2-6. So, it is positively skewed.

Example 2-9: FRM Exam 1999—Question 5/Quant. Analysis

c) X is said to be lognormally distributed if its logarithm $Y = \ln(X)$ is normally distributed.

Example 2-10: FRM Exam 1998—Question 10/Quant. Analysis

c) Using Equation (2.47), $E[X] = \exp[\mu + \frac{1}{2}\sigma^2] = \exp[0 + 0.5 * 0.2^2] = 1.02$.

Example 2-11: FRM Exam 1998—Question 16/Quant. Analysis

d) Normal variables are stable under addition, so that (I) is true. For lognormal variables X_1 and X_2, we know that their logs, $Y_1 = \ln(X_1)$ and $Y_2 = \ln(X_2)$ are normally distributed. Hence, the sum of their logs, or $\ln(X_1) + \ln(X_2) = \ln(X_1 X_2)$ must also be normally distributed. The product is itself lognormal, so that (IV) is true.

Example 2-12: FRM Exam 2000—Question 128/Quant. Analysis

c) Using Equation (2.47), we have $E[X] = \exp[\mu + 0.5\sigma^2] = \exp[0 + 0.5 * 0.5^2] = 1.1331$. Assuming there is no error in the answers listed for the variance, it is sufficient to find the correct answer for the expected value.

Example 2-13: FRM Exam 1999—Question 3/Quant. Analysis

b) Leptokurtic refers to a distribution with fatter tails than the normal, which implies greater kurtosis.

Example 2-14: FRM Exam 2001—Question 68

a) EVT allows risk managers to approximate distributions in the tails beyond the usual VAR confidence levels. Answers (c) and (d) are too general. Answer (b) is also incorrect as EVT is based on historical data instead of scenario analyses.

Appendix: Review of Matrix Multiplication

This appendix briefly reviews the mathematics of matrix multiplication. Say that we have two matrices, A and B that we wish to multiply to obtain the new matrix $C = AB$. The respective dimensions are $(n \times m)$ for A, that is, n rows and m columns, and $(m \times p)$ for B. The number of columns for A must exactly match (or conform) to the number of rows for B. If so, this will result in a matrix C of dimensions $(n \times p)$.

We can write the matrix A in terms of its individual components a_{ij}, where i denotes the row and j denotes the column:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

As an illustration, take a simple example where the matrices are of dimension (2×3) and (3×2).

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$C = AB = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

To multiply the matrices, each row of A is multiplied element-by-element by each column of B. For instance, c_{12} is obtained by taking

$$c_{12} = [a_{11} \quad a_{12} \quad a_{13}] \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}.$$

The matrix C is then

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}$$

Matrix multiplication can be easily implemented in Excel using the function "=MMULT". First, we highlight the cells representing the output matrix C, say f1:g2. Then we enter the function, for instance "=MMULT(a1:c2; d1:e3)", where the first range represents the first matrix A, here 2 by 3, and the second range represents the matrix B, here 3 by 2. The final step is to hit the three keys Control-Shift-Return simultaneously.

Chapter 3

Fundamentals of Statistics

The preceding chapter was mainly concerned with the theory of probability, including distribution theory. In practice, researchers have to find methods to choose among distributions and to estimate distribution parameters from real data. The subject of sampling brings us now to the theory of statistics. Whereas probability assumes the distributions are known, statistics attempts to make inferences from actual data.

Here, we sample from a distribution of a population, say the change in the exchange rate, to make inferences about the population. A fundamental goal for risk management is to estimate the variability of future movements in exchange rates. Additionally, we want to establish whether there is some relationship between the risk factors, for instance, whether movements in the yen/dollar rate are correlated with the dollar/euro rate. Or, we may want to develop decision rules to check whether value-at-risk estimates are in line with subsequent profits and losses.

These examples illustrate two important problems in statistical inference, **estimation** and **tests of hypotheses**. With estimation, we wish to estimate the value of an unknown parameter from sample data. With tests of hypotheses, we wish to verify a conjecture about the data.

This chapter reviews the fundamental tools of statistics theory for risk managers. Section 3.1 discusses the sampling of real data and the construction of returns. The problem of parameter estimation is presented in Section 3.2. Section 3.3 then turns to regression analysis, summarizing important results as well as common pitfalls in their interpretation.

3.1 Real Data

To start with an example, let us say that we observe movements in the daily yen/dollar exchange rate and wish to characterize the distribution of tomorrow's exchange rate.

The risk manager's job is to assess the range of potential gains and losses on a trader's position. He or she observes a sequence of past spot rates $S_0, S_1, \ldots, S_t$, including the latest rate, from which we have to infer the distribution of tomorrow's rate, S_{t+1}.

3.1.1 Measuring Returns

The truly random component in tomorrow's price is not its level, but rather its change relative to today's price. We measure rates of change in the spot price:

$$r_t = (S_t - S_{t-1})/S_{t-1} \tag{3.1}$$

Alternatively, we could construct the logarithm of the price ratio:

$$R_t = \ln[S_t/S_{t-1}] \tag{3.2}$$

which is equivalent to using continuous instead of discrete compounding. This is also

$$R_t = \ln[1 + (S_t - S_{t-1})/S_{t-1}] = \ln[1 + r_t]$$

Because $\ln(1 + x)$ is close to x if x is small, R_t should be close to r_t provided the return is small. For daily data, there is typically little difference between R_t and r_t.

The return defined so far is the **capital appreciation return**, which ignores the income payment on the asset. Define the dividend or coupon as D_t. In the case of an exchange rate position, this is the interest payment in the foreign currency over the holding period. The **total return** on the asset is

$$r_t^{\text{TOT}} = (S_t + D_t - S_{t-1})/S_{t-1} \tag{3.3}$$

When the horizon is very short, the income return is typically very small compared to the capital appreciation return.

The next question is whether the sequence of variables r_t can be viewed as independent observations. If so, one could hypothesize, for instance, that the random variables are drawn from a normal distribution $N(\mu, \sigma^2)$. We could then proceed to estimate μ and σ^2 from the data and use this information to create a distribution for tomorrow's spot price change.

Independent observations have the very nice property that their joint distribution is the product of their marginal distribution, which considerably simplifies the analysis. The obvious question is whether this assumption is a workable approximation. In fact, there are good economic reasons to believe that rates of change on financial prices are close to independent.

The hypothesis of **efficient markets** postulates that current prices convey all relevant information about the asset. If so, any change in the asset price must be due to news events, which are by definition impossible to forecast (otherwise, it would not be news). This implies that changes in prices are unpredictable and hence satisfy our definition of truly random variables. Although this definition may not be strictly true, it usually provides a sufficient approximation to the behavior of financial prices.

This hypothesis, also known as the **random walk** theory, implies that the conditional distribution of returns depends only on current prices, and not on the previous history of prices. If so, technical analysis must be a fruitless exercise, because previous patterns in prices cannot help in forecasting price movements.

If in addition the distribution of returns is constant over time, the variables are said to be **independently and identically distributed** (i.i.d.). This explains why we could consider that the observations r_t are independent draws from the same distribution $N(\mu, \sigma^2)$.

Later, we will consider deviations from this basic model. Distributions of financial returns typically display fat tails. Also, variances are not constant and display some persistence; expected returns can also slightly vary over time.

3.1.2 Time Aggregation

It is often necessary to translate parameters over a given horizon to another horizon. For example, we may have raw data for daily returns, from which we compute a daily volatility that we want to extend to a monthly volatility.

Returns can be easily related across time when we use the log of the price ratio, because the log of a product is the sum of the logs. The two-day return, for example, can be decomposed as

$$R_{02} = \ln[S_2/S_0] = \ln[(S_2/S_1)(S_1/S_0)] = \ln[S_1/S_0] + \ln[S_2/S_1] = R_{01} + R_{12} \tag{3.4}$$

This decomposition is only approximate if we use discrete returns, however.

The expected return and variance are then $E(R_{02}) = E(R_{01}) + E(R_{12})$ and $V(R_{02}) = V(R_{01}) + V(R_{12}) + 2\text{Cov}(R_{01}, R_{12})$. Assuming returns are uncorrelated and have identical distributions across days, we have $E(R_{02}) = 2E(R_{01})$ and $V(R_{02}) = 2V(R_{01})$.

Generalizing over T days, we can relate the moments of the T-day returns R_T to those of the 1-day returns R_1:

$$E(R_T) = E(R_1)T \tag{3.5}$$

$$V(R_T) = V(R_1)T \tag{3.6}$$

Expressed in terms of volatility, this yields the **square root of time rule**:

$$\mathrm{SD}(R_T) = \mathrm{SD}(R_1)\sqrt{T} \tag{3.7}$$

It should be emphasized that this holds only if returns have the same parameters across time and are uncorrelated. With correlation across days, the 2-day variance is

$$V(R_2) = V(R_1) + V(R_1) + 2\rho V(R_1) = 2V(R_1)(1+\rho) \tag{3.8}$$

With trends, or positive autocorrelation, the 2-day variance is greater than the one obtained by the square root of time rule. With mean reversion, or negative autocorrelation, the 2-day variance is less than the one obtained by the square root of time rule.

Key concept:
When successive returns are uncorrelated, the volatility increases as the horizon extends following the square root of time.

3.1.3 Portfolio Aggregation

Let us now turn to aggregation of returns across assets. Consider, for example, an equity portfolio consisting of investments in N shares. Define the number of each share held as q_i with unit price S_i. The portfolio value at time t is then

$$W_t = \sum_{i=1}^{N} q_i S_{i,t} \tag{3.9}$$

We can write the weight assigned to asset i as

$$w_{i,t} = \frac{q_i S_{i,t}}{W_t} \tag{3.10}$$

which by construction sum to unity. Using weights, however, rules out situations with zero net investment, $W_t = 0$, such as some derivatives positions. But we could have positive and negative weights if short selling is allowed, or weights greater than one if the portfolio can be leveraged.

The next period, the portfolio value is

$$W_{t+1} = \sum_{i=1}^{N} q_i S_{i,t+1} \tag{3.11}$$

assuming that the unit price incorporates any income payment. The gross, or dollar, return is then

$$W_{t+1} - W_t = \sum_{i=1}^{N} q_i (S_{i,t+1} - S_{i,t}) \tag{3.12}$$

and the *rate* of return is

$$\frac{W_{t+1} - W_t}{W_t} = \sum_{i=1}^{N} \frac{q_i S_{i,t}}{W_t} \frac{(S_{i,t+1} - S_{i,t})}{S_{i,t}} = \sum_{i=1}^{N} w_{i,t} \frac{(S_{i,t+1} - S_{i,t})}{S_{i,t}} \tag{3.13}$$

The portfolio discrete rate of return is a linear combination of the asset returns,

$$r_{p,t+1} = \sum_{i=1}^{N} w_{i,t} r_{i,t+1} \tag{3.14}$$

The dollar return is then

$$W_{t+1} - W_t = \left[\sum_{i=1}^{N} w_{i,t} r_{i,t+1} \right] W_t \tag{3.15}$$

and has a normal distribution if the individual returns are also normally distributed.

Alternatively, we could express the individual positions in dollar terms,

$$x_{i,t} = w_{i,t} W_t = q_i S_{i,t} \tag{3.16}$$

The dollar return is also, using dollar amounts,

$$W_{t+1} - W_t = \left[\sum_{i=1}^{N} x_{i,t} r_{i,t+1} \right] \tag{3.17}$$

As we have seen in the previous chapter, the variance of the portfolio dollar return is

$$V[W_{t+1} - W_t] = x' \Sigma x \tag{3.18}$$

which, along with the expected return, fully characterizes its distribution. The portfolio VAR is then

$$\text{VAR} = \alpha \sqrt{x' \Sigma x} \tag{3.19}$$

Example 3-1: FRM Exam 1999—Question 4/Quant. Analysis

3-1. A fundamental assumption of the random walk hypothesis of market returns is that returns from one time period to the next are statistically independent. This assumption implies

a) Returns from one time period to the next can never be equal.
b) Returns from one time period to the next are uncorrelated.
c) Knowledge of the returns from one time period does not help in predicting returns from the next time period.
d) Both (b) and (c) are true.

Example 3-2: FRM Exam 1999—Question 14/Quant. Analysis

3-2. Suppose returns are uncorrelated over time. You are given that the volatility over two days is 1.20%. What is the volatility over 20 days?

a) 0.38%
b) 1.20%
c) 3.79%
d) 12.0%

Example 3-3: FRM Exam 1998—Question 7/Quant. Analysis

3-3. Assume an asset price variance increases linearly with time. Suppose the expected asset price volatility for the next two months is 15% (annualized), and for the one month that follows, the expected volatility is 35% (annualized). What is the average expected volatility over the next three months?

a) 22%
b) 24%
c) 25%
d) 35%

Example 3-4: FRM Exam 1997—Question 15/Risk Measurement

3-4. The standard VAR calculation for extension to multiple periods assumes that returns are serially uncorrelated. If prices display trends, the true VAR will be

a) The same as the standard VAR
b) Greater than standard VAR
c) Less than standard VAR
d) Unable to be determined

3.2 Parameter Estimation

Armed with our i.i.d. sample of T observations, we can start estimating the parameters of interest, the sample mean, variance, and other moments.

As in the previous chapter, define x_i as the realization of a random sample. The expected return, or mean, $\mu = E(X)$ can be estimated by the sample mean,

$$m = \hat{\mu} = \frac{1}{T}\sum_{i=1}^{T} x_i \tag{3.20}$$

Intuitively, we assign the same weight of $1/T$ to all observations because they all have the same probability. The variance, $\sigma^2 = E[(X-\mu)^2]$, can be estimated by the sample variance,

$$s^2 = \hat{\sigma}^2 = \frac{1}{(T-1)}\sum_{i=1}^{T}(x_i - \hat{\mu})^2 \tag{3.21}$$

Note that we divide by $T-1$ instead of T. This is because we estimate the variance around an unknown parameter, the mean. So, we have fewer degrees of freedom than otherwise. As a result, we need to adjust s^2 to ensure that its expectation equals the true value. In most situations, however, T is large so that this adjustment is minor.

It is essential to note that these estimated values depend on the particular sample and, hence, have some inherent variability. The sample mean itself is distributed as

$$m = \hat{\mu} \sim N(\mu, \sigma^2/T) \tag{3.22}$$

If the population distribution is normal, this exactly describes the distribution of the sample mean. Otherwise, the central limit theorem states that this distribution is only valid asymptotically, i.e. for large samples.

For the distribution of the sample variance $\hat{\sigma}^2$, one can show that, when X is normal, the following ratio is distributed as a chi-square with $(T-1)$ degrees of freedom

$$\frac{(T-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2(T-1) \tag{3.23}$$

If the sample size T is large enough, the chi-square distribution converges to a normal distribution:

$$\hat{\sigma}^2 \sim N\left(\sigma^2, \sigma^4\frac{2}{(T-1)}\right) \tag{3.24}$$

Using the same approximation, the sample standard deviation has a normal distribution with a standard error of

$$\text{se}(\hat{\sigma}) = \sigma\sqrt{\frac{1}{2T}} \tag{3.25}$$

We can use this information for **hypothesis testing**. For instance, we would like to detect a constant trend in X. Here, the **null hypothesis** is that $\mu = 0$. To answer the question, we use the distributional assumption in Equation (3.22) and compute a standard normal variable as the ratio of the estimated mean to its standard error, or

$$z = \frac{(m - 0)}{\sigma/\sqrt{T}} \tag{3.26}$$

Because this is now a standard normal variable, we would not expect to observe values far away from zero. Typically, we would set the confidence level at 95 percent, which translates into a two-tailed interval for z of $[-1.96, +1.96]$. Roughly, this means that, if the absolute value of z is greater than two, we would reject the hypothesis that m came from a distribution with a mean of zero. We can have some confidence that the true μ is indeed different from zero.

In fact, we do not know the true σ and use the estimated s instead. The distribution is a Student's t with T degrees of freedom:

$$t = \frac{(m - 0)}{s/\sqrt{T}} \tag{3.27}$$

for which the cutoff values can be found from tables, or a spreadsheet. As T increases, however, the distribution tends to the normal.

At this point, we need to make an important observation. Equation (3.22) shows that, when the sample size increases, the standard error of $\hat{\mu}$ shrinks at a rate proportional to $1/\sqrt{T}$. The precision of the estimate increases with a greater number of observations. This result is quite useful to assess the precision of estimates generated from numerical simulations, which are widely used in risk management.

Key concept:
With independent draws, the standard deviation of most statistics is inversely related to the square root of number of observations T. Thus, more observations make for more precise estimates.

Our ability to reject a hypothesis will also improve with T. Note that hypothesis tests are only meaningful when they lead to a rejection. Nonrejection is not informative. It does not mean that we have any evidence in support of the null hypothesis or that we "accept" the null hypothesis. For instance, the test could be badly designed, or not have enough observations. So, we cannot make a statement that we accept a null hypothesis, but instead only say that we reject it.

Example:

The yen/dollar rate We want to characterize movements in the monthly yen/dollar exchange rate from historical data, taken over 1990 to 1999. Returns are defined in terms of continuously compounded changes, as in Equation (3.2). We have $T = 120$, $m = -0.28\%$, and $s = 3.55\%$ (per month).

Using Equation (3.22), we find that the standard error of the mean is approximately $\text{se}(m) = s/\sqrt{T} = 0.32\%$. For the null of $\mu = 0$, this gives a t-ratio of $t = m/\text{se}(m) = -0.28\%/0.32\% = -0.87$. Because this number is less than 2 in absolute value, we cannot reject at the 95 percent confidence level the hypothesis that the mean is zero. This is a typical result for financial series. The mean is not sufficiently precisely estimated.

Next, we turn to the precision in the sample standard deviation. By Equation (3.25), its standard error is $\text{se}(s) = \sigma\sqrt{\frac{1}{(2T)}} = 0.229\%$. For the null of $\sigma = 0$, this gives a z-ratio of $z = s/\text{se}(s) = 3.55\%/0.229\% = 15.5$, which is very high. Therefore, there is much more precision in the measurement of s than in that of m.

We can construct, for instance, 95 percent confidence intervals around the estimated values. These are:

$$[m - 1.96 \times \text{se}(m), m + 1.96 \times \text{se}(m)] = [-0.92\%, +0.35\%]$$

$$[s - 1.96 \times \text{se}(s), s + 1.96 \times \text{se}(s)] = [3.10\%, 4.00\%]$$

So, we could be reasonably confident that the volatility is between 3% and 4%, but we cannot even be sure that the mean is different from zero.

3.3 Regression Analysis

Regression analysis has particular importance for finance professionals, because it can be used to explain and forecast variables of interest.

3.3.1 Bivariate Regression

In a **linear regression**, the **dependent variable** y is projected on a set of N predetermined **independent variables**, x. In the simplest bivariate case we write

$$y_t = \alpha + \beta x_t + \epsilon_t, \qquad t = 1, \ldots, T \tag{3.28}$$

where α is called the **intercept**, or constant, β is called the **slope**, and ϵ is called the **residual**, or **error term**. This could represent a time-series or a cross section.

The **ordinary least squares** (OLS) assumptions are

1. *The errors are independent of x.*
2. *The errors have a normal distribution with zero mean and constant variance, conditional on x.*
3. *The errors are independent across observations.*

Based on these assumptions, the usual methodology is to estimate the coefficients by minimizing the sum of squared errors. Beta is estimated by

$$\hat{\beta} = \frac{1/(T-1)\sum_t (x_t - \bar{x})(y_t - \bar{y})}{1/(T-1)\sum_t (x_t - \bar{x})^2} \tag{3.29}$$

where $\bar{x}$ and $\bar{y}$ correspond to the means of x_t and y_t. Alpha is estimated by

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x} \tag{3.30}$$

Note that the numerator in Equation (3.29) is also the sample covariance between two series x_i and x_j, which can be written as

$$\hat{\sigma}_{ij} = \frac{1}{(T-1)}\sum_{t=1}^{T}(x_{t,i} - \hat{\mu}_i)(x_{t,j} - \hat{\mu}_j) \tag{3.31}$$

To interpret β, we can take the covariance between y and x, which is

$$\text{Cov}(y, x) = \text{Cov}(\alpha + \beta x + \epsilon, x) = \beta \text{Cov}(x, x) = \beta V(x)$$

because ϵ is conditionally independent of x. This shows that the population β is also

$$\beta(y, x) = \frac{\text{Cov}(y, x)}{V(x)} = \frac{\rho(y, x)\sigma(y)\sigma(x)}{\sigma^2(x)} = \rho(y, x)\frac{\sigma(y)}{\sigma(x)} \tag{3.32}$$

The **regression fit** can be assessed by examining the size of the residuals, obtained by subtracting the fitted values $\hat{y}_t$ from y_t,

$$\hat{\epsilon}_t = y_t - \hat{y}_t = y_t - \hat{\alpha} - \hat{\beta}x_t \tag{3.33}$$

and taking the estimated variance as

$$V(\hat{\epsilon}) = \frac{1}{(T-2)}\sum_{t=1}^{T}\hat{\epsilon}_t^2 \tag{3.34}$$

We divide by $T - 2$ because the estimator uses two unknown quantities, $\hat{\alpha}$ and $\hat{\beta}$. Also note that, since the regression includes an intercept, the average value of $\hat{\epsilon}$ has to be exactly zero.

The quality of the fit can be assessed using a unitless measure called the **regression R-square**. This is defined as

$$R^2 = 1 - \frac{\text{SSE}}{\text{SSY}} = 1 - \frac{\sum_t \hat{\epsilon}_t^2}{\sum_t (y_t - \bar{y})^2} \tag{3.35}$$

where SSE is the sum of squared errors, and SSY is the sum of squared deviations of y around its mean. If the regression includes a constant, we always have $0 \leq R^2 \leq 1$. In this case, R-square is also the square of the usual correlation coefficient,

$$R^2 = \rho(y, x)^2 \tag{3.36}$$

The R^2 measures the degree to which the size of the errors is smaller than that of the original dependent variables y. To interpret R^2, consider two extreme cases. If the fit is excellent, the errors will all be zero, and the numerator in Equation (3.35) will be zero, which gives $R^2 = 1$. However, if the fit is poor, SSE will be as large as SSY and the ratio will be one, giving $R^2 = 0$.

Alternatively, we can interpret the R-square by decomposing the variance of $y_t = \alpha + \beta x_t + \epsilon_t$. This gives

$$V(y) = \beta^2 V(x) + V(\epsilon) \tag{3.37}$$

$$1 = \frac{\beta^2 V(x)}{V(y)} + \frac{V(\epsilon)}{V(y)} \tag{3.38}$$

Since the R-square is also $R^2 = 1 - V(\epsilon)/V(y)$, it is equal to $= \beta^2 V(x)/V(y)$, which is the contribution in the variation of y due to β and x.

Finally, we can derive the distribution of the estimated coefficients, which is normal and centered around the true values. For the slope coefficient, $\hat{\beta} \sim N(\beta, V(\hat{\beta}))$, with variance given by

$$V(\hat{\beta}) = V(\hat{\epsilon})\frac{1}{\sum_t (x_t - \bar{x})^2} \tag{3.39}$$

This can be used to test whether the slope coefficient is significantly different from zero. The associated test statistic

$$t = \hat{\beta}/\sigma(\hat{\beta}) \tag{3.40}$$

has a Student's t distribution. Typically, if the absolute value of the statistic is above 2, we would reject the hypothesis that there is no relationship between y and x.

3.3.2 Autoregression

A particularly useful application is a regression of a variable on a lagged value of itself, called **autoregression**

$$y_t = \alpha + \beta_k y_{t-k} + \epsilon_t, \qquad t = 1, \ldots, T \tag{3.41}$$

If the coefficient is significant, previous movements in the variable can be used to predict future movements. Here, the coefficient β_k is known as the kth-order **autocorrelation coefficient**.

Consider for instance a first-order autoregression, where the daily change in the yen/dollar rate is regressed on the previous day's value. A positive coefficient $\hat{\beta}_1$ indicates that a movement up in one day is likely to be followed by another movement up the next day. This would indicate a trend in the exchange rate. Conversely, a negative coefficient indicates that movements in the exchange rate are likely to be reversed from one day to the next. Technical analysts work very hard at identifying such patterns.

As an example, assume that we find that $\hat{\beta}_1 = 0.10$, with zero intercept. One day, the yen goes up by 2%. Our best forecast for the next day is then another upmove of

$$E[y_t] = \beta_1 y_{t-1} = 0.1 \times 2\% = 0.2\%$$

Autocorrelation changes normal patterns in risk across horizons. When there is no autocorrelation, we know that risk increases with the square root of time. With positive autocorrelation, shocks have a longer-lasting effect and risk increases faster than the square root of time.

3.3.3 Multivariate Regression

More generally, the regression in Equation (3.28) can be written, with N independent variables (perhaps including a constant):

$$\begin{bmatrix} y_1 \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \ldots & x_{1N} \\ \vdots & & & & \\ x_{T1} & x_{T2} & x_{T3} & \ldots & x_{TN} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_T \end{bmatrix} \tag{3.42}$$

or in matrix notation,

$$y = X\beta + \epsilon \tag{3.43}$$

The estimated coefficients can be written in matrix notation as

$$\hat{\beta} = (X'X)^{-1}X'y \tag{3.44}$$

and their covariance matrix as

$$V(\hat{\beta}) = \sigma^2(\epsilon)(X'X)^{-1} \tag{3.45}$$

We can extend the t-statistic to a multivariate environment. Say we want to test whether the last m coefficients are jointly zero. Define $\hat{\beta}_m$ as these grouped coefficients and $V_m(\hat{\beta})$ as their covariance matrix. We set up a statistic

$$F = \frac{\hat{\beta}_m' V_m(\hat{\beta})^{-1}\hat{\beta}_m / m}{\text{SSE}/(T-N)} \tag{3.46}$$

which has a so-called F-distribution with m and $T - N$ degrees of freedom. As before, we would reject the hypothesis if the value of F is too large compared to critical values from tables. This setup takes into account the joint nature of the estimated coefficients $\hat{\beta}$.

3.3.4 Example

This section gives the example of a regression of a stock return on the market. This is useful to assess whether movements in the stock can be hedged using stock-market index futures, for instance.

We consider ten years of data for Intel and the S&P 500, using total rates of return over a month. Figure 3-1 plots the 120 combination of returns, or (y_t, x_t). Apparently, there is a positive relationship between the two variables, as shown by the straight line that represents the regression fit $(\hat{y}_t, x_t)$.

Table 3-1 displays the regression results. The regression shows a positive relationship between the two variables, with $\hat{\beta} = 1.35$. This is significantly positive, with a standard error of 0.229 and t-statistic of 5.90. The t-statistic is very high, with an associated probability value (p-value) close to zero. Thus we can be fairly confident of a positive association between the two variables.

This beta coefficient is also called **systematic risk**, or exposure to general market movements. Technology stocks are said to have greater systematic risk than the

FIGURE 3-1 Intel Return vs. S&P Return

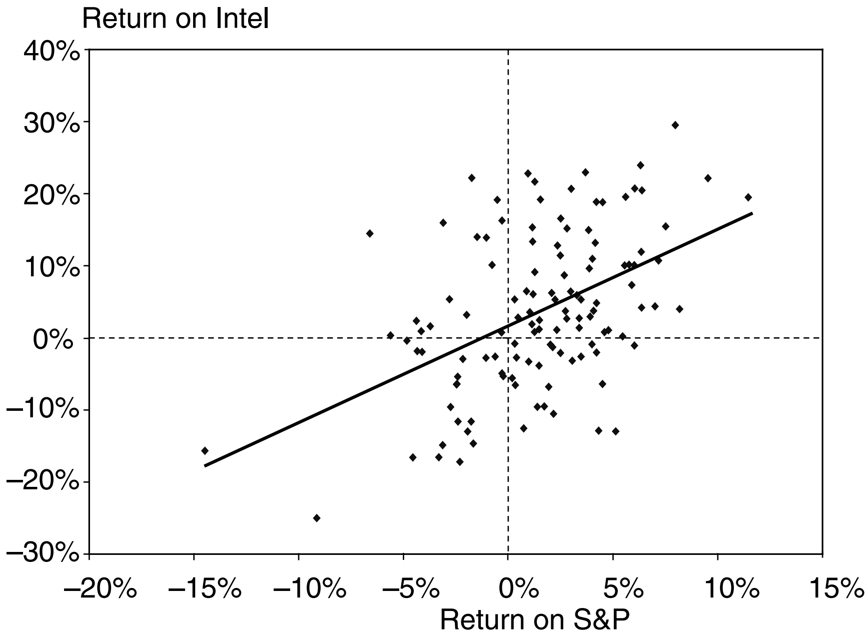


TABLE 3-1 Regression Results $y = \alpha + \beta x$, y = Intel return, x = S&P return

R-square	0.228
Standard error of y	10.94%
Standard error of $\hat{\epsilon}$	9.62%

Coefficient	Estimate	Standard Error	T-statistic	P-value
Intercept $\hat{\alpha}$	0.0168	0.0094	1.78	0.77
Intercept $\hat{\beta}$	1.349	0.229	5.90	0.00

average. Indeed, the slope in Intel's regression is greater than unity. To test whether β is significantly different from one, we can compute a z-score as

$$z = \frac{(\hat{\beta} - 1)}{s(\hat{\beta})} = \frac{(1.349 - 1)}{0.229} = 1.53$$

This is less than the usual cutoff value of 2, so we cannot say for certain that Intel's systematic risk is greater than one.

The R-square of 22.8% can be also interpreted by examining the reduction in dispersion from y to $\hat{\epsilon}$, which is from 10.94% to 9.62%. The R-square can be written as

$$R^2 = 1 - \frac{9.62\%^2}{10.94\%^2} = 22.8\%$$

Thus about 23% of the variance of Intel's returns can be attributed to the market.

3.3.5 Pitfalls with Regressions

As with any quantitative method, the power of regression analysis depends on the underlying assumptions being fulfilled for the particular application. Potential problems of interpretation are now briefly mentioned.

The original OLS setup assumes that the X variables are predetermined (i.e., exogenous or fixed), as in a controlled experiment. In practice, regressions are performed on actual, existing data that do not satisfy these strict conditions. In the previous regression, returns on the S&P are certainly not predetermined.

If the X variables are stochastic, however, most of the OLS results are still valid as long as the X variables are distributed independently of the errors and their distribution does not involve β and σ^2.

Violations of this assumption are serious because they create biases in the slope coefficients. Biases could lead the researcher to come to the wrong conclusion. For instance, we could have measurement error in the X variables, which causes the measured X to be correlated with ϵ. This so-called **errors in the variables** problem causes a downward bias, or reduces the estimated slope coefficients from their true values.[1] Another problem is that of **specification error**. Suppose the true model has N variables but we only use a subset N_1. If the omitted variables are correlated with the included variables, the estimated coefficients will be biased. This is a most serious problem because it is difficult to identify, other than trying other variables in the regression.

Another class of problem is **multicollinearity**. This arises when the X variables are highly correlated. Some of the variables may be superfluous, for example using two currencies that are fixed to each other. As a result, the matrix in Equation (3.44) will be unstable, and the estimated β unreliable. This problem will show up in large standard errors, however. It can be fixed by discarding some of the variables that are highly correlated with others.

The third type of problem has to do with potential biases in the standard errors of the coefficients. These errors are especially serious if standard errors are underestimated, creating a sense of false precision in the regression results and perhaps

[1]Errors in the y variables are not an issue, because they are captured by the error component ϵ.

leading to the wrong conclusions. The OLS approach assumes that the errors are independent across observations. This is generally the case for financial time series, but often not in cross-sectional setups. For instance, consider a cross section of mutual fund returns on some attribute. Mutual fund families often have identical funds, except for the fee structure (e.g., called *A* for a front load, *B* for a deferred load). These funds, however, are invested in the same securities and have the same manager. Thus, their returns are certainly not independent. If we run a standard OLS regression with all funds, the standard errors will be too small. More generally, one has to check that there is no systematic correlation pattern in the residuals. Even with time series, problems can arise with **autocorrelation** in the errors. In addition, the residuals can have different variances across observations, in which case we have **heteroskedasticity**.[2] These problems can be identified by diagnostic checks on the residuals. For instance, the variance of residuals should not be related to other variables in the regression. If some relationship is found, then the model must be improved until the residuals are found to be independent.

Last, even if all the OLS conditions are satisfied, one has to be extremely careful about using a regression for forecasting. Unlike physical systems, which are inherently stable, financial markets are dynamic and relationships can change quickly. Indeed, financial anomalies, which show up as strongly significant coefficients in historical regressions, have an uncanny ability to disappear as soon as one tries to exploit them.

Example 3-5: FRM Exam 1999—Question 2/Quant. Analysis

3-5. Under what circumstances could the explanatory power of regression analysis be overstated?

a) The explanatory variables are not correlated with one another.

b) The variance of the error term decreases as the value of the dependent variable increases.

c) The error term is normally distributed.

d) An important explanatory variable is omitted that influences the explanatory variables included, and the dependent variable.

[2]This is the opposite of the constant variance case, or homoskedasticity.

Example 3-6: FRM Exam 1999—Question 20/Quant. Analysis

3-6. What is the covariance between populations A and B?

A	B
17	22
14	26
12	31
13	29

a) −6.25
b) 6.50
c) −3.61
d) 3.61

Example 3-7: FRM Exam 1999—Question 6/Quant. Analysis

3-7. It has been observed that daily returns on spot positions of the euro against the U.S. dollar are highly correlated with returns on spot holdings of the Japanese yen against the dollar. This implies that

a) When the euro strengthens against the dollar, the yen also tends to strengthen against the dollar. The two sets of returns are not necessarily equal.
b) The two sets of returns tend to be almost equal.
c) The two sets of returns tend to be almost equal in magnitude but opposite in sign.
d) None of the above are true.

Example 3-8: FRM Exam 1999—Question 10/Quant. Analysis

3-8. An analyst wants to estimate the correlation between stocks on the Frankfurt and Tokyo exchanges. He collects closing prices for select securities on each exchange but notes that Frankfurt closes after Tokyo. How will this time discrepancy bias the computed volatilities for individual stocks and correlations between any pair of stocks, one from each market? There will be

a) Increased volatility with correlation unchanged
b) Lower volatility with lower correlation
c) Volatility unchanged with lower correlation
d) Volatility unchanged with correlation unchanged

Example 3-9: FRM Exam 2000—Question 125/Quant. Analysis

3-9. If the F-test shows that the set of X variables explain a significant amount of variation in the Y variable, then

a) Another linear regression model should be tried.
b) A t-test should be used to test which of the individual X variables, if any, should be discarded.
c) A transformation of the Y variable should be made.
d) Another test could be done using an indicator variable to test the significance level of the model.

Example 3-10: FRM Exam 2000—Question 112/Quant. Analysis

3-10. Positive autocorrelation in prices can be defined as

a) An upward movement in price is more than likely to be followed by another upward movement in price.

b) A downward movement in price is more than likely to be followed by another downward movement in price.

c) Both (a) and (b) are correct.

d) Historic prices have no correlation with futures prices.

3.4 Answers to Chapter Examples

Example 3-1: FRM Exam 1999—Question 4/Quant. Analysis

d) Efficient markets implies that the distribution of future returns does not depend on past returns. Hence, returns cannot be correlated. It could happen, however, that return distributions are independent, but that, just by chance, two successive returns are equal.

Example 3-2: FRM Exam 1999—Question 14/Quant. Analysis

c) This is given by $\mathrm{SD}(R_2) \times \sqrt{20/2} = 3.79\%$.

Example 3-3: FRM Exam 1998—Question 7/Quant. Analysis

b) The methodology is the same as for the time aggregation, except that the variance may not be constant over time. The total (annualized) variance is $0.15^2 \times 2 + 0.35^2 \times 1 = 0.1675$ for 3 months, or 0.0558 on average. Taking the square root, we get 0.236, or 24%.

Example 3-4: FRM Exam 1997—Question 15/Risk Measurement

b) This question assumes that VAR is obtained from the volatility using a normal distribution. With trends, or positive correlation between subsequent returns, the 2-day variance is greater than the one obtained from the square root of time rule. See Equation (3.7).

Example 3-5: FRM Exam 1999—Question 2/Quant. Analysis

d) If the true regression includes a third variable z that influences both y and x, the error term will not be conditionally independent of x, which violates one of the assumptions of the OLS model. This will artificially increase the explanatory power of the regression. Intuitively, the variable x will appear to explain more of the variation in y simply because it is correlated with z.

Example 3-6: FRM Exam 1999—Question 20/Quant. Analysis

a) First, compute the average of A and B, which is 14 and 27. Then construct a table as follows.

	A	B	$(A-14)$	$(B-27)$	$(A-14)(B-27)$
	17	22	3	−5	−15
	14	26	0	−1	0
	12	31	−2	4	−8
	13	29	−1	2	−2
Sum	56	108			−25

Summing the last column gives −25, or an average of −6.25.

Example 3-7: FRM Exam 1999—Question 6/Quant. Analysis

a) Positive correlation means that, on average, a positive movement in one variable is associated with a positive movement in the other variable. Because correlation is scale-free, this has no implication for the actual size of movements.

Example 3-8: FRM Exam 1999—Question 10/Quant. Analysis

c) The nonsynchronicity of prices does not alter the volatility, but will induce some error in the correlation coefficient across series. This is similar to the effect of errors in the variables, which biases downward the slope coefficient and the correlation.

Example 3-9: FRM Exam 2000—Question 125/Quant. Analysis

b) The F-test applies to the group of variables but does not say which one is most significant. To identify which particular variable is significant, we use a t-test and discard the variables that do not appear significant.

Example 3-10: FRM Exam 2000—Question 112/Quant. Analysis

c) Positive autocorrelation means that price movements in one direction are more likely to be followed by price movements in the same direction.

Chapter 4

Monte Carlo Methods

The two preceding chapters have dealt with probability and statistics. The former deals with the generation of random variables from known distributions. The second deals with estimation of distribution parameters from actual data. With estimated distributions in hand, we can proceed to the next step, which is the simulation of random variables for the purpose of risk management.

Such simulations, called **Monte Carlo** simulations, are a staple of financial economics. They allow risk managers to build the distribution of portfolios that are far too complex to model analytically.

Simulation methods are quite flexible and are becoming easier to implement with technological advances in computing. Their drawbacks should not be underestimated, however. For all their elegance, simulation results depend heavily on the model's assumptions: the shape of the distribution, the parameters, and the pricing functions. Risk managers need to be keenly aware of the effect that errors in these assumptions can have on the results.

This chapter shows how Monte Carlo methods can be used for risk management. Section 4.1 introduces a simple case with just one source of risk. Section 4.2 shows how to apply these methods to construct value at risk (VAR) measures, as well as to price derivatives. Multiple sources of risk are then considered in Section 4.3.

4.1 Simulations with One Random Variable

Simulations involve creating artificial random variables with properties similar to those of the observed risk factors. These may be stock prices, exchange rates, bond yields or prices, and commodity prices.

4.1.1 Simulating Markov Processes

In efficient markets, financial prices should display a random walk pattern. More precisely, prices are assumed to follow a **Markov process**, which is a particular stochastic process where the whole distribution relies on the current price only. The past history is irrelevant. These processes are built from the following components, described in order of increasing complexity.

- **The Wiener process.** This describes a variable Δz, whose change is measured over the interval Δt such that its mean change is zero and variance proportional to Δt

$$\Delta z \sim N(0, \Delta t) \tag{4.1}$$

  If ϵ is a standard normal variable $N(0, 1)$, this can be written as $\Delta z = \epsilon\sqrt{\Delta t}$. In addition, the increments Δz are independent across time.

- **The Generalized Wiener process.** This describes a variable Δx built up from a Wiener process, with in addition a constant trend a per unit time and volatility b

$$\Delta x = a\Delta t + b\Delta z \tag{4.2}$$

  A particular case is the **martingale**, which is a zero drift stochastic process, $a = 0$. This has the convenient property that the expectation of a future value is the current value

$$E(x_T) = x_0 \tag{4.3}$$

- **The Ito process.** This describes a generalized Wiener process, whose trend and volatility depend on the *current* value of the underlying variable and time

$$\Delta x = a(x, t)\Delta t + b(x, t)\Delta z \tag{4.4}$$

4.1.2 The Geometric Brownian Motion

A particular example of Ito process is the **geometric Brownian motion** (GBM), which is described for the variable S as

$$\Delta S = \mu S\Delta t + \sigma S\Delta z \tag{4.5}$$

The process is geometric because the trend and volatility terms are proportional to the current value of S. This is typically the case for stock prices, for which *rates of returns* appear to be more stationary than raw dollar returns, ΔS. It is also used for currencies. Because $\Delta S/S$ represents the capital appreciation only, abstracting from dividend payments, μ represents the expected total rate of return on the asset minus the dividend yield, $\mu = \mu_{TOTAL} - q$.

Example: A stock price process

Consider a stock that pays no dividends, has an expected return of 10% per annum, and volatility of 20% per annum. If the current price is $100, what is the process for the change in the stock price over the next week? What if the current price is $10?

The process for the stock price is

$$\Delta S = S(\mu\Delta t + \sigma\sqrt{\Delta t} \times \epsilon)$$

where ϵ is a random draw from a standard normal distribution. If the interval is one week, or $\Delta t = 1/52 = 0.01923$, the process is $\Delta S = 100(0.001923 + 0.027735 \times \epsilon)$. With an initial stock price at $100, this gives $\Delta S = 0.1923 + 2.7735\epsilon$. With an initial stock price at $10, this gives $\Delta S = 0.01923 + 0.27735\epsilon$. The trend and volatility are scaled down by a factor of ten.

This model is particularly important because it is the underlying process for the Black-Scholes formula. The key feature of this distribution is the fact that the volatility is proportional to S. This ensures that the stock price will stay positive. Indeed, as the stock price falls, its variance decreases, which makes it unlikely to experience a large downmove that would push the price into negative values. As the limit of this model is a normal distribution for $dS/S = d\ln(S)$, S follows a **lognormal distribution**.

This process implies that, over an interval $T - t = \tau$, the logarithm of the ending price is distributed as

$$ln(S_T) = ln(S_t) + (\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}\,\epsilon \tag{4.6}$$

where ϵ is a standardized normal, $N(0, 1)$ random variable.

Example: A stock price process (continued)

Assume the price in one week is given by $S = \$100\exp(R)$, where R has annual expected value of 10% and volatility of 20%. Construct a 95% confidence interval for S.

The standard normal deviates that corresponds to a 95% confidence interval are $\alpha_{\text{MIN}} = -1.96$ and $\alpha_{\text{MAX}} = 1.96$. In other words, we have 2.5% in each tail.

The 95% confidence band for R is then $R_{\text{MIN}} = \mu\Delta t - 1.96\sigma\sqrt{\Delta t} = 0.001923 - 1.96 \times 0.027735 = -0.0524$ $R_{\text{MAX}} = \mu\Delta t + 1.96\sigma\sqrt{\Delta t} = 0.001923 + 1.96 \times 0.027735 = 0.0563$ This gives $S_{\text{MIN}} = \$100\exp(-0.0524) = \$94.89$, and $S_{\text{MAX}} = \$100\exp(0.0563) = \$105.79$.

The importance of the lognormal assumption depends on the horizon considered. If the horizon is one day only, the choice of the lognormal versus normal assumption does not really matter. It is highly unlikely that the stock price would drop below zero in one day, given typical volatilities. On the other hand, if the horizon is measured in years, the two assumptions do lead to different results. The lognormal distribution is more realistic as it prevents prices form turning negative.

In simulations, this process is approximated by small steps with a normal distribution with mean and variance given by

$$\frac{\Delta S}{S} \sim N(\mu\Delta t, \sigma^2\Delta t) \tag{4.7}$$

To simulate the future price path for S, we start from the current price S_t and generate a sequence of independent standard normal variables ϵ, for $i = 1, 2, \ldots, n$. This can be done easily in an Excel spreadsheet, for instance. The next price S_{t+1} is built as $S_{t+1} = S_t + S_t(\mu\Delta t + \sigma\epsilon_1\sqrt{\Delta t})$. The following price S_{t+2} is taken as $S_{t+1} + S_{t+1}(\mu\Delta t + \sigma\epsilon_2\sqrt{\Delta t})$, and so on until we reach the target horizon, at which point the price $S_{t+n} = S_T$ should have a distribution close to the lognormal.

Table 4-1 illustrates a simulation of a process with a drift (μ) of 0 percent and volatility (σ) of 20 percent over the total interval, which is divided into 100 steps.

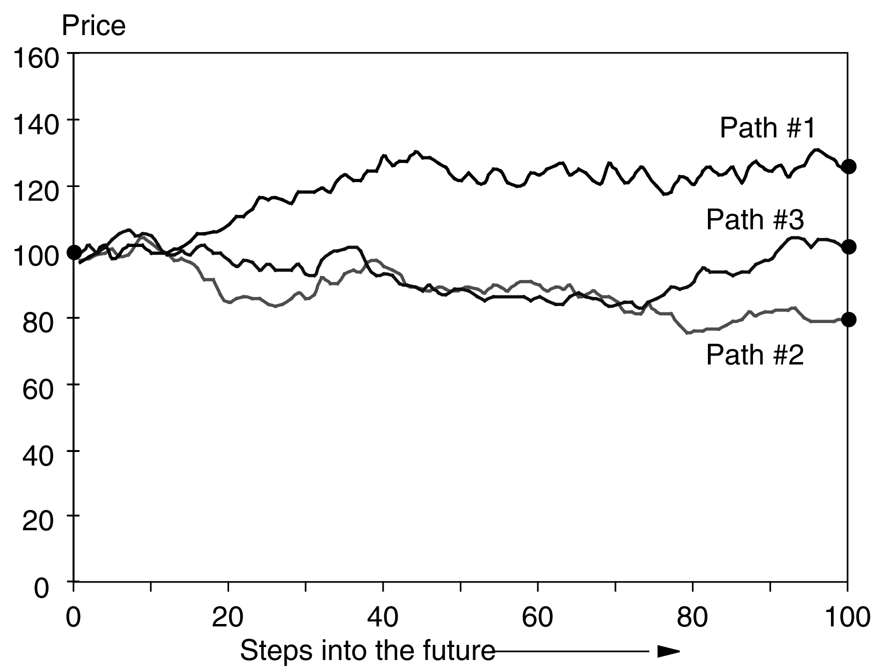
TABLE 4-1 Simulating a Price Path

Step	Random Variable		Price	Price
	Uniform	Normal	Increment	
i	u_i	$\mu\Delta t + \sigma\Delta z$	ΔS_i	S_{t+i}
	=RAND()	=NORMINV(u_i,0.0,0.02)		
0				100.00
1	0.0430	−0.0343	−3.433	96.57
2	0.8338	0.0194	1.872	98.44
3	0.6522	0.0078	0.771	99.21
4	0.9219	0.0284	2.813	102.02
...				
99				124.95
100	0.5563	0.0028	0.354	125.31

The initial price is \$100. The local expected return is $\mu\Delta t = 0.0/100 = 0.0$ and the volatility is $0.20 \times \sqrt{1/100} = 0.02$. The second column shows the realization of a uniform $U(0, 1)$ variable, with the corresponding Excel function. The value for the first step is $u_1 = 0.0430$. The next column transforms this variable into a normal variable with mean 0.0 and volatility of 0.02, which gives −0.0343, showing the Excel function. The price increment is then obtained by multiplying the random variable by the previous price, which gives −\$3.433. This generates a new value of $S_1 = \$96.57$. The process is repeated until the final price of \$125.31 is reached at the 100th step.

This experiment can be repeated as often as needed. Define K as the number of **replications**, or random trials. Figure 4-1 displays the first three trials. Each leads to a simulated final value S_T^k. This generates a distribution of simulated prices S_T. With just one step $n = 1$, the distribution must be normal. As the number of steps n grows large, the distribution tends to a lognormal distribution.

FIGURE 4-1 Simulating Price Paths



While very useful to model stock prices, this model has shortcomings. Price increments are assumed to have a normal distribution. In practice, we observe that price changes have fatter tails than the normal distribution and may also experience changing variance.

In addition, as the time interval Δt shrinks, the volatility shrinks as well. In other words, large discontinuities cannot occur over short intervals. In reality, some assets, such as commodities, experience discrete jumps. This approach, however, is sufficiently flexible to accommodate other distributions.

4.1.3 Simulating Yields

The GBM process is widely used for stock prices and currencies. Fixed-income products are another matter.

Bond prices display long-term reversion to the face value (unless there is default). Such process is inconsistent with the GBM process, which displays no such mean reversion. The volatility of bond prices also changes in a predictable fashion, as duration shrinks to zero. Similarly, commodities often display mean reversion.

These features can be taken into account by modelling bond yields directly in a first step. In the next step, bond prices are constructed from the value of yields and a pricing function. The dynamics of interest rates r_t can be modeled by

$$\Delta r_t = \kappa(\theta - r_t)\Delta t + \sigma r_t{}^{\gamma} \Delta z_t \tag{4.8}$$

where Δz_t is the usual Wiener process. Here, we assume that $0 \leq \kappa < 1, \theta \geq 0, \sigma \geq 0$. If there is only one stochastic variable in the fixed income market Δz, the model is called a **one-factor model**.

This Markov process has a number of interesting features. First, it displays mean reversion to a long-run value of θ. The parameter κ governs the speed of mean reversion. When the current interest rate is high, i.e. $r_t > \theta$, the model creates a negative drift $\kappa(\theta - r_t)$ toward θ. Conversely, low current rates create with a positive drift.

The second feature is the volatility process. This class of model includes the **Vasicek model** when $\gamma = 0$. Changes in yields are normally distributed because δr is a linear function of Δz. This model is particularly convenient because it leads to closed-form solutions for many fixed-income products. The problem, however, is that it could allow negative interest rates because the volatility of the change in rates does not depend on the level.

Equation (4.8) is more general because it includes a power of the yield in the variance function. With $\gamma = 1$, the model is the **lognormal model**.[1] This implies that the

[1]This model is used by RiskMetrics for interest rates.

rate of change in the yield has a fixed variance. Thus, as with the GBM model, smaller yields lead to smaller movements, which makes it unlikely the yield will drop below zero. With $\gamma = 0.5$, this is the **Cox, Ingersoll, and Ross (CIR) model**. Ultimately, the choice of the exponent γ is an empirical issue. Recent research has shown that $\gamma = 0.5$ provides a good fit to the data.

This class of models is known as **equilibrium models**. They start with some assumptions about economic variables and imply a process for the short-term interest rate r. These models generate a predicted term structure, whose shape depends on the model parameters and the initial short rate. The problem with these models is that they are not flexible enough to provide a good fit to today's term structure. This can be viewed as unsatisfactory, especially by most practitioners who argue that they cannot rely on a model that cannot even be trusted to price today's bonds.

In contrast, **no-arbitrage models** are designed to be consistent with today's term structure. In this class of models, the term structure is an input into the parameter estimation. The earliest model of this type was the **Ho and Lee model**

$$\Delta r_t = \theta(t)\Delta t + \sigma\Delta z_t \tag{4.9}$$

where $\theta(t)$ is a function of time chosen so that the model fits the initial term structure. This was extended to incorporate mean reversion in the **Hull and White model**

$$\Delta r_t = [\theta(t) - ar_t]\Delta t + \sigma\Delta z_t \tag{4.10}$$

Finally, the **Heath, Jarrow, and Morton model** goes one step further and allows the volatility to be a function of time.

The downside of these no-arbitrage models, however, is that they impose no consistency between parameters estimated over different dates. They are also more sensitive to outliers, or data errors in bond prices used to fit the term structure.

4.1.4 Binomial Trees

Simulations are very useful to mimic the uncertainty in risk factors, especially with numerous risk factors. In some situations, however, it is also useful to describe the uncertainty in prices with discrete trees. When the price can take one of two steps, the tree is said to be **binomial**.

The binomial model can be viewed as a discrete equivalent to the geometric Brownian motion. As before, we subdivide the horizon T into n intervals $\Delta t = T/n$. At each "node," the price is assumed to go either up with probability p, or down with probability $1 - p$.

The parameters u, d, p are chosen so that, for a small time interval, the expected return and variance equal those of the continuous process. One could choose

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = (1/u), \quad p = \frac{e^{\mu\Delta t} - d}{u - d} \tag{4.11}$$

This matches the mean

$$E[S_1/S_0] = pu + (1 - p)d = \frac{e^{\mu\Delta t} - d}{u - d}u + \frac{u - e^{\mu\Delta t}}{u - d}d = \frac{e^{\mu\Delta t}(u - d) - du + ud}{u - d} = e^{\mu\Delta t}$$

Table 4-2 shows how a binomial tree is constructed.

TABLE 4-2 Binomial Tree

0		1		2		3
						u^3S
					↗	
				u^2S		
			↗		↘	
		uS				u^2dS
	↗		↘		↗	
S				udS		
	↘		↗		↘	
		dS				d^2uS
			↘		↗	
				d^2S		
					↘	
						d^3S

As the number of steps increases, Cox, Ross, and Rubinstein (1979) have shown that the discrete distribution of S_T converges to the lognormal distribution.[2] This model will be used in a later chapter to price options.

[2] Cox, J., Ross S., and Rubinstein M. (1979), Option Pricing: A Simplified Approach, *Journal of Financial Economics* 7, 229–263.

Example 4-1: FRM Exam 1999—Question 18/Quant. Analysis

4-1. If S1 follows a geometric Brownian motion and S2 follows a geometric Brownian motion, which of the following is *true*?

a) Ln(S1 + S2) is normally distributed.
b) S1 × S2 is lognormally distributed.
c) S1 × S2 is normally distributed.
d) S1 + S2 is normally distributed.

Example 4-2: FRM Exam 1999—Question 19/Quant. Analysis

4-2. Considering the one-factor Cox, Ingersoll, and Ross term-structure model and the Vasicek model:

I) Drift coefficients are different.
II) Both include mean reversion.
III) Coefficients of the stochastic term, dz, are different.
IV) CIR is a jump-diffusion model.

a) All of the above are true.
b) I and III are true.
c) II, III, and IV are true.
d) II and III are true.

Example 4-3: FRM Exam 1999—Question 25/Quant. Analysis

4-3. The Vasicek model defines a risk-neutral process for r which is $dr = a(b - r)dt + \sigma dz$, where a, b, and σ are constant, and r represents the rate of interest. From this equation we can conclude that the model is a

a) Monte Carlo-type model
b) Single factor term-structure model
c) Two-factor term-structure model
d) Decision tree model

Example 4-4: FRM Exam 1999—Question 26/Quant. Analysis

4-4. The term $a(b - r)$ in the equation in Question 25 represents which term?

a) Gamma
b) Stochastic
c) Reversion
d) Vega

Example 4-5: FRM Exam 1999—Question 30/Quant. Analysis

4-5. For which of the following currencies would it be most appropriate to choose a lognormal interest rate model over a normal model?

a) USD
b) JPY
c) EUR
d) GBP

Example 4-6: FRM Exam 1998—Question 23/Quant. Analysis

4-6. Which of the following interest rate term-structure models tends to capture the mean reversion of interest rates?

a) $dr = a \times (b - r)dt + \sigma \times dz$
b) $dr = a \times dt + \sigma \times dz$
c) $dr = a \times r \times dt + \sigma \times r \times dz$
d) $dr = a \times (r - b) \times dt + \sigma \times dz$

Example 4-7: FRM Exam 1998—Question 24/Quant. Analysis

4-7. Which of the following is a shortcoming of modeling a bond option by applying Black-Scholes formula to bond prices?

a) It fails to capture convexity in a bond.
b) It fails to capture the pull-to-par phenomenon.
c) It fails to maintain put-call parity.
d) It works for zero-coupon bond options only.

Example 4-8: FRM Exam 2000—Question 118/Quant. Analysis

4-8. Which group of term-structure models do the Ho-Lee, Hull-White and Heath, Jarrow, and Morton models belong to?

a) No-arbitrage models
b) Two-factor models
c) Lognormal models
d) Deterministic models

Example 4-9: FRM Exam 2000—Question 119/Quant. Analysis

4-9. A plausible stochastic process for the short-term rate is often considered to be one where the rate is pulled back to some long-run average level. Which one of the following term-structure models does *not* include this characteristic?

a) The Vasicek model
b) The Ho-Lee model
c) The Hull-White model
d) The Cox-Ingersoll-Ross model

Example 4-10: FRM Exam 2001—Question 76
4-10. A martingale is a
a) Zero-drift stochastic process
b) Chaos-theory-related process
c) Type of time series
d) Mean-reverting stochastic process

4.2 Implementing Simulations

4.2.1 Simulation for VAR

To summarize, the sequence of steps of Monte Carlo methods in risk management follows these steps:

1. Choose a stochastic process (including the distribution and its parameters).
2. Generate a pseudo-sequence of variables $\epsilon_1, \epsilon_2, \ldots \epsilon_n$, from which we compute prices as $S_{t+1}, S_{t+2}, \ldots, S_{t+n} = S_T$.
3. Calculate the value of the portfolio $F_T(S_T)$ under this particular sequence of prices at the target horizon.
4. Repeat steps 2 and 3 as many times as necessary. Call K the number of replications.

These steps create a distribution of values, $F_T^1, \ldots, F_T^K$, which can be sorted to derive the VAR. We measure the cth quantile $Q(F_T, c)$ and the average value $\text{Ave}(F_T)$. If VAR is defined as the deviation from the expected value on the target date, we have

$$\text{VAR}(c) = \text{Ave}(F_T) - Q(F_T, c) \tag{4.12}$$

4.2.2 Simulation for Derivatives

Readers familiar with derivatives pricing will have recognized that this method is similar to the Monte Carlo method for valuing derivatives. In that case, we simply focus on the expected value on the target date discounted into the present:

$$F_t = e^{-r(T-t)} \, \text{Ave}(F_T) \tag{4.13}$$

Thus derivatives valuation focuses on the discounted center of the distribution, while VAR focuses on the quantile on the target date.

Monte Carlo simulations have been long used to price derivatives. As will be seen in a later chapter, pricing derivatives can be done by assuming that the underlying asset grows at the risk-free rate r (assuming no income payment). For instance, pricing an option on a stock with expected return of 20% is done assuming that (1) the stock grows at the risk-free rate of 10% and (2) we discount at the same risk-free rate. This is called the **risk-neutral approach**.

In contrast, risk measurement deals with actual distributions, sometimes called **physical distributions**. For measuring VAR, the risk manager must simulate asset growth using the actual expected return μ of 20%. Therefore, risk management uses physical distributions, whereas pricing methods use risk-neutral distributions. This can create difficulties, as risk-neutral probabilities can be inferred from observed asset prices, unlike not physical probabilities.

It should be noted that simulation methods are not applicable to all types of options. These methods assume that the derivative at expiration can be priced solely as a function of the end-of-period price S_T, and perhaps of its sample path. This is the case, for instance, with an Asian option, where the payoff is a function of the price *averaged* over the sample path. Such an option is said to be **path-dependent**.

Simulation methods, however, cannot be used to price American options, which can be exercised early. The exercise decision should take into account future values of the option. Valuing American options requires modelling such decision process, which cannot be done in a regular simulation approach.

Instead, this requires a **backward recursion**. This method examines whether the option should be exercised starting from the end and working backward in time until the starting time. This can be done using binomial trees.

4.2.3 Accuracy

Finally, we should mention the effect of **sampling variability**. Unless K is extremely large, the empirical distribution of S_T will only be an approximation of the true distribution. There will be some natural variation in statistics measured from Monte Carlo simulations. Since Monte Carlo simulations involve *independent* draws, one can show that the standard error of statistics is inversely related to the square root of K. Thus more simulations will increase precision, but at a slow rate. Accuracy is increased by a factor of ten going from $K = 10$ to $K = 1{,}000$, but then requires going from $K = 1{,}000$ to $K = 100{,}000$ for the same factor of ten.

For VAR measures, the precision is also a function of the selected confidence level. Higher confidence levels generate fewer observations in the left tail and hence less precise VAR measures. A 99% VAR using 1,000 replications should be expected to have only 10 observations in the left tail, which is not a large number. The VAR estimate is derived from the 10th and 11th sorted number. In contrast, a 95% VAR is measured from the 50th and 51th sorted number, which will be more precise.

Various methods are available to speed up convergence.

- **Antithetic Variable Technique** This technique uses twice the same sequence of random draws ϵ_i. It takes the original sequence and changes the sign of all their values. This creates twice the number of points in the final distribution of F_T.
- **Control Variate Technique** This technique is used with trees when a similar option has an analytical solution. Say that f_E is a European option with an analytical solution. Going through the tree yields the values of an American and European option, F_A and F_E. We then assume that the error in F_A is the same as that in F_E, which is known. The adjusted value is $F_A - (F_E - f_E)$.
- **Quasi-Random Sequences** These techniques, also called Quasi Monte Carlo (QMC), create draws that are not independent but instead are designed to fill the sample space more uniformly. Simulations have shown that QMC methods converge faster than Monte Carlo. In other words, for a fixed number of replications K, QMC values will be on average closer to the true value.

  The advantage of traditional MC, however, is that the MC method also provides a standard error, or a measure of precision of the estimate, which is on the order of $1/\sqrt{K}$, because draws are independent. So, we have an idea of how far the estimate might be from the true value, which is useful to decide on the number of replications. In contrast, QMC methods give no measure of precision.

Example 4-11: FRM Exam 1999—Question 8/Quant. Analysis

4-11. Several different estimates of the VAR of an options portfolio were computed using 1,000 independent, lognormally distributed samples of the underlyings. Because each estimate was made using a different set of random numbers, there was some variability in the answers; in fact, the standard deviation of the distribution of answers was about $100,000. It was then decided to re-run the VAR calculation using 10,000 independent samples per run. The standard deviation of the reruns is most likely to be

a) About $10,000
b) About $30,000
c) About $100,000 (i.e., no change from the previous set of runs)
d) Cannot be determined from the information provided

Example 4-12: FRM Exam 1998—Question 34/Quant. Analysis

4-12. You have been asked to find the value of an Asian option on the short rate. The Asian option gives the holder an amount equal to the average value of the short rate over the period to expiration less the strike rate. To value this option with a one-factor binomial model of interest rates, what method would you recommend using?

a) The backward induction method, since it is the fastest
b) The simulation method, using path averages since the option is path-dependent
c) The simulation method, using path averages since the option is path-independent
d) Either the backward induction method or the simulation method, since both methods return the same value

Example 4-13: FRM Exam 1997—Question 17/Quant. Analysis

4-13. The measurement error in VAR, due to sampling variation, should be greater with

a) More observations and a high confidence level (e.g. 99%)
b) Fewer observations and a high confidence level
c) More observations and a low confidence level (e.g. 95%)
d) Fewer observations and a low confidence level

4.3 Multiple Sources of Risk

We now turn to the more general case of simulations with many sources of financial risk. Define N as the number of risk factors. In what follows, we use matrix manipulations to summarize the method.

If the factors S_j are independent, the randomization can be performed independently for each variable. For the GBM model,

$$\Delta S_{j,t} = S_{j,t-1}\mu_j \Delta t + S_{j,t-1}\sigma_j \epsilon_{j,t}\sqrt{\Delta t} \tag{4.14}$$

where the standard normal variables ϵ are independent across time and factor $j = 1, \ldots, N$.

In general, however, risk factors are correlated. The simulation can be adapted by, first, drawing a set of independent variables η, and, second, transforming them into correlated variables ϵ. As an example, with two factors only, we write

$$\epsilon_1 = \eta_1$$

$$\epsilon_2 = \rho\eta_1 + (1-\rho^2)^{1/2}\eta_2 \tag{4.15}$$

Here, ρ is the correlation coefficient between the variables ϵ. Because the ηs have unit variance and are uncorrelated, we verify that the variance of ϵ_2 is one, as required

$$\mathrm{V}(\epsilon_2) = \rho^2\mathrm{V}(\eta_1) + [(1-\rho^2)^{1/2}]^2\mathrm{V}(\eta_2) = \rho^2 + (1-\rho^2) = 1,$$

Furthermore, the correlation between ϵ_1 and ϵ_2 is given by

$$\mathrm{Cov}(\epsilon_1, \epsilon_2) = \mathrm{Cov}(\eta_1, \rho\eta_1 + (1-\rho^2)^{1/2}\eta_2) = \rho\mathrm{Cov}(\eta_1, \eta_1) = \rho$$

Defining ϵ as the *vector* of values, we verified that the covariance matrix of ϵ is

$$V(\epsilon) = \begin{bmatrix} \sigma^2(\epsilon_1) & \mathrm{Cov}(\epsilon_1, \epsilon_2) \\ \mathrm{Cov}(\epsilon_1, \epsilon_2) & \sigma^2(\epsilon_2) \end{bmatrix} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = R$$

Note that this covariance matrix, which is the expectation of squared deviations from the mean, can also be written as

$$V(\epsilon) = E[(\epsilon - E(\epsilon)) \times (\epsilon - E(\epsilon))'] = E(\epsilon \times \epsilon')$$

because the expectation of ϵ is zero. More generally, we need a systematic method to derive the transformation in Equation (4.15) for many risk factors.

4.3.1 The Cholesky Factorization

We would like to generate N joint values of ϵ that display the correlation structure $V(\epsilon) = E(\epsilon\epsilon') = R$. Because the matrix R is a symmetric real matrix, it can be decomposed into its so-called Cholesky factors

$$R = TT' \tag{4.16}$$

where T is a lower triangular matrix with zeros on the upper right corners (above the diagonal). This is known as the **Cholesky factorization**.

As in the previous section, we first generate a vector of independent η, which are standard normal variables. Thus, the covariance matrix is $V(\eta) = I$, where I is the identity matrix with zeros everywhere except on the diagonal.

We then construct the transformed variable $\epsilon = T\eta$. The covariance matrix is now $V(\epsilon) = E(\epsilon\epsilon') = E((T\eta)(T\eta)') = E(T\eta\eta'T') = TE(\eta\eta')T' = TV(\eta)T' = TIT' = TT' = R$. This transformation therefore generates ϵ variables with the desired correlations.

To illustrate, let us go back to our 2-variable case. The correlation matrix can be decomposed into its Cholesky factors as

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{11}a_{21} \\ a_{21}a_{11} & a_{21}^2 + a_{22}^2 \end{bmatrix}$$

To find the entries a_{11}, a_{21}, a_{22}, we solve and substitute as follows

$$a_{11}^2 = 1$$

$$a_{11}a_{21} = \rho$$

$$a_{21}^2 + a_{22}^2 = 1$$

The Cholesky factorization is then

$$\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & (1-\rho^2)^{1/2} \end{bmatrix} \begin{bmatrix} 1 & \rho \\ 0 & (1-\rho^2)^{1/2} \end{bmatrix}$$

Note that this conforms precisely to Equation (4.15):

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & (1-\rho^2)^{1/2} \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$$

In practice, this decomposition yields a number of useful insights. The decomposition will fail if the number of independent factors implied in the correlation matrix is less than N. For instance, if $\rho = 1$, meaning that we have twice the same factor, perhaps two currencies fixed to each other, we have: $a_{11} = 1, a_{21} = 1, a_{22} = 0$. The new variables are then $\epsilon_1 = \eta_1$ and $\epsilon_2 = \eta_1$. The second variable η_2 is totally superfluous.

This type of information can be used to reduce the dimension of the covariance matrix of risk factors. RiskMetrics, for instance, currently has about 400 variables. This translates into a correlation matrix with about 80,000 elements, which is huge. Simulations based on the full set of variables would be inordinately time-consuming. The art of simulation is to design parsimonious experiments that represent the breadth of movements in risk factors.

Example 4-14: FRM Exam 1999—Question 29/Quant. Analysis

4-14. Given the covariance matrix,

$$\Sigma = \begin{bmatrix} 0.09\% & 0.06\% & 0.03\% \\ 0.06\% & 0.05\% & 0.04\% \\ 0.03\% & 0.04\% & 0.06\% \end{bmatrix}$$

let $\Sigma = XX'$, where X is lower triangular, be a Cholesky decomposition. Then the four elements in the upper left-hand corner of $X, x_{11}, x_{12}, x_{21}, x_{22}$, are, respectively,

a) 3.0%, 0.0%, 4.0%, 2.0%
b) 3.0%, 4.0%, 0.0%, 2.0%
c) 3.0%, 0.0%, 2.0%, 1.0%
d) 2.0%, 0.0%, 3.0%, 1.0%

4.4 Answers to Chapter Examples

Example 4-1: FRM Exam 1999—Question 18/Quant. Analysis

b) Both S1 and S2 are lognormally distributed since $d\ln(S1)$ and $d\ln(S2)$ are normally distributed. Since the logarithm of (S1*S2) is also its sum, it is also normally distributed and the variable S1*S2 is lognormally distributed.

Example 4-2: FRM Exam 1999—Question 19/Quant. Analysis

d) Answers II and III are correct. Both models include mean reversion but have different variance coefficients. None includes jumps.

Example 4-3: FRM Exam 1999—Question 25/Quant. Analysis

b) This model postulates only one source of risk in the fixed-income market. This is a single-factor term-structure model.

Example 4-4: FRM Exam 1999—Question 26/Quant. Analysis

c) This represents the expected return with mean reversion.

Example 4-5: FRM Exam 1999—Question 30/Quant. Analysis

b) *(This requires some knowledge of markets)* Currently, yen interest rates are very low, the lowest of the group. This makes it important to choose a model that, starting from current rates, does not allow negative interest rates, such as the lognormal model.

Example 4-6: FRM Exam 1998—Question 23/Quant. Analysis

a) This is also Equation (4.8), assuming all parameters are positive.

Example 4-7: FRM Exam 1998—Question 24/Quant. Analysis

b) The model assumes that prices follow a random walk with a constant trend, which is not consistent with the fact that the price of a bond will tend to par.

Example 4-8: FRM Exam 2000—Question 118/Quant. Analysis

a) These are no-arbitrage models of the term structure, implemented as either one-factor or two-factor models.

Example 4-9: FRM Exam 2000—Question 119/Quant. Analysis

b) Both the Vasicek and CIR models are one-factor equilibrium models with mean reversion. The Hull-White model is a no-arbitrage model with mean reversion. The Ho and Lee model is an early no-arbitrage model without mean-reversion.

Example 4-10: FRM Exam 2001—Question 76

a) A martingale is a stochastic process with zero drift $dx = \sigma dz$, where dz is a Wiener process, i.e. such that $dz \sim N(0, dt)$. The expectation of future value is the current value: $E[x_T] = x_0$, so it cannot be mean-reverting.

Example 4-11: FRM Exam 1999—Question 8/Quant. Analysis

b) Accuracy with independent draws increases with the square root of K. Thus multiplying the number of replications by a factor of 10 will shrink the standard errors from 100,000 to $100{,}000/\sqrt{10}$, or to approximately 30,000.

Example 4-12: FRM Exam 1998—Question 34/Quant. Analysis

b) *(Requires knowledge of derivative products)* Asian options create a payoff that depends on the average value of S during the life of the options. Hence, they are "path-dependent" and do not involve early exercise. Such options must be evaluated using simulation methods.

Example 4-13: FRM Exam 1997—Question 17/Quant. Analysis

b) Sampling variability (or imprecision) increases with (i) fewer observations and (ii) greater confidence levels. To show (i), we can refer to the formula for the precision of the sample mean, which varies inversely with the square root of the number of data points. A similar reasoning applies to (ii). A greater confidence level involves fewer observations in the left tails, from which VAR is computed.

Example 4-14: FRM Exam 1999—Question 29/Quant. Analysis

c) *(Data-intensive)* This involves a Cholesky decomposition. We have $XX' =$

$$\begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{21} & x_{31} \\ 0 & x_{22} & x_{32} \\ 0 & 0 & x_{33} \end{bmatrix} = \begin{bmatrix} x_{11}^2 & x_{11}x_{21} & x_{11}x_{33} \\ x_{21}x_{11} & x_{21}^2 + x_{22}^2 & x_{21}x_{31} + x_{22}x_{32} \\ x_{31}x_{11} & x_{31}x_{21} + x_{32}x_{22} & x_{31}^2 + x_{32}^2 + x_{33}^2 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 0.09\% & 0.06\% & 0.03\% \\ 0.06\% & 0.05\% & 0.04\% \\ 0.03\% & 0.04\% & 0.06\% \end{bmatrix}$$

We then laboriously match each term, $x_{11}^2 = 0.0009$, or $x_{11} = 0.03$. Next, $x_{12} = 0$ since this is in the upper right corner, above the diagonal. Next, $x_{11}x_{21} = 0.0006$, or $x_{21} = 0.02$. Next, $x_{21}^2 + x_{22}^2 = 0.0005$, or $x_{22} = 0.01$.

PART two

Capital Markets

Chapter 5

Introduction to Derivatives

This chapter provides an overview of derivative instruments. Derivatives are contracts traded in private **over-the-counter** (OTC) markets, or on organized exchanges. These instruments are fundamental building blocks of capital markets and can be broadly classified into two categories: linear and nonlinear instruments.

To the first category belong forward contracts, futures, and swaps. These are *obligations* to exchange payments according to a specified schedule. Forward contracts are relatively simple to evaluate and price. So are futures, which are traded on exchanges. Swaps are more complex but generally can be reduced to portfolios of forward contracts. To the second category belong options, which are traded both OTC and on exchanges. These will be covered in the next chapter.

This chapter describes the general characteristics as well as the pricing of linear derivatives. Pricing is the first step toward risk measurement. The second step consists of combining the valuation formula with the distribution of underlying risk factors to derive the distribution of contract values. This will be done later, in the market risk section.

Section 5.1 provides an overview of the size of the derivatives markets. Section 5.2 then presents the valuation and pricing of forwards. Sections 5.3 and 5.4 introduce futures and swap contracts, respectively.

5.1 Overview of Derivatives Markets

A **derivative instrument** can be generally defined as a private contract whose value derives from some underlying asset price, reference rate or index—such as a stock, bond, currency, or a commodity. In addition, the contract must also specify a principal, or **notional** amount, which is defined in terms of currency, shares, bushels, or some other unit. Movements in the value of the derivative are obtained as a function of the notional and the underlying price or index.

In contrast with **securities**, such as stocks and bonds, which are issued to raise capital, derivatives are **contracts**, or private agreements between two parties. Thus the sum of gains and losses on derivatives contracts must be zero; for any gain made by one party, the other party must have suffered a loss of equal magnitude.

At the broadest level, derivatives markets can be classified by the underlying instrument, as well as by type of trading. Table 5-1 describes the size and growth of the

TABLE 5-1 Global Derivatives Markets - 1995-2001 (Billions of U.S. Dollars)

	Notional Amounts	
	March 1995	Dec. 2001
OTC Instruments	**47,530**	**111,115**
Interest rate contracts	**26,645**	**77,513**
Forwards (FRAs)	4,597	7,737
Swaps	18,283	58,897
Options	3,548	10,879
Foreign exchange contracts	**13,095**	**16,748**
Forwards and forex swaps	8,699	10,336
Swaps	1,957	3,942
Options	2,379	2,470
Equity-linked contracts	**579**	**1,881**
Forwards and swaps	52	320
Options	527	1,561
Commodity contracts	**318**	**598**
Others	**6,893**	**14,375**
Exchange-Traded Instruments	**8,838**	**23,799**
Interest rate contracts	**8,380**	**21,758**
Futures	5,757	9,265
Options	2,623	12,493
Foreign exchange contracts	**88**	**93**
Futures	33	66
Options	55	27
Stock-index contracts	**370**	**1,947**
Futures	128	342
Options	242	1,605
Total	**55,910**	**134,914**

Source: Bank for International Settlements global derivatives markets. As of December 2001, the total notional amounts add up to $135 trillion, of which $111 trillion is on OTC markets and $24 trillion on organized exchanges.

The table shows that interest rate contracts are the most widespread type of derivatives, especially swaps. On the OTC market, currency contracts are also widely used, especially outright forwards and **forex swaps**, which are a combination of spot and short-term forward transactions. Among exchange-traded instruments, interest rate futures and options are the most common.

The magnitude of the notional amount of $135 trillion is difficult to grasp. This number is several times the world **gross domestic product (GDP)**, which amounts to approximately $30 trillion. It is also greater than the total outstanding value of stocks and bonds, which is around $70 trillion.

Notional amounts give an indication of equivalent positions in cash markets. For example, a long futures contract on a stock index with a notional of $1 million is equivalent to a cash position in the stock market of the same magnitude.

Notional amounts, however, do not give much information about the risks of the positions. The liquidation value of OTC derivatives contracts, for instance, is estimated at $3.8 trillion, which is only 3 percent of the notional. For futures contracts, which are marked-to-market daily, market values are close to zero. The risk of these derivatives is best measured by the potential change in mark-to-market values over the horizon, or their value at risk (VAR).

5.2 Forward Contracts

5.2.1 Definition

The most common transactions in financial instruments are **spot transactions**, that is, for physical delivery as soon as practical (perhaps in 2 business days or in a week). Historically, grain farmers went to a centralized location to meet buyers for their product.

As markets developed, the farmers realized that it would be beneficial to trade for delivery at some future date. This allowed them to hedge out price fluctuations for the sale of their anticipated production.

This gave rise to **forward contracts**, which are private agreements to exchange a given asset against cash at a fixed point in the future.[1] The terms of the contract are the quantity (number of units or shares), date, and price at which the exchange will be done.

A position which implies buying the asset is said to be **long**. A position to sell is said to be **short**. Note that, since this instrument is a private contract, any gain to one party must be a loss to the other.

These instruments represent contractual obligations, as the exchange must occur whatever happens to the intervening price, unless default occurs. Unlike an option contract, there is no choice in taking delivery or not.

To avoid the possibility of losses, the farmer could enter a forward sale of grain for dollars. By so doing, he locks up a price now for delivery in the future. We then say that the farmer is **hedged** against movements in the price.

We use the notations,

$$
\begin{aligned}
t &= \text{current time} \\
T &= \text{time of delivery} \\
\tau = T - t &= \text{time to maturity} \\
S_t &= \text{current spot price of the asset in dollars} \\
F_t(T) &= \text{current forward price of the asset for delivery at } T \\
&\quad \text{(also written as } F_t \text{ or } F \text{ to avoid clutter)} \\
V_t &= \text{current value of contract} \\
r &= \text{current domestic risk-free rate for delivery at } T \\
n &= \text{quantity, or number of units in contract}
\end{aligned}
$$

The **face amount**, or **principal value** of the contract is defined as the amount nF to pay at maturity, like a bond. This is also called the **notional amount**. We will assume that interest rates are continuously compounded so that the present value of a dollar paid at expiration is $\text{PV}(\$1) = e^{-r\tau}$.

Say that the initial forward price is $F_t = \$100$. A speculator agrees to buy $n = 500$ units for F_t at T. At expiration, the payoff on the forward contract is determined as follows:

---

[1] More generally, any agreement to exchange an asset for another and not only against cash.

(1) The speculator pays $nF = \$50,000$ in cash and receives 500 units of the underlying. (2) The speculator could then sell the underlying at the prevailing spot price S_T, for a profit of $n(S_T - F)$. For example, if the spot price is at $S_T = \$120$, the profit is $500 \times (\$120 - \$100) = \$10,000$. This is also the mark-to-market value of the contract at expiration.

In summary, the value of the forward contract at expiration, for one unit of the underlying asset is

$$V_T = S_T - F \tag{5.1}$$

Here, the value of the contract at expiration is derived from the purchase and **physical delivery** of the underlying asset. There is a payment of cash in exchange for the actual asset.

Another mode of settlement is **cash settlement**. This involves simply measuring the market value of the asset upon maturity, S_T, and agreeing for the "long" to receive $nV_T = n(S_T - F)$. This amount can be positive or negative, involving a profit or loss.

Figures 5-1 and 5-2 present the payoff patterns on long and short positions in a forward contract, respectively. It is important to note that the payoffs are *linear* in the underlying spot price. Also, the positions are symmetrical around the horizontal

FIGURE 5-1 Payoff of Profits on Long Forward Contract

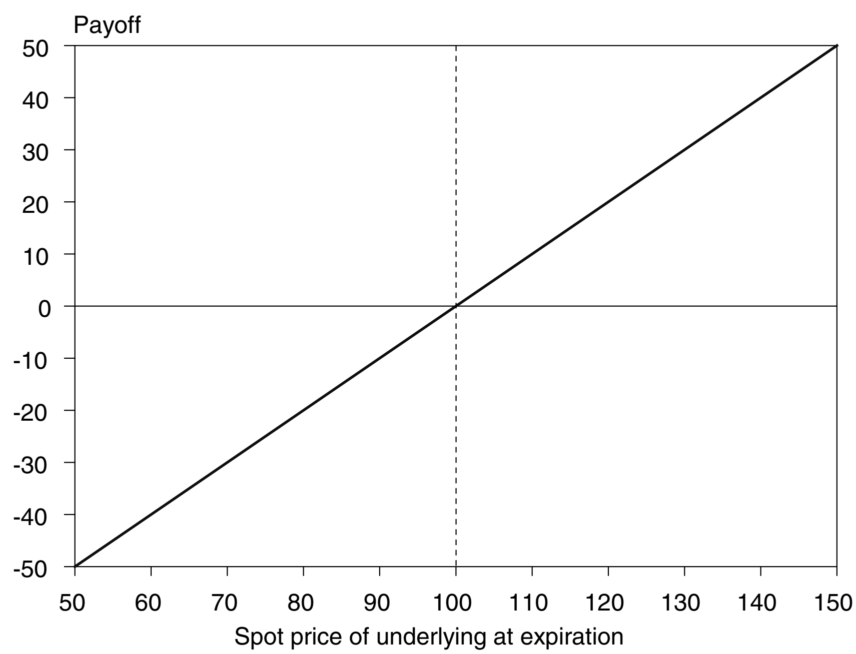
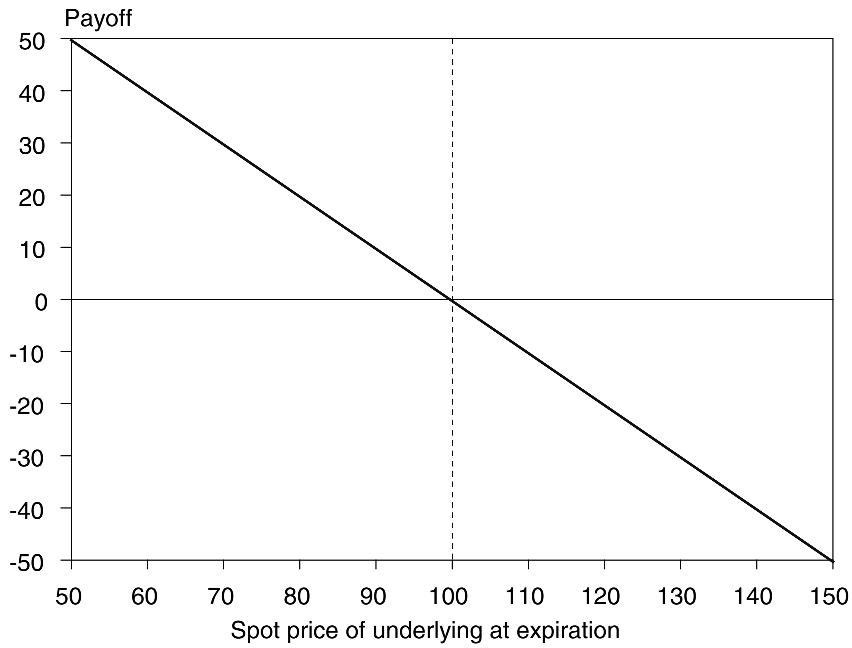


FIGURE 5-2 Payoff of Profits on Short Forward Contract



axis. For a given spot price, the sum of the profit or loss for the long and the short is zero. This reflects the fact that forwards are private contracts between two parties.

5.2.2 Valuing Forward Contracts

When evaluating forward contracts, two important questions arise. First, how is the current forward price F_t determined? Second, what is the current value V_t of an outstanding forward contract?

Initially, we assume that the underlying asset pays no income. This will be generalized in the next section. We also assume no transaction costs, that is, zero bid-ask spread on spot and forward quotations as well as the ability to lend and borrow at the same risk-free rate.

Generally, forward contracts are established so that their initial value is zero. This is achieved by setting the forward price F_t appropriately by a **no-arbitrage relationship** between the cash and forward markets. No-arbitrage is a situation where positions with the same payoffs have the same price. This rules out situations where **arbitrage profits** can exist. Arbitrage is a zero-risk, zero-net investment strategy that still generates profits.

Consider these strategies:

(1) Buy one share/unit of the underlying asset at the spot price S_t and hold until time T.
(2) Enter a forward contract to buy one share/unit of same underlying asset at the forward price F_t; in order to have sufficient funds at maturity to pay F_t, we invest the present value of F_t in an interest-bearing account. This is the present value $F_t e^{-r\tau}$.

The forward price F_t is set so that the initial cost of the forward contract, V_t, is zero.

The two portfolios are economically equivalent because they will be identical at maturity. Each will contain one share of the asset. Hence their up-front cost must be the same:

$$S_t = F_t e^{-r\tau} \tag{5.2}$$

This equation defines the fair forward price F_t such that the initial value of the contract is zero. For instance, assuming $S_t = \$100$, $r = 5\%$, $\tau = 1$, we have $F_t = S_t e^{r\tau} = \$100 \times \exp(0.05 \times 1) = \$105.13$.

We see that the forward rate is higher than the spot rate. This reflects the fact that there is no down payment to enter the forward contract, unlike for the cash position. As a result, the forward price must be higher than the spot price to reflect the time value of money. In practice, this relationship must be tempered by transaction costs.

Abstracting from these costs, any deviation creates an arbitrage opportunity. This can be taken advantage of by buying the cheap asset and selling the expensive one. Assume for instance that $F = \$110$. The fair value is $S_t e^{r\tau} = \$105.13$. We apply the principle of buying low at \$105.13 and selling high at \$110. We can lock in a sure profit by:

(1) Buying the asset spot at \$100
(2) Selling the asset forward at \$110

Because we know we will receive \$110 in one year, we could borrow against this, which brings in \$110 × PV(\$1), or \$104.64. Thus we are paying \$100 and receiving \$104.64 now, for a profit of \$4.64. This would be a blatant arbitrage opportunity, or "money machine."

Now consider a mispricing where $F = \$102$. We apply the principle of buying low at \$102 and selling high at \$105.13. We can lock in a sure profit by:

(1) Short-selling the asset spot at \$100
(2) Buying the asset forward at \$102

Because we know we will have to pay \$102 in one year, this is worth \$102 × PV(\$1), or \$97.03, which we need to invest up front. Thus we are paying \$97.03 and receiving \$100, for a profit of \$2.97.

This transaction involves the **short-sale** of the asset, which is more involved than an outright purchase. When purchasing, we pay \$100 and receive one share of the asset. When short-selling, we borrow one share of the asset and promise to give it back at a future date; in the meantime, we sell it at \$100.[2] When time comes to deliver the asset, we have to buy it on the open market and then deliver it to the counterparty.

5.2.3 Valuing an Off-Market Forward Contract

We can use the same reasoning to evaluate an outstanding forward contract, with a locked-in delivery price of K. In general, such a contract will have non zero value because K differs from the prevailing forward rate. Such a contract is said to be **off-market**.

Consider these strategies:

(1) Buy one share/unit of the underlying asset at the spot price S_t and hold until time T.
(2) Enter a forward contract to buy one share/unit of same underlying asset at the price K; in order to have sufficient funds at maturity to pay K, we invest the present value of K in an interest-bearing account. This present value is also $Ke^{-r\tau}$. In addition, we have to pay the market value of the forward contract, or V_t.

The up-front cost of the two portfolios must be identical. Hence, we must have $V_t + Ke^{-r\tau} = S_t$, or

$$V_t = S_t - Ke^{-r\tau} \tag{5.3}$$

which defines the market value of an outstanding long position.[3] This gains value when the underlying increases in value. A short position would have the reverse sign. Later, we will extend this relationship to the measurement of risk by considering the distribution of the underlying risk factors, S_t and r.

[2] In practice, we may not get full access to the proceeds of the sale when it involves individual stocks. The broker will typically only allow us to withdraw 50% of the cash. The rest is kept as a performance bond should the transaction lose money.

[3] Note that V_t is not the same as the forward price F_t. The former is the value of the contract; the latter refers to a specification of the contract.

For instance, assume we still hold the previous forward contract with $F_t = \$105.13$ and after one month the spot price moves to $S_t = \$110$. The interest has not changed at $r = 5\%$, but the maturity is now shorter by one month, $\tau = 11/12$. The value of the contract is now $V_t = S_t - Ke^{-r\tau} = \$110 - \$105.13\exp(-0.05 \times 11/12) = \$110 - \$100.42 = \$9.58$. The contract is now more valuable than before since the spot price has moved up.

5.2.4 Valuing Forward Contracts With Income Payments

We previously considered a situation where the asset produces no income payment. In practice, the asset may be

- A stock that pays a regular dividend
- A bond that pays a regular coupon
- A stock index that pays a dividend stream that can be approximated by a continuous yield
- A foreign currency that pays a foreign-currency denominated interest rate

Whichever income is paid on the asset, we can usefully classify the payment into **discrete**, that is, fixed dollar amounts at regular points in time, or on a **continuous** basis, that is, accrued in proportion to the time the asset is held. We must assume that the income payment is fixed or is certain. More generally, a storage cost is equivalent to a negative dividend.

We use these definitions:

$$\begin{aligned} D &= \text{discrete (dollar) dividend or coupon payment} \\ r_t^*(T) &= \text{foreign risk-free rate for delivery at } T \\ q_t^*(T) &= \text{dividend yield} \end{aligned}$$

The adjustment is the same for all these payments. We can afford to invest less in the asset up front to get one unit at expiration. This is because the income payment can be reinvested into the asset. Alternatively, we can borrow against the value of the income payment to increase our holding of the asset.

Continuing our example, consider a stock priced at \$100 that pays a dividend of $D = \$1$ in three months. The present value of this payment discounted over three months is $De^{-r\tau} = \$1\exp(-0.05 \times 3/12) = \$0.99$. We only need to put up

$S_t - \text{PV}(D) = \$100.00 - 0.99 = \$99.01$ to get one share in one year. Put differently, we buy 0.9901 fractional shares now and borrow against the (sure) dividend payment of \$1 to buy an additional 0.0099 fractional share, for a total of 1 share.

The pricing formula in Equation (5.2) is extended to

$$F_t e^{-r\tau} = S_t - \text{PV}(D) \tag{5.4}$$

where PV(D) is the present value of the dividend/coupon payments. If there is more than one payment, PV(D) represents the sum of the present values of each individual payment, discounted at the appropriate risk-free rate. With storage costs, we need to *add* the present value of storage costs PV(C) to the right side of Equation (5.4).

The approach is similar for an asset that pays a continuous income, defined per unit time instead of discrete amounts. Holding a foreign currency, for instance, should be done through an interest-bearing account paying interest that accrues with time. Over the horizon τ, we can afford to invest less up front, $S_t e^{-r^*\tau}$ in order to receive one unit at maturity. Hence the forward price should be such that

$$F_t = S_t e^{-r^*\tau} / e^{-r\tau} \tag{5.5}$$

If instead interest rates are annually compounded, this gives

$$F_t = S_t (1 + r)^\tau / (1 + r^*)^\tau \tag{5.6}$$

If $r^* < r$, we have $F_t > S_t$ and the asset trades at a **forward premium**. Conversely, if $r^* > r$, $F_t < S_t$ and the asset trades at a **forward discount**. Thus the forward price is higher or lower than the spot price, depending on whether the yield on the asset is lower than or higher than the domestic risk-free interest rate. Note also that, for this equation to be valid, both the spot and forward prices have to be expressed in dollars, or domestic currency units that correspond to the rate r. Equation (5.5) is also known as **interest rate parity** when dealing with currencies.

> **Key concept:**
> The forward rate differs from the spot rate to reflect the time value of money and the income yield on the underlying asset. It is higher than the spot rate if the yield on the asset is lower than the risk-free interest rate, and vice versa.

The value of an outstanding forward contract is

$$V_t = S_t e^{-r^*\tau} - K e^{-r\tau} \tag{5.7}$$

If F_t is the new, current forward price, we can also write

$$V_t = F_t e^{-r\tau} - K e^{-r\tau} = (F - K)e^{-r\tau} \tag{5.8}$$

This provides a useful alternative formula for the valuation of a forward contract. The intuition here is that we could liquidate the outstanding forward contract by entering a reverse position at the current forward rate. The payoff at expiration is $(F - K)$, which, discounted back to the present, gives Equation (5.8).

Key concept:
The current value of an outstanding forward contract can be found by entering an offsetting forward position and discounting the net cash flow at expiration.

Example 5-1: FRM Exam 1999—Question 49/Capital Markets

5-1. Assume the spot rate for euro against U.S. dollar is 1.05 (i.e. 1 euro buys 1.05 dollars). A U.S. bank pays 5.5% compounded annually for one year for a dollar deposit and a German bank pays 2.5% compounded annually for one year for a euro deposit. What is the forward exchange rate one year from now?

a) 1.0815
b) 1.0201
c) 1.0807
d) 1.0500

Example 5-2: FRM Exam 1999—Question 31/Capital Markets

5-2. Consider an eight-month forward contract on a stock with a price of $98/share. The delivery date is eight months hence. The firm is expected to pay a $1.80/share dividend in four months time. Riskless zero-coupon interest rates (continuously compounded) for different maturities are for less than/equal to 6 months, 4%; for 8 months, 4.5%. The theoretical forward price (to the nearest cent) is

a) 99.15
b) 99.18
c) 100.98
d) 96.20

Example 5-3: FRM Exam 2001—Question 93

5-3. Calculate the price of a 1-year forward contract on gold. Assume the storage cost for gold is \$5.00 per ounce with payment made at the end of the year. Spot gold is \$290 per ounce and the risk free rate is 5%.

a) \$304.86
b) \$309.87
c) \$310.12
d) \$313.17

Example 5-4: FRM Exam 2000—Question 4/Capital Markets

5-4. On Friday, October 4, the spot price of gold was \$378.85 per troy ounce. The price of an April gold futures contract was \$387.20 per troy ounce. (Note: Each gold futures contract is for 100 troy ounces.) Assume that a Treasury bill maturing in April with an "ask yield" of 5.28 percent provides the relevant financing (borrowing or lending rate). Use 180 days as the term to maturity (with continuous compounding and a 365-day year). Also assume that warehousing and delivery costs are negligible and ignore convenience yields. What is the theoretically correct price for the April futures contract and what is the potential arbitrage profit per contract?

a) \$379.85 and \$156.59
b) \$318.05 and \$615.00
c) \$387.84 and \$163.25
d) \$388.84 and \$164.00

Example 5-5: FRM Exam 1999—Question 41/Capital Markets

5-5. Assume a dollar asset provides no income for the holder and an investor can borrow money at risk-free interest rate r, then the forward price F for time T and spot price S at time t of the asset are related. If the investor observes that $F > S \exp[r(T - t)]$, then the investor can take a profit by

a) Borrowing S dollars for a period of $(T - t)$ at the rate of r, buy the asset, and short the forward contract.
b) Borrowing S dollars for a period of $(T - t)$ at the rate of r, buy the asset, and long the forward contract.
c) Selling short the asset and invest the proceeds of S dollars for a period of $(T - t)$ at the rate of r, and short the forward contract.
d) Selling short the asset and invest the proceeds of S dollars for a period of $(T - t)$ at the rate of r, and long the forward contract.

5.3 Futures Contracts

5.3.1 Definitions of Futures

Forward contracts allow users to take positions that are economically equivalent to those in the underlying cash markets. Unlike cash markets, however, they do not involve substantial up-front payments. Thus, forward contracts can be interpreted as having *leverage*.

Leverage is that it creates credit risk for the counterparty. When a speculator buys a stock at the price of $100, the counterparty receives the cash and has no credit risk. Instead, when a speculator enters a forward contract to buy an asset at the price of $105, there is very little up-front payment. In effect the speculator borrows from the counterparty to invest in the asset. There is a risk that if the price of the asset and hence the value of the contract falls sufficiently, the speculator could default.

In response, futures contracts have been structured so as to minimize credit risk for all counterparties. From a market risk standpoint, futures contracts are identical to forward contracts. The pricing relationships are generally similar. Some of the features of futures contracts are now finding their way into OTC forward and swap markets.

Futures contracts are standardized, negotiable, and exchange-traded contracts to buy or sell an underlying asset. They differ from forward contracts as follows.

- *Trading on organized exchanges*
  In contrast to forwards, which are OTC contracts tailored to customers' needs, futures are traded on organized exchanges (either with a physical location or electronic).
- *Standardization*
  Futures contracts are offered with a limited choice of expiration dates. They trade in fixed contract sizes. This standardization ensures an active secondary market for many futures contracts, which can be easily traded, purchased or resold. In other words, most futures contracts have good liquidity. The trade-off is that futures are less precisely suited to the need of some hedgers, which creates basis risk (to be defined later).

- *Clearinghouse*
  Futures contracts are also standardized in terms of the counterparty. After each transaction is confirmed, the clearinghouse basically interposes itself between the buyer and the seller, ensuring the performance of the contract (for a fee). Thus, unlike forward contracts, counterparties do not have to worry about the credit risk of the other side of the trade. Instead, the credit risk is that of the clearinghouse (or the broker), which is generally excellent.
- *Marking-to-market*
  As the clearinghouse now has to deal with the credit risk of the two original counterparties, it has to develop mechanisms to monitor credit risk. This is achieved by daily marking-to-market, which involves settlement of the gains and losses on the contract every day. The goal is to avoid a situation where a speculator loses a large amount of money on a trade and defaults, passing on some of the losses to the clearinghouse.
- *Margins*
  Although daily settlement accounts for past losses, it does not provide a buffer against future losses. This is the goal of **margins**, which represent up-front posting of collateral that provides some guarantee of performance.

Example: Margins for a futures contract

Consider a futures contract on 1000 units of an asset worth $100. A long futures position is economically equivalent to holding $100,000 worth of the asset directly. To enter the futures position, a speculator has to post only $5,000 in margin, for example. This represents the initial value of the equity account.

The next day, the profit or loss is added to the equity account. If the futures price moves down by $3, the loss is $3,000, bringing the equity account down to $5,000 − $3,000 = $2,000. The speculator is then required to post an additional $3,000 of capital. In case he or she fails to meet the **margin call**, the broker has the right to liquidate the position.

Since futures trading is centralized on an exchange, it is easy to collect and report aggregate trading data. **Volume** is the number of contracts traded during the day, which is a flow item. **Open interest** represents the outstanding number of contracts at the close of the day, which is a stock item.

5.3.2 Valuing Futures Contracts

Valuation principles for futures contracts are very similar to those for forward contracts. The main difference between the two types of contracts is that any profit or loss accrues *during* the life of the futures contract instead of all at once, at expiration.

When interest rates are assumed constant or deterministic, forward and futures prices must be equal. With stochastic interest rates, the difference is small, unless the value of the asset is highly correlated with the interest rate.

If the correlation is zero, then it makes no difference whether payments are received earlier or later. The futures price must be the same as the forward price. In contrast, consider a contract whose price is positively correlated with the interest rate. If the value of the contract goes up, it is more likely that interest rates will go up as well. This implies that profits can be withdrawn and reinvested at a higher rate. Relative to forward contracts, this marking-to-market feature is beneficial to long futures position. Because both parties recognize this feature, the futures price must be higher in equilibrium.

In practice, this effect is only observable for interest-rate futures contracts, whose value is *negatively* correlated with interest rates. For these contracts, the futures price must be lower than the forward price. Chapter 8 will explain how to compute the adjustment, called the **convexity effect**.

Example 5-6: FRM Exam 2000—Question 7/Capital Markets

5-6. For assets that are strongly positively correlated with interest rates, which one of the following is *true*?

a) Long-dated forward contracts will have higher prices than long-dated futures contracts.

b) Long-dated futures contracts will have higher prices than long-dated forward contracts.

c) Long-dated forward and long-dated futures prices are always the same.

d) The "convexity effect" can be ignored for long-dated futures contracts on that asset.

5.4 Swap Contracts

Swap contracts are OTC agreements to exchange a *series* of cash flows according to prespecified terms. The underlying asset can be an interest rate, an exchange rate, an equity, a commodity price, or any other index. Typically, swaps are established for longer periods than forwards and futures.

For example, a 10-year currency swap could involve an agreement to exchange every year 5 million dollars against 3 million pounds over the next ten years, in addition to a principal amount of 100 million dollars against 50 million pounds at expiration. The principal is also called **notional principal**.

Another example is that of a 5-year interest rate swap in which one party pays 8% of the principal amount of 100 million dollars in exchange for receiving an interest payment indexed to a floating interest rate. In this case, since both payments are tied to the same principal amount, there is no exchange of principal at maturity.

Swaps can be viewed as a portfolio of forward contracts. They can be priced using valuation formulas for forwards. Our currency swap, for instance, can be viewed as a combination of ten forward contracts with various face values, maturity dates, and rates of exchange. We will give detailed examples in later chapters.

5.5 Answers to Chapter Examples

Example 5-1: FRM Exam 1999—Question 49/Capital Markets

a) Using annual compounding, $(1 + r)^1 = (1 + 0.055) = 1.055$ and $(1 + r^*)^1 = 1.025$. The spot rate of 1.05 is expressed in dollars per euro, $S(\$/EUR)$.

From Equation (5.6), we have $F = S(\$/EUR) \times (1 + r)^\tau/(1 + r^*)^\tau = \$1.05 \times 1.055/1.025 = \$1.08073$. Intuitively, since the euro interest rate is lower than the dollar interest rate, the euro must be selling at a higher price in the forward than in the spot market.

Example 5-2: FRM Exam 1999—Question 31/Capital Markets

a) We need first to compute the PV of the dividend payment, which is $\text{PV}(D) = \$1.8\exp(-0.04 \times 4/12) = \$1.776$. By Equation (5.4), we have $F = [S - \text{PV}(D)]\exp(r\tau)$. Hence, $F = (\$98 - \$1.776)\exp(0.045 \times 8/12) = \$99.15$.

Example 5-3: FRM Exam 2001—Question 93

b) Assuming continuous compounding, the present value factor is $\text{PV} = \exp(-0.05) = 0.951$. Here, the storage cost C is equivalent to a negative dividend and must be evaluated as of now. This gives $\text{PV}(C) = \$5 \times 0.951 = \$4.756$. Generalizing Equation (5.4), we have $F = (S + \text{PV}(C))/\text{PV}(\$1) = (\$290 + \$4.756)/0.951 = \$309.87$. Assuming discrete compounding gives \$309.5, which is close.

Example 5-4: FRM Exam 2000—Question 4/Capital Markets

d) The theoretical forward/futures rate is given by $F = Se^{r\tau} = 378.85 \times \exp(0.0528 \times 180/365) = \$388.844$ with continuous compounding. Discrete compounding gives a close answer, \$388.71. This is consistent with the observation that futures rates must be greater than spot rates when there is no income on the underlying asset. The profit is then $100 \times (388.84 - 387.20) = 164.4$.

Example 5-5: FRM Exam 1999—Question 41/Capital Markets

a) The forward price is too high relative to the fair rate, so we need to sell the forward contract. In exchange, we need to buy the asset. To ensure a zero initial cash flow, we need to borrow the present value of the asset.

Example 5-6: FRM Exam 2000—Question 7/Capital Markets

b) The convexity effect is important for long-dated contracts, so (d) is wrong. This positive correlation makes it more beneficial to have a long futures position since profits can be reinvested at higher rates. Hence the futures price must be higher than the forward price. Note that the relationship assumed here is the opposite to that of Eurodollar futures contracts, where the value of the asset is *negatively* correlated with interest rates.

Chapter 6

Options

This chapter now turns to nonlinear derivatives, or options. As described in Table 5-1, options account for a large part of the derivatives markets. On organized exchanges, options represent $14 trillion out of a total of $24 trillion in derivatives outstanding. Over-the-counter (OTC) options add up to more than $15 trillion.

Although the concept behind these instruments are not new, options have blossomed since the early 1970s, because of a break-through in pricing options, the Black-Scholes formula, and to advances in computing power.

We start with plain, **vanilla** options, calls and puts. These are the basic building blocks of many financial instruments. They are also more common than complicated, **exotic** options.

This chapter describes the general characteristics as well as the pricing of these derivatives. Section 6.1 presents the payoff functions on basic options and combinations thereof. We then discuss option premiums and the Black-Scholes pricing approach in Section 6.2. Next, Section 6.3 briefly summarizes more complex options. Finally, Section 6.4 shows how to value options using a numerical, binomial tree model. We will cover option sensitivities (the "Greeks") in Chapter 15.

6.1 Option Payoffs

6.1.1 Basic Options

Options are instruments that give their holder the right to buy or sell an asset at a specified price until a specified expiration date. The specified delivery price is known as the **delivery price**, **exercise price**, or **strike price**, and is denoted by K.

Options to buy are **call options**; options to sell are **put options**. As options confer a right to the purchaser of the option, but not an obligation, they will be exercised only if they generate profits. In contrast, forwards involve an obligation to either buy or sell and can generate profits or losses. Like forward contracts, options can be either purchased or sold. In the latter case, the seller is said to **write** the option.

Depending on the timing of exercise, options can be classified into European or American options. **European options** can be exercised at maturity only. **American options** can be exercised at any time, before or at maturity. Because American options include the right to exercise at maturity, they must be at least as valuable as European options. In practice, however, the value of this early exercise feature is small, as an investor can generally receive better value by reselling the option on the open market instead of exercising it.

We use these notations, in addition to those in the previous chapter:

$$\begin{aligned} K &= \text{exercise price} \\ c &= \text{value of European call option} \\ C &= \text{value of American call option} \\ p &= \text{value of European put option} \\ P &= \text{value of American put option} \end{aligned}$$

To illustrate, take an option on an asset that currently trades at \$85 with a delivery price of \$100 in one year. If the spot price stays at \$85, the holder of the call will not **exercise** the option, because the option is not profitable with a stock price less than \$100. In contrast, if the price goes to \$120, the holder will exercise the right to buy at \$100, will acquire the stock now worth \$120, and will enjoy a “paper” profit of \$20. This profit can be realized by selling the stock. For put options, a profit accrues if the spot price falls below the exercise price $K = \$100$.

Thus the payoff profile of a long position in the call option at expiration is

$$C_T = \text{Max}(S_T - K, 0) \tag{6.1}$$

The payoff profile of a long position in a put option is

$$P_T = \text{Max}(K - S_T, 0) \tag{6.2}$$

If the current asset price S_t is close to the strike price K, the option is said to be **at-the-money**. If the current asset price S_t is such that the option could be exercised at a profit, the option is said to be **in-the-money**. If the remaining situation, the option is said to be **out-of-the-money**. A call will be in-the-money if $S_t > K$; a put will be in-the-money if $S_t < K$;

As in the case of forward contracts, the payoff at expiration can be cash settled. Instead of actually buying the asset, the contract could simply pay \$20 if the price of the asset is \$120.

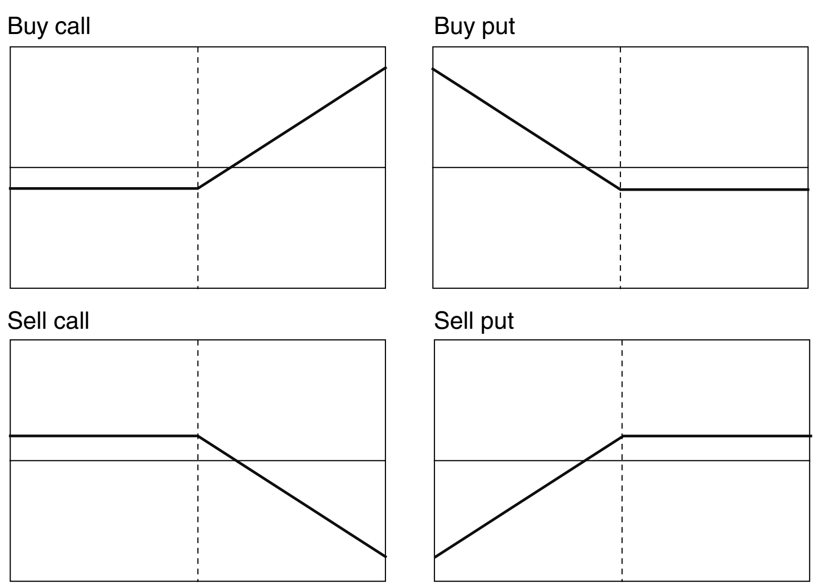
Because buying options can generate only profits (at worst zero) at expiration, an option contract must be a valuable asset (or at worst have zero value). This means that a payment is needed to acquire the contract. This up-front payment, which is much like an insurance premium, is called the option "premium." This premium cannot be negative. An option becomes more expensive as it moves in-the-money.

Thus the payoffs on options must take into account this cost (for long positions) or benefit (for short positions). To be complete, we should translate all option payoffs by the future value of the premium, that is, $ce^{r\tau}$ for European call options.

Figure 6-1 compares the payoff patterns on long and short positions in a call and a put contract. Unlike those of forwards, these payoffs are **nonlinear** in the underlying spot price. Sometimes they are referred to as the "hockey stick" diagrams. This is because forwards are obligations, whereas options are rights. Note that the positions are symmetrical around the horizontal axis. For a given spot price, the sum of the profit or loss for the long and for the short is zero.

So far, we have covered options on cash instruments. Options can also be struck on futures. When exercising a call, the investor becomes long the futures at a price set to the strike price. Conversely, exercising a put creates a short position in the futures contract.

FIGURE 6-1 Profit Payoffs on Long and Short Calls and Puts



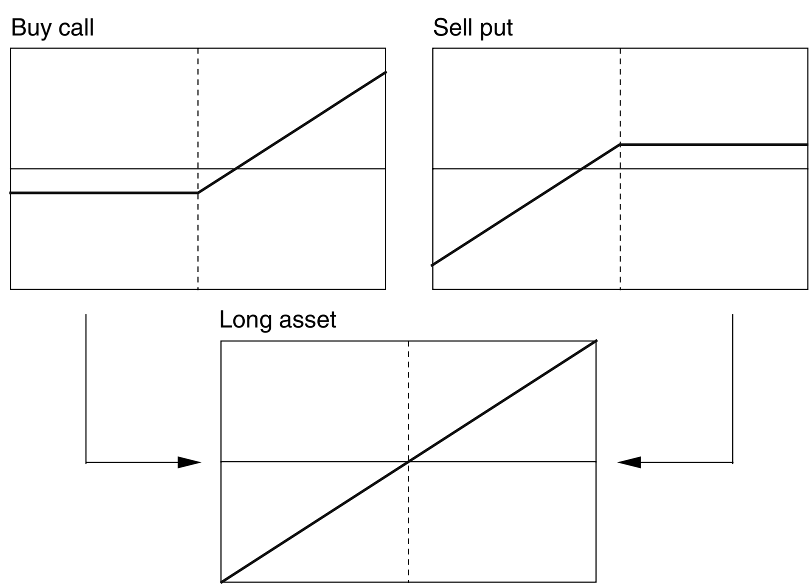
Because positions in futures are equivalent to leveraged positions in the underlying cash instrument, options on cash instruments and on futures are also equivalent. The only conceptual difference lies in the income payment to the underlying instrument. With an option on cash, the income is the dividend or interest on the cash instrument. In contrast, with a futures contract, the economically equivalent stream of income is the riskless interest rate. The intuition is that a futures can be viewed as equivalent to a position in the underlying asset with the investor setting aside an amount of cash equivalent to the present value of F.

Key concept:
With an option on futures, the implicit income is the risk-free rate of interest.

6.1.2 Put-Call Parity

These option payoffs can be used as the basic building blocks for more complex positions. At the most basic level, a long position in the underlying asset (plus some borrowing) can be decomposed into a long call plus a short put, as shown in Figure 6-2. We only consider European options with the same maturity and exercise price. The long call provides the equivalent of the upside while the short put generates the same downside risk as holding the asset.

FIGURE 6-2 Decomposing a Long Position in the Asset



This link creates a relationship between the value of the call and that of the put, also known as **put-call parity**. The relationship is illustrated in Table 6-1, which examines the payoff at initiation and at expiration under the two possible states of the world. We assume no income payment on the underlying asset.

The portfolio consists of a long position in the call (with an outflow of c represented by $-c$), a short position in the put and an investment to ensure that we will be able to pay the exercise price at maturity.

TABLE 6-1 Put-Call Parity

Position:	Initial Payoff	Final Payoff $S_T < K$	Final Payoff $S_T \geq K$
Buy call	$-c$	0	$S_T - K$
Sell put	$+p$	$-(K - S_T)$	0
Invest	$-Ke^{-r\tau}$	K	K
Total	$-c + p - Ke^{-r\tau}$	S_T	S_T

The table shows that the final payoffs are, in the two states of the world, equal to that of a long position in the asset. Hence, to avoid arbitrage, the initial payoff must be equal to the cost of buying the underlying asset, which is S_t. We have $-c+p-Ke^{-r\tau} = -S_t$. More generally, with income paid at the rate of r^*, put-call parity can be written as

$$c - p = Se^{-r^*\tau} - Ke^{-r\tau} = (F - K)e^{-r\tau} \tag{6.3}$$

Because $c \geq 0$ and $p \geq 0$, this relationship can be also used to determine the lower bounds for European calls and puts. Note that the relationship does not hold exactly for American options since there is a likelihood of early exercise, which leads to mismatched payoffs.

Example 6-1. FRM Exam 1999—Question 35/Capital Markets

6-1. According to put-call parity, writing a put is like

a) Buying a call, buying stock, and lending

b) Writing a call, buying stock, and borrowing

c) Writing a call, buying stock, and lending

d) Writing a call, selling stock, and borrowing

Example 6-2. FRM Exam 2000—Question 15/Capital Markets

6-2. A six-month call option sells for $30, with a strike price of $120. If the stock price is $100 per share and the risk-free interest rate is 5 percent, what is the price of a 6-month put option with a strike price of $120?

a) $39.20

b) $44.53

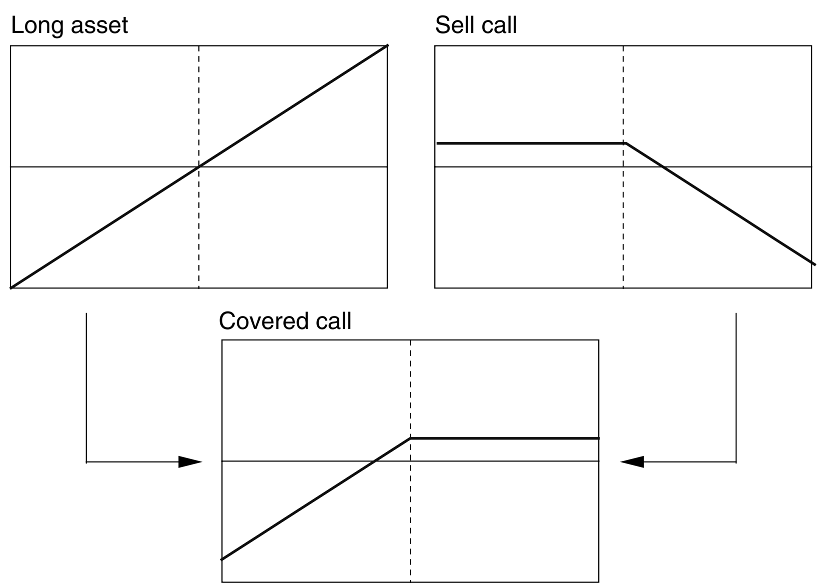
c) $46.28

d) $47.04

6.1.3 Combination of Options

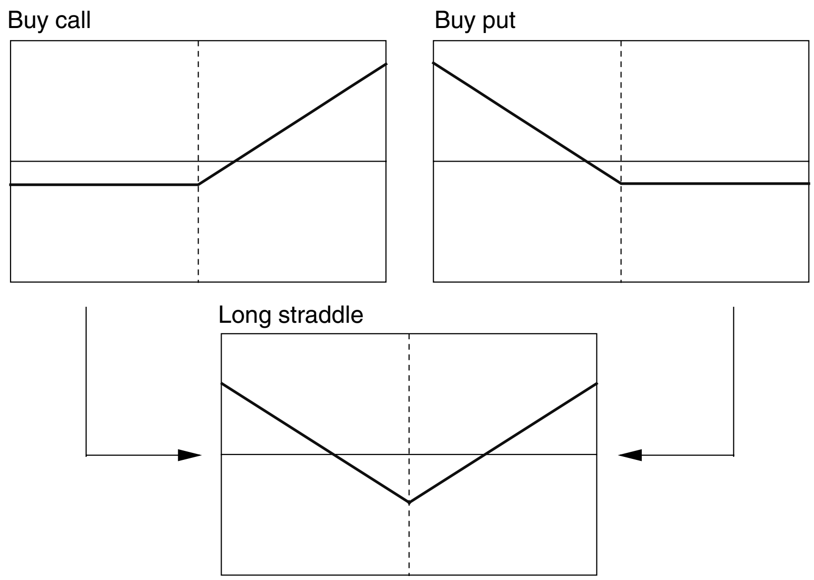
Options can be combined in different ways, either with each other or with the underlying asset. Consider first combinations of the underlying asset and an option. A long position in the stock can be accompanied by a short sale of a call to collect the option premium. This operation, called a **covered call**, is described in Figure 6-3. Likewise, a long position in the stock can be accompanied by a purchase of a put to protect the downside. This operation is called a **protective put**.

FIGURE 6-3 Creating a Covered Call



We can also combine a call and a put with the same or different strike prices and maturities. When the strike prices of the call and the put and their maturities are the same, the combination is referred to as a **straddle**. When the strike prices are different, the combination is referred to as a **strangle**. Since strangles are out-of-the-money, they are cheaper to buy than straddles. Figure 6-4 shows how to construct a long straddle, buying a call and a put with the same maturity and strike price. This position is expected to benefit from a large price move, whether up or down. The reverse position is a short straddle.

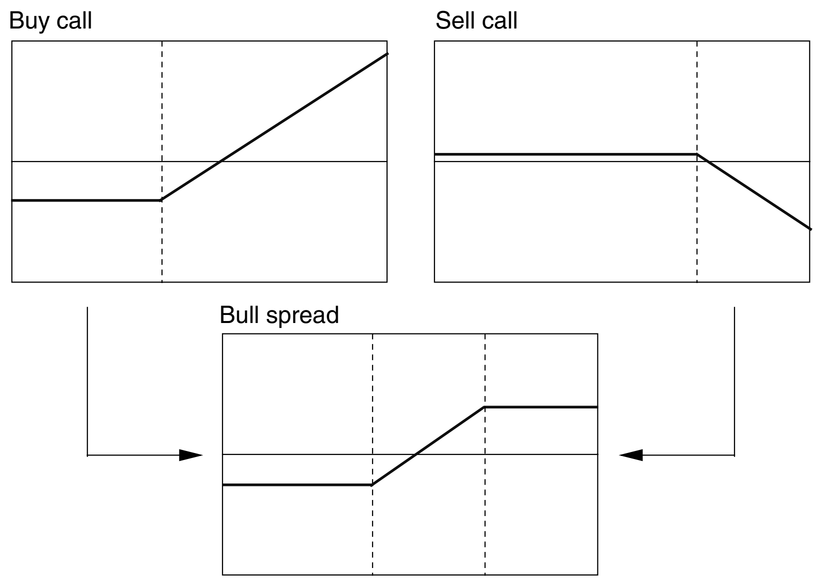
FIGURE 6-4 Creating a Long Straddle



Thus far, we have concentrated on positions involving two classes of options. One can, however, establish positions with one class of options, called **spreads**. Calendar, or **horizontal spreads** correspond to different maturities. **Vertical spreads** correspond to different strike prices. The names of the spreads are derived from the manner in which they are listed in newspapers; time is listed horizontally and strike prices are listed vertically.

For instance, a **bull spread** is positioned to take advantage of an increase in the price of the underlying asset. Conversely, a **bear spread** represents a bet on a falling price. Figure 6-5 shows how to construct a bull(ish) vertical spread with two calls with the same maturity (although this could also be constructed with puts). Here, the spread is formed by buying a call option with a low exercise price K_1 and selling another call with a higher exercise price K_2. Note that the cost of the first call $c(S, K_1)$ must exceed the cost of the second call $c(S, K_2)$, because the first option is more in-the-money than the second. Hence, the sum of the two premiums represents a net cost. At expiration, when $S_T > K_2$, the payoff is $\text{Max}(S_T - K_1, 0) - \text{Max}(S_T - K_2, 0) = (S_T - K_1) - (S_T - K_2) = K_2 - K_1$, which is positive. Thus this position is expected to benefit from an upmove, while incurring only limited downside risk.

FIGURE 6-5 Creating a Bull Spread



Spreads involving more than two positions are referred to as butterfly or sandwich spreads. The latter is the opposite of the former. A **butterfly spread** involves three types of options with the same maturity: a long call at a strike price K_1, two short calls at a higher strike price K_2, and a long call position at an even higher strike price K_3. We can verify that this position is expected to benefit when the underlying asset price stays stable, close to K_2.

Example 6-3. FRM Exam 2001—Question 90

6-3. Which of the following is the riskiest form of speculation using options contracts?

a) Setting up a spread using call options
b) Buying put options
c) Writing naked call options
d) Writing naked put options

Example 6-4. FRM Exam 1999—Question 50/Capital Markets

6-4. A covered call writing position is equivalent to

a) A long position in the stock and a long position in the call option
b) A short put position
c) A short position in the stock and a long position in the call option
d) A short call position

Example 6-5. FRM Exam 1999—Question 33/Capital Markets

6-5. Which of the following will create a bull spread?

a) Buy a put with a strike price of $X = 50$, and sell a put with $K = 55$.
b) Buy a put with a strike price of $X = 55$, and sell a put with $K = 50$.
c) Buy a call with a premium of 5, and sell a call with a premium of 7.
d) Buy a call with a strike price of $X = 50$, and sell a put with $K = 55$.

Example 6-6. FRM Exam 2000—Question 5/Capital Markets

6-6. Consider a bullish spread option strategy of buying one call option with a \$30 exercise price at a premium of \$3 and writing a call option with a \$40 exercise price at a premium of \$1.50. If the price of the stock increases to \$42 at expiration and the option is exercised on the expiration date, the net profit per share at expiration (ignoring transaction costs) will be

a) \$8.50
b) \$9.00
c) \$9.50
d) \$12.50

Example 6-7. FRM Exam 2001—Question 111

6-7. Consider the following bearish option strategy of buying one at-the-money put with a strike price of \$43 for \$6, selling two puts with a strike price of \$37 for \$4 each and buying one put with a strike price of \$32 for \$1. If the stock price plummets to \$19 at expiration, calculate the net profit or loss per share of the strategy.

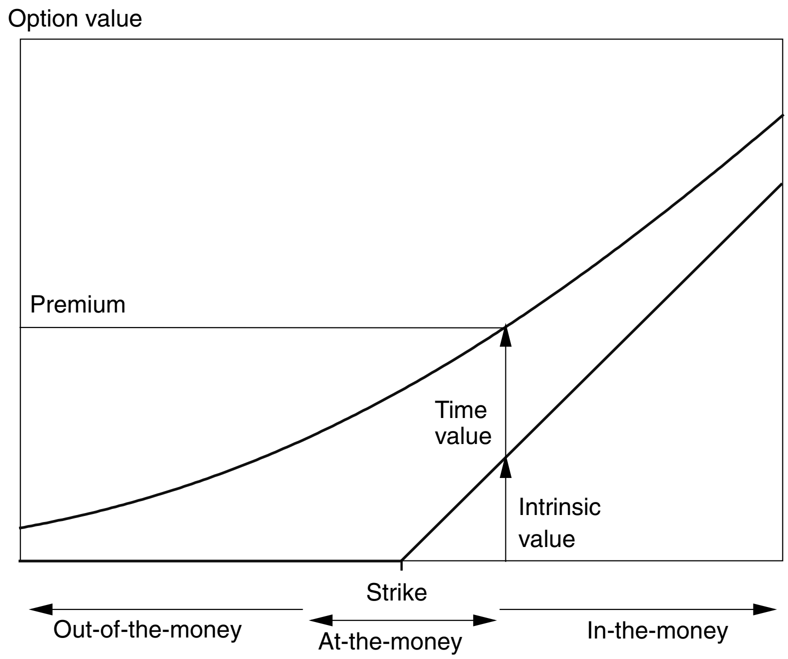
a) −2.00 per share
b) Zero; no profit or loss
c) 1.00 per share
d) 2.00 per share

6.2 Valuing Options

6.2.1 Option Premiums

So far, we have examined the payoffs at expiration only. As important is the instantaneous relationship between the option value and the current price S, which is displayed in Figures 6-6 and 6-7.

FIGURE 6-6 Relationship between Call Value and Spot Price

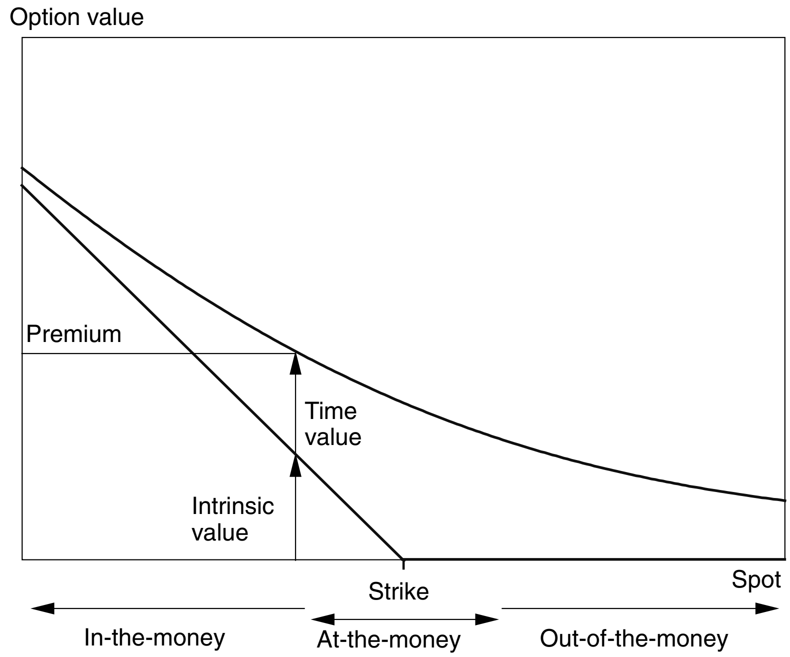


For a call, a higher price S increases the current value of the option, but in a nonlinear, convex fashion. For a put, lower values for S increase the value of the option, also in a convex fashion. As time goes by, the curved line approaches the hockey stick line.

Figures 6-6 and 6-7 decompose the current premium into:

- An **intrinsic value**, which basically consists of the value of the option if exercised today, or $\text{Max}(S_t - K, 0)$ for a call, and $\text{Max}(K - S_t, 0)$ for a put
- A **time value**, which consists of the remainder, reflecting the possibility that the option will create further gains in the future

FIGURE 6-7 Relationship between Put Value and Spot Price



As shown in the figures, options are also classified into:

- **At-the-money**, when the current spot price is close to the strike price
- **In-the-money**, when the intrinsic value is large
- **Out-of-the-money**, when the spot price is much below the strike price for calls and conversely for puts (out-of-the-money options have zero intrinsic value)

We can also identify some general bounds for European options that should always be satisfied; otherwise there would be an arbitrage opportunity (a money machine). For simplicity, assume no dividend. First, the value of a call must be less than, or equal to, the asset price:

$$c \leq C \leq S_t \tag{6.4}$$

In the limit, an option with zero exercise price is equivalent to holding the stock. Second, the value of a call must be greater than, or equal to, the price of the asset minus the present value of the strike price:

$$c \geq S_t - Ke^{-r\tau} \tag{6.5}$$

To prove this, Table 6-2 considers the final payoffs for two portfolios: (1) a long call and (2) a long stock with a loan of K. In each case, an outflow, or payment, is represented with a negative sign. A receipt has a positive sign.

We consider the two states of the world, $S_T < K$ and $S_T \geq K$. In the state where $S_T \geq K$, the call is exercised and the two portfolios have exactly the same value, which is $S_T - K$. In the state where $S_T < K$, however, the second portfolio has a negative value and is worth less than the value of the call, which is zero.

Since the payoffs on the call dominate those on the second portfolio, buying the call must be more expensive. Hence the initial cost of the call c must be greater than, or equal to, the up-front cost of the portfolio, which is $S_t - Ke^{-r\tau}$.

TABLE 6-2 Lower Option Bound for a Call

Position:	Initial Payoff	Final Payoff $S_T < K$	Final Payoff $S_T \geq K$
Buy call	$-c$	0	$S_T - K$
Buy asset	$-S_t$	S_T	S_T
Borrow	$+Ke^{-r\tau}$	$-K$	$-K$
Total	$-S + Ke^{-r\tau}$	$S_T - K < 0$	$S_T - K$

Note that, since $e^{-r\tau} < 1$, we must have $S_t - Ke^{-r\tau} > S_t - K$ before expiration. Thus $S_t - Ke^{-r\tau}$ is a better lower bound than $S_t - K$.

We can also describe upper and lower bounds for put options. The value of a put cannot be worth more than K

$$p \leq P \leq K \tag{6.6}$$

which is the upper bound if the price falls to zero. Using an argument similar to that in Table 6-2, we can show that the value of a European put must satisfy the following lower bound

$$p \geq Ke^{-r\tau} - S_t \tag{6.7}$$

6.2.2 Early Exercise of Options

These relationships can be used to assess the value of early exercise for American options. An American call on a non-dividend-paying stock will never be exercised early. Recall that the choice is not between exercising or not, but rather between exercising the option and selling it on the open market. By exercising, the holder gets exactly $S_t - K$.

¿From Equation (6.5), the current value of a European call must satisfy $c \geq S_t - Ke^{-r\tau}$, which is strictly greater than $S_t - K$. Since the European call is a lower bound on the American call, it is never optimal to exercise early such American options. The American call is always worth more **alive**, that is, nonexercised, than **dead**, that is, exercised. As a result, the value of the American feature is zero and we always have $c_t = C_t$.

The only reason one would want to exercise early a call is to capture a dividend payment. Intuitively, a high income payment makes holding the asset more attractive than holding the option. Thus American options on income-paying assets may be exercised early. Note that this applies also to options on futures, since the implied income stream on the underlying is the risk-free rate.

> **Key concept:**
> An American call option on a non-dividend-paying stock (or asset with no income) should never be exercised early. If the asset pays income, early exercise may occur, with a probability that increases with the size of the income payment.

For an American put, we must have

$$P \geq K - S_t \tag{6.8}$$

because it could be exercised now. Unlike the relationship for calls, this lower bound $K - S_t$ is strictly greater than the lower bound for European puts $Ke^{-r\tau} - S_t$. So, we could have early exercise.

To decide whether to exercise early or not, the holder of the option has to balance the benefit of exercising, which is to receive K now instead of later, against the loss of killing the time value of the option. Because it is better to receive money now than later, it may be worth exercising the put option early.

Thus, American puts on nonincome paying assets may be exercised early, unlike calls. This translates into $p_t \leq P_t$. With an increased income payment on the asset, the probability of early exercise decreases, as it becomes less attractive to sell the asset.

> **Key concept:**
> An American put option on a non-dividend-paying stock (or asset with no income) may be exercised early. If the asset pays income, the possibility of early exercise decreases with the size of the income payments.

Example 6-8. FRM Exam 1998—Question 58/Capital Markets

6-8. Which of the following statements about options on futures is *true*?

a) An American call is equal in value to a European call.
b) An American put is equal in value to a European put.
c) Put-call parity holds for both American and European options.
d) None of the above statements are true.

Example 6-9. FRM Exam 1999—Question 34/Capital Markets

6-9. What is the lower pricing bound for a European call option with a strike price of 80 and one year until expiration? The price of the underlying asset is 90, and the one-year interest rate is 5% per annum. Assume continuous compounding of interest.

a) 14.61
b) 13.90
c) 10.00
d) 5.90

Example 6-10. FRM Exam 1999—Question 52/Capital Markets

6-10. The price of an American call stock option is equal to an otherwise equivalent European call stock option at time t when:

I) The stock pays continuous dividends from t to option expiration T.
II) The interest rates follow a mean-reverting process between t and T.
III) The stock pays no dividends from t to option expiration T.
IV) Interest rates are nonstochastic between t and T.

a) II and IV
b) III only
c) I and III
d) None of the above; an American option is always worth more than a European option.

6.2.3 Black-Scholes Valuation

We now briefly introduce the pricing of conventional European call and put options. Initially, we focus on valuation. We will discuss sensitivities to risk factors later, in Chapter 15 that deals with risk management.

To illustrate the philosophy of option pricing methods, consider a call option on a stock whose price is represented by a binomial process. The initial price of $S_0 = \$100$ can only move up or down, to two values (hence the "bi"), $S_1 = \$150$ or $S_2 = \$50$.

The option is a call with $K = \$100$, and therefore can only take values of $c_1 = \$50$ or $c_2 = \$0$. We assume that the rate of interest is $r = 25\%$, so that a dollar invested now grows to \$1.25 at maturity.

		$S_1 = \$150$	$c_1 = \$50$
	↗		
$S_0 = \$100$			
	↘		
		$S_2 = \$50$	$c_2 = \$0$

The key idea of derivatives pricing is that of **replication**. In other words, we exactly replicate the payoff on the option by a suitable portfolio of the underlying asset plus some borrowing. This is feasible in this simple setup because have 2 states of the world and 2 instruments, the stock and the bond. To prevent arbitrage, the current value of the derivative must be the same as that of the portfolio.

The portfolio consists of n shares and a risk-free investment currently valued at B (a negative value implies borrowing). We set $c_1 = nS_1 + B$, or $\$50 = n\$150 + B$ and $c_2 = nS_2 + B$, or $\$0 = n\$50 + B$ and solve the 2 by 2 system, which gives $n = 0.5$ and $B = -\$25$. At time $t = 0$, the value of the loan is $B_0 = \$25/1.25 = \$20$. The current value of the portfolio is $nS_0 + B_0 = 0.5 \times \$100 - \$20 = \$30$. Hence the current value of the option must be $c_0 = \$30$. This derivation shows the essence of option pricing methods.

Note that we did not need the actual probabilities of an upmove. Furthermore, we could write the current value of the stock as the discounted expected payoff assuming investors were risk-neutral:

$$S_0 = [p \times S_1 + (1 - p) \times S_2]/(1 + r)$$

Solving for $100 = [p \times 150 + (1 - p) \times 50]/1.25$, we find a risk-neutral probability of $p = 0.75$. We now value the option in the same fashion:

$$c_0 = [0.75 \times \$50 + 0.25 \times \$0]/1.25 = \$30$$

This simple example illustrates a very important concept, which is that of **risk-neutral pricing**. We can price the derivative, like the underlying asset, assuming discount rates and growth rates are the same as the risk-free rate.

The Black-Scholes (BS) model is an application of these ideas that provides an elegant closed-form solution to the pricing of European calls. The derivation of the model is based on four assumptions:

Black-Scholes Model Assumptions:

(1) *The price of the underlying asset moves in a continuous fashion.*

(2) *Interest rates are known and constant.*

(3) *The variance of underlying asset returns is constant.*

(4) *Capital markets are perfect* (i.e., short-sales are allowed, there are no transaction costs or taxes, and markets operate continuously).

The most important assumption behind the model is that prices are continuous. This rules out discontinuities in the sample path, such as jumps, which cannot be hedged in this model.

The statistical process for the asset price is modeled by a geometric Brownian motion: over a very short time interval, dt, the logarithmic return has a normal distribution with mean = μdt and variance = $\sigma^2 dt$. The total return can be modeled as

$$dS/S = \mu dt + \sigma dz \tag{6.9}$$

where the first term represents the drift component, and the second is the stochastic component, with dz distributed normally with mean zero and variance dt.

This process implies that the logarithm of the ending price is distributed as

$$\ln(S_T) = \ln(S_0) + (\mu - \sigma^2/2)\tau + \sigma\sqrt{\tau}\,\epsilon \tag{6.10}$$

where ϵ is a $N(0, 1)$ random variable.

Based on these assumptions, Black and Scholes (1972) derived a closed-form formula for European options on a non-dividend-paying stock, called the **Black-Scholes model**. Merton (1973) expanded their model to the case of a stock paying a continuous dividend yield. Garman and Kohlhagen (1983) extended the formula to foreign currencies, reinterpreting the yield as the foreign rate of interest, in what is called the **Garman-Kohlhagen model**. The **Black model** (1976) applies the same formula to options on futures, reinterpreting the yield as the domestic risk-free rate and the spot price as the forward price. In each case, μ represents the capital appreciation return, i.e. without any income payment.

The key point of the analysis is that a position in the option can be replicated by a "delta" position in the underlying asset. Hence, a portfolio combining the asset and the option in appropriate proportions is "locally" risk-free, that is, for small movements in prices. To avoid arbitrage, this portfolio must return the risk-free rate.

As a result, we can directly compute the present value of the derivative as the discounted expected payoff

$$f_t = E_{\mathrm{RN}}[e^{-r\tau}F(S_T)] \tag{6.11}$$

where the underlying asset is assumed to grow at the risk-free rate, and the discounting is also done at the risk-free rate. Here, the subscript RN refers to the fact that the analysis assumes **risk neutrality**. In a risk-neutral world, the expected return on all securities must be the risk-free rate of interest, r. The reason is that risk-neutral investors do not require a risk premium to induce them to take risks. The BS model value can be computed assuming that all payoffs grow at the risk-free rate and are discounted at the same risk-free rate.

This risk-neutral valuation approach is without a doubt the most important tool in derivatives pricing. Before the Black-Scholes breakthrough, Samuelson had derived a very similar model in 1965, but with the asset growing at the rate μ and discounting as some other rate μ^*.[1] Because μ and μ^* are unknown, the Samuelson model was not practical. The risk-neutral valuation is merely an artificial method to obtain the correct solution, however. It does not imply that investors are in fact risk-neutral.

Furthermore, this approach has limited uses for risk management. The BS model can be used to derive the **risk-neutral probability** of exercising the option. For risk management, however, what matters is the actual probability of exercise, also called **physical probability**. This can differ from the BS probability.

In the case of a European call, the final payoff is $F(S_T) = \mathrm{Max}(S_T - K, 0)$. If the asset pays a continuous income of r^*, the current value of the call is given by:

$$c = Se^{-r^*\tau}N(d_1) - Ke^{-r\tau}N(d_2) \tag{6.12}$$

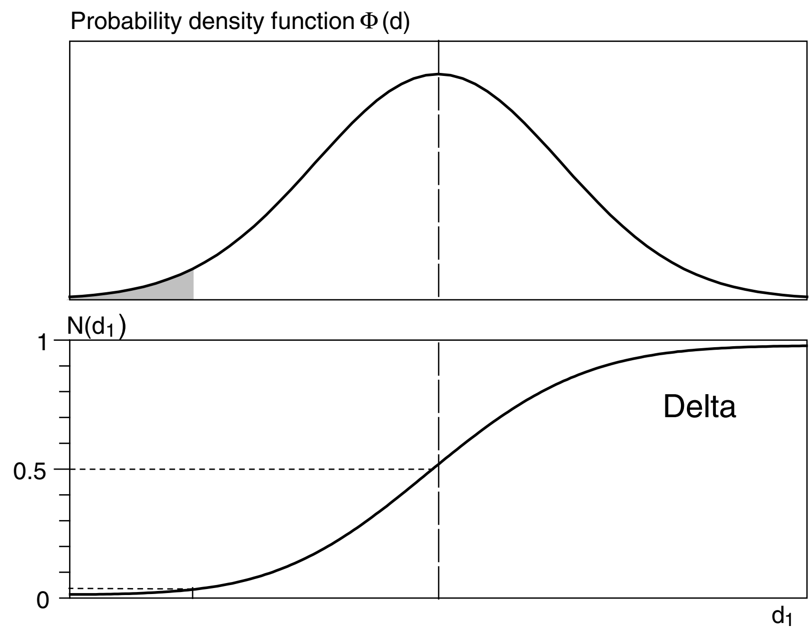
where $N(d)$ is the cumulative distribution function for the standard normal distribution:

$$N(d) = \int_{-\infty}^{d} \Phi(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{1}{2}x^2}dx$$

with Φ defined as the standard normal density function. $N(d)$ is also the area to the left of a standard normal variable with value equal to d, as shown in Figure 6-8. Note that, since the normal density is symmetrical, $N(d) = 1 - N(-d)$, or the area to the left of d is the same as the area to the right of $-d$.

[1] Samuelson, Paul (1965), Rational Theory of Warrant Price, *Industrial Management Review* 6, 13–39.

FIGURE 6-8 Cumulative Distribution Function



The values of d_1 and d_2 are:

$$d_1 = \frac{ln\,(Se^{-r^*\tau}/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

By put-call parity, the European put option value is

$$p = Se^{-r^*\tau}[N(d_1) - 1] - Ke^{-r\tau}[N(d_2) - 1] \tag{6.13}$$

Example: Computing the Black-Scholes value

Consider an at-the-money call on a stock worth $S = \$100$, with a strike price of $K = \$100$ and maturity of six months. The stock has annual volatility of $\sigma = 20\%$ and pays no dividend. The risk-free rate is $r = 5\%$.

First, we compute the present value factor, which is $e^{-r\tau} = \exp(-0.05 \times 6/12) = 0.9753$. We then compute the value of $d_1 = \ln[S/Ke^{-r\tau}]/\sigma\sqrt{\tau} + \sigma\sqrt{\tau}/2 = 0.2475$ and $d_2 = d_1 - \sigma\sqrt{\tau} = 0.1061$. Using standard normal tables or the "=NORMSDIST" Excel function, we find $N(d_1) = 0.5977$ and $N(d_2) = 0.5422$. Note that both values are greater than 0.5 since d_1 and d_2 are both positive. The option is at-the-money. As S is close to K, d_1 is close to zero and $N(d_1)$ close to 0.5.

The value of the call is $c = SN(d_1) - Ke^{-r\tau}N(d_2) = \$6.89$.

The value of the call can also be viewed as an equivalent position of $N(d_1) =$ 59.77% in the stock and some borrowing: $c = \$59.77 - \$52.88 = \$6.89$. Thus this is a leveraged position in the stock.

The value of the put is \$4.42. Buying the call and selling the put costs \$6.89 − \$4.42 = \$2.47. This indeed equals $S - Ke^{-r\tau} = \$100 - \$97.53 = \$2.47$, which confirms put-call parity.

For options on futures, we simply replace S by F, the current futures quote and r^* by r, the domestic risk-free rate. The Black model for the valuation of options on futures gives the following formula:

$$c = [FN(d_1) - KN(d_2)]e^{-r\tau} \tag{6.14}$$

We should note that Equation (6.12) can be reinterpreted in view of the discounting formula in a risk-neutral world, Equation (6.11)

$$c = E_{\text{RN}}[e^{-r\tau}\text{Max}(S_T - K, 0)] = e^{-r\tau}[\int_K^\infty Sf(S)dS] - Ke^{-r\tau}[\int_K^\infty f(S)dS] \tag{6.15}$$

Matching this up with (6.12), we see that the term multiplying K is also the risk-neutral probability of exercising the call, or that the option will end up in-the-money:

$$\text{Risk} - \text{neutral probability of exercise} = N(d_2) \tag{6.16}$$

The variable d_2 is indeed linked to the exercise price. Setting S_T to K in Equation (6.10), we have

$$ln(K) = ln(S_0) + (r - \sigma^2/2)\tau + \sigma\sqrt{\tau}\,\epsilon^*$$

Solving, we find $\epsilon^* = -d_2$. The area to the left of d_2 is therefore the same as the area to the right of ϵ^*, which represents the risk-neutral probability of exercising the call.

It is interesting to take the limit of Equation (6.12) as the option moves more in-the-money, that is, when the spot price S is much greater than K. In this case, d_1 and d_2 become very large and the functions $N(d_1)$ and $N(d_2)$ tend to unity. The value of the call then tends to

$$c(S >> K) = Se^{-r^*\tau} - Ke^{-r\tau} \tag{6.17}$$

which is the valuation formula for a forward contract, Equation (5.6). A call that is deep in-the-money is equivalent to a long forward contract, because we are almost certain to exercise.

Finally, we should note that standard options involve a choice to exchange cash for the asset. This is a special case of an **exchange option**, which involves the surrender of an asset (call it *B*) in exchange for acquiring another (call it *A*). The payoff on such a call is

$$c_T = \text{Max}(S_T^A - S_T^B, 0) \tag{6.18}$$

where S^A and S^B are the respective spot prices. Some financial instruments involve the maximum of the value of two assets, which is equivalent to a position in one asset plus an exchange option:

$$\text{Max}(S_t^A, S_t^B) = S_T^B + \text{Max}(S_T^A - S_T^B, 0) \tag{6.19}$$

Margrabe (1978) has shown that the valuation formula is similar to the usual model, except that K is replaced by the price of asset B (S_B), and the risk-free rate by the yield on asset B (y_B).[2] The volatility σ is now that of the difference between the two assets, which is

$$\sigma_{AB}^2 = \sigma_A^2 + \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B \tag{6.20}$$

These options also involve the correlation coefficient. So, if we have a triplet of options, involving A, B, and the option to exchange B into A, we can compute σ_A, σ_B, and σ_{AB}. This allows us to infer the correlation coefficient. The pricing formula is called the **Margrabe model**.

6.2.4 Market vs. Model Prices

In practice, the BS model is widely used to price options. All of the parameters are observable, except for the volatility. If we observe a market price, however, we can solve for the volatility parameter that sets the model price equal to the market price. This is called the **implied standard deviation** (ISD).

If the model were correct, the ISD should be constant across strike prices. In fact, this is not what we observe. Plots of the ISD against the strike price display what is called a **volatility smile** pattern, meaning that ISDs increase for low and high values of K. This effect has been observed in a variety of markets, and can even change over time. Before the stock market crash of October 1987, for instance, the effect was minor. Since then, it has become more pronounced.

[2] Margrabe, W. (1978), The Value of an Option to Exchange One Asset for Another, *Journal of Finance* 33, 177–186. See also Stulz, R. (1982), Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications, *Journal of Financial Economics* 10, 161–185.

Example 6-11. FRM Exam 2001—Question 91

6-11. Using the Black-Scholes model, calculate the value of a European call option given the following information:
Spot rate = 100; Strike price = 110; Risk-free rate = 10%; Time to expiry = 0.5 years; N(d1) = 0.457185; N(d2) = 0.374163.
a) $10.90
b) $9.51
c) $6.57
d) $4.92

Example 6-12. FRM Exam 1999—Question 55/Capital Markets

6-12. If the Garman-Kohlhagen formula is used for valuing options on a dividend-paying stock, then to be consistent with its assumptions, upon receipt of the dividend, the dividend should be
a) Placed into a noninterest bearing account
b) Placed into an interest bearing account at the risk-free rate assumed in the G-K model
c) Used to purchase more stock of the same company
d) Placed into an interest bearing account, paying interest equal to the dividend yield of the stock

Example 6-13. FRM Exam 1998—Question 2/Quant. Analysis

6-13. In the Black-Scholes expression for a European call option the term used to compute option probability of exercise is
a) d_1
b) d_2
c) $N(d_1)$
d) $N(d_2)$

6.3 Other Option Contracts

The options described so far are standard, plain-vanilla options. Since the 1970s, however, markets have developed more complex option types.

Binary options, also called **digital options** pay a fixed amount, say Q, if the asset price ends up above the strike price

$$c_T = Q \times I(S_T - K) \tag{6.21}$$

where $I(x)$ is an indicator variable that takes the value of 1 if $x > 0$ and 0 otherwise. Because the probability of ending in the money in a risk-neutral world is $N(d_2)$, the initial value of this option is simply

$$c = Qe^{-r\tau}N(d_2) \tag{6.22}$$

These options involve a sharp discontinuity around the strike price. As a result, they are quite difficult to hedge since the value of the option cannot be smoothly replicated by a changing position in the underlying asset.

Another important class of options are barrier options. **Barrier options** are options where the payoff depends on the value of the asset hitting a barrier during a certain period of time. A **knock-out option** disappears if the price hits a certain barrier. A **knock-in option** comes into existence when the price hits a certain barrier.

An example of a knock-out option is the **down-and-out call**. This disappears if S hits a specified level H during its life. In this case, the knock-out price H must be lower than the initial price S_0. The option that appears at H is the **down-and-in call**. With identical parameters, the two options are perfectly complementary. When one disappears, the other appears. As a result, these two options must add up to a regular call option. Similarly, an **up-and-out call** ceases to exist when S reaches $H > S_0$. The complementary option is the **up-and-in call**.

Figure 6-9 compares price paths for the four possible combinations of calls. The left panels involve the same underlying sample path. For the down-and-out call, the only relevant part is the one starting from $S(0)$ until it hits the barrier. In all figures, the dark line describes the relevant price path, during which the option is alive; the grey line describes the remaining path.

The call is not exercised even though the final price S_T is greater than the strike price. Conversely, the down-and-in call comes into existence precisely when the other one dies. Thus at initiation, the value of these two options must add up to a regular European call

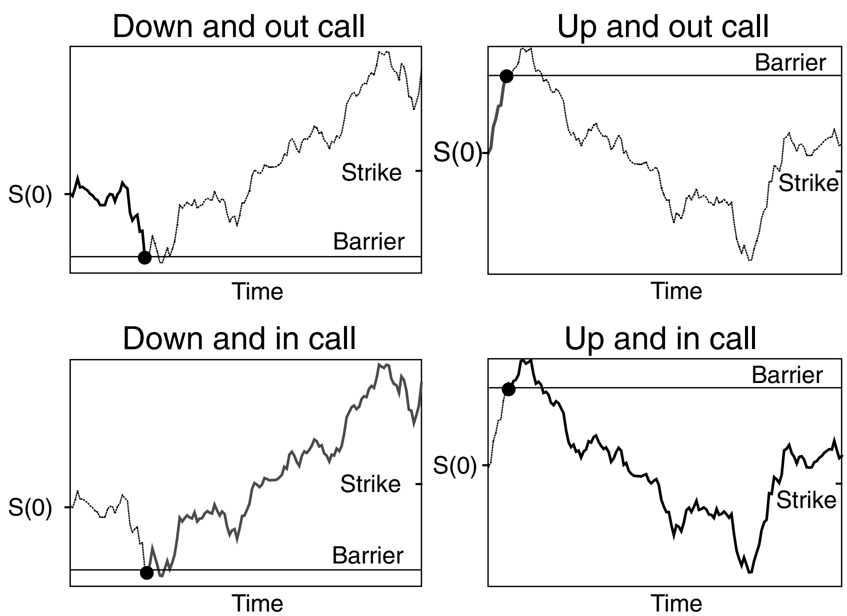
$$c = c_{DO} + c_{DI} \tag{6.23}$$

Because all these values are positive (or at worst zero), the value of c_{DO} and c_{DI} each must be no greater than that of c. A similar reasoning applies to the two options in the right panels.

Similar combinations exist for put options. An **up-and-out put** ceases to exist when S reaches $H > S_0$. A **down-and-out put** ceases to exist when S reaches $H < S_0$.

Barrier options are attractive because they are “cheaper” than the equivalent ordinary option. This, of course, reflects the fact that they are less likely to be exercised than other options. These options are also difficult to hedge due to the fact that a discontinuity arises as the spot price get closer to the barrier. Just above the barrier, the option has positive value. For a very small movement in the asset price, going below the barrier, this value disappears.

FIGURE 6-9 Paths for Knock-out and Knock-in Call Options



Finally, another widely used class of options are Asian options. **Asian options**, or **average rate options**, generate payoffs that depend on the average value of the underlying spot price during the life of the option, instead of the ending value. The final payoff for a call is

$$c_T = \text{Max}(S_{\text{AVE}}(t, T) - K, 0) \tag{6.24}$$

Because an average is less variable than an instantaneous value, such options are “cheaper” than regular options due to lower volatility. In fact, the price of the option can be treated like that of an ordinary option with the volatility set equal to $\sigma/\sqrt{3}$ and an adjustment to the dividend yield.[3] As a result of the averaging process, such

[3] This is only strictly true when the averaging is a geometric average. In practice, average options involve an arithmetic average, for which there is no analytic solution; the lower volatility adjustment is just an approximation.

options are easier to hedge than ordinary options.

Example 6-14. FRM Exam 1998—Question 4/Capital Markets

6-14. A knock-in barrier option is harder to hedge when it is

a) In the money
b) Out of the money
c) At the barrier and near maturity
d) At the inception of the trade

Example 6-15. FRM Exam 1997—Question 10/Derivatives

6-15. Knockout options are often used instead of regular options because

a) Knockouts have a lower volatility.
b) Knockouts have a lower premium.
c) Knockouts have a shorter maturity on average.
d) Knockouts have a smaller gamma.

6.4 Valuing Options by Numerical Methods

Some options have analytical solutions, such as the Black-Scholes models for European vanilla options. For more general options, however, we need to use numerical methods.

The basic valuation formula for derivatives is Equation (6.11), which states that the current value is the discounted present value of expected cash flows, where all assets grow at the risk-free rate and are discounted at the same risk-free rate.

We can use the Monte Carlo simulation methods presented in Chapter 4 to generate sample paths, final option values, and discount them into the present. Such simulation methods can be used for European or even path-dependent options, such as Asian options.

Simulation methods, however, cannot account for the possibility of early exercise. Instead, binomial trees must be used to value American options. As explained previously, the method consists of chopping up the time horizon into n intervals Δt and setting up the tree so that the characteristics of price movements fit the lognormal distribution.

At each node, the initial price S can go up to uS with probability p or down to dS with probability $(1 - p)$. The parameters u, d, p are chosen so that, for a small time interval, the expected return and variance equal those of the continuous process. One could choose, for instance,

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = (1/u), \quad p = \frac{e^{\mu\Delta t} - d}{u - d} \tag{6.25}$$

Since this a risk-neutral process, the total expected return must be equal to the risk-free rate r. Allowing for an income payment of r^*, this gives $\mu = r - r^*$.

The tree is built starting from the current time to maturity, from the left to the right. Next, the derivative is valued by starting at the end of the tree, and working backward to the initial time, from the right to the left.

Consider first a European call option. At time T (maturity) and node j, the call option is worth $\text{Max}(S_{Tj} - K, 0)$. At time $T-1$ and node j, the call option is the discounted expected value of the option at time T and nodes j and $j+1$:

$$c_{T-1,j} = e^{-r\Delta t}[pc_{T,j+1} + (1-p)c_{T,j}] \tag{6.26}$$

We then work backward through the tree until the current time.

For American options, the procedure is slightly different. At each point in time, the holder compares the value of the option *alive* and *dead* (i.e., exercised). The American call option value at node $T-1, j$ is

$$C_{T-1,j} = \text{Max}[(S_{T-1,j} - K), c_{T-1,j}] \tag{6.27}$$

Example: Computing an American option value

Consider an at-the-money call on a foreign currency with a spot price of \$100, a strike price of $K = \$100$, and a maturity of six months. The annualized volatility is $\sigma = 20\%$. The domestic interest rate is $r = 5\%$; the foreign rate is $r^* = 8\%$. Note that we require an income payment for the American feature to be valuable.

First, we divide the period into 4 intervals, for instance, so that $\Delta t = 0.125$. The discounting factor over one interval is $e^{-r\Delta t} = 0.9938$. We then compute:

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.20\sqrt{0.125}} = 1.0733,$$
$$d = (1/u) = 0.9317,$$
$$a = e^{(r-r^*)\Delta t} = e^{(-0.03)0.125} = 0.9963,$$
$$p = \frac{a-d}{u-d} = (0.9963 - 0.9317)/(1.0733 - 0.9317) = 0.4559.$$

The procedure is detailed in Table 6-3. First, we lay out the tree for the spot price, starting with $S = 100$ at time $t = 0$, then $uS = 107.33$ and $dS = 93.17$ at time $t = 1$, and so on.

This allows us to value the European call. We start from the end, at time $t = 4$, and set the call price to $c = S - K = 132.69 - 100.00 = 32.69$ for the highest spot price, 15.19 for the next rate and so on, down to $c = 0$ if the spot price is below $K = 100.00$. At the previous step and highest node, the value of the call is

$$c = 0.9938[0.4559 \times 32.69 + (1 - 0.4559) \times 15.19] = 23.02$$

Continuing through the tree to time 0 yields a European call value of \$4.43. The Black-Scholes formula gives an exact value of \$4.76. Note how close the binomial approximation is, with just 4 steps. A finer partition would quickly improve the approximation.

TABLE 6-3 Computation of American option value

	0	1	2	3	4
Spot Price S_t	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$
					132.69
				123.63	115.19
			115.19	107.33	100.00
		107.33	100.00	93.17	86.81
	100.00	93.17	86.81	80.89	75.36
European Call c_t	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$
					32.69
				23.02	15.19
			14.15	6.88	0.00
		8.10	3.12	0.00	0.00
	4.43	1.41	0.00	0.00	0.00
Exercised Call $S_t - K$					
					32.69
				23.63	15.19
			15.19	7.33	0.00
		7.33	0.00	0.00	0.00
	0.00	0.00	0.00	0.00	0.00
American Call C_t	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$	$\leftarrow$
					32.69
				23.63	15.19
			15.19	7.33	0.00
		8.68	3.32	0.00	0.00
	4.74	1.50	0.00	0.00	0.00

Next, we examine the American call. At time $t = 4$, the values are the same as above since the call expires. At time $t = 3$ and node $j = 4$, the option holder can either keep the call, in which case the value is still \$23.02, or exercise. When exercised, the option payoff is $S - K = 123.63 - 100.00 = 23.63$. Since this is greater than the value of the option alive, the holder should optimally exercise the option. We replace the European option value by \$23.63. Continuing through the tree in the same fashion, we find a starting value of \$4.74. The value of the American call is slightly greater than the European call price, as expected.

6.5 Answers to Chapter Examples

Example 6-1: FRM Exam 1999—Question 35/Capital Markets

b) A short put position is equivalent to a long asset position plus shorting a call. To fund the purchase of the asset, we need to borrow. This is because the value of the call or put is small relative to the value of the asset.

Example 6-2: FRM Exam 2000—Question 15/Capital Markets

d) By put-call parity, $p = c - (S - Ke^{-r\tau}) = 30 - (100 - 120\exp(-0.5 \times 0.5)) = 30 + 17.04 = 47.04$. In the absence of other information, we had to assume these are European options, and that the stock pays no dividend.

Example 6-3. FRM Exam 2001—Question 90

c) Long positions in options can lose at worst the premium, so (b) is wrong. Spreads involve long and short positions in options and have limited downside loss, so (a) is wrong. Writing options exposes the seller to very large losses. In the case of puts, the worst loss is the strike price K, if the asset price goes to zero. In the case of calls, however, the worst loss is in theory unlimited because there is a small probability of a huge increase in S. Between (c) and (d), (c) is the best answer.

Example 6-4: FRM Exam 1999—Question 50/Capital Markets

b) A covered call is long the asset plus a short call. This preserves the downside but eliminates the upside, which is equivalent to a short put.

Example 6-5: FRM Exam 1999—Question 33/Capital Markets

a) The purpose of a bull spread is to create a profit when the underlying price increases. The strategy involves the same options but with different strike prices. It can be achieved with calls or puts. Answer (c) is incorrect as a bull spread based on calls involves buying a call with high premium and selling another with lower premium. Answer (d) is incorrect as it mixes a call and a put. Among the two puts $p(K = \$55)$ must have higher value than $p(K = \$50)$. If the spot price ends up above 55, none of the puts is exercised. The profit must be positive, which implies selling the put with $K = 55$ and buying a put with $K = 50$.

Example 6-6: FRM Exam 2000—Question 5/Capital Markets

a) The proceeds from exercise are $(\$42 - \$30) - (\$42 - \$40) = \$10$. ¿From this should be deducted the net cost of the options, which is $\$3 - \$1.5 = \$1.5$, ignoring the time value of money. This adds up to a net profit of \$8.50.

Example 6-7. FRM Exam 2001—Question 111

d) All of the puts will be exercised, leading to a payoff of $+(43 - 19) - 2(37 - 19) + (32 - 19) = +1$. To this, we add the premiums, or $-6 + 2(4) - 1 = +1$. Ignoring the time value of money, the total payoff is \$2. The same result holds for any value of S lower than 32. The fact that the strategy creates a profit if the price falls explain why it is called *bearish*.

Example 6-8: FRM Exam 1998—Question 58/Capital Markets

d) Futures have an "implied" income stream equal to the risk-free rate. As a result, an American call may be exercised early. Similarly, the American put may be exercised early. Also, the put-call parity only works when there is no possibility of early exercise, or with European options.

Example 6-9: FRM Exam 1999—Question 34/Capital Markets

b) The call lower bound, when there is no income, is $S_t - Ke^{-r\tau} = \$90 - \$80\exp(-0.05 \times 1) = \$90 - \$76.10 = \$13.90$.

Example 6-10: FRM Exam 1999—Question 52/Capital Markets

b) An American call will not be exercised early when there is no income payment on the underlying asset.

Example 6-11. FRM Exam 2001—Question 91

c) We use Equation (6.12) assuming there is no income payment on the asset. This gives $c = SN(d_1) - K\exp(-r\tau)N(d_2) = 100 \times 0.457185 - 110\exp(-0.1 \times 0.5) \times 0.374163 = \$6.568$.

Example 6-12: FRM Exam 1999—Question 55/Capital Markets

c) The GK formula assumes that income payments are reinvested in the stock itself. Answers (a) and (b) assume reinvestment at a zero and risk-free rate, which is incorrect. Answer (d) is not feasible.

Example 6-13: FRM Exam 1998—Question 2/Quant. Analysis

d) This is the term multiplying the present value of the strike price, by Equation (6.13).

Example 6-14: FRM Exam 1998—Question 4/Capital Markets

c) Knock-in or knock-out options involve discontinuities, and are harder to hedge when the spot price is close to the barrier.

Example 6-15: FRM Exam 1997—Question 10/Derivatives

b) Knockouts are no different from regular options in terms of maturity or underlying volatility, but are cheaper than the equivalent European option since they involve a lower probability of final exercise.

Chapter 7

Fixed-Income Securities

The next two chapters provide an overview of fixed-income markets, securities, and their derivatives. Originally, **fixed-income securities** referred to bonds that promise to make fixed coupon payments. Over time, this narrow definition has evolved to include any security that obligates the borrower to make specific payments to the bondholder on specified dates. Thus, a **bond** is a security that is issued in connection with a borrowing arrangement. In exchange for receiving cash, the borrower becomes obligated to make a series of payments to the bondholder.

Fixed-income derivatives are instruments whose value derives from some bond price, interest rate, or other bond market variable. Due to their complexity, these instruments are analyzed in the next chapter.

Section 7.1 provides an overview of the different segments of the bond market. Section 7.2 then introduces the various types of fixed-income securities. Section 7.3 reviews the basic tools for analyzing fixed-income securities, including the determination of cash flows, the measurement of duration, and the term structure of interest rates and forward rates. Because of their importance, mortgage-backed securities (MBSs) are analyzed separately in Section 7.4. The section also discusses collateralized mortgage obligations (CMOs), which illustrate the creativity of financial engineering.

7.1 Overview of Debt Markets

Table 7-1 breaks down the world debt securities market, which was worth $38 trillion at the end of 2001. This includes the **bond markets**, defined as fixed-income securities with remaining maturities beyond one year, and the shorter-term **money markets**, with maturities below one year. The table includes all publicly tradable debt securities sorted by country of issuer and issuer type as of December 2001.

To help sort the various categories of the bond markets, Table 7-2 provides a broad classification of bonds by borrower and currency type. Bonds issued by resident entities and denominated in the domestic currency are called **domestic bonds**. In

TABLE 7-1 Global Debt Securities Markets - 2001 (Billions of U.S. dollars)

Country of Issuer	Domestic	Of which			Int'l	Total
		Public	Financials	Corporates		
United States	15,655	8,703	4,517	2,434	2,395	18,049
Japan	5,820	4,576	570	674	96	5,915
Germany	1,475	686	752	36	643	2,117
Italy	1,362	963	330	70	176	1,537
France	1,050	642	289	119	402	1,452
United Kingdom	925	407	292	227	757	1,682
Canada	571	406	92	73	221	792
Spain	364	266	55	43	72	436
Belgium	315	222	75	18	54	369
Brazil	316	261	52	3	60	375
Korea (South)	305	79	108	118	44	350
Denmark	229	73	144	13	34	263
Sweden	166	85	60	21	89	255
Netherlands	360	159	151	51	569	930
Australia	183	66	68	50	138	321
China	407	291	106	10	13	420
Switzerland	161	56	82	23	16	177
Austria	154	92	59	3	105	259
India	132	131	0	2	4	137
Subtotal	29,950	18,161	7,801	3,988	5,887	35,837
Others	602	703	136	125	1,624	2,226
Total	30,552	18,864	7,936	4,113	7,511	38,063
Of which, Eurozone	5,080	3,029	1,711	340	2,020	7,100

Source: Bank for International Settlements

contrast, **foreign bonds** are those floated by a foreign issuer in the domestic currency and subject to domestic country regulations (e.g., by the government of Sweden in dollars in the United States). **Eurobonds** are mainly placed outside the country of the currency in which they are denominated and are sold by an international syndicate of financial institutions (e.g., a dollar-denominated bond issued by IBM and marketed in London). These should not be confused with Euro-denominated bonds. Foreign bonds and Eurobonds constitute the **international bond market**. **Global bonds** are placed at the same time in the Eurobond and one or more domestic markets with securities fungible between these markets.

TABLE 7-2 Classification of Bond Markets

	By resident	By non-resident
In domestic currency	Domestic Bond	Foreign Bond
In foreign currency	Eurobond	Eurobond

Coupon payment frequencies can differ across markets. For instance, domestic dollar bonds pay interest semiannually. In contrast, Eurobonds pay interest annually only. Because investors are spread all over the world, less frequent coupons lower payment costs.

Going back to Table 7-1, we see that U.S. entities have issued a total of $15,665 billion in domestic bonds and $2,395 billion in international bonds. This leads to a total principal amount of $18,049 billion, which is by far the biggest debt market. Next comes the Eurozone market, with a size of $7,100 billion, and the Japanese market, with $5,915 billion.

The domestic bond market can be further decomposed into the categories representing the public and private bond markets:

- **Government bonds**, issued by central governments, or also called **sovereign bonds** (e.g., by the United States or Argentina)
- **Government agency and guaranteed bonds**, issued by agencies or guaranteed by the central government, (e.g., by Fannie Mae, a U.S. government agency)
- **State and local bonds**, issued by local governments, other than the central government, also known as **municipal bonds** (e.g., by the state or city of New York)
- Bonds issued by private **financial institutions**, including banks, insurance companies, or issuers of asset-backed securities (e.g., by Citibank in the U.S. market)
- **Corporate bonds**, issued by private nonfinancial corporations, including industrials and utilities (e.g., by IBM in the U.S. market)

As Table 7-1 shows, the public sector accounts for more than half of the debt markets. This sector includes sovereign debt issued by emerging countries in their own currencies, e.g. Mexican peso-denominated debt issued by the Mexican government. Few of these markets have long-term issues, because of their history of high inflation, which renders long-term bonds very risky. In Mexico, for instance, the market consists mainly of **Cetes**, which are peso-denominated, short-term Treasury Bills.

The emerging market sector also includes dollar-denominated debt, such as **Brady bonds**, which are sovereign bonds issued in exchange for bank loans, and the **Tesebonos**, which are dollar-denominated bills issued by the Mexican government. Brady bonds are hybrid securities whose principal is collateralized by U.S. Treasury zero-coupon bonds. As a result, there is no risk of default on the principal, unlike on coupon payments.

A large and growing proportion of the market consists of mortgage-backed securities. **Mortgage-backed securities** (MBSs), or mortgage **pass-throughs**, are securities issued in conjunction with mortgage loans, either residential or commercial. Payments on MBSs are repackaged cash flows supported by mortgage payments made by property owners. MBSs can be issued by government agencies as well as by private financial corporations. More generally, **asset-backed securities** (ABSs) are securities whose cash flows are supported by assets such as credit card receivables or car loan payments.

Finally, the remainder of the market represents bonds raised by private, nonfinancial corporations. This sector, large in the United States but smaller in other countries, is growing rather quickly as corporations recognize that bond issuances are a lower-cost source of funds than bank debt. The advent of the common currency, the Euro, is also leading to a growing, more liquid and efficient, corporate bond market in Europe.

7.2 Fixed-Income Securities

7.2.1 Instrument Types

Bonds pay interest on a regular basis, semiannual for U.S. Treasury and corporate bonds, annual for others such as Eurobonds, or quarterly for others. The most common types of bonds are:

- **Fixed-coupon bonds**, which pay a fixed percentage of the principal every period and the principal as a **balloon**, one-time, payment at maturity
- **Zero-coupon bonds**, which pay no coupons but only the principal; their return is derived from price appreciation only
- **Annuities**, which pay a constant amount over time which includes interest plus amortization, or gradual repayment, of the principal;

- **Perpetual bonds** or **consols**, which have no set redemption date and whose value derives from interest payments only
- **Floating-coupon bonds**, which pay interest equal to a reference rate plus a margin, reset on a regular basis; these are usually called **floating-rate notes** (FRN)
- **Structured notes**, which have more complex coupon patterns to satisfy the investor's needs

There are many variations on these themes. For instance, **step-up bonds** have coupons that start at a low rate and increase over time.

It is useful to consider floating-rate notes in more detail. Take for instance a 10-year \$100 million FRN paying semiannually 6-month LIBOR in arrears.[1] Here, **LIBOR** is the London Interbank Offer Rate, a benchmark short-term cost of borrowing for AA credits. Every semester, on the **reset date**, the value of 6-month LIBOR is recorded. Say LIBOR is initially at 6%. At the next coupon date, the payment will be $(\frac{1}{2}) \times \$100 \times 6\% =$ \$3 million. Simultaneously, we record a new value for LIBOR, say 8%. The next payment will then increase to \$4 million, and so on. At maturity, the issuer pays the last coupon plus the principal. Like a cork at the end of a fishing line, the coupon payment "floats" with the current interest rate.

Among structured notes, we should mention **inverse floaters**, which have coupon payments that vary inversely with the level of interest rates. A typical formula for the coupon is $c = 12\% - \text{LIBOR}$, if positive, payable semiannually. Assume the principal is \$100 million. If LIBOR starts at 6%, the first coupon will be $(1/2) \times \$100 \times (12\% - 6\%) = \$3$ million. If after six months LIBOR moves to 8%, the second coupon will be $(1/2) \times \$100 \times (12\% - 8\%) = \$2$ million. The coupon will go to zero if LIBOR moves above 12%. Conversely, the coupon will increase if LIBOR drops. Hence, inverse floaters do best in a falling interest rate environment.

Bonds can also be issued with option features. The most important are:

- **Callable bonds**, where the issuer has the right to "call" back the bond at fixed prices on fixed dates, the purpose being to call back the bond when the cost of issuing new debt is lower than the current coupon paid on the bond

[1] Note that the index could be defined differently. The floating payment could be tied to a Treasury rate, or LIBOR with a different maturity–say 3-month LIBOR. The pricing of the FRN will depend on the index. Also, the coupon will typically be set to LIBOR plus some spread that depends on the creditworthiness of the issuer.

- **Puttable bonds**, where the investor has the right to "put" the bond back to the issuer at fixed prices on fixed dates, the purpose being to dispose of the bond should its price deteriorate
- **Convertible bonds**, where the bond can be converted into the common stock of the issuing company at a fixed price on a fixed date, the purpose being to partake in the good fortunes of the company (these will be covered in Chapter 9 on equities)

The key to analyzing these bonds is to identify and price the option feature. For instance, a callable bond can be decomposed into a long position in a straight bond minus a call option on the bond price. The call feature is unfavorable for investors who will demand a lower price to purchase the bond, thereby increasing its yield. Conversely, a put feature will make the bond more attractive, increasing its price and lowering its yield. Similarly, the convertible feature allows companies to issue bonds at a lower yield than otherwise.

Example 7-1: FRM Exam 1998—Ques:wtion 3/Capital Markets
7-1. The price of an inverse floater
a) Increases as interest rates increase
b) Decreases as interest rates increase
c) Remains constant as interest rates change
d) Behaves like none of the above

Example 7-2: FRM Exam 2000—Ques:wtion 9/Capital Markets
7-2. An investment in a callable bond can be analytically decomposed into a
a) Long position in a noncallable bond and a short position in a put option
b) Short position in a noncallable bond and a long position in a call option
c) Long position in a noncallable bond and a long position in a call option
d) Long position in a noncallable and a short position in a call option

7.2.2 Methods of Quotation

Most bonds are quoted on a **clean price** basis, that is, without accounting for the accrued income from the last coupon. For U.S. bonds, this clean price is expressed as a percent of the face value of the bond with fractions in thirty-seconds, for instance $104 - 12$ or $104 + 12/32$ for the 6.25% May 2030 Treasury bond. Transactions are expressed in number of units, e.g. $20 million face value.

Actual payments, however, must account for the accrual of interest. This is factored into the **gross price**, also known as the **dirty price**, which is equal to the clean price plus accrued interest. In the U.S. Treasury market, accrued interest (AI) is computed on an *actual/actual* basis:

$$\text{AI} = \text{Coupon} \times \frac{\text{Actual number of days since last coupon}}{\text{Actual number of days between last and next coupon}} \tag{7.1}$$

The fraction involves the actual number of days in both the numerator and denominator. For instance, say the 6.25% of May 2030 paid the last coupon on November 15 and will pay the next coupon on May 15. The denominator is, counting the number of days in each month, $15 + 31 + 31 + 29 + 31 + 30 + 15 = 182$. If the trade settles on April 26, there are $15 + 31 + 31 + 29 + 31 + 26 = 163$ days into the period. The accrued is computed from the \$3.125 coupon as

$$\$3.125 \times \frac{163}{182} = \$2.798763$$

The total, gross price for this transaction is:

$$(\$20,000,000/100) \times [(104 + 12/32) + 2.798763] = \$21,434,753$$

Different markets have different day count conventions. A 30/360 convention, for example, considers that all months count for 30 days exactly. The computation of the accrued interest is tedious but must be performed precisely to settle the trades.

We should note that the accrued interest in the **LIBOR** market is based on *actual/360*. For instance, the actual interest payment on a 6% \$1 million loan over 92 days is

$$\$1,000,000 \times 0.06 \times \frac{92}{360} = \$15,333.33$$

Another notable pricing convention is the discount basis for Treasury Bills. These bills are quoted in terms of an annualized discount rate (DR) to the face value, defined as

$$\text{DR} = (\text{Face} - \text{P})/\text{Face} \times (360/t) \tag{7.2}$$

where P is the price and t is the actual number of days. The dollar price can be recovered from

$$P = \text{Face} \times [1 - \text{DR} \times (t/360)] \tag{7.3}$$

For instance, a bill quoted at 5.19% discount with 91 days to maturity could be purchased for

$$\$100 \times [1 - 5.19\% \times (91/360)] = \$98.6881.$$

This price can be transformed into a conventional yield to maturity, using

$$F/P = (1 + y \times t/365) \tag{7.4}$$

which gives 5.33% in this case. Note that the yield is greater than the discount rate because it is a rate of return based on the initial price. Because the price is lower than the face value, the yield must be greater than the discount rate.

Example 7-3: FRM Exam 1998—Ques:wtion 13/Capital Markets

7-3. A U.S. Treasury bill selling for $97,569 with 100 days to maturity and a face value of $100,000 should be quoted on a bank discount basis at

a) 8.75%
b) 8.87%
c) 8.97%
d) 9.09%

7.3 Analysis of Fixed-Income Securities

7.3.1 The NPV Approach

Fixed-income securities can be valued by, first, laying out their cash flows and, second, discounting them at the appropriate discount rate.

This approach can also be used to infer a more convenient measure of value for the bond, which is the bond's own yield. Let us write the market value of a bond P as the present value of future cash flows:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \tag{7.5}$$

where:

C_t = the cash flow (coupon or principal) in period t,

t = the number of periods (e.g., half-years) to each payment,

T = the number of periods to final maturity,

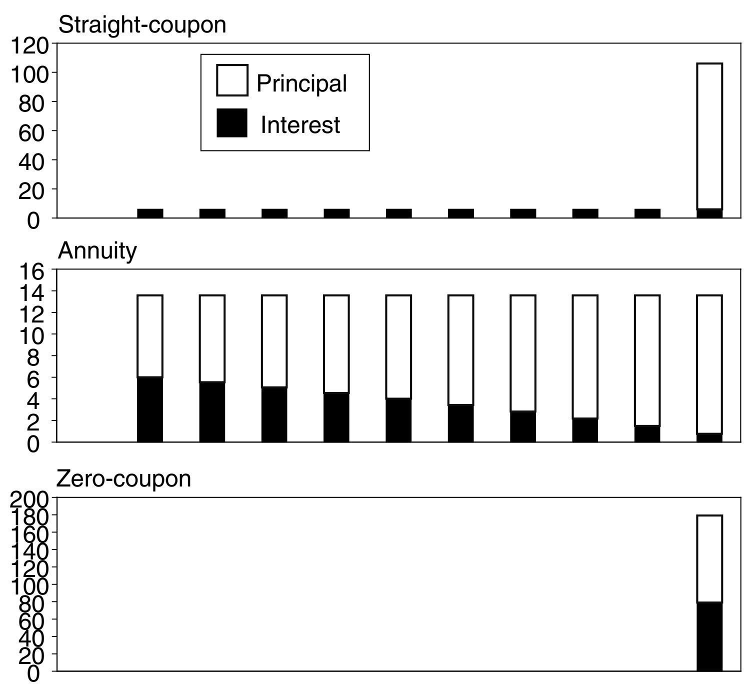
y = the yield to maturity for this particular bond,

P = the price of the bond, including accrued interest.

Here, the yield is the internal rate of return that equates the NPV of the cash flows to the market price of the bond. The yield is also the expected rate of return on the bond, provided all coupons are reinvested at the same rate. For a fixed-rate bond with face value F, the cash flow C_t is cF each period, where c is the coupon rate, plus F upon maturity. Other cash flow patterns are possible.

Figure 7-1 shows the time profile of the cash flows C_t for three bonds with initial market value of \$100, 10 year maturity and 6% annual interest. The figure describes a straight coupon-paying bond, an annuity, and a zero-coupon bond. As long as the cash flows are predetermined, the valuation is straightforward.

FIGURE 7-1 Time Profile of Cash Flows



Problems start to arise when the cash flows are random or when the life of the bond could be changed due to option-like features. In this case, the standard valuation formula in Equation (7.5) fails. More precisely, the yield cannot be interpreted as a reinvestment rate. Particularly important examples are MBSs, which are detailed in a later section.

It is also important to note that we discounted all cash flows at the same rate, y. More generally, the fair value of the bond can be assessed using the term structure of interest rates. Define R_t as the **spot interest rate** for maturity t and this risk class (i.e., same currency and credit risk). The fair value of the bond is then:

$$\hat{P} = \sum_{t=1}^{T} \frac{C_t}{(1 + R_t)^t} \tag{7.6}$$

To assess whether a bond is rich or cheap, we can add a fixed amount s, called the **static spread** to the spot rates so that the NPV equals the current price:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1 + R_t + s)^t} \tag{7.7}$$

All else equal, a bond with a large static spread is preferable to another with a lower spread. It means the bond is cheaper, or has a higher expected rate of return.

It is often simpler to compute a **yield spread** Δy, using yield to maturity, such that

$$P = \sum_{t=1}^{T} \frac{C_t}{(1 + y + \Delta y)^t} \tag{7.8}$$

The static spread and yield spread are conceptually similar, but the former is more accurate since the term structure is not necessarily flat. When the term structure is flat, the two measures are identical.

Table 7-3 gives an example of a 7% coupon, 2-year bond. The term structure environment, consisting of spot rates and par yields, is described on the left side. The right side lays out the present value of the cash flows (PVCF). Discounting the two cash flows at the spot rates gives a fair value of $\hat{P} = \$101.9604$. In fact, the bond is selling at a price of $P = \$101.5000$. So, the bond is cheap.

We can convert the difference in prices to annual yields. The yield to maturity on this bond is 6.1798%, which implies a yield spread of $\Delta y = 6.1798 - 5.9412 = 0.2386\%$. Using the static spread approach, we find that adding $s = 0.2482\%$ to the spot rates gives the current price. The second measure is more accurate than the first. In this case, the difference is small. This will not be the case, however, with longer maturities and irregular yield curves.

TABLE 7-3 Bond Price and Term Structure

	Term Structure		7% Bond PVCF Discounted at		
Maturity (Year) i	Spot Rate R_i	Par Yield y_i	Spot $s = 0$	Yield+YS $\Delta y = 0.2386$	Spot+SS $s = 0.2482$
1	4.0000	4.0000	6.7308	6.5926	6.7147
2	6.0000	5.9412	95.2296	94.9074	94.7853
Sum			101.9604	101.5000	101.5000
Price			101.5000	101.5000	101.5000

Cash flows with different credit risks need to be discounted with different rates. For example, the principal on Brady bonds is collateralized by U.S. Treasury securities and carries no default risk, in contrast to the coupons. As a result, it has become common to separate the discounting of the principal from that of the coupons. Valuation is done in two steps. First, the principal is discounted into a present value using the appropriate Treasury yield. The present value of the principal is subtracted from the market value. Next, the coupons are discounted at what is called the **stripped yield**, which accounts for the credit risk of the issuer.

7.3.2 Duration

Armed with a cash flow profile, we can proceed to compute duration. As we have seen in Chapter 1, **duration** is a measure of the exposure, or sensitivity, of the bond price to movements in yields. When cash flows are fixed, duration is measured as the weighted maturity of each payment, where the weights are proportional to the present value of the cash flows. Using the same notations as in Equation (7.5), recall that **Macaulay duration** is

$$D = \sum_{t=1}^{T} t \times w_t = \sum_{t=1}^{T} t \times \frac{C_t/(1+y)^t}{\sum C_t/(1+y)^t}. \tag{7.9}$$

> **Key concept:**
> Duration can only be viewed as the weighted average time to wait for each payment when the cash flows are predetermined.

More generally, duration is a measure of interest-rate exposure:

$$\frac{dP}{dy} = -\frac{D}{(1+y)}P = -D^{*}P \tag{7.10}$$

where D^{*} is **modified duration**. The second term $D^{*}P$ is also known as the **dollar duration**. Sometimes this sensitivity is expressed in **dollar value of a basis point** (also known as DV01), defined as

$$\frac{dP}{0.01\%} = \text{DVBP} \tag{7.10}$$

For fixed cash flows, duration can be computed using Equation (7.9). Otherwise, we can infer duration from an understanding of the security. Consider a floating-rate note (FRN). Just before the reset date, we know that the coupon will be set to the prevailing interest rate. The FRN is then similar to cash, or a money market instrument, which has no interest rate risk and hence is selling at par with zero duration. Just after the reset date, the investor is locked into a fixed coupon over the accrual period. The FRN is then economically equivalent to a zero-coupon bond with maturity equal to the time to the next reset date.

Key concept:
The duration of a floating-rate note is the time to wait until the next reset period, at which time the FRN should be at par.

Example 7-4: FRM Exam 1999—Ques:wtion 53/Capital Markets

7-4. Consider a 9% annual coupon 20-year bond trading at 6% with a price of 134.41. When rates rise 10bps, price reduces to 132.99, and when rates decrease by 10bps, the price goes up to 135.85. What is the modified duration of the bond?

a) 11.25
b) 10.61
c) 10.50
d) 10.73

Example 7-5: FRM Exam 1998—Ques:wtion 31/Capital Markets

7-5. A 10-year zero-coupon bond is callable annually at par (its face value) starting at the beginning of year six. Assume a flat yield curve of 10%. What is the bond duration?

a) 5 years
b) 7.5 years
c) 10 years
d) Cannot be determined based on the data given

Example 7-6: FRM Exam 1999—Ques:wtion 91/Market Risk

7-6. (Modified) duration of a fixed-rate bond, in the case of flat yield curve, can be interpreted as (where B is the bond price and y is the yield to maturity)

a) $-\frac{1}{B}\frac{\partial B}{\partial y}$
b) $\frac{1}{B}\frac{\partial B}{\partial y}$
c) $-\frac{y}{B}\frac{\partial B}{\partial y}$
d) $\frac{(1+y)}{B}\frac{\partial B}{\partial y}$

Example 7-7: FRM Exam 1997—Ques:wtion 49/Market Risk

7-7. A money markets desk holds a floating-rate note with an eight-year maturity. The interest rate is floating at three-month LIBOR rate, reset quarterly. The next reset is in one week. What is the approximate duration of the floating-rate note?

a) 8 years
b) 4 years
c) 3 months
d) 1 week

7.4 Spot and Forward Rates

In addition to the cash flows, we also need detailed information on the term structure of interest rates to value fixed-income securities and their derivatives. This information is provided by **spot rates**, which are zero-coupon investment rates that start at the current time. From spot rates, we can infer **forward rates**, which are rates that start at a future date. Both are essential building blocks for the pricing of bonds.

Consider for instance a one-year rate that starts in one year. This forward rate is defined as $F_{1,2}$ and can be inferred from the one-year and two-year spot rates, R_1

and R_2. The forward rate is the break-even future rate that equalizes the return on investments of different maturities. An investor has the choice to lock in a 2-year investment at the 2-year rate, or to invest for a term of one year and roll over at the 1-to-2 year forward rate.

The two portfolios will have the same payoff when the future rate $F_{1,2}$ is such that

$$(1 + R_2)^2 = (1 + R_1)(1 + F_{1,2}) \tag{7.12}$$

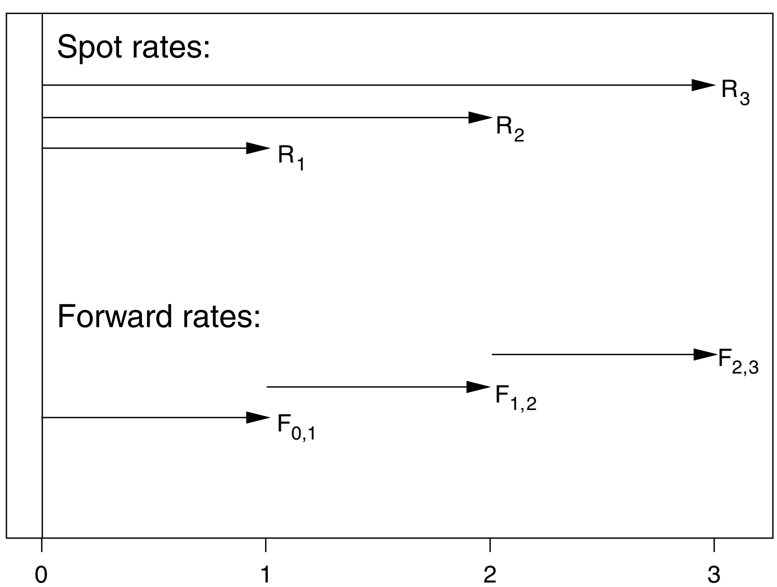
For instance, if $R_1 = 4.00\%$ and $R_2 = 4.62\%$, we have $F_{1,2} = 5.24\%$.

More generally, the T-period spot rate can be written as a geometric average of the spot and forward rates

$$(1 + R_T)^T = (1 + R_1)(1 + F_{1,2})...(1 + F_{T-1,T}) \tag{7.13}$$

where $F_{i,i+1}$ is the forward rate of interest prevailing now (at time t) over a horizon of i to $i + 1$. Table 7-4 displays a sequence of spot rates, forward rates, and par yields, using annual compounding. The first three years of this sequence are displayed in Figure 7-2.

FIGURE 7-2 Spot and Forward Rates



Forward rates allow us to project future cash flows that depend on future rates. The $F_{1,2}$ forward rate, for example, can be taken as the market's expectation of the second coupon payment on an FRN with annual payments and resets. We will also show later that positions in forward rates can be taken easily with derivative instruments.

TABLE 7-4 Spot, Forward Rates and Par Yields

Maturity (Year) i	Spot Rate R_i	Forward Rate $F_{i-1,i}$	Par Yield y_i	Discount Function $D(t_i)$
1	4.000	4.000	4.000	0.9615
2	4.618	5.240	4.604	0.9136
3	5.192	6.350	5.153	0.8591
4	5.716	7.303	5.640	0.8006
5	6.112	7.712	6.000	0.7433
6	6.396	7.830	6.254	0.6893
7	6.621	7.980	6.451	0.6383
8	6.808	8.130	6.611	0.5903
9	6.970	8.270	6.745	0.5452
10	7.112	8.400	6.860	0.5030

Forward rates have to be positive, otherwise there would be an arbitrage opportunity. We abstract from transaction costs and assume we can invest and borrow at the same rate. For instance, $R_1 = 11.00\%$ and $R_2 = 4.62\%$ gives $F_{1,2} = -1.4\%$. This means that $(1 + R_1) = 1.11$ is greater than $(1 + R_2)^2 = 1.094534$. To take advantage of this discrepancy, we could borrow \$1 million for two years and invest it for one year. After the first year, the proceeds are kept in cash, or under the proverbial mattress, for the second period. The investment gives \$1,110,000 and we have to pay back \$1,094,534 only. This would create a profit of \$15,466 out of thin air, which is highly unlikely in practice. Interest rates must be positive for the same reason.

The forward rate can be interpreted as a measure of the slope of the term structure. We can, for instance, expand both sides of Equation (7.12). After neglecting cross-product terms, we have

$$F_{1,2} \approx R_2 + (R_2 - R_1) \tag{7.14}$$

Thus, with an upward sloping term structure, R_2 is above R_1, and $F_{1,2}$ will also be above R_2.

We can also show that in this situation, the spot rate curve is above the par yield curve. Consider a bond with 2 payments. The 2-year par yield y_2 is implicitly defined from:

$$P = \frac{cF}{(1+y_2)} + \frac{(cF+F)}{(1+y_2)^2} = \frac{cF}{(1+R_1)} + \frac{(cF+F)}{(1+R_2)^2}$$

where P is set to par $P = F$. The par yield can be viewed as a weighted average of spot rates. In an upward-sloping environment, par yield curves involve coupons that are discounted at shorter and thus lower rates than the final payment. As a result, the par yield curve lies below the spot rate curve.

For a formal proof, consider a 2-period par bond with a face value of \$1 and coupon of y_2. We can write the price of this bond as

$$\begin{aligned}
1 &= y_2/(1+R_1) + (1+y_2)/(1+R_2)^2 \\
(1+R_2)^2 &= y_2(1+R_2)^2/(1+R_1) + (1+y_2) \\
(1+R_2)^2 &= y_2(1+F_{1,2}) + (1+y_2) \\
2R_2 + R_2^2 &= y_2(1+F_{1,2}) + y_2 \\
y_2 &= R_2(2+R_2)/(2+F_{1,2})
\end{aligned}$$

In an upward-sloping environment, $F_{1,2} > R_2$ and thus $y_2 < R_2$.

When the spot rate curve is flat, the spot curve is identical to the par yield curve and to the forward curve. In general, the curves differ. Figure 7-3a displays the case of an upward sloping term structure. It shows the yield curve is below the spot curve while the forward curve is above the spot curve. With a downward sloping term structure, as shown in Figure 7-3b, the yield curve is above the spot curve, which is above the forward curve.

Example 7-8: FRM Exam 1998—Ques:wtion 39/Capital Markets

7-8. Which of the following statements about yield curve arbitrage is *true*?

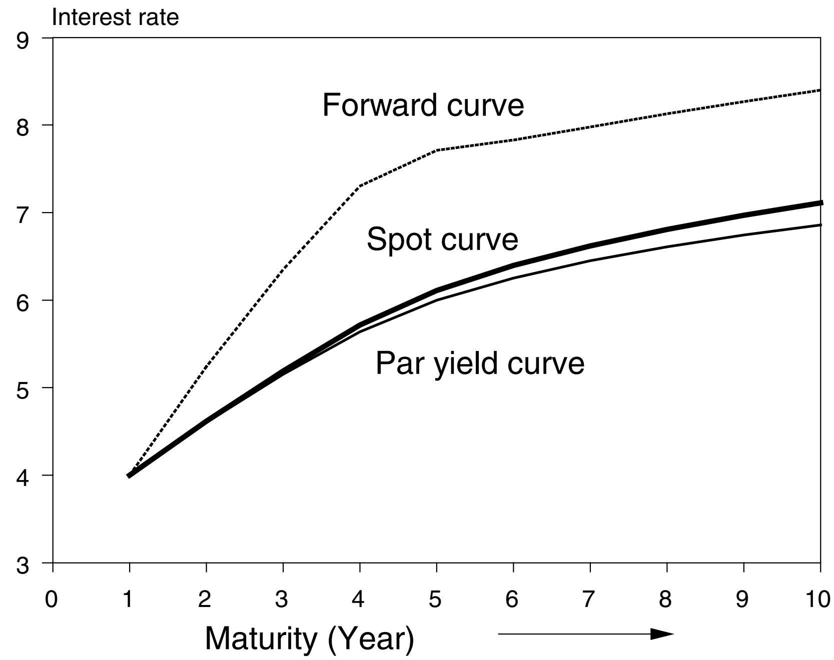
a) No-arbitrage conditions require that the zero-coupon yield curve is either upward sloping or downward sloping.

b) It is a violation of the no-arbitrage condition if the one-year interest rate is 10% or more, higher than the 10-year rate.

c) As long as all discount factors are less than one but greater than zero, the curve is arbitrage free.

d) The no-arbitrage condition requires all forward rates be nonnegative.

FIGURE 7-3a Upward-Sloping Term Structure



Example 7-9: FRM Exam 1997—Ques:wtion 1/Quantitative Techniques

7-9. Suppose a risk manager has made the mistake of valuing a zero-coupon bond using a swap (par) rate rather than a zero-coupon rate. Assume the par curve is upward sloping. The risk manager is therefore

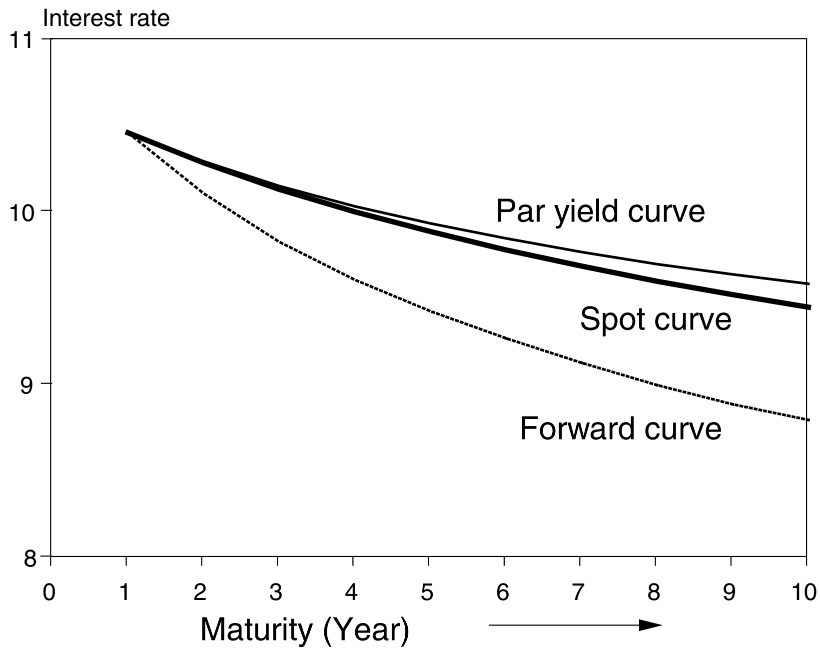
a) Indifferent to the rate used
b) Over-estimating the value of the bond
c) Under-estimating the value of the bond
d) Lacking sufficient information

Example 7-10: FRM Exam 1999—Ques:wtion 1/Quant. Analysis

7-10. Suppose that the yield curve is upward sloping. Which of the following statements is *true*?

a) The forward rate yield curve is above the zero-coupon yield curve, which is above the coupon-bearing bond yield curve.
b) The forward rate yield curve is above the coupon-bearing bond yield curve, which is above the zero-coupon yield curve.
c) The coupon-bearing bond yield curve is above the zero-coupon yield curve, which is above the forward rate yield curve.
d) The coupon-bearing bond yield curve is above the forward rate yield curve, which is above the zero-coupon yield curve.

FIGURE 7-3b Downward-Sloping Term Structure



7.5 Mortgage-Backed Securities

7.5.1 Description

Mortgage-backed securities represent claims on repackaged mortgage loans. Their basic cash-flow patterns start from an annuity, where the homeowner makes a monthly fixed payment that covers principal and interest.

Whereas mortgage loans are subject to credit risk, due to the possibility of default by the homeowner, most traded securities have third-party guarantees against credit risk. For instance, MBSs issued by Fannie Mae, an agency that is sponsored by the U.S. government, carry a guarantee of full interest and principal payment, even if the original borrower defaults.

Even so, MBSs are complex securities due to the uncertainty in their cash flows. Consider the traditional fixed-rate mortgage. Homeowners have the possibility of making early payments of principal. This represents a long position in an option. In some cases, these prepayments are random, for instance when the homeowner sells the home due to changing job or family conditions. In other cases, these prepayments are more predictable. When interest rates fall, prepayments increase as homeowners can refinance at a lower cost. This is similar to the rational early exercise of American call options.

Generally, these factors affect refinancing patterns:

- *Age of the loan*: Prepayment rates are generally low just after the mortgage loan has been issued and gradually increase over time until they reach a stable, or "seasoned," level. This effect is known as **seasoning**.
- *Spread between the mortgage rate and current rates*: Like a callable bond, there is a greater benefit to refinancing if it achieves a significant cost saving.
- *Refinancing incentives*: The smaller the costs of refinancing, the more likely homeowners will refinance often.
- *Previous path of interest rates*: Refinancing is more likely to occur if rates have been high in the past but recently dropped. In this scenario, past prepayments have been low but should rise sharply. In contrast, if rates are low but have been so for a while, most of the principal will already have been prepaid. This path dependence is usually referred to as **burnout**.
- *Level of mortgage rates*: Lower rates increase affordability and turnover.
- *Economic activity*: An economic environment where more workers change job location creates greater job turnover, which is more likely to lead to prepayments.
- *Seasonal effects*: There is typically more home-buying in the Spring, leading to increased prepayments in early Fall.

The prepayment rate is summarized into what is called the **conditional prepayment rate (CPR)**, which is expressed in annual terms. This number can be translated into a monthly number, known as the **single monthly mortality (SMM) Rate** using the adjustment:

$$(1 - \text{SMM})^{12} = (1 - \text{CPR}) \tag{7.15}$$

For instance, if CPR = 6% annually, the monthly proportion of principal paid early will be SMM $= 1 - (1 - 0.06)^{1/12} = 0.005143$, or 0.514% monthly. For a loan with a beginning monthly balance (BMB) of BMB = \$50,525 and a scheduled principal payment of SP = \$67, the prepayment will be $0.005143 \times (\$50{,}525 - \$67) = \$260$.

To price the MBS, the portfolio manager should describe the schedule of prepayments during the remaining life of the bond. This depends on many factors, including the age of the loan.

Prepayments can be described using an industry standard, known as the **Public Securities Association (PSA)** prepayment model. The PSA model assumes a CPR of

0.2% for the first month, going up by 0.2% per month for the next 30 months, until 6% thereafter. Formally, this is:

$$\text{CPR} = \text{Min}[6\% \times (t/30), 6\%] \qquad (7.16)$$

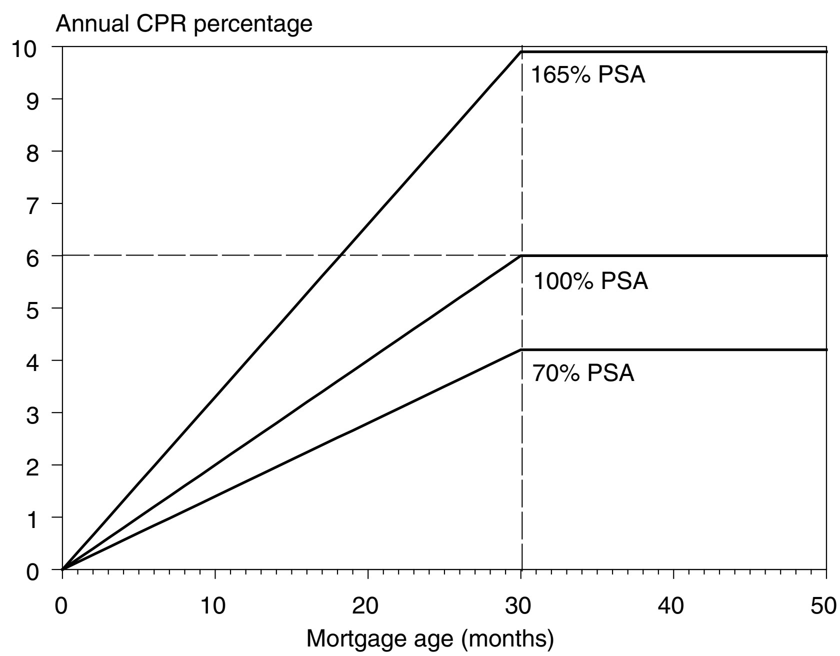
This pattern is described in Figure 7-4 as the 100% PSA speed. By convention, prepayment patterns are expressed as a percentage of the PSA speed, for example 165% for a faster pattern and 70% PSA for a slower pattern.

Example:

Computing the CPR Consider an MBS issued 20 months ago with a speed of 150% PSA. What are the CPR and SMM?

The PSA speed is $\text{Min}[6\% \times (20/30), 6\%] = 0.04$. Applying the 150 factor, we have $\text{CPR} = 150\% \times 0.04 = 0.06$. This implies $\text{SMM} = 0.514\%$.

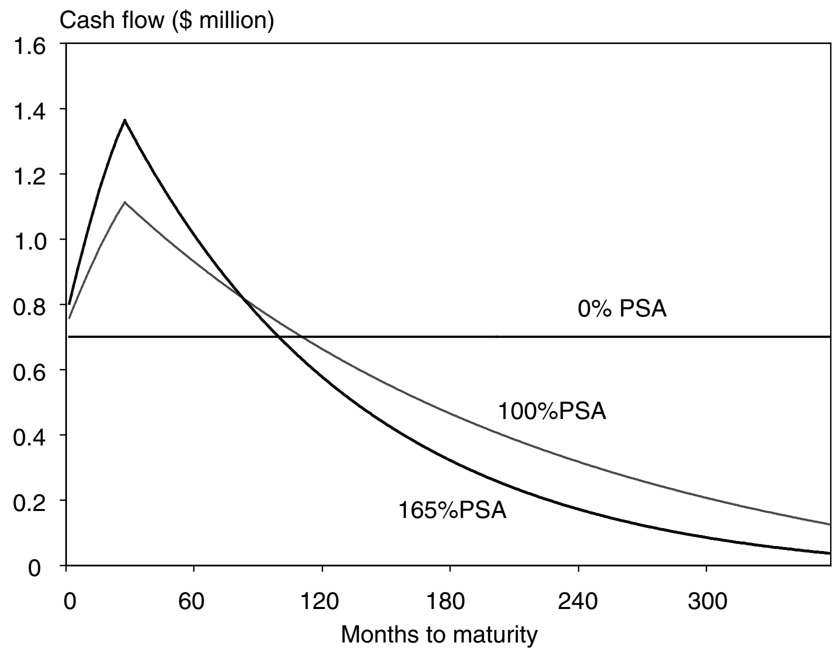
FIGURE 7-4 Prepayment Pattern



The next step is to project cash flows based on the prepayment speed pattern. Figure 7-5 displays cash-flow patterns for a 30-year MBS with a face amount of $100 million, 7.5% interest rate, and three months into its life. The horizontal, "0% PSA" line, describes the annuity pattern without any prepayment. The "100% PSA" line describes an increasing pattern of cash flows, peaking in 27 months and decreasing thereafter. This point corresponds to the stabilization of the CPR at 6%. This pattern is more marked for the "165% PSA" line, which assumes a faster prepayment speed.

Early prepayments create less payments later, which explains why the 100% PSA line is initially greater than the 0% line, then lower later as the principal has been paid off more quickly.

FIGURE 7-5 Cash Flows on MBS for Various PSA



Example 7-11: FRM Exam 1999—Ques:wtion 51/Capital Markets 7-11. Suppose the annual prepayment rate CPR for a mortgage-backed security is 6%. What is the corresponding single-monthly mortality rate SMM? a) 0.514% b) 0.334% c) 0.5% d) 1.355%

Example 7-12: FRM Exam 1998—Ques:wtion 14/Capital Markets

7-12. In analyzing the monthly prepayment risk of Mortgage-backed securities, an annual prepayment rate (CPR) is converted into a monthly prepayment rate (SMM). Which of the following formulas should be used for the conversion?

a) $SMM = (1 - CPR)^{1/12}$
b) $SMM = 1 - (1 - CPR)^{1/12}$
c) $SMM = 1 - (CPR)^{1/12}$
d) $SMM = 1 + (1 - CPR)^{1/12}$

Example 7-13: FRM Exam 1999—Ques:wtion 87/Market Risk

7-13. A CMO bond class with a duration of 50 means that

a) It has a discounted cash flow weighted average life of 50 years.
b) For a 100 bp change in yield, the bond's price will change by roughly 50%.
c) For a 1 bp change in yield, the bond's price will change by roughly 5%.
d) None of the above is correct.

Example 7-14: FRM Exam 1998—Ques:wtion 18/Capital Markets

7-14. Which of the following risks are common to both mortgage-backed securities and emerging market Brady bonds?

I. Interest rate risk
II. Prepayment risk
III. Default risk
IV. Political risk

a) I only
b) II and III only
c) I and III only
d) III and IV only

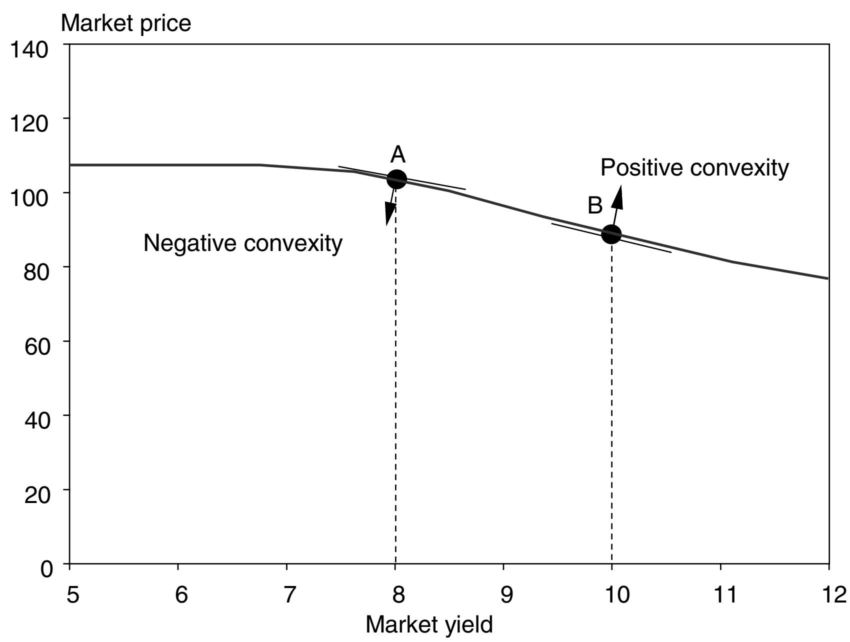
7.5.2 Prepayment Risk

Like other bonds, mortgage-backed securities are subject to market risk, due to fluctuations in interest rates. They are also, however, subject to **prepayment risk**, which is the risk that the principal will be repaid early.

Consider for instance an 8% MBS, which is illustrated in Figure 7-6. If rates drop to 6%, homeowners will rationally prepay early to refinance the loan. Because the average life of the loan is shortened, this is called **contraction risk**. Conversely, if rates increase to 10%, homeowners will be less likely to refinance early, and prepayments will slow down. Because the average life of the loan is extended, this is called **extension risk**.

As shown in Figure 7-6, these features create "negative convexity", which reflects the fact that the investor in an MBS is short an option. At point B, interest rates are very high and there is little likelihood that the homeowner will refinance early. The option is nearly worthless and the MBS behaves like a regular bond, with positive convexity. At point A, the option pattern starts to affect the value of the MBS. Shorting an option creates negative gamma, or convexity.

FIGURE 7-6 Negative Convexity of MBSs



This changing cash-flow pattern makes standard duration measures unreliable. Instead, sensitivity measures are computed using **effective duration** and **effective convexity**, as explained in Chapter 1. The measures are based on the estimated price of the MBS for three yield values, making suitable assumptions about how changes in rates should affect prepayments.

Table 7-5 shows an example. In each case, we consider an upmove and downmove of 25bp. In the first, "unchanged" panel, the PSA speed is assumed to be constant at 165 PSA. In the second, "changed" panel, we assume a higher PSA speed if rates drop and lower speed if rates increase. When rates drop, the MBS value goes up but not as much as if the prepayment speed had not changed, which reflects contraction risk. When rates increase, the MBS value drops by more than if the prepayment speed had not changed, which reflects extension risk.

TABLE 7-5 Computing Effective Duration and Convexity

	Initial	Unchanged PSA		Changed PSA	
Yield	7.50%	+25bp	−25bp	+25bp	−25bp
PSA		165PSA	165PSA	150PSA	200PSA
Price	100.125	98.75	101.50	98.7188	101.3438
Duration		5.49y		5.24y	
Convexity		0		−299	

As we have seen in Chapter 1, **effective duration** is measured as

$$D^E = \frac{P(y_0 - \Delta y) - P(y_0 + \Delta y)}{(2P_0 \Delta y)} \tag{7.17}$$

Effective convexity is measured as

$$C^E = \left[\frac{P(y_0 - \Delta y) - P_0}{(P_0 \Delta y)} - \frac{P_0 - P(y_0 + \Delta y)}{(P_0 \Delta y)} \right] \Big/ \Delta y \tag{7.18}$$

In the first, "unchanged" panel, the effective duration is 5.49 years and convexity close to zero. In the second, "changed" panel, the effective duration is 5.24 years and convexity is negative, as expected, and quite large.

> **Key concept:**
> Mortgage-backed securities have negative convexity, which reflects the short position in an option granted to the homeowner to repay early. This creates extension risk when rates increase or contraction risk when rates fall.

The option feature in MBSs increases their yield. To ascertain whether the securities represent good value, portfolio managers need to model the option component. The approach most commonly used is the **option-adjusted spread** (OAS).

Starting from the **static spread**, the OAS method involves running simulations of various interest rate scenarios and prepayments to establish the option cost. The OAS is then

$$\text{OAS} = \text{Static spread} - \text{Option cost} \tag{7.19}$$

which represents the net richness or cheapness of the MBS. Within the same risk class, a security trading at a high OAS is preferable to others.

The OAS is more stable over time than the spread, because the latter is affected by the option component. This explains why during market rallies (i.e., when long-term Treasury yields fall) yield spreads on current coupon mortgages often widen. These mortgages are more likely to be prepaid early, which makes them less attractive. Their option cost increases, pushing up the yield spread.

Example 7-15: FRM Exam 1999—Ques:wtion 44/Capital Markets

7-15. The following are reasons that a prepayment model will not accurately predict future mortgage prepayments. Which of these will have the greatest effect on the convexity of mortgage pass-throughs?

a) Refinancing incentive
b) Seasoning
c) Refinancing burnout
d) Seasonality

Example 7-16: FRM Exam 1999—Ques:wtion 40/Capital Markets

7-16. Which attribute of a bond is *not* a reason for using effective duration instead of modified duration?

a) Its life may be uncertain.
b) Its cash flow may be uncertain.
c) Its price volatility tends to decline as maturity approaches.
d) It may include changes in adjustable rate coupons with caps or floors.

Example 7-17: FRM Exam 2001—Ques:wtion 95

7-17. The option-adjusted duration of a callable bond will be close to the duration of a similar non-callable bond when the

a) Bond trades above the call price.
b) Bond has a high volatility.
c) Bond trades much lower than the call price.
d) Bond trades above parity.

7.5.3 Financial Engineering and CMOs

The MBS market has grown enormously in the last twenty years in the United States and is growing fast in other markets. MBSs allow capital to flow from investors to borrowers, principally homeowners, in an efficient fashion.

One major drawback of MBSs, however, is their negative convexity. This makes it difficult for investors, such as pension funds, to invest in MBSs because the life of these instruments is uncertain, making it more difficult to match the duration of pension assets to the horizon of pension liabilities.

In response, the finance industry has developed new classes of securities based on MBSs with more appealing characteristics. These are the **collateralized mortgage obligations (CMOs)**, which are new securities that redirect the cash flows of an MBS pool to various segments.

Figure 7-7 illustrates the process. The cash flows from the MBS pool go into a **special-purpose vehicle (SPV)**, which is a legal entity that issues different claims, or **tranches** with various characteristics, like slices in a pie. These are structured so that the cash flow from the first tranche, for instance, is more predictable than the original cash flows. The uncertainty is then pushed into the other tranches.

Starting from an MBS pool, financial engineering creates securities that are better tailored to investors' needs. It is important to realize, however, that the cash flows and risks are fully preserved. They are only redistributed across tranches. Whatever transformation is brought about, the resulting package must obey basic laws of conservation for the underlying securities and package of resulting securities. We must have the same cash flows at each point in time, except for fees paid to the issuer. As a result, we must have

(1) The same market value
(2) The same risk profile

As Lavoisier, the French chemist who was executed during the French revolution said,

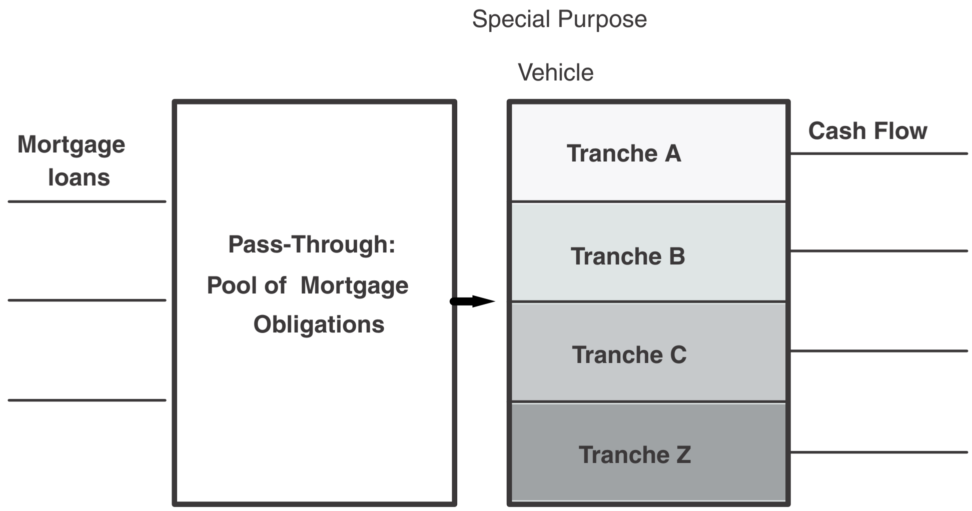
Rien ne se perd, rien ne se crée (nothing is lost, nothing is created).

In particular, the weighted duration and convexity of the portfolio of tranches must add up to the original duration and convexity. If Tranche A has less convexity than the underlying securities, the other tranches must have more convexity.

Similar structures apply to **collateralized bond obligations** (CBOs), **collateralized loan obligations** (CLOs), **collateralized debt obligations** (CDOs), which are a set of tradable bonds backed by bonds, loans, or debt (bonds and loans), respectively. These structures rearrange credit risk and will be explained in more detail in a later chapter.

As an example of a two-tranche structure, consider a claim on a regular 5-year, 6% coupon $100 million note. This can be split up into a floater, that pays LIBOR on a

FIGURE 7-7 Creating CMO Tranches



notional of \$50 million, and an inverse floater, that pays 12% − LIBOR on a notional of \$50 million. The coupon on the inverse floater cannot go below zero: Coupon = Max(12%−LIBOR, 0). This imposes another condition on the floater Coupon = Min(LIBOR, 12%).

We verify that the cash flows exactly add up to the original. For each coupon payment, we have, in millions

$$\$50 \times \text{LIBOR} + \$50 \times (12\% - \text{LIBOR}) = \$100 \times 6\% = \$6.$$

At maturity, the total payments of twice \$50 million add up to \$100 million.

We can also decompose the risk of the original structure into that of the two components. Assume a flat term structure for the original note. Say the duration of the original 5-year note is $D = 4.5$ years. The portfolio dollar duration is:

$$\$50,000,000 \times D_F + \$50,000,000 \times D_{\text{IF}} = \$100,000,000 \times D$$

Just before a reset, the duration of the floater is close to zero $D_F = 0$. Hence, the duration of the inverse floater must be $D_{\text{IF}} = (\$100,000,000/\$50,000,000) \times D = 2 \times D$, or twice that of the original note. Note that the duration is much greater than the maturity of the note. This illustrates the point that duration is an interest rate sensitivity measure. When cash flows are uncertain, duration is not necessarily related to maturity. Intuitively, the first tranche, the floater, has zero risk so that all of the risk must be absorbed into the second tranche, which must have a duration of 9 years. The total risk of the portfolio is conserved.

This analysis can be easily extended to inverse floaters with greater leverage. Suppose the coupon the coupon is tied to twice LIBOR, for example 18% − 2 × LIBOR. The principal must be allocated in the amount x, in millions, for the floater and $100 - x$ for the inverse floater so that the coupon payment is preserved. We set

$$x \times LIBOR + (100 - x) \times (18\% - 2 \times LIBOR) = \$6$$

$$[x - (100 - x)2] \times LIBOR + (100 - x) \times 18\% = \$6$$

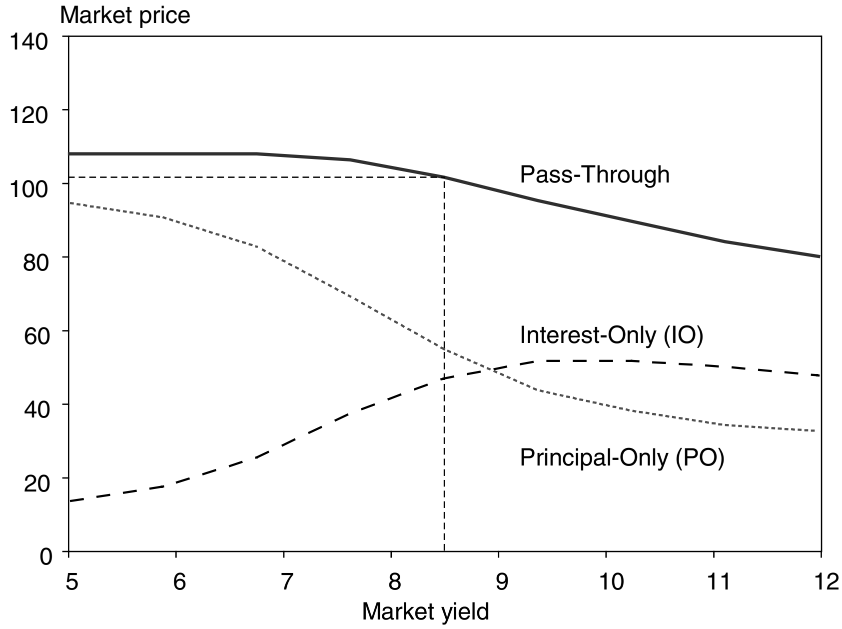
This can only be satisfied if $3x - 200 = 0$, or if $x = \$66.67$ million. Thus, two-thirds of the notional must be allocated to the floater, and one-third to the inverse floater. The inverse floater now has three times the duration of the original note.

Key concept:
Collateralized mortgage obligations (CMOs) rearrange the total cash flows, total value, and total risk of the underlying securities. At all times, the total cash flows, value, and risk of the tranches must equal those of the collateral. If some tranches are less risky than the collateral, others must be more risky.

When the collateral is a mortgage-backed security, CMOs can be defined by prioritizing the payment of principal into different tranches. This is defined as **sequential-pay tranches**. Tranche A, for instance, will receive the principal payment on the whole underlying mortgages first. This creates more certainty in the cash flows accruing to Tranche A, which makes it more appealing to some investors. Of course, this is to the detriment of others. After principal payments to Tranche A are exhausted, Tranche B then receives all principal payments on the underlying MBS. And so on for other tranches.

Another popular construction is the IO/PO structure. An **interest-only (IO)** tranche receives only the interest payments on the underlying MBS. The **principal-only (PO)** tranche then receives only the principal payments. As before, the market value of the IO and PO must exactly add to that of the MBS. Figure 7-8 describes the price behavior of the IO and PO. Note that the vertical addition of the two components always equals the value of the MBS.

FIGURE 7-8 Creating an IO and PO from an MBS



To analyze the PO, it is useful to note that the sum of all principal payments is constant (because we have no default risk). Only the timing is uncertain. In contrast, the sum of all interest payments depends on the timing of principal payments. Later principal payments create greater total interest payments.

If interest rates fall, principal payments will come early, which reflects contraction risk. Because the principal is paid earlier and the discount rate decreases, the PO should appreciate sharply in value. On the other hand, the faster prepayments mean less interest payments over the life of the MBS, which is unfavorable to the IO. the IO should depreciate.

Conversely, if interest rates rise, slower prepayments will slow down, which reflects extension risk. Because the principal is paid later and the discount rate increases, the PO should lose value. On the other hand, the slower prepayments mean more interest payments over the life of the MBS, which is favorable to the IO. The IO appreciates in value, up to the point where the higher discount rate effect dominates. Thus, IOs are bullish securities with negative duration, as shown in Figure 7-8.

Example 7-18: FRM Exam 2000—Ques:wtion 13/Capital Markets

7-18. A CLO is generally

a) A set of loans that can be traded individually in the market
b) A pass-through
c) A set of bonds backed by a loan portfolio
d) None of the above

Example 7-19: FRM Exam 2000—Ques:wtion 121/Quant. Analysis

7-19. Which one of the following long positions is more exposed to an increase in interest rates?

a) A Treasury Bill
b) 10-year fixed-coupon bond
c) 10-year floater
d) 10-year reverse floater

Example 7-20: FRM Exam 1998—Ques:wtion 32/Capital Markets

7-20. A 10-year reverse floater pays a semiannual coupon of 8% minus 6-month LIBOR. Assume the yield curve is 8% flat, the current 10-year note has a duration of 7 years, and the interest rate on the note was just reset. What is the duration of the note?

a) 6 months
b) Shorter than 7 years
c) Longer than 7 years
d) 7 years

Example 7-21: FRM Exam 1999—Ques:wtion 79/Market Risk

7-21. Suppose that the coupon and the modified duration of a 10-year bond priced to par are 6.0% and 7.5, respectively. What is the approximate modified duration of a 10-year inverse floater priced to par with a coupon of $18\% - 2 \times$ LIBOR?

a) 7.5
b) 15.0
c) 22.5
d) 0.0

Example 7-22: FRM Exam 2000—Ques:wtion 3/Capital Markets

7-22. How would you describe the typical price behavior of a low premium mortgage pass-through security?

a) It is similar to a U.S. Treasury bond.

b) It is similar to a plain vanilla corporate bond.

c) When interest rates fall, its price increase would exceed that of a comparable duration U.S. Treasury bond.

d) When interest rates fall, its price increase would lag that of a comparable duration U.S. Treasury bond.

7.6 Answers to Chapter Examples

Example 7-1: FRM Exam 1998—Ques:wtion 3/Capital Markets

b) As interest rates increase, the coupon decreases. In addition, the discount factor increases. Hence, the value of the note must decrease even more than a regular fixed-coupon bond.

Example 7-2: FRM Exam 2000—Ques:wtion 9/Capital Markets

d) With a callable bond the issuer has the option to call the bond early if its price would otherwise go up. Hence, the investor is short an option. A long position in a callable bond is equivalent to a long position in a noncallable bond plus a short position in a call option.

Example 7-3: FRM Exam 1998—Ques:wtion 13/Capital Markets

a) $DR = (\text{Face} - \text{Price})/\text{Face} \times (360/t) = (\$100{,}000 - \$97{,}569)/\$100{,}000 \times (360/100) = 8.75\%$. Note that the yield is 9.09%, which is higher.

Example 7-4: FRM Exam 1999—Ques:wtion 53/Capital Markets

b) Using Equation (7.8), we have $D^* = -(dP/P)/dy = [(135.85 - 132.99)/134.41]/[0.001 \times 2] = 10.63$. This is also a measure of effective duration.

Example 7-5: FRM Exam 1998—Ques:wtion 31/Capital Markets

c) Because this is a zero-coupon bond, it will always trade below par, and the call should never be exercised. Hence its duration is the maturity, 10 years.

Example 7-6: FRM Exam 1999—Ques:wtion 91/Market Risk
a) By Equation (7.8).

Example 7-7: FRM Exam 1997—Ques:wtion 49/Market Risk
d) Duration is not related to maturity when coupons are not fixed over the life of the investment. We know that at the next reset, the coupon on the FRN will be set at the prevailing rate. Hence, the market value of the note will be equal to par at that time. The duration or price risk is only related to the time to the next reset, which is 1 week here.

Example 7-8: FRM Exam 1998—Ques:wtion 39/Capital Markets
d) Discount factors need to be below one, as interest rates need to be positive, but in addition forward rates also need to be positive.

Example 7-9: FRM Exam 1997—Ques:wtion 1/Quantitative Techniques
b) If the par curve is rising, it must be below the spot curve. As a result, the discounting will use rates that are too low, thereby overestimating the bond value.

Example 7-10: FRM Exam 1999—Ques:wtion 1/Quant. Analysis
a) See Figures 7-3a an 7-3b. The coupon yield curve is an average of the spot, zero-coupon curve, hence has to lie below the spot curve when it is upward-sloping. The forward curve can be interpreted as the spot curve plus the slope of the spot curve. If the latter is upward sloping, the forward curve has to be above the spot curve.

Example 7-11: FRM Exam 1999—Ques:wtion 51/Capital Markets
a) Using $(1 - 6\%) = (1 - \text{SMM})^{12}$, we find SMM = 0.51%.

Example 7-12: FRM Exam 1998—Ques:wtion 14/Capital Markets
b) As $(1 - \text{SMM})^{12} = (1 - \text{CPR})$.

Example 7-13: FRM Exam 1999—Ques:wtion 87/Market Risk
b) Discounted cash flows are not useful for CMOs because they are uncertain. So, duration is a measure of interest rate sensitivity. We have $(dP/P) = D^{*}dy = 50 \times 1\% = 50\%$.

Example 7-14: FRM Exam 1998—Ques:wtion 18/Capital Markets
c) MBSs are subject to I, II, III (either homeowner or agency default). Brady bonds are subject to I, III, IV. Neither is exposed to currency risk.

Example 7-15: FRM Exam 1999—Ques:wtion 44/Capital Markets
a) The question is which factor has the greatest effect on the interest rate convexity, or increases the prepayment rate when rates fall . Seasoning and seasonality are not related to interest rates. Burnout lowers the prepayment rate. So, refinancing incentives is the remaining factor that affects most the option feature.

Example 7-16: FRM Exam 1999—Ques:wtion 40/Capital Markets
c) Effective convexity is useful when the cash flows are uncertain. All attributes are reasons for using effective convexity, except that the price risk decreases as maturity gets close. This holds for a regular coupon-paying bond anyway.

Example 7-17: FRM Exam 2001—Ques:wtion 95
c) This question is applicable to MBSs as well as callable bonds. From Figure 7-6, we see that the callable bond behaves like a straight bond when market yields are high, or when the bond price is low. So, (c) is correct and (a) and (d) must be wrong.

Example 7-18: FRM Exam 2000—Ques:wtion 13/Capital Markets
c) Like a CMO, a CLO represents a set of tradable securities backed by some collateral, in this case a loan portfolio.

Example 7-19: FRM Exam 2000—Ques:wtion 121/Quant. Analysis
d) Risk is measured by duration. Treasury bills and floaters have very small duration. A 10-year fixed-rate note will have a duration of perhaps 8 years. In contrast, an inverse (or reverse) floater has twice the duration.

Example 7-20: FRM Exam 1998—Ques:wtion 32/Capital Markets
c) The duration is normally about 14 years. Note that if the current coupon is zero, the inverse floater behaves like a zero-coupon bond with a duration of 10 years.

Example 7-21: FRM Exam 1999—Ques:wtion 79/Market Risk

c) Following the same reasoning as above, we must divide the fixed-rate bonds into 2/3 FRN and 1/3 inverse floater. This will ensure that the inverse floater payment is related to twice LIBOR. As a result, the duration of the inverse floater must be 3 times that of the bond.

Example 7-22: FRM Exam 2000—Ques:wtion 3/Capital Markets

d) MBSs are unlike regular bonds, Treasuries, or corporates, because of their negative convexity. When rates fall, homeowners prepay early, which means that the price appreciation is less than that of comparable duration regular bonds.

Chapter 8

Fixed-Income Derivatives

This chapter turns to the analysis of fixed-income derivatives. These are instruments whose value derives from a bond price, interest rate, or other bond market variable. As discussed in Chapter 5, fixed-income derivatives account for the largest proportion of the global derivatives markets. Understanding fixed-income derivatives is also important because many fixed-income securities have derivative-like characteristics.

This chapter focuses on the use of fixed-income derivatives, as well as their pricing. Pricing involves finding the fair market value of the contract. For risk management purposes, however, we also need to assess the range of possible movements in contract values. This will be further examined in the chapters on market risk and in Chapter 21, when discussing credit exposure.

Section 8.1 discusses interest rate forward contracts, also known as forward rate agreements. Section 8.2 then turns to the discussion of interest rate futures, covering Eurodollar and Treasury Bond futures. Although these products are dollar-based, similar products exist on other capital markets. Swaps are analyzed in Section 8.3. Swaps are very important instruments due to their widespread use. Finally, interest rate options are covered in Section 8.4, including caps and floors, swaptions, and exchange-traded options.[1]

8.1 Forward Contracts

Forward Rate Agreements (FRAs) are over-the-counter financial contracts that allow counterparties to lock in an interest rate starting at a future time. The buyers of an FRA lock in a borrowing rate, the sellers lock in a lending rate. In other words, the "long" benefits from an increase in rates and the short benefits from a fall in rates.

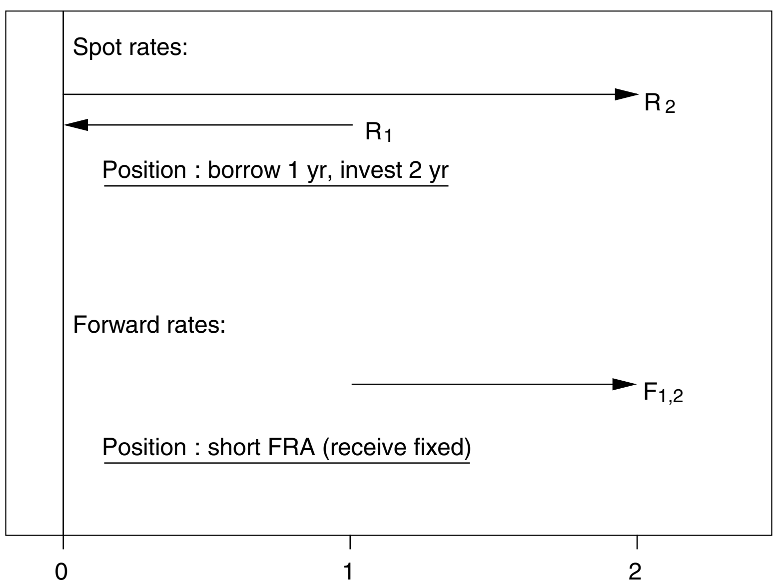
[1] The reader should be aware that this chapter is very technical.

As an example, consider an FRA that settles in one month on 3-month LIBOR. Such FRA is called 1×4. The first number corresponds to the first settlement date, the second to the time to final maturity. Call τ the period to which LIBOR applies, 3 months in this case. On the settlement date, in one month, the payment to the long involves the net value of the difference between the spot rate S_T (the prevailing 3-month LIBOR rate) and of the locked-in forward rate F The payoff is $S_T - F$, as with other forward contracts, present valued to the first settlement date. This gives

$$V_T = (S_T - F) \times \tau \times \text{Notional} \times \text{PV}(\$1) \tag{8.1}$$

where $\text{PV}(\$1) = \$1/(1+S_T\tau)$. The amount is cash settled. Figure 8-1 shows how a short position in an FRA, which locks in an investing rate, is equivalent to borrowing short-term to finance a long-term investment. In both cases, there is no up-front investment. The duration is equal to the difference between the durations of the two legs. From Equation (8.1), the duration is τ and dollar duration $\text{DD} = \tau \times \text{Notional} \times \text{PV}(\$1)$.

FIGURE 8-1 Decompositions of an FRA



Example: Using an FRA

A company will receive $100 million in 6 months to be invested for a 6-month period. The Treasurer is afraid rates will fall, in which case the investment return will be lower. The company could sell a 6 × 12 FRA on \$100 million at the rate of $F = 5\%$. This locks in an investment rate of 5% starting in six months.

When the FRA expires in 6 months, assume that the prevailing 6-month spot rate is $S_T = 4\%$. This will lower the investment return on the cash received, which is the scenario the Treasurer feared. Using Equation (8.1), the FRA has a payoff of $V_T = -(4\% - 5\%) \times (6/12) \times \$100$ million $= \$500{,}000$, which multiplied by the present value factor gives \$490,196. In effect, this payment offsets the lower return that the company would otherwise receive on a floating investment, guaranteeing a return equal to the forward rate.

This contract is also equivalent to borrowing the present value of \$100 million for 6 months and investing the proceeds for 12 months. Thus its duration is $D_{12} - D_6 = 12 - 6 = 6$ months.

Key concept:
A short FRA position is similar to a long position in a bond. Its duration is positive and equal to the difference between the two maturities.

Example 8-1: FRM Exam 2001—Question 70/Capital Markets

8-1. Consider the following 6 × 9 FRA. Assume the buyer of the FRA agrees to a contract rate of 6.35% on a notional amount of \$10 million. Calculate the settlement amount of the seller if the settlement rate is 6.85%. Assume a 30/360 day count basis.

a) $-12,500$
b) $-12,290$
c) $+12,500$
d) $+12,290$

Example 8-2: FRM Exam 2001—Question 73/Capital Markets

8-2. The following instruments are traded, on an ACT/360 basis: 3-month deposit (91 days), at 4.5%
3 × 6 FRA (92 days), at 4.6%
6 × 9 FRA (90 days), at 4.8%
9 × 12 FRA (92 days), at 6%
What is the 1-year interest rate on an ACT/360 basis?

a) 5.19%
b) 5.12%
c) 5.07%
d) 4.98%

Example 8-3: FRM Exam 1998—Question 54/Capital Markets

8-3. Roughly estimate the DV01 for a 2 × 5 CHF 100 million FRA in which a trader will pay fixed and receive floating rate.

a) CHF 1,700

b) CHF (1,700)

c) CHF 2,500

d) CHF (2,500)

8.2 Futures

Whereas FRAs are over-the-counter contracts, futures are traded on organized exchanges. We will cover the most important types of futures contracts, Eurodollar and T-bond futures.

8.2.1 Eurodollar Futures

Eurodollar futures are futures contracts tied to a forward LIBOR rate. Since their creation on the Chicago Mercantile Exchange, Eurodollar futures have spread to equivalent contracts such as Euribor futures (denominated in euros),[2] Euroswiss futures (denominated in Swiss francs), Euroyen futures (denominated in Japanese yen), and so on. These contracts are akin to FRAs involving 3-month forward rates starting on a wide range of dates, from near dates to ten years into the future.

The formula for calculating the price of one contract is

$$P_t = 10{,}000 \times [100 - 0.25(100 - \mathrm{FQ}_t)] = 10{,}000 \times [100 - 0.25 F_t] \qquad (8.2)$$

where FQ_t is the quoted Eurodollar futures price. This is quoted as 100.00 minus the interest rate F_t, expressed in percent, that is, $\mathrm{FQ}_t = 100 - F_t$. The 0.25 factor represents the 3-month maturity, or 0.25 years. For instance, if the market quotes $\mathrm{FQ}_t = 94.47$, the contract price is $P = 10{,}000[100 - 0.25 \times 5.53] = \$98{,}175$. At expiration, the contract price settles to

$$P_T = 10{,}000 \times [100 - 0.25 S_T] \qquad (8.3)$$

[2] Euribor futures are based on the European Bankers Federations' Euribor Offered Rate (EBF Euribor). The contracts differ from Euro LIBOR futures, which are based on the British Bankers' Association London Interbank Offer Rate (BBA LIBOR), but are much less active.

where S_T is the 3-month Eurodollar spot rate prevailing at T. Payments are cash settled.

As a result, F_t can be viewed as a 3-month forward rate that starts at the maturity of the futures contract. The formula for the contract price may look complicated but in fact is structured so that an increase in the interest rate leads to a decrease in the price of the contract, as is usual for fixed-income instruments. Also, since the change in the price is related to the interest rate by a factor of 0.25, this contract has a constant duration of 3 months. The DV01 is DV01 $= \$10,000 \times 0.25 \times 0.01 = \$25$.

Example: Using Eurodollar futures

As in the previous section, the Treasurer wants to hedge a future investment of \$100 million in 6 months for a 6-month period. He or she should sell Eurodollar futures to generate a gain if rates fall. If the futures contract trades at $\text{FQ}_t = 95.00$, the dollar value of the contract is $P = 10{,}000 \times [100 - 0.25(100 - 95)] = \$987,500$. The duration of the Eurodollar futures is three months; that of the company's investment is six months.

Using the ratio of dollar durations, the number of contracts to sell is

$$N = \frac{D_V V}{D_F P} = \frac{0.50 \times \$100,000,000}{0.25 \times \$987,500} = 202.53$$

Rounding, the Treasurer needs to sell 203 contracts.

Chapter 5 has explained that the pricing of forwards is similar to those of futures, except when the value of the futures contract is strongly correlated with the reinvestment rate. This is the case with Eurodollar futures.

Interest rate futures contracts are designed to move like a bond, that is, lose value when interest rates increase. The correlation is negative. This implies that when interest rates rise, the futures contract loses value and in addition funds have to be provided precisely when the borrowing cost or reinvestment rate is higher. Conversely when rates drop, the contract gains value and the profits can be withdrawn but are now reinvested at a lower rate. Relative to forward contracts, this marking-to-market feature is *disadvantageous* to long futures positions. This has to be offset by a lower value for the futures contract price. Given that $P_t = 10,000 \times [100 - 0.25 \times F_t]$, this implies a higher Eurodollar futures rate F_t.

The difference is called the **convexity adjustment** and can be described as[3]

$$\textbf{Futures Rate} = \textbf{Forward Rate} + (1/2)\sigma^2 t_1 t_2 \tag{8.4}$$

where σ is the volatility of the change in the short-term rate, t_1 is the time to maturity of the futures contract, and t_2 is the maturity of the rate underlying the futures contract.

Example: Convexity adjustment

Consider a 10-year Eurodollar contract, for which $t_1 = 10$, $t_2 = 10.25$. The maturity of the futures contract itself is 10 years and that of the underlying rate is 10 years plus three months.

Typically, $\sigma = 1\%$, so that the adjustment is $(1/2)0.01^2 \times 10 \times 10.25 = 0.51\%$. So, if the forward price is 6%, the equivalent futures rate would be 6.51%. Note that the effect is significant for long maturities only. Changing t_1 to one year and t_2 to 1.25, for instance, reduces the adjustment to 0.006%, which is negligible.

Example 8-4: FRM Exam 1998—Question 7/Capital Markets

8-4. What are the differences between forward rate agreements (FRAs) and Eurodollar Futures?

I. FRAs are traded on an exchange, whereas Eurodollar Futures are not.

II. FRAs have better liquidity than Eurodollar Futures.

III. FRAs have standard contract sizes, whereas Eurodollar Futures do not.

a) I only

b) I and II only

c) II and III only

d) None of the above

Example 8-5: FRM Exam 1998—Question 40/Capital Markets

8-5. Roughly, how many 3-month LIBOR Eurodollar Futures contracts are needed to hedge a long 100 million position in 1-year U.S. Treasury Bills?

a) Short 100

b) Long 4,000

c) Long 100

d) Short 400

[3] This formula is derived from the Ho-Lee model. See for instance Hull (2000), *Options, Futures, and Other Derivatives,* Upper Saddle River, NJ: Prentice-Hall.

Example 8-6: FRM Exam 2000—Question 7/Capital Markets

8-6. For assets that are strongly positively correlated with interest rates, which one of the following is *true*?

a) Long-dated forward contracts will have higher prices than long-dated futures contracts.

b) Long-dated futures contracts will have higher prices than long-dated forward contracts.

c) Long-dated forward and long-dated futures prices are always the same.

d) The "convexity effect" can be ignored for long-dated futures contracts on that asset.

8.2.2 T-bond Futures

T-bond futures are futures contracts tied to a pool of Treasury bonds that consists of all bonds with a remaining maturity greater than 15 years (and noncallable within 15 years). Similar contracts exist on shorter rates, including 2-, 5-, and 10-year Treasury notes. Treasury futures also exist in other markets, including Canada, the United Kingdom, Eurozone, and Japanese government bonds.

Futures contracts are quoted like T-bonds, for example 97-02, in percent plus thirty-seconds, with a notional of \$100,000. Thus the price of the contract would be $\$100{,}000 \times (97 + 2/32)/100 = \$97{,}062.50$. The next day, if yields go up and the quoted price falls to 96-0, the new price would be \$965,000, and the loss on a long position would be $P_2 - P_1 = -\$1{,}062.50$.

It is important to note that the T-bond futures contract is settled by physical delivery. To ensure interchangeability between the deliverable bonds, the futures contract uses a **conversion factor** (CF) for delivery. This factor multiplies the futures price for payment to the short and attempts to equalize the net cost of delivering the eligible bonds.

The conversion factor is needed due to the fact that bonds trade at widely different prices. High coupon bonds trade at a premium, low coupon bonds at a discount. Without this adjustment, the party with the short position (the"short") would always deliver the same, cheap bond and there would be little exchangeability between bonds. This exchangeability minimizes the possibility of market squeezes. A **squeeze** occurs when holders of the short position cannot acquire or borrow the securities required for delivery under the terms of the contract.

So, the "short" delivers a bond and receives the quoted futures price times a CF that is specific to the delivered bond (plus accrued interest). The "short" picks the bond that minimizes the net cost,

$$\text{Cost} = \text{Price} - \text{Futures Quote} \times \text{CF} \tag{8.5}$$

The bond with the lowest net cost is called **cheapest to deliver** (CTD).

In practice, the CF is set by the exchange at initiation of the contract. It is computed by discounting the bond cash flows at a notional 6% rate, assuming a flat term structure. So, high coupon bonds receive a high conversion factor.

The net cost calculations are illustrated in Table 8-1 for three bonds. The 10 5/8% coupon bond has a high factor, at 1.4533. The 5 1/2% bond in contrast has a factor less than one. Note how the CF adjustment brings the cost of all bonds much closer to each other than their original prices. Still, small differences remain due to the fact that the term structure is not perfectly flat at 6%.[4] The first bond is the CTD.

TABLE 8-1 Calculation of CTD

Bond	Price	Futures	CF	Cost
8 7/8% Aug 2017	127.094	97.0625	1.3038	0.544
10 5/8% Aug 2015	141.938	97.0625	1.4533	0.877
5 1/2% Nov 2028	91.359	97.0625	0.9326	0.839

As a first approximation, this CTD bond drives the characteristics of the futures contract. The equilibrium futures price is given by

$$F_t e^{-r\tau} = S_t - \text{PV}(D) \tag{8.6}$$

where S_t is the gross price of the CTD and PV(D) is the present value of the coupon payments. This has to be further divided by the conversion factor for this bond. The duration of the futures contract is also given by that of the CTD. In fact, these relations are only approximate because the "short" has an *option* to deliver the cheapest of a

[4] The adjustement is not perfect when current yields are far from 6%, or when the term structure is not flat, or when bonds do not trade at their theoretical prices. When rates are below 6%, discounting cash flows at 6% creates an downside bias for CF that increases for longer-term bonds. This tends to favor short-term bonds for delivery. When the term structure is upward sloping, the opposite occurs, and there is a tendency for long-term bonds to be delivered. Every so often, the exchange changes the coupon on the notional to reflect market conditions. The recent fall in yields explains why, for instance, the *Chicago Board of Trade* changed the notional coupon from 8% to 6% in 1999.

group of bonds. The value of this delivery option should depress the futures price since the party who is long the futures is also short the option, which is unfavorable. Unfortunately, this complex option is not easy to evaluate.

Example 8-7: FRM Exam 2000—Question 11/Capital Markets

8-7. The Chicago Board of Trade has reduced the notional coupon of its Treasury futures contracts from 8% to 6%. Which of the following statements are likely to be *true* as a result of the change?

a) The cheapest to deliver status will become more unstable if yields hover near the 6% range.

b) When yields fall below 6%, higher duration bonds will become cheapest to deliver, whereas lower duration bonds will become cheapest to deliver when yields range above 6%.

c) The 6% coupon would decrease the duration of the contract, making it a more effective hedge for the long end of the yield curve. d) There will be no impact at all by the change.

8.3 Swaps

Swaps are agreements by two parties to exchange cash flows in the future according to a prearranged formula. Interest rate swaps have payments tied to an interest rate. The most common type of swap is the **fixed-for-floating** swap, where one party commits to pay a fixed percentage of notional against a receipt that is indexed to a floating rate, typically LIBOR. The risk is that of a change in the level of rates.

Other types of swaps are **basis swaps**, where both payments are indexed to a floating rate. For instance, the swap can involve exchanging payments tied to 3-month LIBOR against a 3-month Treasury Bill rate. The risk is that of a change in the spread between the reference rates.

8.3.1 Definitions

Consider two counterparties, A and B, that can raise funds either at fixed or floating rates, $100 million over ten years. A wants to raise floating, and B wants to raise fixed.

Table 8-2a displays capital costs. Company A has an **absolute advantage** in the two markets as it can raise funds at rates systematically lower than B. Company A, however, has a **comparative advantage** in raising fixed as the cost is 1.2% lower than for B. In contrast, the cost of raising floating is only 0.70% lower than for B. Conversely, company B must have a comparative advantage in raising floating.

TABLE 8-2a Cost of Capital Comparison

Company	Fixed	Floating
A	10.00%	LIBOR + 0.30%
B	11.20%	LIBOR + 1.00%

This provides a rationale for a swap that will be to the mutual advantage of both parties. If both companies directly issue funds in their final desired market, the total cost will be LIBOR + 0.30% (for A) and 11.20% (for B), for a total of LIBOR + 11.50%. In contrast, the total cost of raising capital where each has a comparative advantage is 10.0% (for A) and LIBOR + 1.00% (for B), for a total of LIBOR + 11.00%. The gain to both parties from entering a swap is 11.50% − 11.00% = 0.50%. For instance, the swap described in Tables 8-2b and 8-2c splits the benefit equally between the two parties.

TABLE 8-2b Swap to Company A

Operation	Fixed	Floating
Issue debt	Pay 10.00%	
Enter swap	Receive 10.00%	Pay LIBOR + 0.05%
Net		Pay LIBOR + 0.05%
Direct cost		Pay LIBOR + 0.30%
Savings		0.25%

Company A issues fixed debt at 10.00%, and then enters a swap whereby it promises to pay LIBOR + 0.05% in exchange for receiving 10.00% fixed payments. Its effective funding cost is therefore LIBOR + 0.05%, which is less than the direct cost by 25bp.

TABLE 8-2c Swap to Company B

Operation	Floating	Fixed
Issue debt	Pay LIBOR + 1.00%	
Enter swap	Receive LIBOR + 0.05%	Pay 10.00%
Net		Pay 10.95%
Direct cost		Pay 11.20%
Savings		0.25%

Similarly, Company B issues floating debt at LIBOR + 1.0%, and then enters a swap whereby it receives LIBOR + 0.05% in exchange for paying 10.0% fixed. Its effective funding cost is therefore 10.95%, which is less than the direct cost by 25bp. Both parties benefit from the swap.

In terms of actual cash flows, payments are typically *netted* against each other. For instance, if the first LIBOR rate is at 9% assuming annual payments, Company A would be owed 10% × \$100 = \$1 million, and have to pay 9.05% × \$100 = \$0.905 million. This gives a net receipt of \$95,000. There is no need to exchange principals since both involve the same amount.

8.3.2 Quotations

Swaps are often quoted in terms of spreads relative to the yield of similar-maturity Treasury notes. For instance, a dealer may quote 10-year swap spreads as 31/34bp against LIBOR. If the current note yield is 6.72, this means that the dealer is willing to pay $6.72 + 0.31 = 7.03\%$ against receiving LIBOR, or that the dealer is willing to receive $6.72 + 0.34 = 7.06\%$ against paying LIBOR. Of course, the dealer makes a profit from the spread, which is rather small, at 3bp only. Swap rates are quoted for AA-rated counterparties. For lower rated counterparties the spread would be higher.

8.3.3 Pricing

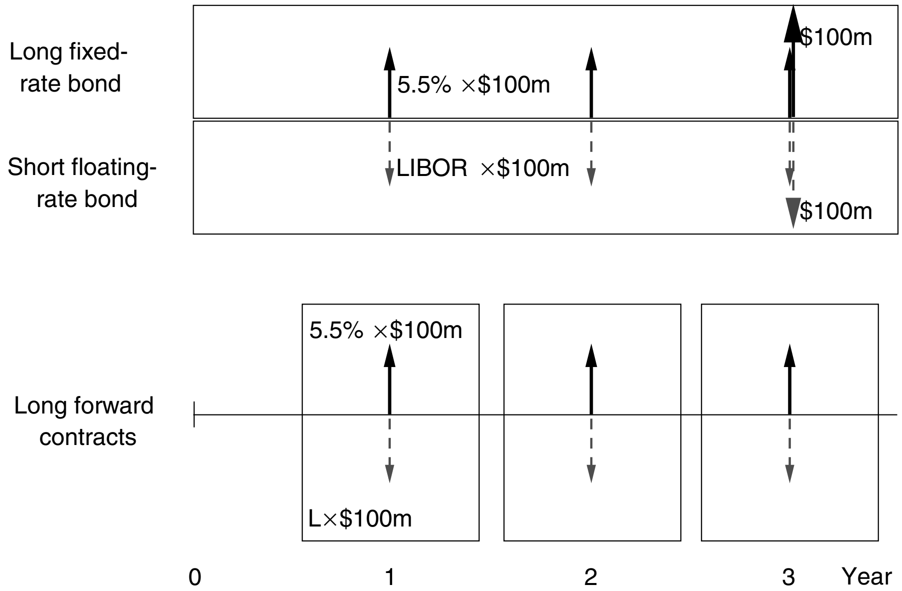
Consider, for instance, a 3-year \$100 million swap, where we receive a fixed coupon of 5.50% against LIBOR. Payments are annual and we ignore credit spreads. We can price the swap using either of two approaches, taking the difference between two bond prices or valuing a sequence of forward contracts. This is illustrated in Figure 8-2.

This swap is equivalent to a long position in a fixed-rate, 5.5% 3-year bond and a short position in a 3-year floating-rate note (FRN). If B_F is the value of the fixed-rate bond and B_f is the value of the FRN, the value of the swap is $V = B_F - B_f$.

The value of the FRN should be close to par. Just before a reset, B_f will behave exactly like a cash investment, as the coupon for the next period will be set to the prevailing interest rate. Therefore, its market value should be close to the face value. Just after a reset, the FRN will behave like a bond with a 6-month maturity. But overall, fluctuations in the market value of B_f should be small.

Consider now the swap value. If at initiation the swap coupon is set to the prevailing par yield, B_F is equal to the face value, $B_F = 100$. Because $B_f = 100$ just before

FIGURE 8-2 Alternative Decompositions for Swap Cash Flows



the reset on the floating leg, the value of the swap is zero, $V = B_F - B_f = 0$. This is like a forward contract at initiation.

After the swap is consummated, its value will be affected by interest rates. If rates fall, the swap will move in the money, since it receives higher coupons than prevailing market yields. B_F will increase whereas B_f will barely change.

Thus the duration of a receive-fixed swap is similar to that of a fixed-rate bond, including the fixed coupons and principal at maturity. This is because the duration of the floating leg is close to zero. The fact that the principals are not exchanged does not mean that the duration computation should not include the principal. Duration should be viewed as an interest rate sensitivity.

Key concept:
A position in a receive-fixed swap is equivalent to a long position in a bond with similar coupon characteristics and maturity offset by a short position in a floating-rate note. Its duration is close to that of the fixed-rate note.

We now value the 3-year swap using term-structure data from the preceding chapter. The time is just before a reset, so $B_f = \$100$ million. We compute B_F (in millions) as

$$B_F = \frac{\$5.5}{(1+4.000\%)} + \frac{\$5.5}{(1+4.618\%)^2} + \frac{\$105.5}{(1+5.192\%)^3} = \$100.95$$

The outstanding value of the swap is therefore $V = \$100.95 - \$100 = \$0.95$ million.

Alternatively, the swap can be valued as a sequence of forward contracts. Recall that the valuation of a unit position in a long forward contract is given by

$$V_i = (F_i - K)\exp(-r_i \tau_i) \tag{8.7}$$

where F_i is the market forward rate, K the prespecified rate, and r_i the spot rate for time τ_i, using continuous compounding.

Extending this to multiple maturities, the swap can be valued as

$$V = \sum_i n_i (F_i - K)\exp(-r_i \tau_i) \tag{8.8}$$

where n_i is the notional amount for maturity i. Since the contract increases in value if market rates, i.e., F_i, go up, this corresponds to a pay-fixed position.

We have to adapt this to our receive-fixed swap and annual compounding. Using the forward rates listed in Table 7-4, we find

$$V = -\frac{\$100(4.000\% - 5.50\%)}{(1+4.000\%)} - \frac{\$100(5.240\% - 5.50\%)}{(1+4.618\%)^2} - \frac{\$100(6.350\% - 5.50\%)}{(1+5.192\%)^3}$$

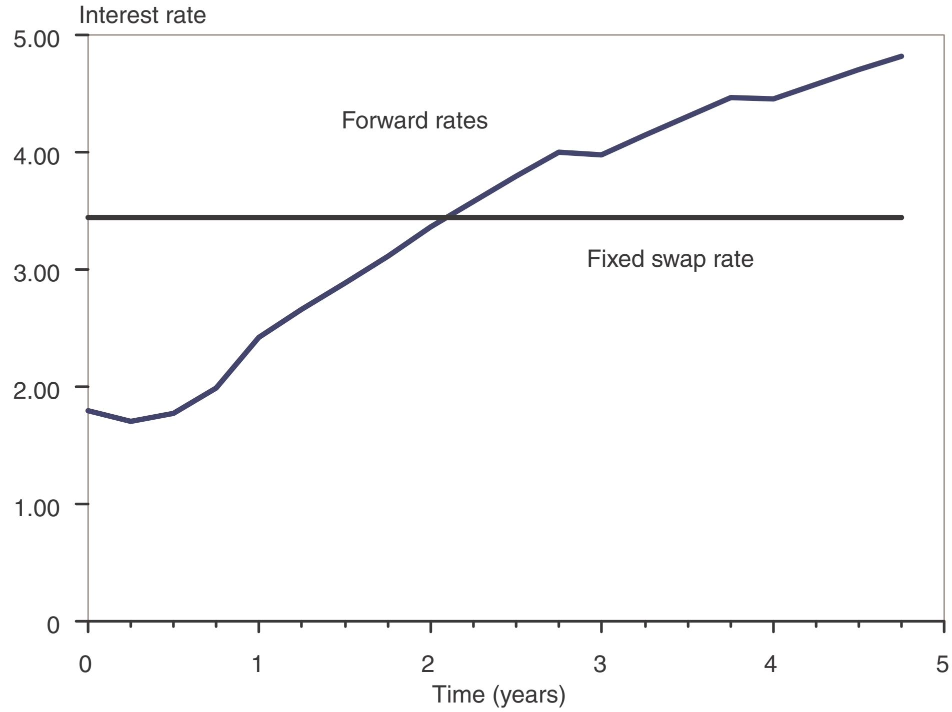
$$V = +1.4423 + 0.2376 - 0.7302 = \$0.95 \text{ million}$$

This is identical to the previous result, as should be. The swap is in-the-money primarily because of the first payment, which pays a rate of 5.5% whereas the forward rate is only 4.00%.

Thus, interest rate swaps can be priced and hedged using a sequence of forward rates, such as those implicit in Eurodollar contracts. In practice, the practice of daily marking-to-market futures induces a slight convexity bias in futures rates, which have to be adjusted downward to get forward rates.

Figure 8-3 compares a sequence of quarterly forward rates with the five-year swap rate prevailing at the same time. Because short-term forward rates are less than the swap rate, the near payments are in-the-money. In contrast, the more distant payments are out-of-the-money. The current market value of this swap is zero, which implies that all the near-term positive values must be offset by distant negative values.

FIGURE 8-3 Sequence of Forward Rates and Swap Rate



Example 8-8: FRM Exam 2000—Question 55/Credit Risk

8-8. Bank One enters into a 5-year swap contract with Mervin Co. to pay LIBOR in return for a fixed 8% rate on a nominal principal of $100 million. Two years from now, the market rate on three-year swaps at LIBOR is 7%; at this time Mervin Co. declares bankruptcy and defaults on its swap obligation. Assume that the net payment is made only at the end of each year for the swap contract period. What is the market value of the loss incurred by Bank One as result of the default?

a) $1.927 million
b) $2.245 million
c) $2.624 million
d) $3.011 million

Example 8-9: FRM Exam 1999—Question 42/Capital Markets

8-9. A multinational corporation is considering issuing a fixed-rate bond. However, by using interest swaps and floating-rate notes, the issuer can achieve the same objective. To do so, the issuer should consider

a) Issuing a floating-rate note of the same maturity of and enter into an interest rate swap paying fixed and receiving float

b) Issuing a floating-rate note of the same maturity of and enter into an interest rate swap paying float and receiving fixed

c) Buying a floating-rate note of the same maturity of and enter into an interest rate swap paying fixed and receiving float

d) Buying a floating-rate note of the same maturity of and enter into an interest rate swap paying float and receiving fixed

Example 8-10: FRM Exam 1998—Question 46/Capital Markets

8-10. Which of the following positions has the same exposure to interest rates as the receiver of the floating rate on a standard interest rate swap?

a) Long a floating-rate note with the same maturity

b) Long a fixed-rate note with the same maturity

c) Short a floating-rate note with the same maturity

d) Short a fixed-rate note with the same maturity

Example 8-11: FRM Exam 1999—Question 59/Capital Markets

8-11. *(Complex)* If an interest rate swap is priced off the Eurodollar futures strip without correcting the rates for convexity, the resulting arbitrage can be exploited by a

a) Receive-fixed swap + short Eurodollar futures position

b) Pay-fixed swap + short Eurodollar futures position

c) Receive-fixed swap + long Eurodollar futures position

d) Pay-fixed swap + long Eurodollar futures position

8.4 Options

There is a large variety of fixed-income options. We will briefly describe here caps and floors, swaptions, and exchange-traded options. In addition to these stand alone instruments, fixed-income options are embedded in many securities. For instance, a callable bond can be viewed as a regular bond plus a short position in an option.

When considering fixed-income options, the underlying can be a yield or a price. Due to the negative price-yield relationship, a call option on a bond can also be viewed as a put option on the underlying yield.

8.4.1 Caps and Floors

A **cap** is a call option on interest rates with unit value

$$C_T = \text{Max}[i_T - i_C, 0] \tag{8.9}$$

where i_C is the cap rate and i_T is the rate prevailing at maturity.

In practice, caps are issued jointly with the issuance of floating-rate notes that pay LIBOR plus a spread on a periodic basis for the term of the note. By purchasing the cap, the issuer ensures that the cost of capital will not exceed the capped rate. Such caps are really a combination of individual options, called **caplets**.

The payment on each caplet is determined by C_T, the notional, and an accrual factor. Payments are made in **arrears**, that is, at the end of the period. For instance, take a one-year cap on a notional of \$1 million and a 6-month LIBOR cap rate of 5%. The agreement period is from January 15 to the next January with a reset on July 15. Suppose that on July 15, LIBOR is at 5.5%. On the following January, the payment is

$$\$1 \text{ million} \times (0.055 - 0.05)(184/360) = \$2,555.56$$

using $Actual/360$ interest accrual. If the cap is used to hedge an FRN, this would help to offset the higher coupon payment, which is now 5.5%.

A **floor** is a put option on interest rates with value

$$P_T = \text{Max}[i_F - i_T, 0] \tag{8.10}$$

where i_F is the floor rate. A **collar** is a combination of buying a cap and selling a floor. This combination decreases the net cost of purchasing the cap protection.

When the cap and floor rates converge to the same value, the overall debt cost becomes fixed instead of floating. By put-call parity, we have

$$\text{Long Cap}(i_C = K) - \text{Short Floor}(i_F = K) = \text{Long Pay} - \text{Fixed Swap} \tag{8.11}$$

Caps are typically priced using a variant of the Black model, assuming that interest rate changes are lognormal. The value of the cap is set equal to a portfolio of K caplets, which are European-style individual options on different interest rates with regularly spaced maturities

$$c = \sum_{k=1}^{K} c_k \tag{8.12}$$

Each caplet is priced according to the Black model, per dollar and year

$$c_k = [FN(d_1) - KN(d_2)]\text{PV}(\$1) \tag{8.13}$$

where F is the current forward rate for the period t_k to t_{k+1}, K is the cap rate, and PV($1) is the discount factor to time t_{k+1}. To obtain a dollar amount, we must adjust for the notional amount as well as the length of the accrual period.

The volatility entering the function, σ, is that of the forward rate between now and the expiration of the option contract, that is, at t_k. Generally, volatilities are quoted as one number for all caplets within a cap, which is called **flat volatilities**.

$$\sigma_k = \sigma$$

Alternatively, volatilities can be quoted separately for each forward rate in the caplet, which is called **spot volatilities**.

Example: Computing the value of a cap

Consider the previous cap on $1 million at the capped rate of 5%. Assume a flat term structure at 5.5% and a volatility of 20% pa. The reset is on July 15, in 181 days; the accrual period is 184 days.

Since the term structure is flat, the six-month forward rate starting in six months is also 5.5%. First, we compute the present value factor, which is $\text{PV}(\$1) = 1/(1 + 0.055 \times 365/360) = 0.9472$, and the volatility, which is $\sigma\sqrt{\tau} = 0.20\sqrt{181/360} = 0.1418$.

We then compute the value of $d_1 = \ln[F/K]/\sigma\sqrt{\tau} + \sigma\sqrt{\tau}/2 = \ln[0.055/0.05]/0.1418 + 0.1418/2 = 0.7430$ and $d_2 = d_1 - \sigma\sqrt{\tau} = 0.7430 - 0.1418 = 0.6012$. We find $N(d_1) = 0.7713$ and $N(d_2) = 0.7261$. The value of the call is $c = [FN(d_1) - KN(d_2)]\text{PV}(\$1) = 0.5789\%$. Finally, the total price of the call is $\$1\text{million} \times 0.5789\% \times (184/360) = \$2{,}959$.

Figure 8-3 can be taken as an illustration of the sequence of forward rates. If the cap rate is the same as the prevailing swap rate, the cap is said to be *at-the-money*. In the figure, the near caplets are out-of-the-money because $F_i < K$. The distant caplets, however, are in-the-money.

Example 8-12: FRM Exam 1999—Question 54/Capital Markets
8-12. The cap-floor parity can be stated as
a) Short cap + Long floor = Fixed-rate bond.
b) Long cap + Short floor = Fixed swap.
c) Long cap + Short floor = Floating-rate bond.
d) Short cap + Short floor = Interest rate collar.

Example 8-13: FRM Exam 1999—Question 60/Capital Markets
8-13. For a 5-year ATM cap on the 3-month LIBOR, what can be said about the individual caplets, in a downward sloping term-structure environment?
a) The short maturity caplets are ITM, long maturity caplets are OTM.
b) The short maturity caplets are OTM, long maturity caplets are ITM.
c) All the caplets are ATM.
d) The moneyness of the individual caplets also depends on the volatility term structure.

8.4.2 Swaptions

Swaptions are OTC options that give the buyer the right to enter a swap at a fixed point in time at specified terms, including a fixed coupon rate.

These contracts take many forms. A **European swaption** is exercisable on a single date at some point in the future. On that date, the owner has the right to enter a swap with a specific rate and term. Consider for example a "1Y x 5Y" swaption. This gives the owner the right to enter in one year a long or short position in a 5-year swap.

A fixed-term **American swaption** is exercisable on any date during the exercise period. In our example, this would be during the next year. If, for instance, exercise occurs after six months, the swap would terminate in 5 years and six months. So, the termination date of the swap depends on the exercise date. In contrast, a **contingent American swaption** has a prespecified termination date, for instance exactly six years from now. Finally, a **Bermudan option** gives the holder the right to exercise on a specific set of dates during the life of the option.

As an example, consider a company that, in one year, will issue 5-year floating-rate debt. The company wishes to swap the floating payments into fixed payments.

The company can purchase a swaption that will give it the right to create a 5-year pay-fixed swap at the rate of 8%. If the prevailing swap rate in one year is higher than 8%, the company will exercise the swaption, otherwise not. The value of the option at expiration will be

$$P_T = \text{Max}[V(i_T) - V(i_K), 0] \tag{8.14}$$

where $V(i)$ is the value of a swap to pay a fixed rate i, i_T is the prevailing swap rate at swap maturity, and i_K is the locked-in swap rate. This contract is called a European 6/1 put swaption, or 1 into 5-year payer option.

Such a swap is equivalent to an option on a bond. As this swaption creates a profit if rates rise, it is akin to a one-year put option on a 6-year bond. Conversely, a swaption that gives the right to receive fixed is akin to a call option on a bond. Table 8-3 summarizes the terminology for swaps, caps and floors, and swaptions.

Swaptions are typically priced using a variant of the Black model, assuming that interest rates are lognormal. The value of the swaption is then equal to a portfolio of options on different interest rates, all with the same maturity. In practice, swaptions are traded in terms of volatilities instead of option premiums.

TABLE 8-3 Summary of Terminology for OTC Swaps and Options

Product	Buy (long)	Sell (short)
Fixed/Floating Swap	Pay fixed Receive floating	Pay floating Receive fixed
Cap	Pay premium Receive Max$(i - i_C, 0)$	Receive premium Pay Max$(i - i_C, 0)$
Floor	Pay premium Receive Max$(i_F - i, 0)$	Receive premium Pay Max$(i_F - i, 0)$
Put Swaption (payer option)	Pay premium Option to pay fixed and receive floating	Receive premium If exercised, receive fixed and pay floating
Call Swaption (receiver option)	Pay premium Option to pay floating and receive fixed	Receive premium If exercised, receive floating and pay fixed

Example 8-14: FRM Exam 1997—Question 18/Derivatives

8-14. The price of an option that gives you the right to receive fixed on a swap will decrease as

a) Time to expiry of the option increases.
b) Time to expiry of the swap increases.
c) The swap rate increases.
d) Volatility increases.

Example 8-15: FRM Exam 2000—Question 10/Capital Markets

8-15. Consider a 2 into 3-year Bermudan swaption (i.e., an option to obtain a swap that starts in 2 years and matures in 5 years). Consider the following statements:

I. A lower bound on the Bermudan price is a 2 into 3-year European swaption.
II. An upper bound on the Bermudan price is a cap that starts in 2 years and matures in 5 years.
III. A lower bound on the Bermudan price is a 2 into 5-year European option.

Which of the following statements is (are) *true*?

a) I only
b) II only
c) I and II
d) III only

8.4.3 Exchange-Traded Options

Among exchange-traded fixed-income options, we describe options on Eurodollar futures and on T-bond futures.

Options on Eurodollar futures give the owner the right to enter a long or short position in Eurodollar futures at a fixed price. The payoff on a put option, for example, is

$$P_T = \text{Notional} \times \text{Max}[K - \text{FQ}_T, 0] \times (90/360) \qquad (8.15)$$

where K is the strike price and FQ_T the prevailing futures price quote at maturity. In addition to the cash payoff, the option holder enters a position in the underlying futures. Since this is a put, it creates a short position after exercise, with the counterparty taking the opposing position. Note that, since futures are settled daily, the value of the contract is zero.

Since the futures price can also be written as $FQ_T = 100 - i_T$ and the strike price as $K = 100 - i_C$, the payoff is also

$$P_T = \text{Notional} \times \text{Max}[i_T - i_C, 0] \times (90/360) \tag{8.16}$$

which is equivalent to that of a cap on rates. Thus a put on Eurodollar futures is equivalent to a caplet on LIBOR.

In practice, there are minor differences in the contracts. Options on Eurodollar futures are American style instead of European style. Also, payments are made at the expiration date of Eurodollar futures options instead of in arrears.

Options on T-Bond futures give the owner the right to enter a long or short position in futures at a fixed price. The payoff on a call option, for example, is

$$C_T = \text{Notional} \times \text{Max}[F_T - K, 0] \tag{8.17}$$

An investor who thinks that rates will fall, or that the bond market will rally, could buy a call on T-Bond futures. In this manner, he or she will participate in the upside, without downside risk.

8.5 Answers to Chapter Examples

Example 8-1: FRM Exam 2001—Question 70/Capital Markets

b) The seller of an FRA agrees to receive fixed. Since rates are now higher than the contract rate, this contract must show a loss. The loss is $\$10,000,000 \times (6.85\% - 6.35\%) \times (90/360) = \$12,500$ when paid in arrears, i.e. in 9 months. On the settlement date, i.e. in 6 months, the loss is $\$12,500/(1 + 6.85\%0.25) = \$12,290$.

Example 8-2: FRM Exam 2001—Question 73/Capital Markets

c) The 1-year spot rate can be inferred from the sequence of 3-month spot and consecutive 3-month forward rates. We can compute the future value factor for each leg:
for 3-mo, $(1 + 4.5\% \times 91/360) = 1.011375$,
for 3×6, $(1 + 4.6\% \times 92/360) = 1.011756$,
for 6×9, $(1 + 4.8\% \times 90/360) = 1.01200$,
for 9×12, $(1 + 6.0\% \times 92/360) = 1.01533$.
The product is $1.05142 = (1 + r \times 365/360)$, which gives $r = 5.0717\%$.

Example 8-3: FRM Exam 1998—Question 54/Capital Markets

c) The duration is $5 - 2 = 3$ months. If rates go up, the position generates a profit. So the DV01 must be positive and $100 \times 0.01\% \times 0.25 = 2{,}500$.

Example 8-4: FRM Exam 1998—Question 7/Capital Markets

d) FRAs are OTC contracts, so (I) is wrong. Since Eurodollar futures are the most active contracts in the world, liquidity is excellent and (II) is wrong. Eurodollar contracts have fixed contract sizes, $1 million, so (III) is wrong.

Example 8-5: FRM Exam 1998—Question 40/Capital Markets

d) We need to short Eurodollars in an amount that accounts for the notional and durations of the inventory and hedge. The duration of the 1-year Treasury Bills is 1 year. The DV01 of Eurodollar futures is $\$1,000,000 \times 0.25 \times 0.0001 = \$25$. The DV01 of the portfolio is $\$100,000,000 \times 1.00 \times 0.0001 = \$10,000$. This gives a ratio of 400. Alternatively, $(V_P/V_F) \times (D_P/D_F) = (100/1) \times (1/0.25) = 400$.

Example 8-6: FRM Exam 2000—Question 7/Capital Markets

b) For assets whose value is *negatively* related to interest rates, such as Eurodollar futures, the futures rate must be higher than the forward rate. Because rates and prices are inversely related, the futures price quote is lower than the forward price quote. The question deals with a situation where the correlation is *positive*, rather than negative. Hence, the futures price quote must be above the forward price quote.

Example 8-7: FRM Exam 2000—Question 11/Capital Markets

a) The goal of the CF is to equalize differences between various deliverable bonds. In the extreme, if we discounted all bonds using the current term structure, the CF would provide an exact offset to all bond prices, making all of the deliverable bonds equivalent. This reduction from 8% to 6% notional reflects more closely recent interest rates. It will lead to more instability in the CTD, which is exactly the effect intended. (b) is not correct as yields lower than 6% imply that the CF for long-term bonds is lower than otherwise. This will tend to favor bonds with high conversion factors, or shorter bonds. Also, a lower coupon increases the duration of the contract, so (c) is not correct.

Example 8-8: FRM Exam 2000—Question 55/Credit Risk

c) Using Equation (8.8) for three remaining periods, we have the discounted value of the net interest payment, or (8% − 7%)$100,000,000 = $1,000,000, discounted at 7%, which is $934, 579 + $873, 439 + $816, 298 = $2, 624, 316.

Example 8-9: FRM Exam 1999—Question 42/Capital Markets

a) Receiving a floating rate on the swap will offset the payment on the note, leaving a net obligation at a fixed rate.

Example 8-10: FRM Exam 1998—Question 46/Capital Markets

d) Paying fixed on the swap is the same as being short a fixed-rate note.

Example 8-11: FRM Exam 1999—Question 59/Capital Markets

a) *(Complex)* A receive-fixed swap is equivalent to a long position in a bond, which can be hedged by a short Eurodollar position. Conversely, a pay-fixed swap is hedged by a long Eurodollar position. So, only (a) and (d) are correct. The convexity adjustment should correct futures rates downward. Without this adjustment, forward rates will be too high. This implies that the valuation of a pay-fixed swap is too high. To arbitrage this, we should short the asset that is priced too high, i.e. enter a receive-fixed swap, and buy the position that is cheap, i.e. take a short Eurodollar position.

Example 8-12: FRM Exam 1999—Question 54/Capital Markets

a) With the same strike price, a short cap/long floor loses money if rates increase, which is equivalent to a long position in a fixed-rate bond.

Example 8-13: FRM Exam 1999—Question 60/Capital Markets

a) In a downward-sloping rate environment, forward rates are higher for short maturities. Caplets involves the right to buy at the same fixed rate for all caplets. Hence short maturities are ITM.

Example 8-14: FRM Exam 1997—Question 18/Derivatives

c) The value of a call increases with the maturity of the call and the volatility of the underlying asset value (which here also increases with the maturity of the swap contract). So (a) and (d) are wrong. In contrast, the value of the right to receive an asset at K decreases as K increases.

Example 8-15: FRM Exam 2000—Question 10/Capital Markets

c) A swaption is a one-time option that can be exercised either at one point in time (European), at any point during the exercise period (American), or on a discrete set of dates during the exercise period (Bermudan). As such the Bermudan option must be more valuable than the European option, *ceteris paribus*. Also, a cap is a series of options. As such, it must be more valuable than any option that is exercisable only once. Answers (I) and (II) match the exercise date of the option and the final maturity. Answer (III), in contrast, describes an option that matures in 7 years, so cannot be compared with the original swaption.

Chapter 9

Equity Markets

Having covered fixed-income instruments, we now turn to equities and equity linked instruments. Equities, or common stocks, represent ownership shares in a corporation.

Due to the uncertainty in their cash flows, as well as in the appropriate discount rate, equities are much more difficult to value than fixed-income securities. They are also less amenable to the quantitative analysis that is used in fixed-income markets. Equity derivatives, however, can be priced reasonably precisely in relation to underlying stock prices.

Section 9.1 introduces equity markets and presents valuation methods. Section 9.2 briefly discusses convertible bonds and warrants. These differ from the usual equity options in that exercising them creates new shares. In contrast, the exercise of options on individual stocks simply transfers shares from one counterpart to another. Section 9.3 then provides an overview of important equity derivatives, including stock index futures, stock options, stock index options, and equity swaps. As the basic valuation methods have been covered in a previous chapter, this section instead focuses on applications.

9.1 Equities

9.1.1 Overview

Common stocks, also called **equities**, are securities that represent ownership in a corporation. Bonds are senior to equities, that is, have a prior claim on the firm's assets in case of bankruptcy. Hence equities represent **residual claims** to what is left of the value of the firm after bonds, loans, and other contractual obligations have been paid off.

Another important feature of common stocks is their **limited liability**, which means that the most shareholders can lose is their original investment. This is unlike owners of unincorporated businesses, whose creditors have a claim on the personal assets of the owner should the business turn bad.

Table 9-1 describes the global equity markets. The total market value of common stocks was worth approximately $35 trillion at the end of 1999. The United States accounts for the largest proportion, followed by the Eurozone, Japan, and the United Kingdom.

TABLE 9-1 Global Equity Markets - 1999 (Billions of U.S. Dollars)

United States	15,370
Eurozone	5,070
Japan	4,693
United Kingdom	2,895
Other Europe	1,589
Other Pacific	1,216
Canada	763
Developed	31,594
Emerging	2,979
World	34,573

Source: Morgan Stanley Capital International

Preferred stocks differ from common stock because they promise to pay a specific stream of dividends. So, they behave like a perpetual bond, or consol. Unlike bonds, however, failure to pay these dividends does not result in bankruptcy. Instead, the corporation cannot pay dividends to common stock holders until the preferred dividends have been paid out. In other words, preferred stocks are junior to bonds, but senior to common stocks.

With **cumulative preferred dividends**, all current and previously postponed dividends must be paid before any dividends on common stock shares can be paid. Preferred stocks usually have no voting rights.

Unlike interest payments, preferred stocks dividends are not tax-deductible expenses. Preferred stocks, however, have an offsetting tax advantage. Corporations that receive preferred dividends only pay taxes on 30% of the amount received, which lowers their income tax burden. As a result, most preferred stocks are held by corporations. The market capitalization of preferred stocks is much lower than that of common stocks, as seen from the IBM example below. Trading volumes are also much lower.

Example: IBM Preferred Stock

IBM issued 11.25 million preferred shares in June 1993. These are traded as 45 million "depositary" shares, each representing one-fourth of the preferred, under the ticker "IBM-A" on the NYSE. Dividends accrue at the rate of $7.50 per annum, or $1.875 per depositary share.

As of April 2001, the depositary shares were trading at $25.4, within a narrow 52-week trading range of [$25.00, $26.25]. Using the valuation formula for a consol, the shares trade at an implied yield of 7.38%. The total market capitalization of the IBM-A shares amounts to approximately $260 million. In comparison, the market value of the common stock is $214,602 million, which is more than 800 times larger.

9.1.2 Valuation

Common stocks are extremely difficult to value. Like any other asset, their value derives from their future benefits, that is, from their stream of future cash flows (i.e., dividend payments) or future stock price.

We have seen that valuing Treasury bonds is relatively straightforward, as the stream of cash flows, coupon and principal payments, can be easily laid out and discounted into the present.

This is an entirely different affair for common stocks. Consider for illustration a "simple" case where a firm pays out a dividend D over the next year that grows at the constant rate of g. We ignore the final stock value and discount at the constant rate of r, such that $r > g$. The firm's value, P, can be assessed using the net present value formula, like a bond

$$
\begin{aligned}
P &= \sum_{t=1}^{\infty} C_t/(1+r)^t \\
&= \sum_{t=1}^{\infty} D(1+g)^{(t-1)}/(1+r)^t \\
&= [D/(1+r)] \sum_{t=0}^{\infty} [(1+g)/(1+r)]^t \\
&= [D/(1+r)] \times \left[\frac{1}{1-(1+g)/(1+r)} \right] \\
&= [D/(1+r)] \times [(1+r)/(r-g)]
\end{aligned}
$$

This is also the so-called “Gordon-growth” model,

$$P = \frac{D}{r - g} \tag{9.1}$$

as long as the discount rate exceeds the growth rate of dividends, $r > g$.

The problem with equities is that the growth rate of dividends is uncertain and that, in addition, it is not clear what the required discount rate should be. To make things even harder, some companies simply do not pay any dividend and instead create value from the appreciation of their share price.

Still, this valuation formula indicates that large variations in equity prices can arise from small changes in the discount rate or in the growth rate of dividends, explaining the large volatility of equities.

More generally, the risk and expected return of the equity depends on the underlying business fundamentals as well as on the amount of leverage, or debt in the capital structure.

For financial intermediaries for which the value of underlying assets can be measured precisely, we can value the equity based on the capital structure. In this situation, however, the equity is really valued as a derivative on the underlying assets.

Example 9-1: FRM Exam 1998—Question 50/Capital Markets

9-1. A hedge fund leverages its $100 million of investor capital by a factor of three and invests it into a portfolio of junk bonds yielding 14%. If its borrowing costs are 8%, what is the yield on investor capital?

a) 14%
b) 18%
c) 26%
d) 42%

9.1.3 Equity Indices

It is useful to summarize the performance of a group of stocks by an index. A **stock index** summarizes the performance of a representative group of stocks. Most commonly, this is achieved by mimicking the performance of a buy-and-hold strategy where each stock is weighted by its market capitalization.

Define R_i as the price appreciation return from stock i, from the initial price P_{i0} to the final price P_{i1}. N_i is the number of shares outstanding, which is fixed over the period. The portfolio value at the initial time is $\sum_i N_i P_{i0}$. The performance of the index is computed from the rate of change in the portfolio value

$$
\begin{aligned}
R_{M1} &= [\sum_i N_i P_{i1} - (\sum_i N_i P_{i0})]/(\sum_i N_i P_{i0}) \\
&= [\sum_i N_i (P_{i1} - P_{i0})]/(\sum_i N_i P_{i0}) \\
&= [\sum_i N_i P_{i0} (P_{i1} - P_{i0})/P_{i0}]/(\sum_i N_i P_{i0}) \\
&= \sum_i [N_i P_{i0}/(\sum_i N_i P_{i0})](P_{i1} - P_{i0})/P_{i0} \\
&= \sum_i [w_i](P_{i1} - P_{i0})/P_{i0}
\end{aligned}
$$

Here, $N_i P_{i0}$ is the market capitalization of stock i, and $w_i = [N_i P_{i0}/(\sum_i N_i P_{i0})]$ is the market-cap weight of stock i in the index. This gives

$$R_{M1} = \sum_i w_i R_{i1} \tag{9.2}$$

From this, the level of the index can be computed, starting from I_0, as

$$I_1 = I_0 \times (1 + R_{M1}) \tag{9.3}$$

and so on for the next periods. Thus, most stock indices are constructed using **market value weights**, also called **capitalization weights**.

Notable exceptions are the Dow and Nikkei 225 indices, which are **price weighted**, or simply involve a summation of share prices for companies in the index. Among international indices, the German DAX is also unusual because it includes dividend payments. These indices can be used to assess general market risk factors for equities.

9.2 Convertible Bonds and Warrants

9.2.1 Definitions

We now turn to convertible bonds and warrants. While these instruments have option like features, they differ from regular options. When a call option is exercised, for instance, the "long" purchases an outstanding share from the "short." There is no net creation of shares. In contrast, the exercise of convertible bonds, of warrants, (and of executive stock options) entails the creation of new shares, as the option is sold by the corporation itself. In this case, the existing shares are said to be **diluted** by the creation of new shares.

Warrants are long-term call options issued by a corporation on its own stock. They are typically created at the time of a bond issue, but they trade separately from the bond to which they were originally attached. When a warrant is exercised, it results in a cash inflow to the firm which issues more shares.

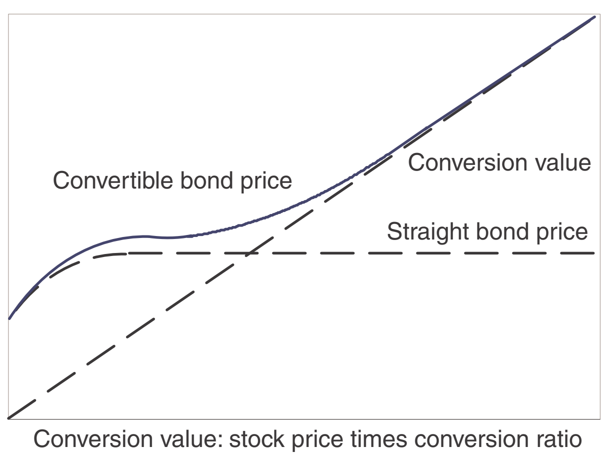
Convertible bonds are bonds issued by a corporation that can be converted into equity at certain times using a predetermined exchange ratio. They are equivalent to a regular bond plus a warrant. This allows the company to issue debt with a lower coupon than otherwise.

For example, a bond with a **conversion ratio** of 10 allows its holder to convert one bond with par value of $1,000 into 10 shares of the common stock. The **conversion price**, which is really the strike price of the option, is $1,000/10 = $100. The corporation will typically issue the convertible deep out of the money, for example when the stock price is at $50. When the stock price moves, for instance to $120, the bond can be converted into stock for an immediate option profit of ($120 − $100) × 10 = $200.

Figure 9-1 describes the relationship between the value of the convertible bond and the **conversion value**, defined as the current stock price times the conversion ratio. The convertible bond value must be greater than the price of an otherwise identical straight bond and the conversion value.

For high values of the stock price, the firm is unlikely to default and the straight bond price is constant, reflecting the discounting of cash flows at the risk-free rate. In this situation, it is almost certain the option will be exercised and the convertible value is close to the conversion value. For low values of the stock price, the firm is likely to

FIGURE 9-1 Convertible Bond Price and Conversion Value



default and the straight bond price drops, reflecting the likely loss upon default. In this situation, it is almost certain the option will not be exercised, and the convertible value is close to the straight bond value. In the intermediate region, the convertible value depends on both the conversion and straight bond values. The convertible is also sensitive to interest rate risk.

Example: A Convertible Bond

Consider a 8% annual coupon, 10-year convertible bond with face value of $1,000. The yield on similar maturity straight debt issued by the company is currently 8.50%, which gives a current value of straight debt of $967. The bond can be converted into common stock at a ratio of 10-to-1.

Assume first that the stock price is $50. The conversion value is then $500, much less than the straight debt value of $967. This corresponds to the left area of Figure 9-1. If the convertible trades at $972, its promised yield is 8.42%. This is close to the yield of straight debt, as the option has little value.

Assume next that the stock price is $150. The conversion value is then $1,500, much higher than the straight debt value of $967. This corresponds to the right area of Figure 9-1. If the convertible trades at $1,505, its promised yield is 2.29%. In this case, the conversion option is in-the-money, which explains why the yield is so low.

9.2.2 Valuation

Warrants can be valued by adapting standard option pricing models to the dilution effect of new shares. Consider a company with N outstanding shares and M outstanding warrants, each allowing the holder to purchase γ shares at the fixed price of K. At origination, the value of the firm includes the warrant, or

$$V_0 = NS_0 + MW_0 \tag{9.4}$$

where S_0 is the initial stock price just before issuing the warrant, and W_0 is the up-front value of the warrant.

After dilution, the total value of the firm includes the value of the firm before exercise (including the original value of the warrants) plus the proceeds from exercise, i.e. $V_T + M\gamma K$. The number of shares then increases to $N + \gamma M$. The total payoff to the warrant holder is

$$W_T = \gamma\left(\frac{V_T + M\gamma K}{N + \gamma M} - K\right) \tag{9.5}$$

which must be positive. After simplification, this is also

$$W_T = \gamma \left(\frac{V_T - NK}{N + \gamma M} \right) = \frac{\gamma}{N + \gamma M}(V_T - NK) = \frac{\gamma N}{N + \gamma M}\left(\frac{V_T}{N} - K \right) \tag{9.6}$$

which is equivalent to $n = \frac{\gamma N}{N+\gamma M}$ options on the stock price. The warrant can be valued by standard option models with the asset value equal to the stock price plus the warrant proceeds, multiplied by the factor n,

$$W_0 = n \times c\left(S_0 + \frac{M}{N} W_0, K, \tau, \sigma, r, d \right) \tag{9.7}$$

with the usual parameters and the unit asset value is $\frac{V_0}{N} = S_0 + \frac{M}{N} W_0$. This must be solved iteratively since W_0 appears on both sides. If, however, M is small relative to the current float, or number of outstanding shares N, the formula reduces to a simple call option in the amount γ

$$W_0 = \gamma \; c(S_0, K, \tau, \sigma, r, d) \tag{9.8}$$

Example: Pricing a Convertible Bond

Consider a zero-coupon, 10-year convertible bond with face value of \$1,000. The yield on similar maturity straight debt issued by the company is currently 8.158%, using continuous compounding, which gives a straight debt value of \$442.29.

The bond can be converted into common stock at a ratio of 10-to-1 at expiration only. This gives a strike price of $K = \$100$. The current stock price is \$60. The stock pays no dividend and has annual volatility of 30%. The risk-free rate is 5%, also continuously compounded.

Ignoring dilution effects, the Black-Scholes model gives an option value of \$216.79. So, the theoretical value for the convertible bond is $P = \$442.29 + \$216.79 = \$659.08$. If the market price is lower than \$659, the convertible is said to be cheap. This, of course, assumes that the pricing model and input assumptions are correct.

One complication is that most convertibles are also callable at the discretion of the firm. Convertible securities can be called for several reasons. First, an issue can be called to force conversion into common stock when the stock price is high enough. Bondholders have typically a month during which they can still convert, in which case this is a **forced conversion**. This call feature gives the corporation more control over conversion and allows it to raise equity capital.

Second, the call may be exercised when the option value is worthless and the firm can refinance its debt at a lower coupon. This is similar to the call of a non-convertible bond, except that the convertible must be *busted*, which occurs when the stock price is much lower than the conversion price.

Example 9-2: FRM Exam 1997—Question 52/Market Risk
9-2. A convertible bond trader has purchased a long-dated convertible bond with a call provision. Assuming there is a 50% chance that this bond will be converted into stock, which combination of stock price and interest rate level would constitute the *worst* case scenario?
a) Decreasing rates and decreasing stock prices
b) Decreasing rates and increasing stock prices
c) Increasing rates and decreasing stock prices
d) Increasing rates and increasing stock prices

Example 9-3: FRM Exam 2001—Question 119
9-3. A corporate bond with face value of $100 is convertible at $40 and the corporation has called it for redemption at $106. The bond is currently selling at $115 and the stock's current market price is $45. Which of the following would a bondholder most likely do?
a) Sell the bond
b) Convert the bond into common stock
c) Allow the corporation to call the bond at 106
d) None of the above

Example 9-4: FRM Exam 2001—Question 117
9-4. What is the main reason why convertible bonds are generally issued with a call?
a) To make their analysis less easy for investors
b) To protect against unwanted takeover bids
c) To reduce duration
d) To force conversion if in-the-money

9.3 Equity Derivatives

Equity derivatives can be traded on over-the-counter markets as well as organized exchanges. We only consider a limited range of popular instruments.

9.3.1 Stock Index Futures

Stock index futures are actively traded all over the world. In fact, the turnover corresponding to the notional amount is often greater than the total amount of trading in physical stocks in the same market. The success of thee contracts can be explained by their versatility for risk management. Stock index futures allow investors to manage their exposure to broad stock market movements. Speculators can take efficiently directional bets, on the upside or downside. Hedgers can protect the value of their investments.

Perhaps the most active contract is the S&P 500 futures contract on the Chicago Mercantile Exchange (CME). The contract notional is defined as \$250 times the index level. Table 9-2 displays quotations as of December 31, 1999.

TABLE 9-2 Sample S&P Futures Quotations

Maturity	Open	Settle	Change	Volume	Open Interest
March	1480.80	1484.20	+3.40	34,897	356,791
June	1498.00	1503.10	+3.60	410	8,431

The table shows that most of the volume was concentrated in the "near" contract, that is, March in this case. Translating the trading volume in number of contracts into a dollar equivalent, we find $\$250 \times 1484.2 \times 34,897$, which gives \$12.9 billion. In 2001, average daily volume was worth \$35 billion, which is close to the trading volume of \$42 billion on the New York Stock Exchange (NYSE).

We can also compute the daily profit on a long position, which would have been $\$250 \times (+3.40)$, or \$850. This is rather small, as the daily move was $+3.4/1480.8$, which is only 0.23%. The typical daily standard deviation is about 1%, which gives a typical profit or loss of \$3,710.50.

These contracts are cash settled. They do not involve delivery of the underlying stocks at expiration. In terms of valuation, the futures contract is priced according to the usual cash-and-carry relationship,

$$F_t e^{-r\tau} = S_t e^{-y\tau} \tag{9.9}$$

where y is now the dividend yield defined per unit time. For instance, the yield on the S&P was $y = 0.94$ percent per annum.

Here, we assume that the dividend yield is known in advance and paid on a continuous basis. In general, this is not necessarily the case but can be viewed as a good approximation. With a large number of firms in the index, dividends will be spread reasonably evenly over the quarter.

To check if the futures contract was fairly valued, we need the spot price, $S = 1469.25$; the short-term interest rate, $r = 5.3\%$; and the number of days to maturity, which was 76 (to March 16). Note that rates are not continuously compounded. The present value factor is $PV(\$1) = 1/(1+r\tau) = 1/(1+5.3\%(76/365)) = 0.9891$. Similarly, the present value of the dividend stream is $1/(1 + y\tau) = 1/(1 + 0.94\%(76/365)) = 0.9980$. The fair price is then

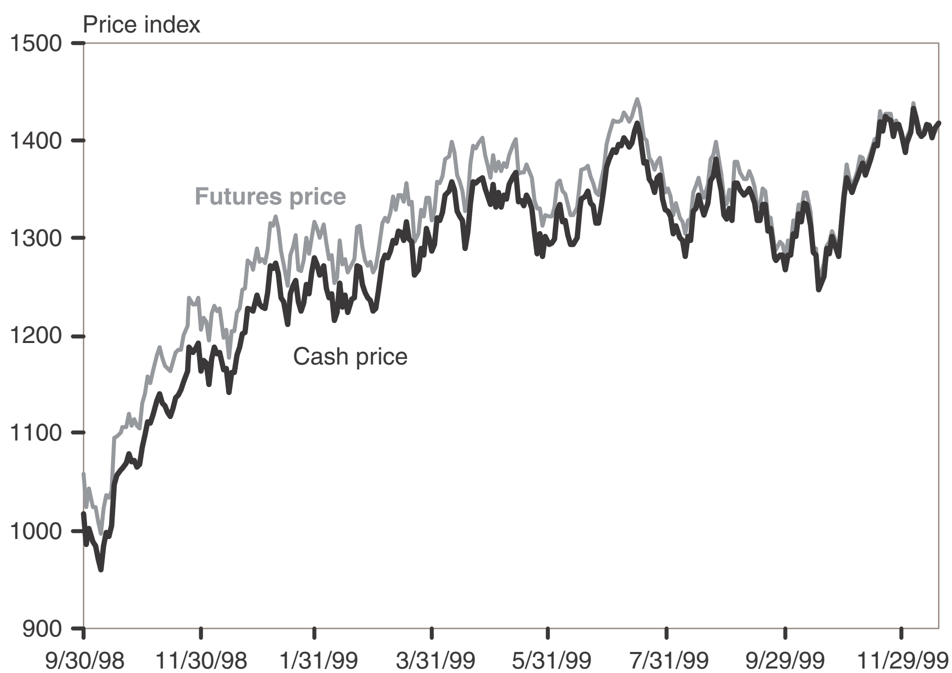
$$F = [S/(1 + y\tau)]\,(1 + r\tau) = [1469.25 \times 0.9980]/0.9891 = 1482.6$$

This is rather close to the settlement value of $F = 1484.2$. The discrepancy is probably due to the fact that the quotes were not measured simultaneously.

Figure 9-2 displays the convergence of futures and cash prices for the December 1999 S&P 500 futures contract traded on the CME. The futures price is always the spot price. The correlation between the two prices is very high, reflecting the cash-and-carry relationship in Equation (9.9).

Because financial institutions engage in stock index arbitrage, we would expect the cash-and-carry relationship to hold very well, One notable exception was during the market crash of October 19, 1987. The market lost more than 20% in a single day. Throughout the day, however, futures prices were more up-to-date than cash prices because of execution delays and closing in cash markets. As a result, the S&P stock index futures value was very cheap compared with the underlying cash market. Arbitrage, however, was made difficult due to chaotic market conditions.

FIGURE 9-2 Futures and Cash Prices for S&P500 Futures



Example 9-5: FRM Exam 1998—Question 9/Capital Markets

9-5. To prevent arbitrage profits, the theoretical future price of a stock index should be fully determined by which of the following?

I. Cash market price
II. Financing cost
III. Inflation
IV. Dividend yield

a) I and II only
b) II and III only
c) I, II and IV only
d) All of the above

Example 9-6: FRM Exam 2000—Question 12/Capital Markets

9-6. Suppose the price for a 6-month S&P index futures contract is 552.3. If the risk-free interest rate is 7.5% per year and the dividend yield on the stock index is 4.2% per year, and the market is complete and there is no arbitrage, what is the price of the index today?

a) 543.26
b) 552.11
c) 555.78
d) 560.02

9.3.2 Single Stock Futures

In late 2000, the United States passed legislation authorizing trading in **single stock futures**, which are futures contracts on individual stocks. Such contracts were already trading in Europe and elsewhere. In the United States, electronic trading started in November 2002.[1]

Each contract gives the obligation to buy or sell 100 shares of the underlying stock. Delivery involves physical settlement. Relative to trading in the underlying stocks, single stock futures have many advantages. Positions can be established more efficiently due to their low margin requirements, which are generally 20% of the cash value. Margin for stocks are higher. Also, short selling eliminates the costs and inefficiencies associated with the stock loan process. Other than physical settlement, these contracts trade like stock index futures.

[1] Two electronic exchanges are currently competing, "OneChicago", a joint venture of Chicago exchanges, and "Nasdaq Liffe", a joint venture of NASDAQ, the main electronic stock exchange in the United States, and Liffe, the U.K. derivatives exchange.

9.3.3 Equity Options

Options can be traded on individual stocks, on stock indices, or on stock index futures. In the United States, stock options trade, for example, on the Chicago Board Options Exchange (CBOE). Each option gives the right to buy or sell a round lot of 100 shares. Exercise of stock options involves physical delivery, or the exchange of the underlying stock.

Traded options are typically American-style, so their valuation should include the possibility of early exercise. In practice, however, their values do not differ much from those of European options, which can be priced by the Black-Scholes model. When the stock pays no dividend, the values are the same. For more precision, we can use numerical models such as binomial trees to take into account dividend payments.

The most active index options in the United States are options on the S&P 100 and S&P 500 index traded on the CBOE. The former are American-style, while the latter are European-style. These options are cash settled, as it would be too complicated to deliver a basket of 100 or 500 underlying stocks. Each contract is for $100 times the value of the index. European options on stock indices can be priced using the Black-Scholes formula, using y as the dividend yield on the index as we have done in the previous section for stock index futures.

Finally, options on S&P 500 stock index futures are also popular. These give the right to enter a long or short futures position at a fixed price. Exercise is cash settled.

9.3.4 Equity Swaps

Equity swaps are agreements to exchange cash flows tied to the return on a stock market index in exchange for a fixed or floating rate of interest. An example is a swap that provides the return on the S&P 500 index every six months in exchange for payment of LIBOR plus a spread. The swap will be typically priced so as to have zero value at initiation. Equity swaps can be valued as portfolios of forward contracts, as in the case of interest rate swaps. We will later see how to price currency swaps. The same method can be used for equity swaps.

These swaps are used by investment managers to acquire exposure to, for example, an emerging market without having to invest in the market itself. In some cases, these swaps can also be used to defeat restrictions on foreign investments.

9.4 Answers to Chapter Examples

Example 9-1: FRM Exam 1998—Question 50/Capital Markets

c) The fund borrows \$200 million and invest \$300 million, which creates a yield of \$300 × 14% = \$42 million. Borrowing costs are \$200 × 8% = \$16 million, for a difference of \$26 million on equity of \$100 million, or 26%. Note that this is a yield, not expected rate of return if we expect some losses from default. This higher yield also implies higher risk.

Example 9-2: FRM Exam 1997—Question 52/Market Risk

c) Abstracting from the convertible feature, the value of the fixed-coupon bond will fall if rates increase; also, the value of the convertible feature falls as the stock price decreases.

Example 9-3: FRM Exam 2001—Question 119

a) The conversion rate is expressed here in terms of the conversion price. The conversion rate for this bond is \$100 into \$40, or 1 bond into 2.5 shares. Immediate conversion will yield 2.5 × \$45 = \$112.5. The call price is \$106. Since the market price is higher than the call price and the conversion value, and the bond is being called, the best value is achieved by selling the bond.

Example 9-4: FRM Exam 2001—Question 117

d) Companies issue convertible bonds because the coupon is lower than for regular bonds. In addition, these bonds are callable in order to force conversion into the stock at a favorable ratio. In the previous question, for instance, conversion would provide equity capital to the firm at the price of \$40, while the market price is at \$45.

Example 9-5: FRM Exam 1998—Question 9/Capital Markets

c) The futures price depends on S, r, y, and time to maturity. The rate of inflation is not in the cash-and-carry formula, although it is embedded in the nominal interest rate.

Example 9-6: FRM Exam 2000—Question 12/Capital Markets

a) This is the cash-and-carry relationship, solved for S. We have $Se^{-y\tau} = Fe^{-r\tau}$, or $S = 552.3 \times \exp(-7.5/200)/\exp(-4.2/200) = 543.26$. We verify that the forward price is greater than the spot price since the dividend yield is less than the risk-free rate.

Chapter 10

Currencies and Commodities Markets

This chapter turns to currency and commodity markets. The foreign exchange markets are by far the largest financial markets in the world, with daily turnover estimated at $1,210 billion in 2001. The **forex** markets consist of the spot, forward, options, futures, and swap markets.

Commodity markets consist of agricultural products, metals, energy, and other products. They are traded cash and through derivatives instruments. Commodities differ from financial assets as their holding provides an implied benefit known as convenience yield but also incurs storage costs.

Section 10.1 presents a brief introduction to currency markets. Contracts such as futures, forward, and options have been developed in previous chapters and do not require special treatment. In contrast, currency swaps are analyzed in some detail in Section 10.2 due to their unique features and importance. Section 10.3 then discusses commodity markets.

10.1 Currency Markets

The global currency markets are without a doubt the most active financial markets in the world. Their size and growth is described in Table 10-1. This trading activity dwarfs that of bond or stock markets. In comparison, the daily trading volume on the New York Stock Exchange (NYSE) is approximately $40 billion.

Even though the largest share of these transaction is between dealers, or with other financial institutions, the volume of trading with other, nonfinancial institutions is still quite large, at $156 billion daily.

Spot transactions are exchanges of two currencies for settlement as soon as practical, typically in two business days. They account for about 40% of trading volume.

TABLE 10-1 Activity in Global Currency Markets Average Daily Trading Volume (Billions of U.S. Dollars)

Year	Spot	Forwards & forex swaps	Total
1989	350	240	590
1992	416	404	820
1995	517	673	1,190
1998	592	898	1,490
2001	399	811	1,210
Of which, between:			
Dealers			689
Financials			329
Others			156

Source: Bank for International Settlements surveys.

Other transactions are outright forward contracts and forex swaps. **Outright forward contracts** are agreements to exchange two currencies at a future date, and account for about 9% of the total market. **Forex swaps** involve two transactions, an exchange of currencies on a given date and a reversal at a later date, and account for 51% of the total market.[1]

In addition to these contracts, there is also some activity in forex options ($60 billion daily) and exchange-traded derivatives ($9 billion daily), as measured in April 2001. The most active currency futures are traded on the Chicago Mercantile Exchange (CME) and settled by physical delivery. Options on currencies are available over-the-counter (OTC), on the Philadelphia Stock Exchange (PHLX), and are also cash settled. The CME also trades options on currency futures.

As we have seen before, currency forwards, futures, and options can be priced according to standard valuation models, specifying the income payment to be a continuous flow defined by the foreign interest rate, r^*.

Currencies are generally quoted in **European terms**, that is, in units of the foreign currency per dollar. The yen, for example, could be quoted as 120 yen per U.S. dollar. Two notable exceptions are the British pound (sterling) and the euro, which are quoted in **American terms**, that is in dollars per unit of the foreign currency The pound, for example, could be quoted as 1.6 dollar per pound.

[1]Forex swaps are typically of a short-term nature and should not be confused with long-term currency swaps, which involve a stream of payments over longer horizons.

10.2 Currency Swaps

Currency swaps are agreements by two parties to exchange a stream of cash flows in different currencies according to a prearranged formula.

10.2.1 Definitions

Consider two counterparties, company A and company B that can raise funds either in dollars or in yen, $100 million or Y10 billion at the current rate of 100Y/$, over ten years. Company A wants to raise dollars, and company B wants to raise yen. Table 10-2a displays borrowing costs. This example is similar to that of interest rate swaps, except that rates now apply to different currencies.

Company A has an **absolute advantage** in the two markets as it can raise funds at rates systematically lower than company B. Company B, however, has a **comparative advantage** in raising dollars as the cost is only 0.50% higher than for company A, compared to the relative cost of 1.50% in yen. Conversely, company A must have a comparative advantage in raising yen.

TABLE 10-2a Cost of Capital Comparison

Company	Yen	Dollar
A	5.00%	9.5%
B	6.50%	10.0%

This provides the basis for a swap which will be to the mutual advantage of both parties. If both institutions directly issue funds in their final desired market, the total cost will be 9.5% (for A) and 6.5% (for B), for a total of 16.0%. In contrast, the total cost of raising capital where each has a comparative advantage is 5.0% (for A) and 10.0% (for B), for a total of 15.0%. The gain to both parties from entering a swap is 16.0 − 15.0 = 1.00%. For instance, the swap described in Tables 10-2b and 10-2c splits the benefit equally between the two parties.

TABLE 10-2b Swap to Company A

Operation	Yen	Dollar
Issue debt	Pay yen 5.0%	
Enter swap	Receive yen 5.0%	Pay dollar 9.0%
Net		Pay dollar 9.0%
Direct cost		Pay dollar 9.5%
Savings		0.50%

Company A issues yen debt at 5.0%, then enters a swap whereby it promises to pay 9.0% in dollar in exchange for receiving 5.0% yen payments. Its effective funding cost is therefore 9.0%, which is less than the direct cost by 50bp.

TABLE 10-2c Swap to Company B

Operation	Dollar	Yen
Issue debt	Pay dollar 10.0%	
Enter swap	Receive dollar 9.0%	Pay yen 5.0%
Net		Pay yen 6.0%
Direct cost		Pay yen 6.5%
Savings		0.50%

Similarly, company B issues dollar debt at 10.0%, then enters a swap whereby it receives 9.0% dollar in exchange for paying 5.0% yen. If we add up the difference in dollar funding cost of 1.0% to the 5.0% yen funding costs, the effective funding cost is therefore 6.0%, which is less than the direct cost by 50bp.[2] Both parties benefit from the swap.

While payments are typically netted for an interest rate swap, since they are in the same currency, this is not the case for currency swaps. At initiation and termination, there is exchange of principal in different currencies. Full interest payments are also made in different currencies. For instance, assuming annual payments, company A will receive 5.0% on a notional of Y10b, which is Y500 million in exchange for paying 9.0% on a notional of $100 million, or $9 million every year.

10.2.2 Pricing

Consider now the pricing of the swap to company A. This involves receiving 5.0% yen in exchange for paying 9.0% dollars. As with interest rate swaps, we can price the swap using either of two approaches, taking the difference between two bond prices or valuing a sequence of forward contracts.

This swap is equivalent to a long position in a fixed-rate, 5% 10-year yen denominated bond and a short position in a 10-year 9% dollar denominated bond. The value of the swap is that of a long yen bond minus a dollar bond. Defining S as the dollar price of the yen and P and P^* as the dollar and yen bond, we have:

$$V = S(\$/Y)P^*(Y) - P(\$) \qquad (10.1)$$

[2] Note that B is somewhat exposed to currency risk, as funding costs cannot be simply added when they are denominated in different currencies. The error, however, is of second-order magnitude.

Here, we indicate the value of the yen bond by an asterisk, P^*.

In general, the bond value can be written as $P(c, r, F)$ where the coupon is c, the yield is r and the face value is F. Our swap is initially worth (in millions)

$$V = (1/100)P(5\%, 5\%, Y10000) - P(9\%, 9\%, \$100) = (\$1/Y100)Y10000 - \$100 = \$0$$

Thus, the initial value of the swap is zero. Here, we assumed a flat term structure for both countries and no credit risk.

We can identify conditions under which the swap will be in-the-money. This will happen:

(1) If the value of the yen S appreciates
(2) If the yen interest rate r^* falls
(3) If the dollar interest rate r goes up

Thus the swap is exposed to three risk factors, the spot rate, and two interest rates. The latter exposures are given by the duration of the equivalent bond.

Key concept:
A position in a receive-foreign currency swap is equivalent to a long position in a foreign currency bond offset by a short position in a dollar bond.

The swap can be alternatively valued as a sequence of forward contracts. Recall that the valuation of a forward contract on one yen is given by

$$V_i = (F_i - K)\exp(-r_i\tau_i) \tag{10.2}$$

using continuous compounding. Here, r_i is the dollar interest rate, F_i is the prevailing forward rate (in \$/yen), K is the locked-in rate of exchange defined as the ratio of the dollar to yen payment on this maturity. Extending this to multiple maturities, the swap is valued as

$$V = \sum_i n_i(F_i - K)\exp(-r_i\tau_i) \tag{10.3}$$

where n_iF_i is the dollar value of the yen payments translated at the forward rate and the other term n_iK is the dollar payment in exchange.

Table 10-3 compares the two approaches for a 3-year swap with annual payments. Market rates have now changed and are $r = 8\%$ for U.S. yields, $r^* = 4\%$ for yen yields. We assume annual compounding. The spot exchange rate has moved from 100Y/\$ to 95Y/\$, reflecting a depreciation of the dollar (or appreciation of the yen).

TABLE 10-3 Pricing a Currency Swap

	Specifications		
	Notional Amount (millions)	Swap Coupon	Market Yield
Dollar	\$100	9%	8%
Yen	Y10,000	5%	4%
Exchange rate:			
initial	100Y/\$		
market	95Y/\$		

Valuation Using Bond Approach (millions)						
	Dollar Bond			Yen Bond		
Time (year)	Dollar Payment	PV(\$1)	PV(CF)	Yen Payment	PV(Y1)	PV(CF)
1	9	0.9259	8.333	500	0.9615	480.769
2	9	0.8573	7.716	500	0.9246	462.278
3	109	0.7938	86.528	10500	0.8890	9334.462
Total			\$102.58			Y10,277.51
Swap (\$)			−\$102.58			\$108.18
Value						\$5.61

Valuation Using Forward Contract Approach (millions)						
Time (year)	Forward Rate (Y/\$)	Yen Receipt (Y)	Yen Receipt (\$)	Dollar Payment (\$)	Difference CF (\$)	PV(CF) (\$)
1	91.48	500	5.47	−9.00	−3.534	−3.273
2	88.09	500	5.68	−9.00	−3.324	−2.850
3	84.83	10500	123.78	−109.00	14.776	11.730
Value						\$5.61

The middle panel shows the valuation using the difference between the two bonds. First, we discount the cash flows in each currency at the newly prevailing yield. This gives $P = \$102.58$ for the dollar bond and Y10,277.51 for the yen bond. Translating the latter at the new spot rate of Y95, we get \$108.18. The swap is now valued at \$108.18 − \$102.58, which is a positive value of $V = \$5.61$ million. The appreciation of the swap is principally driven by the appreciation of the yen.

The bottom panel shows how the swap can be valued by a sequence of forward contracts. First, we compute the forward rates for the three maturities. For example, the 1-year rate is $95 \times (1 + 4\%)/(1 + 8\%) = 91.48\ Y/\$$, by interest rate parity. Next, we convert each yen receipt into dollars at the forward rate, for example Y500 million in one year, which is \$5.47 million. This is offset against a payment of \$9 million, for a net planned cash outflow of −\$3.53 million. Discounting and adding up the planned cash flows, we get $V = \$5.61$ million, which must be exactly equal to the value found using the alternative approach.

Example 10-1: FRM Exam 1999—Question 37/Capital Markets

10-1. The table below shows quoted fixed borrowing rates (adjusted for taxes) in two different currencies for two different firms:

	Yen	Pounds
Company A	2%	4%
Company B	3%	6%

Which of the following is *true*?

a) Company A has a comparative advantage borrowing in both yen and pounds.
b) Company A has a comparative advantage borrowing in pounds.
c) Company A has a comparative advantage borrowing in yen.
d) Company A can arbitrage by borrowing in yen and lending in pounds.

Example 10-2: FRM Exam 2001—Question 67

10-2. Consider the following currency swap: Counterparty A swaps 3% on \$25 million for 7.5% on 20 million Sterling. There are now 18 months remaining in the swap, the term structures of interest rates are flat in both countries with dollar rates currently at 4.25% and Sterling rates currently at 7.75%. The current \$/Sterling exchange rate is \$1.65. Calculate the value of the swap. Use continuous compounding. Assume 6 months until the next annual coupon and use current market rates to discount.

a) $-\$1,237,500$
b) $-\$4,893,963$
c) $-\$9,068,742$
d) $-\$8,250,000$

10.3 Commodities

10.3.1 Products

Commodities are typically traded on exchanges. Contracts include spot, futures, and options on futures. There is also an OTC market for long-term commodity swaps, where payments are tied to the price of a commodity against a fixed or floating rate.

Commodity contracts can be classified into:

- **Agricultural products**, including grains and oilseeds (corn, wheat, soybean) food and fiber (cocoa, coffee, sugar, orange juice)
- **Livestock and meat** (cattle, hogs)
- **Base metals** (aluminum, copper, nickel, and zinc)
- **Precious metals** (gold, silver, platinum), and
- **Energy products** (natural gas, heating oil, unleaded gasoline, crude oil)

The **Goldman Sachs Commodity Index** (GSCI) is a broad index of commodity price performance, containing 49% energy products, 9% industrial/base metals, 3% precious metals, 28% agricultural products, and 12% livestock products. The CME trades futures and options contracts on the GSCI.

In the last five years, active markets have developed for **electricity products**, electricity futures for delivery at specific locations, for instance California/Oregon border (COB), Palo Verde, and so on. These markets have mushroomed following the deregulation of electricity prices, which has led to more variability in electricity prices.

More recently, OTC markets and exchanges have introduced **weather derivatives**, where the payout is indexed to temperature or precipitation. On the CME, for instance, contract payouts are based on the "Degree Day Index" over a calendar month. This index measures the extent to which the daily temperature deviates from the average. These contracts allow users to hedge situations where their income is negatively affected by extreme weather. Markets are also evolving in newer products, such as indices of consumer bankruptcy and catastrophe insurance contracts.

Such commodity markets allow participants to exchange risks. Farmers, for instance, can sell their crops at a fixed price on a future date, insuring themselves against variations in crop prices. Likewise, consumers can buy these crops at a fixed price.

10.3.2 Pricing of Futures

Commodities differ from financial assets in two notable dimensions: they may be expensive, even impossible, to store and they may generate a flow of benefits that are not directly measurable.

The first dimension involves the cost of carrying a physical inventory of commodities. For most financial instruments, this cost is negligible. For bulky commodities, this cost may be high. Other commodities, like electricity cannot be stored easily.

The second dimension involves the benefit from holding the physical commodity. For instance, a company that manufactures copper pipes benefits from an inventory of copper which is used up in its production process. This flow is also called **convenience yield** for the holder. For a financial asset, this flow would be a monetary income payment for the investor.

Consider the first factor, storage cost only. The cash-and-carry relationship should be modified as follows. We compare two positions. In the first, we buy the commodity spot plus pay up front the present value of storage costs PV(C). In the second, we enter a forward contract and invest the present value of the forward price. Since the two positions are identical at expiration, they must have the same initial value:

$$F_t e^{-r\tau} = S_t + \text{PV}(C) \tag{10.4}$$

where $e^{-r\tau}$ is the present value factor. Alternatively, if storage costs are incurred per unit time and defined as c, we can restate this relationship as

$$F_t e^{-r\tau} = S_t e^{c\tau} \tag{10.5}$$

Due to these costs, the forward rate should be much greater than the spot rate, as the holder of a forward contract benefits not only from the time value of money but also from avoiding storage costs.

Example: Computing the forward price of gold

Let us use data from December 1999. The spot price of gold is $S = \$288$, the 1-year interest rate is $r = 5.73\%$ (continuously compounded), and storage costs are \$2 per ounce per year, paid up front. The fair price for a 1-year forward contract should be $F = [S + \text{PV}(C)]e^{r\tau} = [\$288 + \$2]e^{5.73\%} = \$307.1$.

Let us now turn to the convenience yield, which can be expressed as y per unit time. In fact, y represents the net benefit from holding the commodity, after storage costs. Following the same reasoning as before, the forward price on a commodity should be given by

$$F_t e^{-r\tau} = S_t e^{-y\tau} \tag{10.6}$$

where $e^{-y\tau}$ is an actualization factor. This factor may have an economically identifiable meaning, reflecting demand and supply conditions in the cash and futures markets. Alternatively, it can be viewed as a *plug-in* that, given F, S, and $e^{-r\tau}$, will make Equation (10.6) balance.

FIGURE 10-1 Spot and Futures Prices for Crude Oil

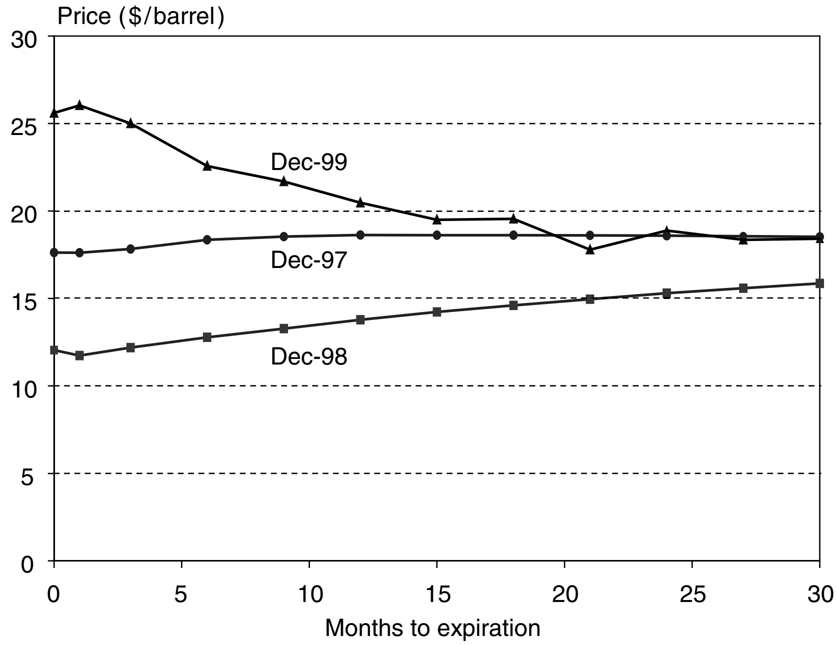


Figure 10-1, for example, displays the shape of the term structure of spot and futures prices for the New York Mercantile Exchange (NYMEX) crude oil contract. On December 1997, the term structure is relatively flat. On December 1998, the term structure becomes strongly upward sloping. Part of this slope can be explained by the time value of money (the term $e^{-r\tau}$ in the equation). In contrast, the term structure is downward sloping on December 1999. This can be interpreted in terms of a large convenience yield from holding the physical asset (in other words, the term $e^{-y\tau}$ in the equation dominates).

Let us focus for example on the 1-year contract. Using $S = \$25.60$, $F = \$20.47$, $r = 5.73\%$ and solving for y,

$$y = r - \frac{1}{\tau}\ln(F/S) \tag{10.7}$$

we find $y = 28.10\%$, which is quite large. In fact, variations in y can be substantial. Just one year before, a similar calculation would have given $y = -9\%$, which implies a negative convenience yield, or a storage cost.

Table 10-4 displays futures prices for selected contracts. Futures prices are generally increasing with maturity, reflecting the time value of money, storage cost and low convenience yields. There are some irregularities, however, reflecting anticipated

TABLE 10-4 Futures Prices as of December 30, 1999

Maturity	Corn	Sugar	Copper	Gold	Nat.Gas	Gasoline
Jan			85.25	288.5		.6910
Mar	204.5	18.24	86.30	290.6	2.328	.6750
July	218.0	19.00	87.10	294.9	2.377	.6675
Sept	224.0	19.85	87.90	297.0	2.418	.6245
Dec	233.8	18.91	88.45	300.1	2.689	
Mar01	241.5	18.90	88.75	303.2	2.494	
...						
Dec01	253.5			312.9	2.688	

imbalances between demand and supply. For instance, gasoline futures prices increase in the summer due to increased driving. Natural gas displays the opposite pattern, where prices increase during the winter due to the demand for heating. Agricultural products can also be highly seasonal. In contrast, futures prices for gold are going up monotonically with time, since this is a perfectly storable good.

10.3.3 Futures and Expected Spot Prices

An interesting issue is whether today's futures price gives the best forecast of the future spot price. If so, it satisfies the **expectations hypothesis**, which can be written as:

$$F_t = E_t[S_T] \tag{10.8}$$

The reason this relationship may hold is as follows. Say that the 1-year oil futures price is $F = \$20.47$. If the market forecasts that oil prices in one year will be at \$25, one could make a profit by going long a futures contract at the cheap futures price of $F = \$20.47$, waiting a year, then buying oil at \$20.47, and reselling it at the higher price of \$25. In other words, deviations from this relationship imply **speculative profits**.

To be sure, these profits are not risk-free. Hence, they may represent some compensation for risk. For instance, if the market is dominated by producers who want to hedge by selling oil futures, F will be abnormally low compared with expectations. Thus the relationship between futures prices and expected spot prices can be complex.

For financial assets for which the arbitrage between cash and futures is easy, the futures or forward rate is solely determined by the cash-and-carry relationship, i.e. the interest rate and income on the asset. For commodities, however, the arbitrage may not be so easy. As a result, the futures price may deviate from the cash-and-carry relationship through this convenience yield factor. Such prices may reflect expectations of futures spot prices, as well as speculative and hedging pressures.

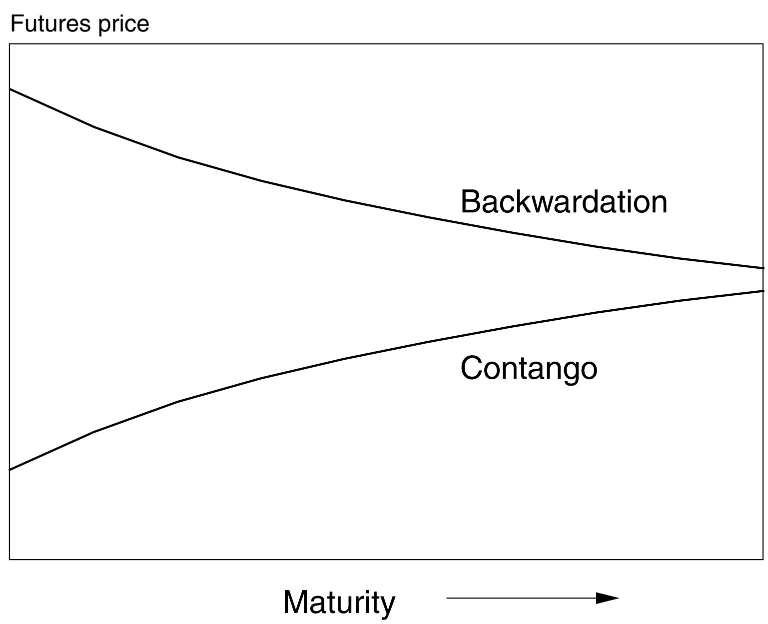
A market is said to be in **contango** when the futures price trades at a premium relative to the spot price, as shown in Figure 10-2. Using Equation (10.7), this implies that the convenience yield is smaller than the interest rate $y < r$.

A market is said to be in **backwardation** (or inverted) when forward prices trade at a discount relative to spot prices. This implies that the convenience yield is greater than the interest rate $y > r$. In other words, a high convenience yields puts a higher price on the cash market, as there is great demand for immediate consumption of the commodity.

With backwardation, the futures price tends to increase as the contract nears maturity. In such a situation, a **roll-over strategy** should be profitable, provided prices do not move too much. This involves by buying a long maturity contract, waiting, and then selling it at a higher price in exchange for buying a cheaper, longer-term contract.

This strategy is comparable to **riding the yield curve** when positively sloped. This involves buying long maturities and waiting to have yields fall due to the passage of

FIGURE 10-2 Patterns of Contango and Backwardation



time. If the shape of the yield curve does not change too much, this will generate a capital gain from bond price appreciation.

This was basically the strategy followed by Metallgesellschaft Refining & Marketing (MGRM), the U.S. subsidiary of Metallgesellschaft, which rolled over purchases of WTI crude oil futures as a hedge against OTC sales to customers. The problem was that the basis $S - F$, which had been generally positive, turned negative, creating losses for the company. In addition, these losses caused cash flow, or liquidity problems. MGRM ended up liquidating the positions, which led to a realized loss of $1.3 billion.

Example 10-3: FRM Exam 1999—Question 32/Capital Markets

10-3. The spot price of corn on April 10th is 207 cents/bushels. The futures price of the September contract is 241.5 cents/bushels. If hedgers are net short, which of the following statements is *most* accurate concerning the expected spot price of corn in September?

a) The expected spot price of corn is higher than 207.
b) The expected spot price of corn is lower than 207.
c) The expected spot price of corn is higher than 241.5.
d) The expected spot price of corn is lower than 241.5.

Example 10-4: FRM Exam 1998—Question 24/Capital Markets

10-4. In commodity markets, the complex relationships between spot and forward prices are embodied in the commodity price curve. Which of the following statements is *true*?

a) In a backwardation market, the discount in forward prices relative to the spot price represents a positive yield for the commodity supplier.
b) In a backwardation market, the discount in forward prices relative to the spot price represents a positive yield for the commodity consumer.
c) In a contango market, the discount in forward prices relative to the spot price represents a positive yield for the commodity supplier.
d) In a contango market, the discount in forward prices relative to the spot price represents a positive yield for the commodity consumer.

Example 10-5: FRM Exam 1998—Question 48/Capital Markets

10-5. If a commodity is more expensive for immediate delivery than for future delivery, the commodity curve is said to be in

a) Contango
b) Backwardation
c) Reversal
d) None of the above

Example 10-6: FRM Exam 1997—Question 45/Market Risk

10-6. In the commodity markets being long the future and short the cash exposes you to which of the following risks?
a) Increasing backwardation
b) Increasing contango
c) Change in volatility of the commodity
d) Decreasing convexity

Example 10-7: FRM Exam 1998—Question 27/Capital Markets

10-7. Metallgesellschaft AG's oil hedging program used a *stack-and-roll* strategy that eventually led to large losses. What can be said about this strategy? The strategy involved
a) Buying short-dated futures or forward contracts to hedge long-term exposure, hence expecting the short-term oil price would not decline
b) Buying short-dated futures or forward contracts to hedge long-term exposure, hence expecting the short-term oil price would decline
c) Selling short-dated futures or forward contracts to hedge long-term exposure, hence expecting the short-term oil price would not decline
d) Selling short-dated futures or forward contracts to hedge long-term exposure, hence expecting the short-term oil price would decline

10.4 Answers to Chapter Examples

Example 10-1: FRM Exam 1999—Question 37/Capital Markets

b) A company can only have a comparative advantage in one currency, that with the greatest difference in capital cost, 2% for pounds versus 1% for yen.

Example 10-2: FRM Exam 2001—Question 67

c) As in Table 10-3, we use the bond valuation approach. The receive-dollar swap is equivalent to a long position in the dollar bond and a short position in the sterling bond.

	Dollar Bond			Sterling Bond		
Time (year)	Dollar Payment	PV($1) (4.25%)	PV(CF) (dollars)	Sterling Payment	PV(GBP1) (7.75%)	PV(CF) (sterling)
1	750,000	0.97897	734,231	1,500,000	0.96199	1,442,987
2	25,750,000	0.93824	24,159,668	21,500,000	0.89025	19,140,432
Total			24,893,899			20,583,418
Dollars			+$24,893,899			−$33,962,640
Value						−$9,068,742

Example 10-3: FRM Exam 1999—Question 32/Capital Markets

c) If hedgers are net short, they are selling corn futures even if it involves a risk premium such that the selling price is lower than the expected future spot price. Thus the expected spot price of corn is higher than the futures price. Note that the current spot price is irrelevant.

Example 10-4: FRM Exam 1998—Question 24/Capital Markets

b) First, forward prices are only at a discount versus spot prices in a backwardation market. The high spot price represents a convenience yield to the consumer of the product, who holds the physical asset.

Example 10-5: FRM Exam 1998—Question 48/Capital Markets

b) Backwardation means that the spot price is greater than futures price.

Example 10-6: FRM Exam 1997—Question 45/Market Risk

a) Shorting the cash exposes the position to increasing cash prices, assuming, for instance, fixed futures prices, hence increasing backwardation.

Example 10-7: FRM Exam 1998—Question 27/Capital Markets

a) Because MG was selling oil forward to clients, it had to hedge by buying short-dated futures oil contracts. In theory, price declines in one market were to be offset by gains in another. In futures markets, however, losses are realized immediately, which may lead to liquidity problems (and did so). Thus, the expectation was that oil prices would stay constant.

PART three

Market Risk Management

Chapter 11

Introduction to Market Risk Measurement

This chapter provides an introduction to the measurement of market risk. Market risk is primarily measured with **value at risk** (VAR). VAR is a statistical measure of downside risk that is simple to explain. VAR measures the *total* portfolio risk, taking into account portfolio diversification and leverage.

In theory, risk managers should report the entire distribution of profits and losses over the specified horizon. In practice, this distribution is summarized by one number, the worst loss at a specified confidence level, such as 99 percent. VAR, however, is only one of the measures that risk managers focus on. It should be complemented by **stress testing**, which identifies potential losses under extreme market conditions, which are associated with much higher confidence levels.

Section 16.1 gives a brief overview of the history of risk measurement systems. Section 16.2 then shows how to compute VAR for a very simple portfolio. It also discusses caveats, or pitfalls to be aware of when interpreting VAR numbers. Section 16.3 turns to the choice of VAR parameters, that is, the confidence level and horizon. Next, Section 16.4 describes the broad components of a VAR system. Section 16.5 shows to complement VAR by stress tests. Finally, Section 16.6 shows how VAR methods, primarily developed for financial institutions, are now applied to measures of cash flow at risk.

11.1 Introduction to Financial Market Risks

Market risk measurement attempts to quantify the risk of losses due movements in financial market variables. The variables include interest rates, foreign exchange rates, equities, and commodities. Positions can include cash or derivative instruments.

In the past, risks were measured using a variety of ad hoc tools, none of which was satisfactory. These included **notional amounts**, **sensitivity measures**, and **scenarios**. While these measures provide some intuition of risk, they do not measure what matters, that is, the downside risk for the total portfolio. They fail to take into account correlations across risk factors. In addition, they do not account for the probability of adverse moves in the risk factors.

Consider for instance a 5-year **inverse floater**, which pays a coupon equal to 16 percent minus twice current LIBOR, if positive, on a notional principal of $100 million. The initial market value of the note is $100 million. This investment is extremely sensitive to movements in interest rates. If rates go up, the present value of the cash flows will drop sharply. In addition, discount rate also increases. The combination of a decrease in the numerator terms and an increase in the denominator terms will push the price down sharply.

The question is, how much could an investor lose on this investment over a specified horizon? The *notional amount* only provide an indication of the potential loss. The worst case scenario is one where interest rates rise above 8 percent. In this situation, the coupon will drop to zero and the bond becomes a deeply-discounted bond. Discounting at 8 percent, the value of the bond will drop to $68 million. This gives a loss of $\$100 - \$68 = \$32$ million, which is much less than the notional.

A *sensitivity measure* such as duration is more helpful. As we have seen in Chapter 7, the bond has three times the duration of a similar 5-year note. This gives a modified duration of $D = 3 \times 4 = 12$ years. This duration measure reveals the extreme sensitivity of the bond to interest rates but does not answer the question of whether such a disastrous movement in interest rates is likely. It also ignores the nonlinearity between the note price and yields.

Scenario analysis provides some improvement, as it allows the investor to investigate nonlinear, extreme effects in price. But again, the method does not associate the loss with a probability.

Another general problem is that these sensitivity or scenario measures do not allow the investor to aggregate risk across different markets. Let us say that this investor also holds a position in a bond denominated in Euros. Do the risks add up, or diversify each other?

The great beauty of value at risk (VAR) is that it provides a neat answer to all these questions. One number aggregates the risks across the whole portfolio, taking into account leverage and diversification, and providing a risk measure with an associated probability.

If the worst increase in yield at the 95% level is 1.645, we can compute VAR as

$$\text{VAR} = \text{Market value} \times \text{Modified Duration} \times \text{Worst yield increase} \qquad (11.1)$$

This gives VAR $= \$100 \times 12 \times 0.0165 = \$19.8$ millions. Or, we could reprice the note on the target date under the worst increase in yield scenario.

The investor can now make a statement such as *the worst loss at the 95% confidence level is approximately $20 million*, with appropriate caveats. This is a huge improvement over traditional risk measurement methods, as it expresses risk in an intuitive fashion, bringing risk transparency to the masses.

The VAR revolution started in 1993 when it was endorsed by the Group of Thirty (G-30) as part of "best practices" for dealing with derivatives. The methodology behind VAR, however, is not new. It results from a merging of finance theory, which focuses on the pricing and sensitivity of financial instruments, and statistics, which studies the behavior of the risk factors. As Table 11-1 shows, VAR could not have happened without its predecessor tools. VAR revolutionized risk management by applying consistent firm-wide risk measures to the market risk of an institution. These methods are now extended to credit risk, operational risk, and the holy grail of integrated, or firm-wide, risk management.

TABLE 11-1 The Evolution of Analytical Risk-Management Tools

1938	Bond duration
1952	Markowitz mean-variance framework
1963	Sharpe's capital asset pricing model
1966	Multiple factor models
1973	Black-Scholes option pricing model, "Greeks"
1988	Risk-weighted assets for banks
1993	Value at Risk
1994	RiskMetrics
1997	CreditMetrics, CreditRisk+
1998	Integration of credit and market risk
1998	Risk budgeting

11.2 VAR as Downside Risk

11.2.1 VAR: Definition

VAR is a summary measure of the downside risk, expressed in dollars. A general definition is

> *VAR is the maximum loss over a target horizon such that there is a low, prespecified probability that the actual loss will be larger.*

Consider for instance a position of $4 billion short the yen, long the dollar. This position corresponds to a well-known hedge fund that took a bet that the yen would fall in value against the dollar. How much could this position lose over a day?

To answer this question, we could use 10 years of historical daily data on the yen/dollar rate and simulate a daily return. The simulated daily return in dollars is then

$$R_t(\$) = Q_0(\$)[S_t - S_{t-1}]/S_{t-1} \tag{11.2}$$

where Q_0 is the current dollar value of the position and S is the spot rate in yen per dollar measured over two consecutive days.

For instance, for two hypothetical days $S_1 = 112.0$ and $S_2 = 111.8$. We then have a hypothetical return of

$$R_2(\$) = \$4,000\text{million} \times [111.8 - 112.0]/112.0 = -\$7.2\text{million}$$

So, the simulated return over the first day is $-\$7.2$ million. Repeating this operation over the whole sample, or 2,527 trading days, creates a time-series of fictitious returns, which is plotted in Figure 11-1.

We can now construct a frequency distribution of daily returns. For instance, there are four losses below $160 million, three losses between $160 million and $120 million, and so on. The histogram, or frequency distribution, is graphed in Figure 11-2. We can also order the losses from worst to best return.

We now wish to summarize the distribution by one number. We could describe the quantile, that is, the level of loss that will not be exceeded at some high **confidence level**. Select for instance this confidence level as $c = 95$ percent. This corresponds to a **right-tail probability**. We could as well define VAR in terms of a **left-tail probability**, which we write as $p = 1 - c$.

FIGURE 11-1 Simulated Daily Returns

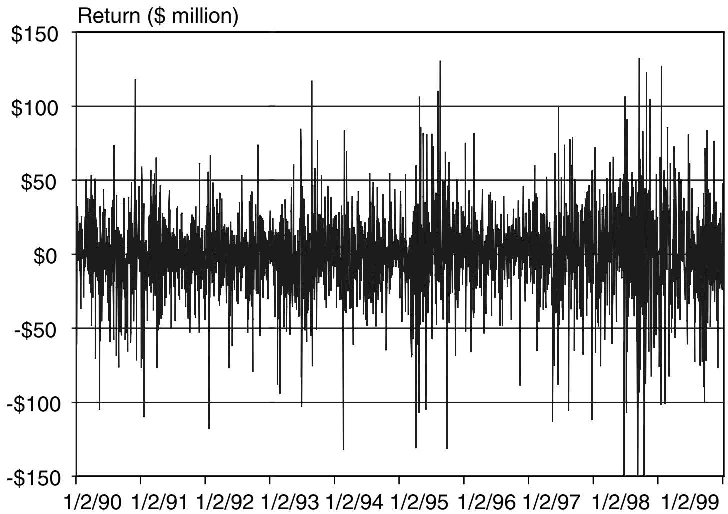
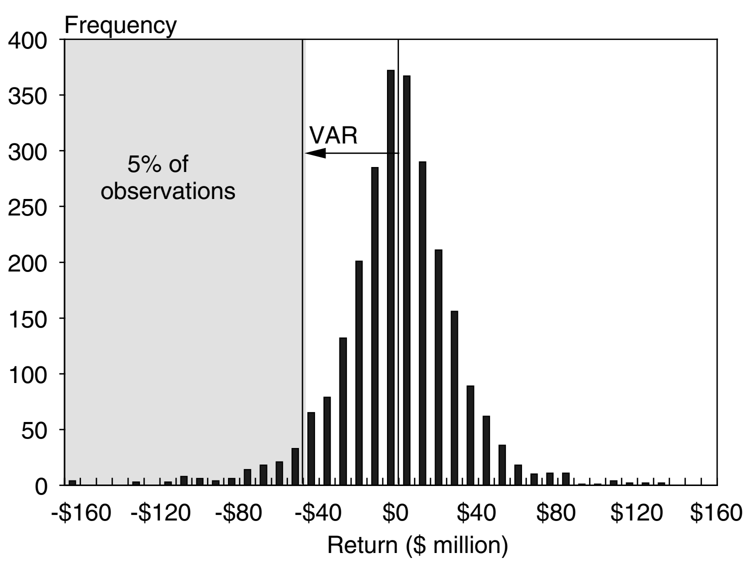


FIGURE 11-2 Distribution of Daily Returns



Defining x as the dollar profit or loss, VAR can be defined implicitly from

$$c = \int_{-\mathrm{VAR}}^{\infty} x f(x) dx \qquad (11.3)$$

Note that VAR measures a loss and therefore taken as a positive number. When the outcomes are discrete, VAR is the smallest loss such that the right-tail probability is at least c.

Sometimes, VAR is reported as the deviation between the mean and the quantile. This second definition is more consistent than the usual one. Because it considers the deviation between two values on the target date, it takes into account the time value of money. In most applications, however, the time horizon is very short and the mean, or expected profit is close to zero. As a result, the two definitions usually give similar values.

In this hedge fund example, we want to find the cutoff value R^* such that the probability of a loss worse than R^* is $p = 1 - c = 5$ percent. With a total of $T = 2,527$ observations, this corresponds to a total of $pT = 0.05 \times 2527 = 126$ observations in the left tail. We pick from the ordered distribution the cutoff value, which is $R^* =$ \$47.1 million. We can now make a statement such as:

> *The maximum loss over one day is about \$47 million at the 95 percent confidence level.*

This vividly describes risk in a way that notional amounts or exposures cannot convey.

From the confidence level, we can determine the number of expected exceedences n over a period of N days:

$$n = p \times N \qquad (11.4)$$

Example 11-1: FRM Exam 1999—Question 89/Market Risk

11-1. What is the correct interpretation of a \$3 million overnight VAR figure with 99% confidence level? The institution

a) Can be expected to lose at most \$3 million in 1 out of next 100 days
b) Can be expected to lose at least \$3 million in 95 out of next 100 days
c) Can be expected to lose at least \$3 million in 1 out of next 100 days
d) Can be expected to lose at most \$6 million in 2 out of next 100 days

11.2.2 VAR: Caveats

VAR is a useful summary measure of risk. Its application, however, is subject to some caveats.

- *VAR does not describe the worst loss.* This is not what VAR is designed to measure. Indeed we would expect the VAR number to be exceeded with a frequency of p, that is 5 days out of a hundred for a 95 percent confidence level. This is perfectly normal. In fact, backtesting procedures are designed to check whether the frequency of exceedences is in line with p.
- *VAR does not describe the losses in the left tail.* VAR does not say anything about the distribution of losses in its left tail. It just indicates the probability of such a value occurring. For the same VAR number, however, we can have very different distribution shapes. In the case of Figure 11-2, the average value of the losses worse than $47 million is around $74 million, which is 60 percent worse than the VAR. So, it would be unusual to sustain many losses beyond $200 million.

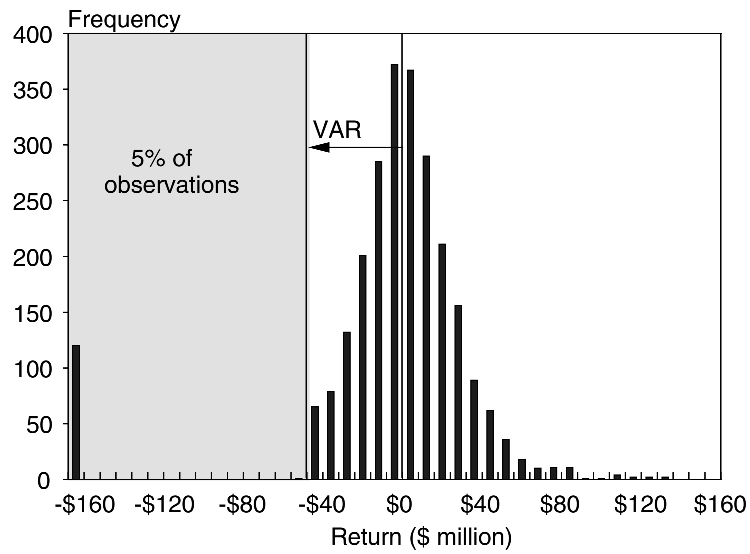
Instead, Figure 11-3 shows a distribution with the same VAR, but with 125 occurrences of large losses beyond $160 million. This graph shows that, while the VAR number is still $47 million, there is a high probability of sustaining very large losses.

- *VAR is measured with some error.* The VAR number itself is subject to normal sampling variation. In our example, we used ten years of daily data. Another sample period, or a period of different length, will lead to a different VAR number. Different statistical methodologies or simplifications can also lead to different VAR numbers. One can experiment with sample periods and methodologies to get a sense of the precision in VAR. Hence, it is useful to remember that there is limited precision in VAR numbers. What matters is the first-order magnitude.

11.2.3 Alternative Measures of Risk

The conventional VAR measure is the *quantile* of the distribution measured in dollars. This single number is a convenient summary, but its very simplicity may be dangerous. We have seen in Figure 11-3 that the same VAR can hide very different distribution patterns. The appendix reviews desirable properties for risk measures and shows that VAR may be inconsistent under some conditions. In particular, the VAR of a

FIGURE 11-3 Altered Distribution with Same VAR



portfolio can be greater than the sum of subportfolios VARs. If so, merging portfolios can increase risk, which is a strange result.

Alternative measures of risk are

- *The entire distribution* In our example, VAR is simply one quantile in the distribution. The risk manager, however, has access to the whole distribution and could report a range of VAR numbers for increasing confidence levels.
- *The conditional VAR* A related concept is the expected value of the loss when it exceeds VAR. This measures the average of the loss conditional on the fact that it is greater than VAR. Define the VAR number as q. Formally, the **conditional VAR** (CVAR) is

$$E[X \mid X < q] = \int_{-\infty}^{q} xf(x)dx \Big/ \int_{-\infty}^{q} f(x)dx \tag{11.5}$$

Note that the denominator represents the probability of a loss exceeding VAR, which is also c. This ratio is also called **expected shortfall**, **tail conditional expectation**, **conditional loss**, or **expected tail loss**. It tells us how much we could lose if we are "hit" beyond VAR. For example, for our yen position, this value is

$$\text{CVAR} = \$74 \text{ million}$$

This is measured as the average loss beyond the $47 million VAR.

- *The standard deviation* A simple summary measure of the distribution is the usual standard deviation (SD)

$$\mathrm{SD}(X) = \sqrt{\frac{1}{(N-1)}\sum_{i=1}^{N}[x_i - E(X)]^2} \tag{11.6}$$

The advantage of this measure is that it takes into account all observations, not just the few around the quantile. Any large negative value, for example, will affect the computation of the variance, increasing SD(X). If we are willing to take a stand on the shape of the distribution, say normal or Student's t, we do know that the standard deviation is the most efficient measure of dispersion. For example, for our yen position, this value is

$$\mathrm{SD} = \$29.7 \text{ million}$$

Using a normal approximation and $\alpha = 1.645$, we get a VAR estimate of $49 million, which is not far from the empirical quantile of $47 million. Under these conditions, VAR inherits all properties of the standard deviation. In particular, the SD of a portfolio must be smaller than the sum of the SDs of subportfolios.

The disadvantage of the standard deviation is that it is symmetrical and cannot distinguish between large losses or gains. Also, computing VAR from SD requires a distributional assumption, which may not be valid.

- *The semi-standard deviation* This is a simple extension of the usual standard deviation that considers only data points that represent a loss. Define N_L as the number of such points. The measure is

$$\mathrm{SD}_L(X) = \sqrt{\frac{1}{(N_L-1)}\sum_{i=1}^{N}[\mathrm{Min}(x_i, 0) - E(X)]^2}$$

where the data are averaged over N_L. In practice, this is rarely used.

Example 11-2: FRM Exam 1998—Question 22/Capital Markets

11-2. Considering arbitrary portfolios A and B, and their combined portfolio C, which of the following relationships *always* holds for VARs of A, B, and C?

a) $\mathrm{VAR}_A + \mathrm{VAR}_B = \mathrm{VAR}_C$

b) $\mathrm{VAR}_A + \mathrm{VAR}_B \geq \mathrm{VAR}_C$

c) $\mathrm{VAR}_A + \mathrm{VAR}_B \leq \mathrm{VAR}_C$

d) None of the above

11.3 VAR: Parameters

To measure VAR, we first need to define two quantitative parameters, the confidence level and the horizon.

11.3.1 Confidence Level

The higher the confidence level c, the greater the VAR measure. Varying the confidence level provides useful information about the return distribution and potential extreme losses. It is not clear, however, whether one should stop at 99%, 99.9%, 99.99% and so on. Each of these values will create an increasingly larger loss, but less likely.

Another problem is that, as c increases, the number of occurrences below VAR shrinks, leading to poor measures of large but unlikely losses. With 1000 observations, for example, VAR can be taken as the 10th lowest observation for a 99% confidence level. If the confidence level increases to 99.9%, VAR is taken from the lowest observation only. Finally, there is no simple way to estimate a 99.99% VAR from this sample.

The choice of the confidence level depends on the use of VAR. For most applications, VAR is simply a benchmark measure of downside risk. If so, what really matters is *consistency* of the VAR confidence level across trading desks or time.

In contrast, if the VAR number is being used to decide how much capital to set aside to avoid bankruptcy, then a high confidence level is advisable. Obviously, institutions would prefer to go bankrupt very infrequently. This **capital adequacy** use, however, applies to the overall institution and not to trading desks.

Another important point is that VAR models are only useful insofar as they can be verified. This is the purpose of backtesting, which systematically checks whether the frequency of losses exceeding VAR is in line with $p = 1 - c$. For this purpose, the risk manager should not choose a value of c that is too high. Picking, for instance, $c = 99.99\%$ should lead, on average, to one exceedence out of 10,000 trading days, or 40 years. In other words, it is going to be impossible to verify if the true probability associated with VAR is indeed 99.99 percent.

For all these reasons, the usual recommendation is to pick a confidence level that is not too high, such as 95 to 99 percent.

11.3.2 Horizon

The longer the horizon (T), the greater the VAR measure. This extrapolation depends on two factors, the behavior of the risk factors, and the portfolio positions.

To extrapolate from a one-day horizon to a longer horizon, we need to assume that returns are independently and identically distributed. This allows us to transform a daily volatility to a multiple-day volatility by multiplication by the square root of time. We also need to assume that the distribution of daily returns is unchanged for longer horizons, which restricts the class of distribution to the so-called "stable" family, of which the normal is a member. If so, we have

$$\text{VAR}(T \text{ days}) = \text{VAR}(1 \text{ day}) \times \sqrt{T} \tag{11.8}$$

This requires (1) the distribution to be invariant to the horizon (i.e., the same α, as for the normal), (2) the distribution to be the same for various horizons (i.e., no time decay in variances), and (3) innovations to be independent across days.

Key concept:
VAR can be extended from a $1-$day horizon to T days by multiplication by the square root of time. This adjustement is valid with i.i.d. returns that have a normal distribution.

The choice of the horizon also depends on the characteristics of the portfolio. If the positions change quickly, or if exposures (e.g., option deltas) change as underlying prices change, increasing the horizon will create "slippage" in the VAR measure.

Again, the choice of the horizon depends on the use of VAR. If the purpose is to provide an accurate benchmark measure of downside risk, the horizon should be relatively short, ideally less than the average period for major portfolio rebalancing.

In contrast, if the VAR number is being used to decide how much capital to set aside to avoid bankruptcy, then a long horizon is advisable. Institutions will want to have enough time for corrective action as problems start to develop.

In practice, the horizon cannot be less than the frequency of reporting of profits and losses. Typically, banks measure P&L on a daily basis, and corporates on a longer interval (ranging from daily to monthly). This interval is the minimum horizon for VAR.

Another criteria relates to the backtesting issue. Shorter time intervals create more data points matching the forecast VAR with the actual, subsequent P&L. As the power of the statistical tests increases with the number of observations, it is advisable to have a horizon as short as possible.

For all these reasons, the usual recommendation is to pick a horizon that is as short as feasible, for instance 1 day for trading desks. The horizon needs to be appropriate to the asset classes and the purpose of risk management. For institutions such as pension funds, for instance, a 1-month horizon may be more appropriate.

For **capital adequacy purposes**, institutions should select a high confidence level and a long horizon. There is a trade-off, however, between these two parameters. Increasing one or the other will increase VAR.

Example 11-3: FRM Exam 1997—Question 7/Risk Measurement

11-3. To convert VAR from a one-day holding period to a ten-day holding period the VAR number is generally multiplied by

a) 2.33
b) 3.16
c) 7.25
d) 10.00

Example 11-4: FRM Exam 2001—Question 114

11-4. Rank the following portfolios from least risky to most risky. Assume 252 trading days a year and there are 5 trading days per week.

Portfolio	VAR	Holding Period in Days	Confidence Interval
1	10		99
2	10		95
3	10	10	99
4	10	10	95
5	10	15	99
6	10	15	95

a) 5,3,6,1,4,2
b) 3,4,1,2,5,6
c) 5,6,1,2,3,4
d) 2,1,5,6,4,3

11.3.3 Application: The Basel Rules

The Basel market risk charge requires VAR to be computed with the following parameters:

a. A horizon of 10 trading days, or two calendar weeks

b. A 99 percent confidence interval

c. An observation period based on at least a year of historical data and updated at least once a quarter

The **Market Risk Charge** (MRC) is measured as follows:

$$\mathrm{MRC}_t^{IMA} = \mathrm{Max}\left(k \frac{1}{60} \sum_{i=1}^{60} \mathrm{VAR}_{t-i},\ \mathrm{VAR}_{t-1} \right) + SRC_t \qquad (11.9)$$

which involves the average of the market VAR over the last 60 days, times a supervisor-determined multiplier k (with a minimum value of 3), as well as yesterday's VAR, and a specific risk charge SRC.[1]

The Basel Committee allows the 10-day VAR to be obtained from an extrapolation of 1-day VAR figures. Thus VAR is really

$$\mathrm{VAR}_t(10, 99\%) = \sqrt{10} \times \mathrm{VAR}_t(1, 99\%)$$

Presumably, the 10-day period corresponds to the time required for corrective action by bank regulators should an institution start to run into trouble. Presumably as well, the 99 percent confidence level corresponds to a low probability of bank failure due to market risk. Even so, one occurrence every 100 periods implies a high frequency of failure. There are $52/2 = 26$ two-week periods in one year. Thus, one failure should be expected to happen every $100/26 = 3.8$ years, which is still much too frequent. This explains why the Basel Committee has applied a multiplier factor, $k \geq 3$ to guarantee further safety.

[1]The specific risk charge is designed to provide a buffer against losses due to idiosyncractic factors related to the individual issuer of the security. It includes the risk that an individual debt or equity moves by more or less than the general market, as well as event risk. Consider for instance a corporate bond issued by Ford Motor, a company with a credit rating of "BBB". component should capture the effect of movements in yields for an index of BBB-rated corporate bonds. In contrast, the SRC should capture the effect of credit downgrades for Ford. The SRC can be computed from the VAR of sub-portfolios of debt and equity positions that contain specific risk.

Example 11-5: FRM Exam 1997—Question 16/Regulatory

11-5. Which of the following quantitative standards is *not* required by the Amendment to the Capital Accord to Incorporate Market Risk?

a) Minimum holding period of 10 days

b) 99th percentile, one-tailed confidence interval

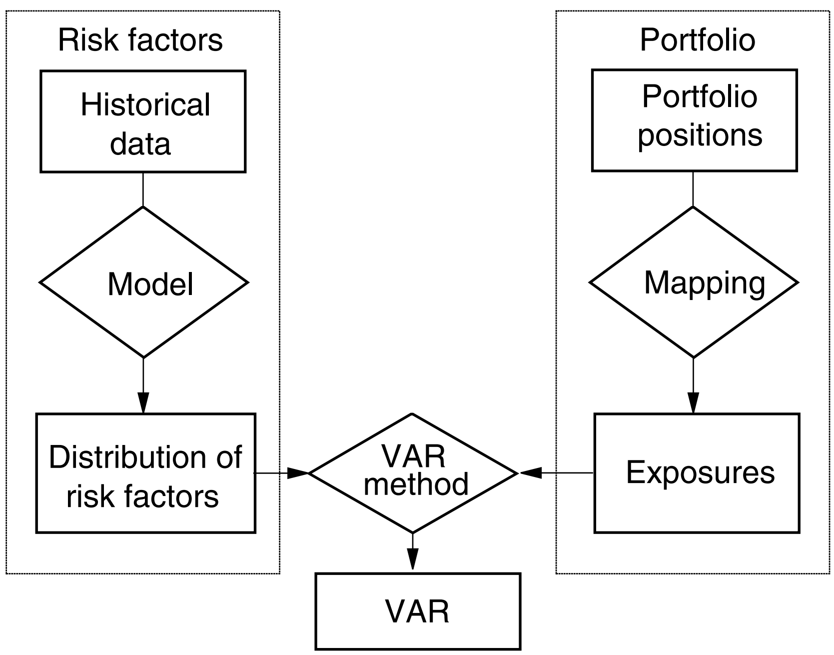
c) Minimum historical observation period of two years

d) Update of data sets at least quarterly

11.4 Elements of VAR Systems

We now turn to the analysis of elements of a VAR system. As described in Figure 11-4, a VAR system combines the following steps:

FIGURE 11-4 Elements of a VAR System



1. From market data, choose the distribution of risk factors (e.g., normal, empirical, or other).
2. Collect the portfolio positions and map them onto the risk factors.
3. Choose a VAR method (delta-normal, historical, Monte Carlo) and compute the portfolio VAR. These methods will be explained in a subsequent chapter.

11.4.1 Portfolio Positions

We start with portfolio positions. The assumption will be that the positions are constant over the horizon. This, of course, cannot be true in an environment where traders turn over their portfolio actively. Rather, it is a simplification.

The true risk can be greater or lower than the VAR measure. It can be greater if VAR is based on close-to-close positions that reflect lower trader limits. If traders take more risks during the day, the true risk will be greater than indicated by VAR. Conversely, the true risk can be lower if management enforces loss limits, in other words cuts down the risk that traders can take if losses develop.

Example 11-6: FRM Exam 1997—Question 23/Regulatory

11-6. The standard VAR calculation for extension to multiple periods also assumes that positions are fixed. If risk management enforces loss limits, the true VAR will be

a) The same
b) Greater than calculated
c) Less than calculated
d) Unable to be determined

11.4.2 Risk Factors

The **risk factors** represent a subset of all market variables that adequately span the risks of the current, or allowed, portfolio. There are literally tens of thousands of securities available, but a much more restricted set of useful risk factors.

The key is to choose market factors that are adequate for the portfolio. For a simple fixed-income portfolio, one bond market risk factor may be enough. In contrast, for a highly leveraged portfolio, multiple risk factors are needed. For an option portfolio, volatilities should be added as risk factors. In general, the more complex the portfolio, the greater the number of risk factors that should be used.

11.4.3 VAR Methods

Similarly, the choice of the method depends on the nature of the portfolio. For a fixed-income portfolio, a linear method may be adequate. In contrast, if the portfolio contains options, we need to include nonlinear effects. For simple, plain vanilla options, we may be able to approximate their price behavior with a first and second derivative (delta and gamma). For more complex options, such as digital or barrier options, this may not be sufficient.

This is why risk management is as much an art as a science. Risk managers need to make reasonable approximations to come up with a cost-efficient measure of risk. They also need to be aware of the fact that traders could be induced to find "holes" in the risk management system.

A VAR system alone will not provide effective protection against market risk. It needs to be used in combination with limits on notionals and on exposures and, in addition, should be supplemented by stress tests.

Example 11-7: FRM Exam 1997—Question 9/Regulatory

11-7. A trading desk has limits only in outright foreign exchange and outright interest rate risk. Which of the following products can not be traded within the current limit structure?

a) Vanilla interest rate swaps, bonds, and interest rate futures

b) Interest rate futures, vanilla interest rate swaps, and callable interest rate swaps

c) Repos and bonds

d) Foreign exchange swaps, and back-to-back exotic foreign exchange options

11.5 Stress-Testing

As shown in the yen example in Figure 11-2, VAR does not purport to measure the worst-ever loss that could happen. It should be complemented by **stress-testing**, which aims at identifying situations that could create extraordinary losses for the institution.

Stress-testing is a key risk management process, which includes (i) scenario analysis, (ii) stressing models, volatilities and correlations, and (iii) developing policy responses. **Scenario analysis** submits the portfolio to large movements in financial market variables. These scenarios can be created:

- *Moving key variables one at a time*, which is a simple and intuitive method. Unfortunately, it is difficult to assess realistic comovements in financial variables. It is unlikely that all variables will move in the worst possible direction at the same time.

- *Using historical scenarios*, for instance the 1987 stock market crash, the devaluation of the British pound in 1992, the bond market debacle of 1984, and so on.
- *Creating prospective scenarios*, for instance working through the effects, direct and indirect, of a U.S. stock market crash. Ideally, the scenario should be tailored to the portfolio at hand, assessing the worst thing that could happen to current positions.

The goal of stress-testing is to identify areas of potential vulnerability. This is not to say that the institution should be totally protected against every possible contingency, as this would make it impossible to take any risk. Rather, the objective of stress-testing and management response should be to ensure that the institution can withstand likely scenarios without going bankrupt.

Example 11-8: FRM Exam 1997—Question 4/Risk Measurement

11-8. The use of scenario analysis allows one to
a) Assess the behavior of portfolios under large moves.
b) Research market shocks which occurred in the past.
c) Analyze the distribution of historical P/L in the portfolio.
d) Perform effective backtesting.

Example 11-9: FRM Exam 1998—Question 20/Regulatory

11-9. VAR measures should be supplemented by portfolio stress-testing because
a) VAR measures indicate that the minimum loss will be the VAR; they don't indicate how large the losses can be.
b) Stress-testing provides a precise maximum loss level.
c) VAR measures are correct only 95% of the time.
d) Stress-testing scenarios incorporate reasonably probable events.

Example 11-10: FRM Exam 2000—Question 105/Market Risk

11-10. Value-at-risk (VAR) analysis should be complemented by stress-testing because stress testing
a) Provides a maximum loss, expressed in dollars
b) Summarizes the expected loss over a target horizon within a minimum confidence interval
c) Assesses the behavior of portfolio at a 99 percent confidence level
d) Identifies losses that go beyond the normal losses measured by VAR

11.6 Cash Flow at Risk

VAR methods have been developed to measure the mark-to-market risk of commercial bank portfolios. By now, these methods have spread to other financial institutions (e.g., investment banks, savings and loans), and the investment management industry (e.g., pension funds).

In each case, the objective function is the market value of the portfolio, assuming fixed positions. VAR methods, however, are now also spreading to other sectors (e.g., corporations), where the emphasis is on periodic earnings. **Cash flow at risk** (CFAR) measures the worst shortfall in cash flows due to unfavorable movements in market risk factors. This involves quantities, Q, unit revenues, P, and unit costs, C. Simplifying, we can write

$$\text{CF} = Q \times (P - C) \tag{11.10}$$

Suppose we focus on the exchange rate, S, as the market risk factor. Each of these variables can be affected by S. Revenues and costs can be denominated in the foreign currency, partially or wholly. Quantities can also be affected by the exchange rate through foreign competition effects. Because quantities are random, this creates **quantity uncertainty**. The risk manager needs to model the relationship between quantities and risk factors. Once this is done, simulations can be used to project the cash-flow distribution and identify the worst loss at some confidence level. Next, the firm can decide whether to hedge and if so, the best instrument to use.

A classic example is the value of a farmer's harvest, say corn. At the beginning of the year, costs are fixed and do not contribute to risk. The price of corn and the size of harvest in the fall, however, are unknown. Suppose price movements are primarily driven by supply shocks, such as the weather. If there is a drought during the summer, quantities will fall and prices will increase. Conversely if there is an exceptionally abundant harvest. Because of the negative correlation between Q and P, total revenues will fluctuate less than if quantities were fixed. Such relationships need to be factored into the risk measurement system because they will affect the hedging program.

11.7 Answers to Chapter Examples

Example 11-1: FRM Exam 1999—Question 89/Market Risk

c) There will be a loss worse than VAR in, on average, $n = 1\% \times 100 = 1$ day out of 100.

Example 11-2: FRM Exam 1998—Question 22/Capital Markets

d) This is the correct answer given the "always" requirement and the fact that VAR is not always subadditive. Otherwise, (b) is not a bad answer, but it requires some additional distributional assumptions.

Example 11-3: FRM Exam 1997—Question 7/Risk Measurement

b) Square root of 10 is 3.16.

Example 11-4: FRM Exam 2001—Question 114

a) We assume a normal distribution and i.i.d. returns, which lead to the square root of time rule and compute the daily standard deviation. For instance, for portfolio 1, $T = 5$, and $\sigma = 10/(\sqrt{5}2.33) = 1.922$. This gives, respectively, 1.922, 2.719, 1.359, 1.923, 1.110, 1.570. So, portfolio 5 has the lowest risk and so on.

Example 11-5: FRM Exam 1997—Question 16/Regulatory

c) The Capital Accord requires a minimum historical observation period of one year.

Example 11-6: FRM Exam 1997—Question 23/Regulatory

c) Less than calculated. Loss limits cut down the positions as losses accumulate. This is similar to a long position in an option, where the delta increases as the price increases, and vice versa. Long positions in options have shortened left tails, and hence involve less risk than an unprotected position.

Example 11-7: FRM Exam 1997—Question 9/Regulatory

b) Callable interest rate swaps involve options, for which there is no limit. Also note that back-to-back options are perfectly hedged and have no market risk.

Example 11-8: FRM Exam 1997—Question 4/Risk Measurement

a) Stress-testing evaluates the portfolio under large moves in financial variables.

Example 11-9: FRM Exam 1998—Question 20/Regulatory

a) The goal of stress-testing is to identify losses that go beyond the "normal" losses measured by VAR.

Example 11-10: FRM Exam 2000—Question 105/Market Risk

d) Stress testing identifies low-probability losses beyond the usual VAR measures. It does not, however, provide a maximum loss.

Appendix: Desirable Properties for Risk Measures

The purpose of a risk measure is to summarize the entire distribution of dollar returns X by one number, $\rho(X)$. Artzner et al. (1999) list four desirable properties of risk measures for capital adequacy purposes.[2]

- **Monotonicity:** if $X_1 \leq X_2$, $\rho(X_1) \geq \rho(X_2)$.
  In other words, if a portfolio has systematically lower values than another (in each state of the world), it must have greater risk.
- **Translation Invariance:** $\rho(X + k) = \rho(X) - k$.
  In other words, adding cash k to a portfolio should reduce its risk by k. This reduces the lowest portfolio value. As with X, k is measured in dollars.
- **Homogeneity:** $\rho(bX) = b\rho(X)$.
  In other words, increasing the size of a portfolio by a factor b should scale its risk measure by the same factor b. This property applies to the standard deviation.[3]
- **Subadditivity:** $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.
  In other words, the risk of a portfolio must be less than the sum of separate risks. Merging portfolios cannot increase risk.

The usefulness of these criteria is that they force us to think about ideal properties and, more importantly, potential problems with simplified risk measures. Indeed, Artzner et al. show that the quantile-based VAR measure fails to satisfy the last property. They give some pathological examples of positions that combine to create portfolios with larger VAR. They also show that the conditional VAR, $E[-X \mid X \leq -\text{VAR}]$, satisfies all these desirable coherence properties.

Assuming a normal distribution, however, the standard deviation-based VAR satisfies the subadditivity property. This is because the volatility of a portfolio is less than the sum of volatilities: $\sigma(X_1 + X_2) \leq \sigma(X_1) + \sigma(X_2)$. We only have a strict equality when the correlation is perfect (positive for long positions). More generally, this property holds for **elliptical distributions**, for which contours of equal density are ellipsoids.

[2]See Artzner, P., Delbaen F., Eber J.-M., and Heath D. (1999), Coherent Measures of Risk. *Mathematical Finance*, 9 (July), 203–228.

[3]This assumption, however, may be questionable in the case of huge portfolios that could not be liquidated without substantial market impact. Thus, it ignores liquidity risk.

Example: Why VAR is not necessarily subadditive

Consider a trader with an investment in a corporate bond with face value of $100,000 and default probability of 0.5%. Over the next period, we can either have no default, with a return of zero, or default with a loss of $100,000. The payoffs are thus −$100,000 with probability of 0.5% and +$0 with probability 99.5%. Since the probability of getting $0 is greater than 99%, the VAR at the 99 percent confidence level is $0, without taking the mean into account. This is consistent with the definition that VAR is the smallest loss such that the right-tail probability is at least 99%.

Now, consider a portfolio invested in three bonds (A,B,C) with the same characteristics and independent payoffs. The VAR numbers add up to $\sum_i \text{VAR}_i = \$0$. To compute the portfolio VAR, we tabulate the payoffs and probabilities:

State	Bonds	Probability	Payoff
No default		$0.995 \times 0.995 \times 0.995 = 0.9850749$	$0
1 default	A,B,C	$3 \times 0.005 \times 0.995 \times 0.995 = 0.0148504$	−$100,000
2 defaults	AB,AC,BC	$3 \times 0.005 \times 0.005 \times 0.995 = 0.0000746$	−$200,000
3 defaults	ABC	$0.005 \times 0.005 \times 0.005 = 0.0000001$	−$300,000

Here, the probability of zero or one default is $0.9851 + 0.0148 = 99.99\%$. The portfolio VAR is therefore $100,000, which is the lowest number such that the probability exceeds 99%. Thus the portfolio VAR is greater than the sum of individual VARs. In this example, VAR is not subadditive. This is an undesirable property because it creates disincentives to aggregate the portfolio, since it appears to have higher risk.

Admittedly, this example is a bit contrived. Nevertheless, it illustrates the danger of focusing on VAR as a sole measure of risk. The portfolio may be structured to display a low VAR. When a loss occurs, however, this may be a huge loss. This is an issue with asymmetrical positions, such as short positions in options or undiversified portfolios exposed to credit risk.

Chapter 12

Identification of Risk Factors

The first step in the measurement of market risk is the identification of the key drivers of risk. These include fixed income, equity, currency, and commodity risks. Later chapters will discuss in more detail the quantitative measurement of risk factors as well as the portfolio risk.

Section 12.g1 presents a general overview of market risks. Downside risk can be viewed as resulting from two sources, exposure and the risk factor. This decomposition is essential because it separates risk into a component over which the risk manager has control (exposure) and another component that is exogenous (the risk factors).

Section 12.g2 illustrates this decomposition in the context of a simple asset, a fixed-coupon bond. An important issue is whether the exposure is constant. If so, the distribution of asset returns can be obtained from a simple transformation of the underlying risk-factor distribution. If not, the measurement of market risk becomes more complex. This section also discusses general and specific risk.

Next, Section 12.g3 discusses discontinuities in returns and event risk. Macroeconomic events can be traced, for instance, to political and economic policies in emerging markets, but also in industrial countries. A related form of financial risk that applies to all instruments is liquidity risk, which is covered in Section 4. This can take the form of asset liquidity risk or funding risk.

12.1 Market Risks

Market risk is the risk of fluctuations in portfolio values because of movements in the level or volatility of market prices.

12.1.1 Absolute and Relative Risk

It is useful to distinguish between absolute and relative risks.

- **Absolute risk** is measured in terms of shortfall relative to the initial value of the investment, or perhaps an alternative investment in cash. It should be expressed in dollar terms (or in the relevant base currency). Let us use the standard deviation as the risk measure and define P as the initial portfolio value and R_P as the rate of return. Absolute risk in dollar terms is

$$\sigma(\Delta P) = \sigma(\Delta P/P) \times P = \sigma(R_P) \times P \tag{12.1}$$

- **Relative risk** is measured relative to a benchmark index and represents active management risk. Defining B as the benchmark, the deviation is $e = R_P - R_B$. In dollar terms, this is $e \times P$. The risk is

$$\sigma = [\sigma(R_P - R_B)] \times P = [\sigma(\Delta P/P - \Delta B/B)] \times P = \omega \times P \tag{12.2}$$

  where ω is called **tracking error volatility** (TEV).

For example, if a portfolio returns -6% over the year but the benchmark dropped by -10%, the excess return is positive $e = -6\% - (-10\%) = 4\%$, even though the absolute performance is negative. On the other hand, a portfolio could return 6%, which is good using absolute measures, but not so good if the benchmark went up by 10%.

Using absolute or relative risk depends on how the trading or investment operation is judged. For bank trading portfolios or hedge funds, market risk is measured in absolute terms. These are sometimes called **total return funds**. For institutional portfolio managers that are given the task of beating a benchmark or peer group, market risk should be measured in relative terms.

To evaluate the performance of portfolio managers, the investor should look not only at the average return, but also the risk. The **Sharpe ratio** (SR) measures the ratio of the average rate of return, $\mu(R_P)$, in excess of the risk-free rate R_F, to the absolute risk

$$SR = [\mu(R_P) - R_F]/\sigma(R_P) \tag{12.3}$$

The **information ratio** (IR) measures the ratio of the average rate of return in excess of the benchmark to the TEV

$$IR = [\mu(R_P) - \mu(R_B)]/\omega \tag{12.4}$$

Table 12-1 gives some examples using annual data, which is the convention for performance measurement. Assume the interest rate is 3%. The Sharpe Ratio of the portfolio is SR $= (-6\% - 3\%)/30\% = -0.30$, which is bad because it is negative and large. In contrast, the Information Ratio is IR $= (-6\% - (-10\%))/8\% = 0.5$, which is positive. It reflects the performance relative to the benchmark. This number is typical of the performance of the top 25th percentile of money managers and is considered "good."[1]

TABLE 12-1 Absolute and Relative Performance

	Average	Volatility	Performance
Cash	3%	0%	
Portfolio *P*	-6%	30%	$SR = -0.30$
Benchmark *B*	-10%	20%	$SR = -0.65$
Deviation *e*	4%	8%	$IR = 0.5$

12.1.2 Directional and Nondirectional Risk

Market risk can be further classified into directional and nondirectional risks.

- **Directional risks** involve exposures to the direction of movements in major financial market variables. These directional exposures are measured by first-order or linear approximations such as
  - **Beta** for exposure to general stock market movements
  - **Duration** for exposure to the level of interest rates
  - **Delta** for exposure of options to the price of the underlying asset
- **Nondirectional risks** involve other remaining exposures, such as nonlinear exposures, exposures to hedged positions or to volatilities. These nondirectional exposures are measured by exposures to differences in price movements, or quadratic exposures such as
  - **Basis risk** when dealing with differences in prices or in interest rates
  - **Residual risk** when dealing with equity portfolios
  - **Convexity** when dealing with second-order effects for interest rates
  - **Gamma** when dealing with second-order effects for options
  - **Volatility risk** when dealing with volatility effects

This classification is to some extent arbitrary. Generally, it is understood that directional risks are greater than nondirectional risks. Some strategies avoid first-order, directional risks and instead take positions in nondirectional risks in the hope of controlling risks better.

[1] See Grinold and Kahn (2000), *Active Portfolio Management*, McGraw-Hill, New York.

Limiting risk also limits rewards, however. As a result, these strategies are often highly leveraged in order to multiply gains from taking nondirectional bets. Perversely, this creates other types of risks, such as liquidity risk and model risk. This strategy indeed failed for long-term capital management (LTCM), a highly leveraged hedge fund that purported to avoid directional risks. Instead, the fund took positions in relative value trades, such as duration-matched short Treasuries, long other fixed-income assets, and in option volatilities. This strategy failed spectacularly.

12.1.3 Market vs. Credit Risk

Market risk is usually measured separately from another major source of financial risk, which is credit risk. **Credit risk** originates from the fact that counterparties may be unwilling or unable to fulfill their contractual obligations. At the most basic level, it involves the risk of default on the asset, such as a loan, bond, or some other security or contract.

When the asset is traded, however, market risk also reflects credit risk—take a corporate bond, for example. Some of the price movement may be due to movements in risk-free interest rates, which is pure market risk. The remainder will reflect the market's changing perception of the likelihood of default. Thus, for traded assets, there is no clear-cut delineation of market and credit risk. Some arbitrary classification must take place.

12.1.4 Risk Interaction

Although it is convenient to categorize risks into different, separately defined, buckets, risk does not occur in isolation. Consider, for instance, a simple transaction whereby a trader purchases 1 million worth of British Pound (BP) spot from Bank A. The current rate is $1.5/BP, for settlement in two business days. So, our bank will have to deliver $1.5 million in two days in exchange for receiving BP 1 million.

This simple transaction involves a series of risks.

- *Market risk:* During the day, the spot rate could change. Say that after a few hours the rate moves to $1.4/BP. The trader cuts the position and enters a spot sale with another bank, Bank B. The million pounds is now worth only $1.4 million, for a loss of $100,000 to be realized in two days. The loss is the change in the market value of the investment.

- *Credit risk:* The next day, Bank B goes bankrupt. The trader must now enter a new, replacement trade with Bank C. If the spot rate has dropped from $1.4/BP to $1.35/BP, the gain of $50,000 on the spot sale with Bank B is now at risk. The loss is the change in the market value of the investment, if positive. Thus there is interaction between market and credit risk.
- *Settlement risk:* Our bank wires the $1.5 million to Bank A in the morning, who defaults at noon and does not deliver the promised BP 1 million. This is also known as **Herstatt risk** because this German bank defaulted on such obligations in 1974, potentially destabilizing the whole financial system. The loss is now the whole principal in dollars.
- *Operational risk:* Suppose that our bank wired the $1.5 million to a wrong bank, Bank D. After two days, our back office gets the money back, which is then wired to Bank A plus compensatory interest. The loss is the interest on the amount due.

12.2 Sources of Loss: A Decomposition

12.2.1 Exposure and Uncertainty

The potential for loss for a plain fixed-coupon bond can be decomposed into the effect of (modified) duration D^* and the yield. **Duration** measures the sensitivity of the bond return to changes in the interest rate.

$$\Delta P = -(D^* P) \times \Delta y \tag{12.5}$$

The dollar exposure is D^*P, which is the **dollar duration**. Figure 12-1 shows how the nonlinear pricing relationship is approximated by the duration line, whose slope is $-(D^*P)$.

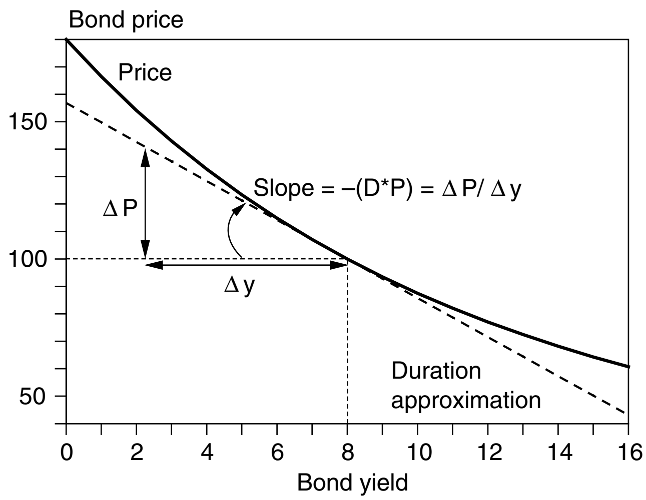
This illustrates the general principle that losses can occur because of a combination of two factors:

- The exposure to the factor, or dollar duration (a choice variable)
- The movement in the factor itself (which is external to the portfolio)

This linear characterization also applies to **systematic risk** and option **delta**. We can, for instance, decompose the return on stock i, R_i into a component due to the market R_M and some residual risk, which we ignore for now because its effect washes out in a large portfolio:

$$R_i = \alpha_i + \beta_i \times R_M + \epsilon_i \approx \beta_i \times R_M \tag{12.6}$$

FIGURE 12-1 Duration as an Exposure



We ignore the constant α_i because it does not contribute to risk, as well as the residual ϵ_i, which is diversified. Note that R_i is expressed here in terms of **rate of return** and, hence, has no dimension. To get a change in a dollar price, we write

$$\Delta P_i = R_i P_i \approx (\beta P_i) \times R_M \tag{12.7}$$

Similarly, the change in the value of a derivative f can be expressed in terms of the change in the price of the underlying asset S,

$$df = \Delta \times dS \tag{12.8}$$

To avoid confusion, we use the conventional notations of Δ for the first partial derivative of the option. Changes are expressed in infinitesimal amounts df and dS.

Equations (12.5), (12.6), and (12.8) all reveal that the change in value is linked to an **exposure** coefficient and a change in market variable:

$$\text{Market Loss} = \text{Exposure} \times \text{Adverse Movement in FinancialVariable}$$

To have a loss, we need to have some exposure *and* an unfavorable move in the risk factor. Traditional risk management methods focus on the exposure term. The drawback is that one does not incorporate the probability of an adverse move, and there is no aggregation of risk across different sources of financial risk.

12.2.2 Specific Risk

The previous section has shown how to explain the movement in individual bond, stock, or derivatives prices as a function of a general market factor. Consider, for instance, the driving factors behind changes in a stock's price:

$$\Delta P_i = (\beta P_i) \times R_M + (\epsilon_i P_i) \tag{12.9}$$

The mapping procedure in risk management replaces the stock by its dollar exposure (βP_i) on the general, market risk factor. But this leaves out the specific risk, ϵ_i.

Specific risk can be defined as risk that is due to issuer-specific price movements, after accounting for general market factors. Taking the variance of both sides of Equation (12.6), we have

$$V[\Delta P_i] = (\beta_i P_i)^2 V[R_M] + V[\epsilon_i P_i] \tag{12.10}$$

The first term represents general market risk, the second, specific risk.

Increasing the amount of detail (or granularity) in the general risk factors should lead to smaller residual, specific risk. For instance, we could model general risk by taking a market index plus industry indices. As the number of market factors increases, specific risk should decrease. Hence, specific risk can only be understood relative to the definition of market risk.

Example 12-1: FRM Exam 1997—Question 16/Market Risk

12-1. The risk of a stock or bond that is *not* correlated with the market (and thus can be diversified) is known as

a) Interest rate risk
b) FX risk
c) Model risk
d) Specific risk

12.3 Discontinuity and Event Risk

12.3.1 Continuous Processes

As seen in the previous section, market risk can be ascribed to movements in the risk factor(s) and in the exposure, or payoff function. If movements in bond yields are smooth, bond prices will also move in a smooth fashion. These continuous movements can be captured well from historical data.

This smoothness characteristic can be expressed in mathematical form as a **Brownian motion**. Formally, the variance of changes in prices over shrinking time intervals has to shrink at the same rate as the length of the time interval, giving

$$\lim_{\Delta t \to 0} V[\Delta P/P] = \sigma^2 dt \tag{12.11}$$

where σ is a finite volatility. Such process allows continuous hedging, or replication, of an option, which leads to the Black-Scholes model. In practice, movements are small enough that effective hedging can occur on a daily basis.

12.3.2 Jump Process

A much more dangerous process is a discontinuous **jump process**, where large movements occur over a small time interval. These discontinuities can create large losses. Furthermore, their probability is difficult to establish because they occur rarely in historical data.

Figure 12-2 depicts a notable discontinuity, which is the 20% drop in the S&P index on October 19, 1987. Prior to that, movements in the index were relatively smooth.

Such discontinuities are inherently difficult to capture. In theory, simulations could modify the usual continuous stochastic processes by adding a jump component occurring with a predefined frequency and size. In practice, the process parameters are difficult to estimate and there is not much point in trying to quantify what is essentially a stress-testing exercise.

Discontinuities in the portfolio series can occur for another reason: The payoff itself can be discontinuous. Figure 12-3 gives the example of a binary option, which

FIGURE 12-2 Jump in U.S. Stock Price Index

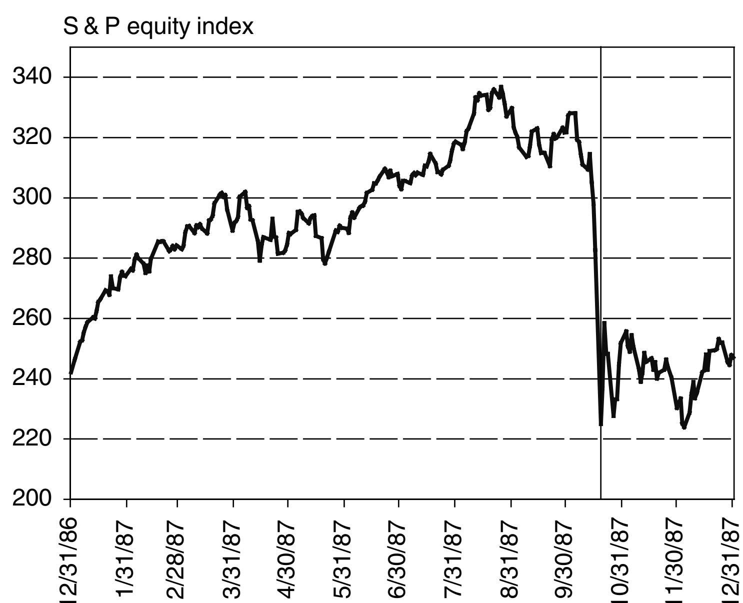
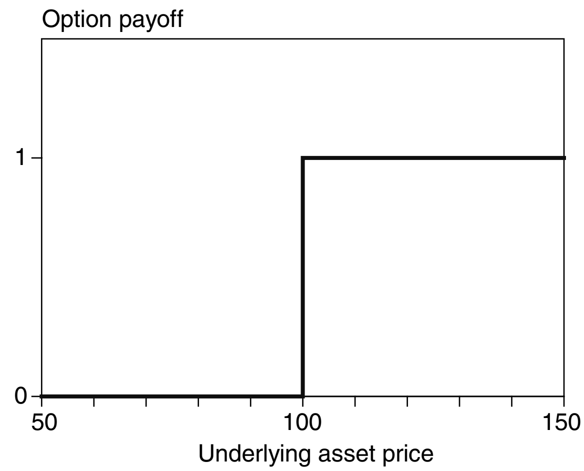


FIGURE 12-3 Discontinuous Payoff: Binary Option



pays $1 if the underlying price is above the strike price and pays zero otherwise. Such an option will create a discontinuous pattern in the portfolio, even if the underlying asset price is perfectly smooth. These options are difficult to hedge because of the instability of the option delta around the strike price. In other words, they have very high gamma at that point.

12.3.3 Event Risk

Discontinuities can occur for a number of reasons. Most notably, there was no immediately observable explanation for the stock market crash of 1987. It was argued that the crash was caused by the "unsustainable" run-up in prices during the year, as well as sustained increases in interest rates. The problem is that all of this information was available to market observers well before the crash. Perhaps the crash was due to the unusual volume of trading, which overwhelmed trading mechanisms, creating further uncertainty as prices dropped.

In many other cases, the discontinuity is due to an observable event. **Event risk** can be characterized as the risk of loss because of an observable political or economic event. These include

- *Changes in governments* leading to changes in economic policies
- *Changes in economic policies*, such as default, capital controls, inconvertibility, changes in tax laws, expropriations, and so on

- *Coups, civil wars, invasions*, or other signs of political instability
- *Currency devaluations*, which are usually accompanied by other drastic changes in market variables

These risks often originate from **emerging markets**,[2] although this is by no means universal. Developing countries have time and again displayed a disturbing tendency to interfere with capital flows.

There is no simple method to deal with event risk, since almost by definition they are unique events. To protect the institution against such risk, risk managers could consult with economists. Political risk insurance is also available for some markets, which should give some measure of the perceived risk.

Setting up prospective events is an important part of stress testing. Even so, recent years have demonstrated that markets seem to be systematically taken by surprise. Precious few seem to have anticipated the Russian default, for instance.

Example: the Argentina Turmoil

Argentina is a good example of political risk in emerging markets. Up to 2001, the Argentine peso was fixed to the U.S. dollar at a one-to-one exchange rate. The government had promised it would defend the currency at all costs. Argentina, however, suffered from the worst economic crisis in decades, compounded by the cost of excessive borrowing.

In December 2001, Argentina announced it would stop paying interest on its $135 billion foreign debt. This was the largest sovereign default recorded so far. Economy Minister Cavallo also announced sweeping restrictions on withdrawals from bank deposits to avoid capital flight.

On December 20, President Fernando de la Rua resigned after 25 people died in street protest and rioting. President Duhalde took office on January 2 and devalued the currency on January 6. The exchange rate promptly moved from 1 peso/dollar to more than 3 pesos.

Such moves could have been factored into risk management systems by scenario analysis. What was totally unexpected, however, was the government's announcement

[2] The term "emerging stock market" was coined by the International Finance Corporation (IFC), in 1981. IFC defines an emerging stock market as one located in a developing country. Using the World Bank's definition, this includes all countries with a GNP per capita less than $8,625 in 1993.

that it would treat differentially bank loans and deposits. Dollar-denominated bank deposits were converted into devalued pesos, but dollar-denominated bank loans were converted into pesos at a one-to-one rate. This mismatch rendered much of the banking system technically insolvent, because loans (bank assets) overnight became less valuable than deposits (bank liabilities). Whereas risk managers had contemplated the market risk effect of a devaluation, few had considered this possibility of such political actions.

Example 12-2: FRM Exam 2001—Question 122

12-2. What is the most important consequence of an option having a discontinuous payoff function?

a) An increase in operational risks, as the expiry price can be contested or manipulated if close to a point of discontinuity

b) When the underlying is close to the points of discontinuity, a very high gamma

c) Difficulties to assess the correct market price at expiry

d) None of the above

12.4 Liquidity Risk

Liquidity risk is usually viewed as a component of market risk. Lack of liquidity can cause the failure of an institution, even when it is technically solvent. We will see in the chapters on regulation that commercial banks have an inherent liquidity imbalance between their assets (long-term loans) and their liabilities (bank deposits) that provides a rationale for deposit insurance.

The problem with liquidity risk is that it is less amenable to formal analysis than traditional market risk. The industry is still struggling with the measurement of liquidity risk. Often, liquidity risk is loosely factored into VAR measures, for instance by selectively increasing volatilities. These adjustments, however, are mainly ad-hoc. Some useful lessons have been learned from the near failure of LTCM. These are discussed in a report by the Counterparty Risk Management Policy Group (CRMPG), which is described in Chapter 26.

Liquidity risk consists of both asset liquidity risk and funding liquidity risk.

- **Asset liquidity risk**, also called **market/product liquidity risk**, arises when transactions cannot be conducted at quoted market prices due to the size of the required trade relative to normal trading lots.
- **Funding liquidity risk**, also called **cash-flow risk**, arises when the institution cannot meet payment obligations.

These two types of risk interact with each other if the portfolio contains illiquid assets that must be sold at distressed prices. Funding liquidity needs can be met from (i) sales of cash, (ii) sales of other assets, and (iii) borrowings.

Asset liquidity risk can be managed by setting limits on certain markets or products and by means of diversification. Funding liquidity risk can be managed by proper planning of cash-flow needs, by setting limits on cash flow gaps, and by having a robust plan in place for raising fresh funds should the need arise.

Asset liquidity can be measured by a price-quantity function, which describes how the price is affected by the quantity transacted. Highly liquid assets, such as major currencies or Treasury bonds, are characterized by

- **Tightness**, which is a measure of the divergence between actual transaction prices and quoted mid-market prices
- **Depth**, which is a measure of the volume of trades possible without affecting prices too much (e.g. at the bid/offer prices), and is in contrast to **thinness**
- **Resiliency**, which is a measure of the speed at which price fluctuations from trades are dissipated

In contrast, illiquid markets are those where transactions can quickly affect prices. This includes assets such as exotic OTC derivatives or emerging-market equities, which have low trading volumes. All else equal, illiquid assets are more affected by current demand and supply conditions and are usually more volatile than liquid assets.

Illiquidity is both asset-specific and market-wide. Large-scale changes in market liquidity seem to occur on a regular basis, most recently during the bond market rout of 1994 and the credit crisis of 1998. Such crises are characterized by a **flight to quality**, which occurs when there is a shift in demand away from low-grade securities toward high-grade securities. The low-grade market then becomes illiquid with depressed prices. This is reflected in an increase in the yield spread between corporate and government issues.

Even government securities can be affected differentially. The yield spread can widen between off-the-run securities and corresponding on-the-run securities. **On-the-run** securities are those that are issued most recently and hence are more active and liquid. Other securities are called **off-the-run**. Consider, for instance, the latest issued 30-year U.S. Treasury bond. This **benchmark** bond is called on-the-run, until another 30-year bond is issued, at which time it becomes off-the-run. Because these securities are very similar in terms of market and credit risk, this yield spread is a measure of the **liquidity premium**.

Example 12-3: FRM Exam 1997—Question 54/Market Risk
12-3. "Illiquid" describes an instrument that
a) Does not trade in an active market
b) Does not trade on any exchange
c) Can not be easily hedged
d) Is an over-the-counter (OTC) product

Example 12-4: FRM Exam 1998—Question 7/Credit Risk
12-4. *(This requires some knowledge of markets.)* Which of the following products has the least liquidity?
a) U.S. on-the-run Treasuries
b) U.S. off-the-run Treasuries
c) Floating-rate notes
d) High-grade corporate bonds

Example 12-5: FRM Exam 1998—Question 6/Capital Markets
12-5. A finance company is interested in managing its balance sheet liquidity risk *(funding risk)*. The most productive means of accomplishing this is by
a) Purchasing marketable securities
b) Hedging the exposure with Eurodollar futures
c) Diversifying its sources of funding
d) Setting up a reserve

Example 12-6: FRM Exam 2000—Question 74/Market Risk

12-6. In a market crash the following are usually *true*?

I. Fixed-income portfolios hedged with short U.S. government bonds and futures lose less than those hedged with interest rate swaps given equivalent durations.

II. Bid offer spreads widen because of lower liquidity.

III. The spreads between off-the-run bonds and benchmark issues widen.

a) I, II & III

b) II & III

c) I & III

d) None of the above

Example 12-7: FRM Exam 2000—Question 83/Market Risk

12-7. Which one of the following statements about liquidity risk in derivatives instruments is *not true*?

a) Liquidity risk is the risk that an institution may not be able to, or cannot easily, unwind or offset a particular position at or near the previous market price because of inadequate market depth or disruptions in the marketplace.

b) Liquidity risk is the risk that the institution will be unable to meet its payment obligations on settlement dates or in the event of margin calls.

c) Early termination agreements can adversely impact liquidity because an institution may be required to deliver collateral or settle a contract early, possibly at a time when the institution may face other funding and liquidity pressures.

d) An institution that participates in the exchange-traded derivatives markets has potential liquidity risks associated with the early termination of derivatives contracts.

12.5 Answers to Chapter Examples

Example 12-1: FRM Exam 1997—Question 16/Market Risk

d) Specific risk represents the risk that is not correlated with market-wide movements.

Example 12-2: FRM Exam 2001—Question 122

b) Answer (c) is not correct since the correct market price can be set at expiration as a function of the underlying spot price. The main problem is that the delta changes very quickly close to expiration when the spot price hovers around the strike price. This high gamma feature makes it very difficult to implement dynamic hedging of options with discontinuous payoffs, such as binary options.

Example 12-3: FRM Exam 1997—Question 54/Market Risk

a) Illiquid instruments are ones that do not trade actively. Answers (b) and (d) are not correct as OTC products, which do not trade on exchanges, such as Treasuries, can be quite liquid. The lack of easy hedging alternatives does not imply the instrument itself is illiquid.

Example 12-4: FRM Exam 1998—Question 7/Credit Risk

c) *(This requires some knowledge of markets.)* Ranking these assets in decreasing order of asset liquidity, we have (a), (b), (d), and (c). Floating-rate notes are typically issued in smaller amounts and have customized payment schedules. As a result, they are typically less liquid than the other securities.

Example 12-5: FRM Exam 1998—Question 6/Capital Markets

c) Managing balance-sheet liquidity risk involves the ability to meet cash-flow needs as required. This can be met by keeping liquid assets or being able to raise fresh funds easily. Answer (a) is not correct because it substitutes cash for marketable securities, which is not an improvement. Hedging with Eurodollar futures does not decrease potential cash-flow needs. Setting up a reserve is simply an accounting entry.

Example 12-6: FRM Exam 2000—Question 74/Market Risk

b) In a crash, bid offer spreads widen, as do liquidity spreads. Answer I is incorrect because Treasuries usually rally more than swaps, which leads to *greater* losses for a portfolio short Treasuries than swaps.

Example 12-7: FRM Exam 2000—Question 83/Market Risk

d) Answer (a) refers to asset liquidity risk; answers (b) and (c) to funding liquidity risk. Answer (d) is incorrect since exchange-traded derivatives are marked-to-market daily and hence can be terminated at any time without additional cash-flow needs.

Chapter 13

Sources of Risk

We now turn to a systematic analysis of the major financial market risk factors. Currency, fixed-income, equity, and commodities risk are analyzed in Sections 13.1, 13.2, 13.3, and 13.4, respectively. Currency risk refers to the volatility of floating exchange rates and devaluation risk, for fixed currencies. Fixed-income risk relates to term-structure risk, global interest rate risk, real yield risk, credit spread risk, and prepayment risk. Equity risk can be described in terms of country risk, industry risk, and stock-specific risk. Commodity risk includes volatility risk, convenience yield risk, delivery and liquidity risk. These first four sections are mainly descriptive.

Finally, Section 13.5 discusses simplifications in risk models. We explain how the multitude of risk factors can be summarized into a few essential drivers. Such factor models include the diagonal model, which decomposes returns into a market-wide factor and residual risk.

13.1 Currency Risk

Currency risk arises from potential movements in the value of foreign currencies. This includes currency-specific volatility, correlations across currencies, and devaluation risk. Currency risk arises in the following environments.

- In a *pure currency float*, the external value of a currency is free to move, to depreciate or appreciate, as pushed by market forces. An example is the dollar/euro exchange rate.
- In a *fixed currency system*, a currency's external value is fixed (or pegged) to another currency. An example is the Hong Kong dollar, which is fixed against the U.S. dollar. This does not mean there is no risk, however, due to possible readjustments in the **parity value**, called devaluations or revaluations.
- In a *change in currency regime*, a currency that was previously fixed becomes flexible, or vice versa. For instance, the Argentinian peso was fixed against the dollar

until 2001, and floated thereafter. Changes in regime can also lower currency risk, as in the recent case of the euro.[1]

13.1.1 Currency Volatility

Table 13-1 compares the RiskMetrics volatility forecasts for a group of 21 currencies.[2] Ten of these correspond to "industrial countries," the others to "emerging" markets.

These numbers are standard deviations, adapted from value-at-risk (VAR) forecasts by dividing by 1.645. The table reports daily, monthly, and annualized (from monthly) standard deviations at the end of 2002 and 1996. Across developed

TABLE 13-1 Currency Volatility Against U.S. Dollar (Percent)

Currency/	Code	End 1999			End 1996
Country		Daily	Monthly	Annual	Annual
Argentina	ARS	0.663	3.746	12.98	0.42
Australia	AUD	0.405	2.310	8.00	8.50
Canada	CAD	0.403	1.863	6.45	3.60
Switzerland	CHF	0.495	2.664	9.23	10.16
Denmark	DKK	0.421	2.275	7.88	7.78
Britain	GBP	0.398	2.165	7.50	9.14
Hong Kong	HKD	0.004	0.016	0.05	0.26
Indonesia	IDR	0.356	2.344	8.12	1.61
Japan	JPY	0.613	3.051	10.57	6.63
Korea	KRW	0.434	2.279	7.89	4.49
Mexico	MXN	0.511	2.615	9.06	6.94
Malaysia	MYR	0.000	0.001	0.01	1.60
Norway	NOK	0.477	2.608	9.03	7.60
New Zealand	NZD	0.631	3.140	10.88	7.89
Philippines	PHP	0.303	1.423	4.93	0.57
Sweden	SEK	0.431	2.366	8.20	6.38
Singapore	SGD	0.230	1.304	4.52	1.79
Thailand	THB	0.286	1.544	5.35	1.23
Taiwan	TWD	0.166	0.981	3.40	0.94
South Africa	ZAR	1.050	4.915	17.03	8.37
Euro	EUR	0.422	2.284	7.91	8.26

[1] As of 2003, the Eurozone includes a block of 12 countries, Austria, Belgium/Luxembourg, Finland, France, Germany, Ireland, Italy, Netherlands, Portugal, and Spain. Greece joined on January 1, 2001. Currency risk is not totally eliminated, however, as there is always a possibility that the currency union could dissolve.

[2] For updates, see www.riskmetrics.com.

markets, volatility typically ranges from 6 to 11 percent per annum. The Canadian dollar is notably lower at 4–5 percent volatility.

Some currencies, such as the Hong Kong dollar have very low volatility, reflecting their pegging to the dollar. This does not mean that they have low risk, however. They are subject to **devaluation risk**, which is the risk that the currency peg could fail. This has happened to Thailand and Indonesia, which in 1996 had low volatility but converted to a floating exchange rate regime, which had higher volatility in 2002.

Example 13-1: FRM Exam 1997—Question 10/Market Risk

13-1. Which currency pair would you expect to have the lowest volatility?

a) USD/EUR
b) USD/CAD
c) USD/JPY
d) USD/MXN

13.1.2 Correlations

Next, we briefly describe the correlations between these currencies against the U.S. dollar. Generally, correlations are low, mostly in the range of -0.10 to 0.20. This indicates substantial benefits from holding a well-diversified currency portfolio.

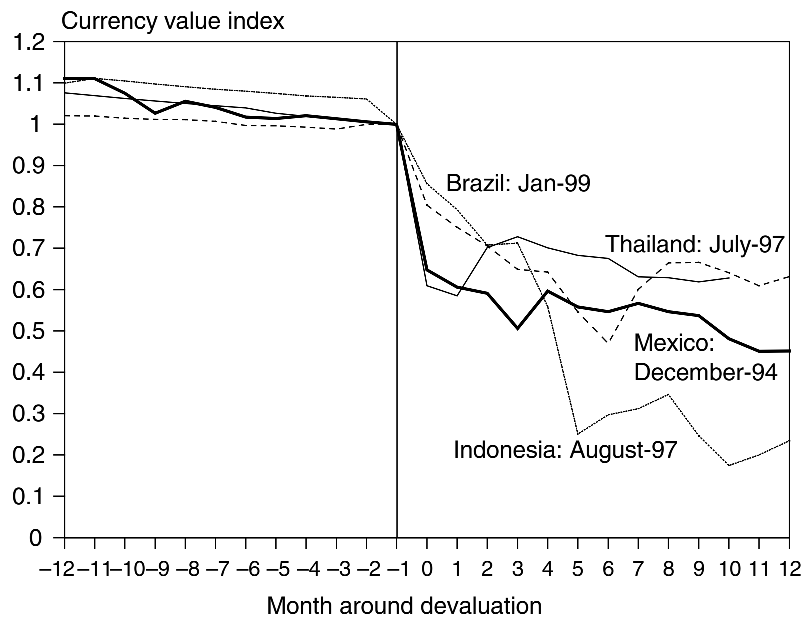
There are, however, blocks of currencies with very high correlations. European currencies, such as the DKK, SEK, NOK, CHF, have high correlation with each other and the Euro, on the order of 0.90. The GBP also has high correlations with European currencies, around 0.60-0.70. As a result, investing across European currencies does little to diversify risk, from the viewpoint of a U.S. dollar-based investor.

13.1.3 Devaluation Risk

Next, we examine the typical impact of a currency devaluation, which is illustrated in Figure 13-1. Each currency has been scaled to a unit value at the end of the month just before the devaluation. In previous months, we observe only small variations in exchange rates. In contrast, the devaluation itself leads to a dramatic drop in value ranging from 20% to an extreme 80% in the case of the rupiah.

Currency risk is also related to other financial risks, in particular interest rate risk. Often, interest rates are raised in an effort to stem the depreciation of a currency, resulting in a positive correlation between the currency and the bond market. These interactions should be taken into account when designing scenarios for stress-tests.

FIGURE 13-1 Effect of Currency Devaluation



13.1.4 Cross-Rate Volatility

Exchange rates are expressed relative to a base currency, usually the dollar. The **cross rate** is the exchange rate between two currencies other than the reference currency. For instance, say that S_1 represents the dollar/pound rate and that S_2 represents the dollar/euro (EUR) rate. Then the euro/pound rate is given by the ratio

$$S_3(EUR/BP) = \frac{S_1(\$/BP)}{S_2(\$/EUR)} \tag{13.1}$$

Using logs, we can write

$$ln[S_3] = ln[S_1] - ln[S_2] \tag{13.2}$$

The volatility of the cross rate is

$$\sigma_3^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2 \tag{13.3}$$

Thus we could infer the correlation from the triplet of variances. Note that this assumes both the numerator and denominator are in the same currency. Otherwise, the log of the cross rate is the sum of the logs, and the negative sign in Equation (13.3) must be changed to a positive sign.

Example 13-2: FRM Exam 1997—Question 14/Market Risk

13-2. What is the implied correlation between JPY/EUR and EUR/USD when given the following volatilities for foreign exchange rates?

JPY/USD at 8%

JPY/EUR at 10%

EUR/USD at 6%.

a) 60%

b) 30%

c) −30%

d) −60%

13.2 Fixed-Income Risk

Fixed-income risk arises from potential movements in the level and volatility of bond yields. Figure 13-2 plots U.S. Treasury yields on a typical range of maturities at monthly intervals since 1986. The graph shows that yield curves move in complicated fashion, which creates **yield curve risk**.

13.2.1 Factors Affecting Yields

Yield volatility reflects economic fundamentals. For a long time, the primary determinant of movements in interest rates was **inflationary expectations**. Any perceived

FIGURE 13-2 Movements in the U.S. Yield Curve

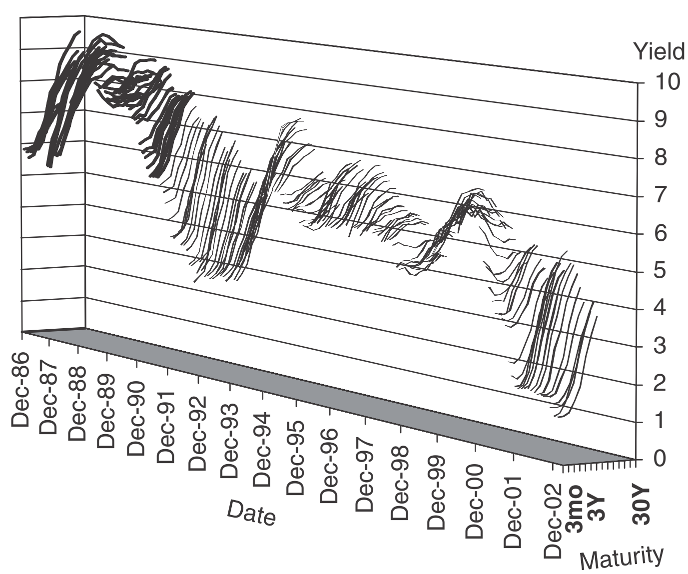
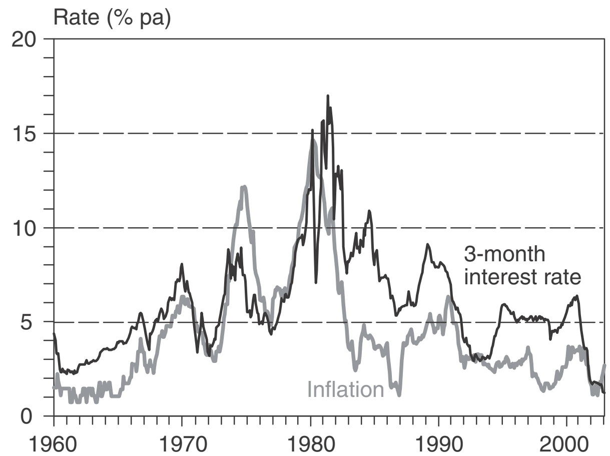


FIGURE 13-3 Inflation and Interest Rates



increase in the predicted rate of inflation will make bonds with fixed nominal coupons less attractive, thereby increasing their yield.

Figure 13-3 compares the level of short-term U.S. interest rates with the concurrent level of inflation. The graphs show that most of the movements in nominal rates can be explained by inflation. In more recent years, however, inflation has been subdued.

Figure 13-2 has shown complex movements in the term structure of interest rates. It would be convenient if these movements could be summarized by a small number of variables. In practice, market observers focus on a long-term rate (say the yield on the 10-year note) and a short-term rate (say the yield on a 3-month bill). These two rates usefully summarize movements in the term structure, which are displayed in Figure 13-4. Shaded areas indicate periods of U.S. economic recessions.

FIGURE 13-4 Movements in the Term Structure

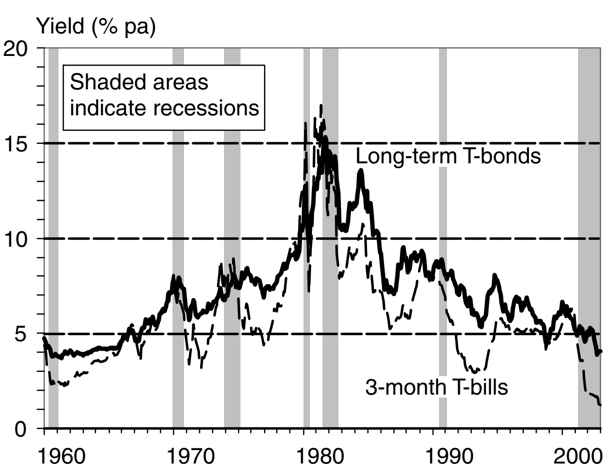
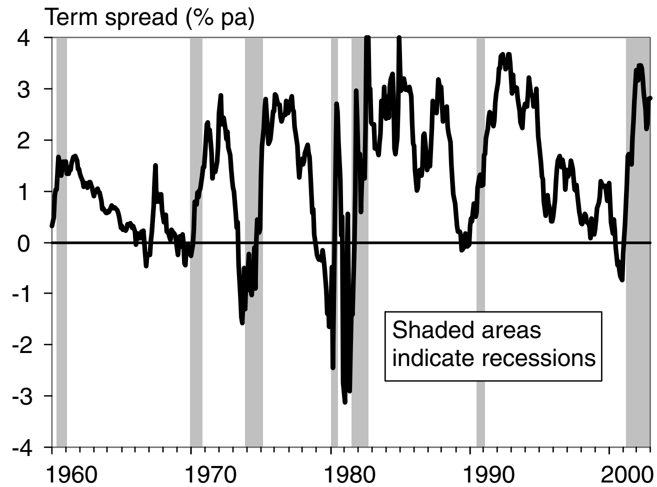


FIGURE 13-5 Term Structure Spread



Generally, the two rates move in tandem, although the short-term rate displays more variability. The **term spread** is defined as the difference between the long rate and the short rate. Figure 13-5 relates the term spread to economic activity. As the graph shows, periods of recessions usually witness an increase in the term spread. Slow economic activity decreases the demand for capital, which in turn decreases short-term rates and increases the term spread.

13.2.2 Bond Price and Yield Volatility

Table 13-2 compares the RiskMetrics volatility forecasts for U.S. bond prices. The data are recorded as of December 31, 2002 and December 31, 1996. The table includes Eurodeposits, fixed swap rates, and zero-coupon Treasury rates, for maturities ranging from 30 day to 30 years. Volatilities are reported at a daily and monthly horizon. Monthly volatilities are also annualized by multiplying by the square root of twelve.

Short-term deposits have very little price risk. Volatility increases with maturity. The price risk of 10-year bonds is around 10% annually, which is similar to that of floating currencies. The risk of 30-year bonds is higher, at 20-30%, which is similar to that of equities.

Risk can be measured as either return volatility or yield volatility. Using the duration approximation, the volatility of the rate of return in the bond price is

$$\sigma\left(\frac{\Delta P}{P}\right) = |\, D^* \,| \times \sigma(\Delta y) \tag{13.4}$$

TABLE 13-2 U.S. Fixed-Income Price Volatility (Percent)

Type/ Maturity	Code	Yield Level	End 2002 Daily	Mty	Annual	End 1996 Annual
Euro-30d	R030	1.360	0.002	0.012	0.04	0.05
Euro-90d	R090	1.353	0.005	0.030	0.10	0.08
Euro-180d	R180	1.348	0.009	0.064	0.22	0.19
Euro-360d	R360	1.429	0.030	0.188	0.65	0.58
Swap-2Y	S02	1.895	0.110	0.634	2.20	1.57
Swap-3Y	S03	2.428	0.184	1.027	3.56	2.59
Swap-4Y	S04	2.865	0.257	1.429	4.95	3.59
Swap-5Y	S05	3.224	0.329	1.836	6.36	4.70
Swap-7Y	S07	3.815	0.454	2.535	8.78	6.69
Swap-10Y	S10	4.434	0.643	3.613	12.52	9.82
Zero-2Y	Z02	1.593	0.107	0.631	2.18	1.64
Zero-3Y	Z03	1.980	0.172	0.999	3.46	2.64
Zero-4Y	Z04	2.372	0.248	1.428	4.95	3.69
Zero-5Y	Z05	2.773	0.339	1.935	6.70	4.67
Zero-7Y	Z07	3.238	0.458	2.603	9.02	6.81
Zero-9Y	Z09	3.752	0.576	3.259	11.29	8.64
Zero-10Y	Z10	3.989	0.637	3.600	12.47	9.31
Zero-15Y	Z15	4.247	0.894	5.018	17.38	13.82
Zero-20Y	Z20	4.565	1.132	6.292	21.80	17.48
Zero-30Y	Z30	5.450	1.692	9.170	31.77	23.53

Here, we took the absolute value of duration since the volatility of returns and of yield changes must be positive.

Price volatility nearly always increases with duration. Yield volatility, on the other hand, may be more intuitive because it corresponds to the usual representation of the term structure of interest rates.

When changes in yields are normally distributed, the term $\sigma(\Delta y)$ is constant: This is the **normal model**. Instead, RiskMetrics reports a volatility of relative changes in yields, where $\sigma(\frac{\Delta y}{y})$ is constant: This is the **lognormal model**. The RiskMetrics forecast can be converted into the usual volatility of yield changes:

$$\sigma(\Delta y) = y \times \sigma(\Delta y/y) \tag{13.5}$$

Table 13-3 displays volatilities of relative and absolute yield changes. Yield volatility for swaps and zeros is much more constant across maturity, ranging from 0.9 to 1.2 percent per annum.

It should be noted that the square root of time adjustment for the volatility is more questionable for bond prices than for most other assets because bond prices must converge to their face value as maturity nears (barring default). This effect is important for short-term bonds, whose return volatility pattern is distorted by the

TABLE 13-3 U.S. Fixed-Income Yield Volatility, 2002 (Percent)

Type/ Maturity	Code	Yield Level	$\sigma(dy/y)$ Daily	Mty	Annual	$\sigma(dy)$ Daily	Mty	Annual
Euro-30d	R030	1.360	1.580	9.584	33.20	0.021	0.130	0.45
Euro-90d	R090	1.353	1.240	7.866	27.25	0.017	0.106	0.37
Euro-180d	R180	1.348	1.267	8.321	28.83	0.017	0.112	0.39
Euro-360d	R360	1.429	1.883	11.177	38.72	0.027	0.160	0.55
Swap-2Y	S02	1.895	2.546	13.993	48.47	0.048	0.265	0.92
Swap-3Y	S03	2.428	2.264	12.247	42.42	0.055	0.297	1.03
Swap-4Y	S04	2.865	2.061	11.158	38.65	0.059	0.320	1.11
Swap-5Y	S05	3.224	1.901	10.370	35.92	0.061	0.334	1.16
Swap-7Y	S07	3.815	1.619	8.883	30.77	0.062	0.339	1.17
Swap-10Y	S10	4.434	1.409	7.827	27.11	0.062	0.347	1.20
Zero-2Y	Z02	1.593	2.916	16.576	57.42	0.046	0.264	0.91
Zero-3Y	Z03	1.980	2.583	14.681	50.86	0.051	0.291	1.01
Zero-4Y	Z04	2.372	2.384	13.541	46.91	0.057	0.321	1.11
Zero-5Y	Z05	2.773	2.263	12.847	44.50	0.063	0.356	1.23
Zero-7Y	Z07	3.238	1.913	10.825	37.50	0.062	0.351	1.21
Zero-9Y	Z09	3.752	1.650	9.309	32.25	0.062	0.349	1.21
Zero-10Y	Z10	3.989	1.556	8.766	30.37	0.062	0.350	1.21
Zero-15Y	Z15	4.247	1.376	7.694	26.65	0.058	0.327	1.13
Zero-20Y	Z20	4.565	1.223	6.776	23.47	0.056	0.309	1.07
Zero-30Y	Z30	5.450	1.037	5.603	19.41	0.057	0.305	1.06

convergence to face value. It is less of an issue, however, for long-term bonds, as long as the horizon is much shorter than the bond maturity.

This explains why the volatility of short-term Eurodeposits appears to be out of line with the others. The concept of monthly risk of a 30-day deposit is indeed fuzzy, since by the end of the VAR horizon, the deposit will have matured, having therefore zero risk. Instead this can be interpreted as an investment in a 30-day deposit that is held for one day only and rolled over the next day into a fresh 30-day deposit.

Example 13-3: FRM Exam 1999—Question 86/Market Risk

13-3. For purposes of computing the market risk of a U.S. Treasury bond portfolio, it is easiest to measure

a) Yield volatility because yields have positive skewness

b) Price volatility because bond prices are positively correlated

c) Yield volatility for bonds sold at a discount and price volatility for bonds sold at a premium to par

d) Yield volatility because it remains more constant over time than price volatility, which must approach zero as the bond approaches maturity

Example 13-4: FRM Exam 1999—Question 80/Market Risk

13-4. BankEurope has a $20,000,000.00 position in the 6.375% AUG 2027 US Treasury Bond. The details on the bond are

Market Price	98 8/32
Accrued	1.43%
Yield	6.509%
Duration	13.133
Modified duration	12.719
Yield volatility	12%

What is the daily VAR of this position at the 95% confidence level (assume there are 250 business days in a year)?

a) $291,400
b) $203,080
c) $206,036
d) $206,698

13.2.3 Correlations

Table 13-4 displays correlation coefficients for all maturity pairs at a 1-day horizon. First, it should be noted that the Eurodeposit block behaves somewhat differently from the zero-coupon Treasury block. Correlations between these two blocks are relatively lower than others. This is because Eurodeposit rates contain credit risk. Variations in the credit spread will create additional noise relative to movements among pure Treasury yield.

Within each block, correlations are generally very high, suggesting that yields are affected by a common factor. If the yield curve were to move in strict parallel fashion, all correlations should be equal to one. In practice, the yield curve displays more com-

TABLE 13-4 U.S. Fixed-Income Price Correlations, 2002 (Daily)

	R030	R090	R180	R360	Z02	Z03	Z04	Z05	Z07	Z09	Z10	Z15	Z20
R030	1.000												
R090	0.786	1.000											
R180	0.690	0.894	1.000										
R360	0.372	0.544	0.814	1.000									
Z02	0.142	0.299	0.614	0.840	1.000								
Z03	0.121	0.269	0.592	0.836	0.992	1.000							
Z04	0.100	0.237	0.563	0.820	0.972	0.994	1.000						
Z05	0.080	0.206	0.532	0.797	0.943	0.977	0.995	1.000					
Z07	0.098	0.219	0.534	0.794	0.933	0.969	0.988	0.995	1.000				
Z09	0.117	0.231	0.530	0.783	0.912	0.949	0.970	0.979	0.994	1.000			
Z10	0.143	0.251	0.534	0.772	0.890	0.928	0.950	0.959	0.982	0.997	1.000		
Z15	0.123	0.226	0.509	0.754	0.863	0.906	0.933	0.946	0.973	0.991	0.996	1.000	
Z20	0.098	0.193	0.471	0.720	0.817	0.865	0.898	0.916	0.948	0.971	0.980	0.994	1.000
Z30	0.022	0.082	0.318	0.554	0.601	0.663	0.709	0.743	0.789	0.827	0.848	0.889	0.935

plex patterns but remains relatively smooth. This implies that movements in adjoining maturities are highly correlated. For instance, the correlation between the 9-year zero and 10-year zero is 0.997, which is very high. zero is not very Correlations are the lowest for maturities further apart, for instance 0.601 between the 2-year and 30-year zero.

These high correlations give risk managers an opportunity to simplify the number of risk factors they have to deal with. Suppose, for instance, that the portfolio consists of global bonds in 17 different currencies. Initially, the risk manager decides to keep 14 risk factors in each market. This leads to a very large number of correlations within, but also across all markets. With 17 currencies, and 14 maturities, for instance, the total number of assets is $n = 17 \times 14 = 238$. The correlation matrix has $n \times (n-1) = 238 \times 237 = 56{,}406$ elements off the diagonal. Surely some of this information is superfluous.

The matrix in Table 13-4 can be simplified using principal components. **Principal components** is a statistical technique that extracts linear combinations of the original variables that explain the highest proportion of diagonal components of the matrix. For this matrix, the first principal component explains 94% of the total variance and has similar weights on all maturities. Hence, it could be called a **level risk factor**. The second principal component explains 4% of the total variance. As it is associated with opposite positions on short and long maturities, it could be called a **slope risk factor** (or twist). Sometimes a third factor is found that represents **curvature risk factor**, or a **bend risk factor** (also called a butterfly).

Previous research has indeed found that, in the United States and other fixed-income markets, movements in yields could be usefully summarized by two to three factors that typically explain over 95 percent of the total variance.

Example 13-5: FRM Exam 2000—Question 96/Market Risk

13-5. Which one of the following statements about historic U.S. Treasury yield curve changes is *true*?

a) Changes in long-term yields tend to be larger than in short-term yields.

b) Changes in long-term yields tend to be of approximately the same size as changes in short-term yields.

c) The same size yield change in both long-term and short-term rates tends to produce a larger price change in short-term instruments when all securities are trading near par.

d) The largest part of total return variability of spot rates is due to parallel changes with a smaller portion due to slope changes and the residual due to curvature changes.

13.2.4 Global Interest Rate Risk

Different fixed-income markets create their own sources of risk. Volatility patterns, however, are similar across the globe. To illustrate, Table 13-5 shows price and yield volatilities for 17 fixed-income markets, focusing only on 10-year zeros.

The level of yields falls within a remarkably narrow range, 4 to 6 percent. This reflects the fact that yields are primarily driven by **inflationary expectations**, which have become similar across all these markets. Indeed central banks across all these countries have proved their common determination to keep inflation in check. Two notable exceptions are South Africa, where yields are at 10.7% and Japan where yields are at 0.9%. These two countries are experiencing much higher and lower inflation, respectively, than the rest of the group.

The table also shows that most countries have an annual volatility of yield changes around 0.6 to 1.2 percent. Again, Japan is an exception, which suggests that the volatility of yields is not independent of the level of yields.

In fact, we would expect this volatility to decrease as yields drop toward zero and to be higher when yields are higher. The Cox, Ingersoll, and Ross (1985) model

TABLE 13-5 Global Fixed-Income Volatility, 2002 (Percent)

Country	Code	Yield	Price Vol.			Yield Vol. $\sigma(dy)$		
		Level	Daily	Mty	Annual	Daily	Mty	Annual
Austrl.	AUD	5.236	0.676	3.660	12.68	0.066	0.353	1.22
Belgium	BEF	4.453	0.352	1.995	6.91	0.035	0.196	0.68
Canada	CAD	4.950	0.426	2.438	8.45	0.042	0.237	0.82
Germany	DEM	4.306	0.349	1.967	6.81	0.035	0.194	0.67
Denmark	DKK	4.563	0.307	1.765	6.12	0.031	0.174	0.60
Spain	ESP	4.399	0.359	2.024	7.01	0.036	0.198	0.69
France	FRF	4.383	0.351	1.952	6.76	0.035	0.192	0.67
Britain	GBP	4.415	0.333	1.848	6.40	0.033	0.181	0.63
Ireland	IEP	4.456	0.353	1.950	6.75	0.035	0.191	0.66
Italy	ITL	4.582	0.348	1.999	6.93	0.034	0.194	0.67
Japan	JPY	0.918	0.171	1.153	3.99	0.015	0.096	0.33
Nether.	NLG	4.335	0.356	1.985	6.88	0.035	0.194	0.67
New Zl.	NZD	6.148	0.477	2.741	9.49	0.047	0.272	0.94
Sweden	SEK	4.812	0.361	2.055	7.12	0.036	0.204	0.71
U.S.	USD	3.989	0.637	3.600	12.47	0.062	0.350	1.21
S.Afr.	ZAR	10.650	0.535	3.358	11.63	0.055	0.337	1.17
Euro	EUR	4.306	0.352	1.978	6.85	0.035	0.195	0.68

of the term structure (CIR), for instance, posits that movements in yields should be proportional to the square root of the yield level:

$$\sigma\left(\frac{\Delta y}{\sqrt{y}}\right) = \text{constant} \tag{13.6}$$

Thus neither the normal nor the lognormal model is totally appropriate.

Finally, correlations are very high across continental European bond markets that are part of the euro. For example, the correlation between French and German bonds is above 0.975. These markets are now moving in synchronization, as monetary policy is dictated by the **European Central Bank** (ECB). Eurozone bonds only differ in terms of credit risk. Otherwise, correlations across other bond markets are in the range of 0.00 to 0.50. The correlation between US and yen bonds is very small; US and German bonds have a correlation close to 0.71.

13.2.5 Real Yield Risk

So far, the analysis has only considered **nominal interest rate risk**, as most bonds represent obligations in nominal terms, i.e. in dollars for the coupon and principal payment. Recently, however, many countries have issued inflation-protected bonds, which make payments that are fixed in real terms but indexed to the rate of inflation.

In this case, the source of risk is **real interest rate risk**. This real yield can be viewed as the internal rate of return that will make the discounted value of promised real bond payments equal to the current real price. This is a new source of risk, as movements in real interest rates may not correlate perfectly with movements in nominal yields.

Example: Real and Nominal Yields

Consider for example the 10-year Treasury Inflation Protected (TIP) note paying a 3% coupon in real terms. coupons are paid semiannually. The actual coupon and principal payments are indexed to the increase in Consumer Price Index (CPI).

The TIP is now trading at a clean real price of 108-23+. Discounting the coupon payments and the principal gives a real yield of $r = 1.98\%$. Note that since the bond is trading at a premium, the real yield must be lower than the coupon.

Projecting the rate of inflation at $\pi = 2\%$, semiannually compounded, we infer the projected nominal yield as $(1 + y/200) = (1 + r/200)(1 + \pi/200)$, which gives 4.00%. This is the same order of magnitude as the current nominal yield on the

10-year Treasury note, which is 3.95%. The two bonds have a very different risk profile, however. If the rate of inflation is 5% instead of 2%, the TIP will pay approximately 5% plus 2%, while the yield on the regular note is predetermined.

Example 13-6: FRM Exam 1997—Question 42/Market Risk

13-6. What is the relationship between yield on the current inflation-proof bond issued by the U.S. Treasury and a standard Treasury bond with similar terms?
a) The yields should be about the same.
b) The yield of the inflation bond should be approximately the yield on the treasury minus the real interest.
c) The yield of the inflation bond should be approximately the yield on the treasury plus the real interest.
d) None of the above is correct.

13.2.6 Credit Spread Risk

Credit spread risk is the risk that yields on duration-matched credit-sensitive bond and Treasury bonds could move differently. The topic of credit risk will be analyzed in more detail in the "Credit Risk" section of this book. Suffice to say that the credit spread represent a compensation for the loss due to default, plus perhaps a risk premium that reflects investor risk aversion.

A position in a credit spread can be established by investing in credit-sensitive bonds, such as corporates, agencies, mortgage-backed securities (MBSs), and shorting Treasuries with the appropriate duration. This type of position benefits from a stable or shrinking credit spread, but loses from a widening of spreads. Because credit spreads cannot turn negative, their distribution is asymmetric, however. When spreads are tight, large moves imply increases in spreads rather than decreases. Thus positions in credit spreads can be exposed to large losses.

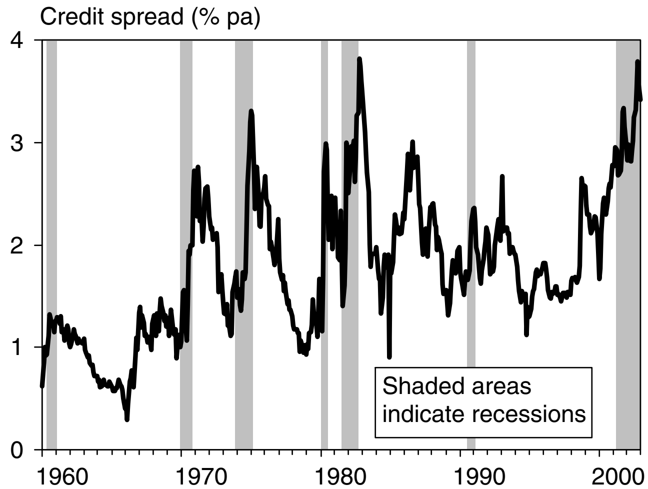
Figure 13-6 displays the time-series of credit spreads since 1960. The graph shows that credit spreads display cyclical patterns, increasing during a recession and decreasing during economic expansions. Greater spreads during recessions reflect the greater number of defaults during difficult times.

Because credit spreads cannot turn negative, their distribution is asymmetric. When spreads are tight, large moves are typically increases, rather than decreases.

13.2.7 Prepayment Risk

Prepayment risk arises in the context of home mortgages when there is uncertainty

FIGURE 13-6 Credit Spreads



about whether the homeowner will refinance his loan early. It is a prominent feature of **mortgage-backed securities** the investor has granted the borrower an option to repay the debt early.

This option, however, is much more complex than an ordinary option, due to the multiplicity of factors involved. We have seen in Chapter 7 that it depends on the age of the loan (seasoning), the current level of interest rates, the previous path of interest rates (burnout), economic activity, and seasonal patterns.

Assuming that the prepayment model adequately captures all these features, investors can evaluate the attractiveness of MBSs by calculating their **option-adjusted spread** (OAS). This represents the spread over the equivalent Treasury minus the cost of the option component.

Example 13-7: FRM Exam 1999—Question 71/Market Risk

13-7. An investor holds mortgage interest-only strips (IO) backed by Fannie Mae 7 percent coupon. She wants to hedge this position by shorting Treasury interest strips off the 10-year on-the-run. The curve steepens as the 1-month rate drops, while the 6-month to 10-year rates remain stable. What will be the effect on the value of this portfolio?

a) Both the IO and the hedge will appreciate in value.

b) The IO and the hedge value will be almost unchanged (a very small appreciation is possible).

c) The change in value of both the IO and hedge cannot be determined without additional details.

d) The IO will depreciate, but the hedge will appreciate.

Example 13-8: FRM Exam 1999—Question 73/Market Risk
13-8. A fund manager attempting to beat his LIBOR-based funding costs, holds pools of adjustable rate mortgages (ARMs) and is considering various strategies to lower the risk. Which of the following strategies will *not* lower the risk?
a) Enter into a total rate of return swap swapping the ARMs for LIBOR plus a spread.
b) Short U.S. government Treasuries.
c) Sell caps based on the projected rate of mortgage paydown.
d) All of the above.

13.3 Equity Risk

Equity risk arises from potential movements in the value of stock prices. We will show that we can usefully decompose the total risk into a marketwide risk and stock-specific risk.

13.3.1 Stock Market Volatility

Table 13-6 compares the RiskMetrics volatility forecasts for a group of 31 stock markets. The selected indices are those most recognized in each market, for example the S&P 500 in the US, Nikkei 225 in Japan, and FTSE-100 in Britain. Most of these have an associated futures contract, so positions can be taken in cash markets or, equivalently, in futures. Nearly all of these indices are weighted by market capitalization.

We immediately note that risk is much greater than for currencies, typically ranging from 12 to 40 percent. Emerging markets have higher volatility. These markets are less diversified and are exposed to greater fluctuations in economic fundamentals.

Concentration refers to the proportion of the index due to the biggest stocks. In Finland, for instance, half of the index represents one firm only, Nokia. This lack of diversification invariably creates more volatility.

TABLE 13-6 Equity Volatility (Percent)

Stock Market Country	Code	End 2002 Daily	Monthly	Annual	End 1996 Annual
Argentina	ARS	1.921	10.06	34.8	22.1
Austria	ATS	0.771	4.17	14.4	11.7
Australia	AUD	0.662	3.58	12.4	13.4
Belgium	BEF	1.453	8.41	29.1	9.3
Canada	CAD	0.841	5.09	17.6	13.8
Switzerland	CHF	1.401	8.34	28.9	11.1
Germany	DEM	2.576	13.89	48.1	18.6
Denmark	DKK	1.062	6.77	23.5	12.5
Spain	ESP	1.497	8.81	30.5	15.0
Finland	FIM	1.790	10.65	36.9	14.5
France	FRF	1.691	10.59	36.7	16.1
Britain	GBP	1.498	8.41	29.1	11.1
Hong Kong	HKD	1.007	5.57	19.3	17.3
Indonesia	IDR	1.218	7.45	25.8	14.4
Ireland	IEP	1.081	6.53	22.6	10.0
Italy	ITL	1.575	9.07	31.4	17.0
Japan	JPY	1.299	7.18	24.9	19.9
Korea	KRW	1.861	9.40	32.6	25.5
Mexico	MXN	0.925	5.87	20.3	17.5
Malaysia	MYR	0.709	3.81	13.2	12.7
Netherlands	NLG	1.911	11.55	40.0	14.8
Norway	NOK	1.160	6.80	23.5	13.3
New Zealand	NZD	0.480	2.79	9.7	10.1
Philippines	PHP	0.807	4.49	15.6	16.2
Portugal	PTE	0.879	5.82	20.2	6.9
Sweden	SEK	1.612	9.91	34.3	16.9
Singapore	SGD	0.817	4.72	16.4	11.9
Thailand	THB	0.680	4.39	15.2	29.7
Taiwan	TWD	1.317	7.72	26.7	15.3
U.S.	USD	1.214	7.42	25.7	12.9
South Africa	ZAR	0.023	0.72	2.5	11.9

Example 13-9: FRM Exam 1997—Question 43/Market Risk

13-9. Which of the following statements about the S&P 500 index is *true*?

I. The index is calculated using market prices as weights.

II. The implied volatilities of options of the same maturity on the index are different.

III. The stocks used in calculating the index remain the same for each year.

IV. The S&P 500 represents only the 500 largest U.S. corporations.

a) II only

b) I and II only

c) II and III only

d) III and IV only

13.3.2 Forwards and Futures

The forward or futures price on a stock index or individual stock can be expressed as

$$F_t e^{-r\tau} = S_t e^{-y\tau} \tag{13.7}$$

where $e^{-r\tau}$ is the present value factor in the base currency and $e^{-y\tau}$ is the discounted value of dividends. For the stock index, this is usually approximated by the dividend yield y, which is taken to be paid continuously as there are many stocks in the index (even though dividend payments may be "lumpy" over the quarter). For an individual stock, we can write the right-hand side as $S_t e^{-y\tau} = S_t - I$, where I is the present value of dividend payments.

Example 13-10: FRM Exam 1997—Question 44/Market Risk

13-10. A trader runs a cash and future arbitrage book on the S&P 500 index. Which of the following are the *major* risk factors?

I. Interest rate
II. Foreign exchange
III. Equity price
IV. Dividend assumption risk

a) I and II only
b) I and III only
c) I, III, and IV only
d) I, II, III, and IV

13.4 Commodity Risk

Commodity risk arises from potential movements in the value of commodity contracts, which include agricultural products, metals, and energy products.

13.4.1 Commodity Volatility Risk

Table 13-7 displays the volatility of the commodity contracts currently covered by the RiskMetrics system. These can be grouped into *base metals* (aluminum, copper, nickel, zinc), *precious metals* (gold, platinum, silver), and *energy products* (natural gas, heating oil, unleaded gasoline, crude oil–West Texas Intermediate).

Among base metals, spot volatility ranged from 13 to 28 percent per annum in 2002, on the same order of magnitude as equity markets. Precious metals are in the

TABLE 13-7 Commodity Volatility (Percent)

Commodity	Code	End 2002			End 1996
Term		Daily	Monthly	Annual	Annual
Aluminium, spot	ALU.C00	0.702	3.85	13.3	16.8
3-month	ALU.C03	0.621	3.46	12.0	15.8
15-month	ALU.C15	0.528	2.99	10.3	13.9
27-month	ALU.C27	0.493	2.72	9.4	13.5
Copper, spot	COP.C00	0.850	4.45	15.4	35.4
3-month	COP.C03	0.824	4.30	14.9	24.9
15-month	COP.C15	0.788	4.04	14.0	21.5
27-month	COP.C27	0.736	3.84	13.3	22.7
Nickel, spot	NIC.C00	1.451	8.11	28.1	22.7
3-month	NIC.C03	1.392	7.78	27.0	22.1
15-month	NIC.C15	1.202	7.07	24.5	22.7
Zinc, spot	ZNC.C00	1.118	5.56	19.3	12.4
3-month	ZNC.C03	1.060	5.22	18.1	11.5
15-month	ZNC.C15	0.895	4.41	15.3	11.6
27-month	ZNC.C27	0.841	4.11	14.2	13.1
Gold, spot	GLD.C00	0.969	4.41	15.3	5.5
Platinum, spot	PLA.C00	0.811	4.54	15.7	6.5
Silver, spot	SLV.C00	1.095	5.12	17.7	18.1
Natural gas, 1m	GAS.C01	2.882	15.66	54.3	95.8
3-month	GAS.C03	2.846	13.56	47.0	55.2
15-month	GAS.C06	1.343	7.62	26.4	34.4
27-month	GAS.C12	1.145	6.48	22.5	25.7
Heating oil, 1m	HTO.C01	2.196	10.39	36.0	34.4
3-month	HTO.C03	1.905	9.24	32.0	26.2
6-month	HTO.C06	1.489	7.46	25.9	23.5
12-month	HTO.C12	1.284	6.07	21.0	22.7
Unleaded gas, 1m	UNL.C01	2.859	14.08	48.8	31.0
3-month	UNL.C03	2.132	9.85	34.1	26.2
6-month	UNL.C06	1.665	8.01	27.7	23.5
Crude oil, 1m	WTI.C01	2.147	10.11	35.0	32.8
3-month	WTI.C03	1.885	8.87	30.7	29.6
5-month	WTI.C06	1.621	7.54	26.1	28.1
12-month	WTI.C12	1.296	6.02	20.8	28.9

same range. Energy products, in contrast, are much more volatile with numbers ranging from 35 to a high of 53 percent per annum in 2002. This is due to the fact that energy products are less storable than metals and, as a result, are much more affected by variations in demand and supply.

13.4.2 Forwards and Futures

The forward or futures price on a commodity can be expressed as

$$F_t e^{-r\tau} = S_t e^{-y\tau} \tag{13.8}$$

where $e^{-r\tau}$ is the present value factor in the base currency and $e^{-y\tau}$ includes a **convenience yield** y (net of storage cost). This represents an implicit flow benefit from holding the commodity, as was explained in Chapter 6.

While this convenience yield is conceptually similar to that of a dividend yield on a stock index, it cannot be measured as regular income. Rather, it should be viewed as a "plug-in" that, given F, S, and $e^{-r\tau}$, will make Equation (13.8) balance. Further, it can be quite volatile.

As Table 13-7 shows, forward prices for all these commodities are less volatile for longer maturities. This decreasing term structure of volatility is more marked for energy products and less so for base metals. Forward prices are not reported for precious metals. Their low storage costs and no convenience yields implies stable volatilities across contract maturities, as for currency forwards.

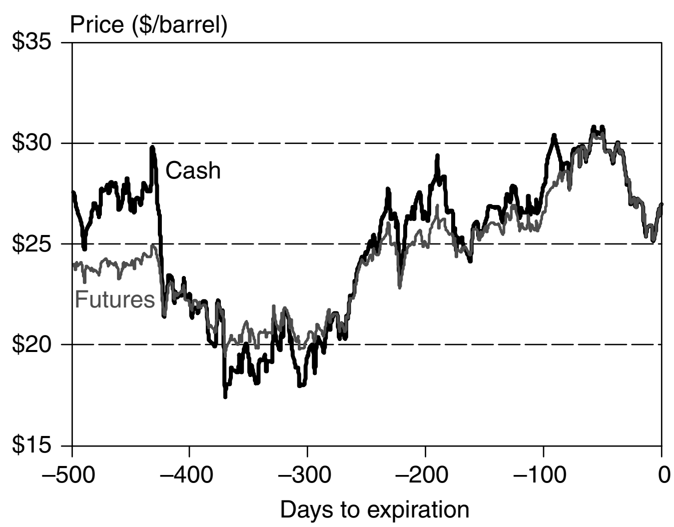
In terms of risk management, movements in futures prices are much less tightly related to spot prices than for financial contracts. This is illustrated in Table 13-8, which displays correlations for copper contracts (spot, 3-, 15-, 27-month) as well as for natural gas and crude oil contracts (1-, 3-, 6-, 12-month). For copper, the cash/15-month correlation is 0.995. For natural gas and oil, the 1-month/12-month correlation is 0.575 and 0.787, respectively. These are much lower numbers. Thus variations in the basis are much more important for energy products than for financial products, or even metals. This is confirmed by Figure 13-7, which compares the spot and futures prices for crude oil.

Recall that the graph describing stock index futures in Chapter 5 showed the future to be systematically above, and converging to, the cash price. Here the picture is totally different. There is much more variation in the basis between the spot and futures prices for crude oil. The market switches from backwardation ($S > F$) to contango ($S < F$). As a result, the futures contract represents a separate risk factor.

TABLE 13-8 Correlations across Maturities

Copper	COP.C00	COP.C03	COP.C15	COP.C27
COP.C00	1			
COP.C03	.999	1		
COP.C15	.995	.995	1	
COP.C27	.992	.993	.998	1
Nat.Gas	GAS.C01	GAS.C03	GAS.C06	GAS.C12
GAS.C01	1			
GAS.C03	.860	1		
GAS.C06	.718	.734	1	
GAS.C12	.575	.445	.852	1
Crude Oil	WTI.C01	WTI.C03	WTI.C06	WTI.C12
WTI.C01	1			
WTI.C03	.960	1		
WTI.C06	.904	.973	1	
WTI.C12	.787	.871	.954	1

FIGURE 13-7 Futures and Spot for Crude Oil



13.4.3 Delivery and Liquidity Risk

In addition to traditional market sources of risk, positions in commodity futures are also exposed to delivery and liquidity risks. Asset liquidity risk is due to the relative low volume in some of these markets, relative to other financial products.

Also, taking delivery or having to deliver on a futures contract that is carried to expiration is costly. Transportation, storage and insurance costs can be quite high.

Futures delivery also requires complying with the type and location of the commodity that is to be delivered.

Example 13-11: FRM Exam 1997—Question 12/Market Risk

13-11. Which of the following products should have the highest expected volatility?

a) Crude oil
b) Gold
c) Japanese Treasury Bills
d) EUR/CHF

Example 13-12: FRM Exam 1997—Question 23/Market Risk

13-12. Identify the *major* risks of being short $50 million of gold two weeks forward and being long $50 million of gold one year forward.

I. Gold liquidity squeeze
II. Spot risk
III. Gold lease rate risk
IV. USD interest rate risk
a) II only
b) I, II, and III only
c) I, III, and IV only
d) I, II, III, and IV

13.5 Risk Simplification

The fundamental idea behind modern risk measurement methods is to aggregate the portfolio risk at the highest level. In practice, it would be too complex to model each of them individually. Instead, some simplification is required, such as the **diagonal model** proposed by Professor William Sharpe. This was initially applied to stocks, but the methodology can be used in any market.

13.5.1 Diagonal Model

The diagonal model starts with a statistical decomposition of the return on stock i into a marketwide return and an idiosyncratic risk. The diagonal model adds the assumption that all specific risks are uncorrelated. Hence, any correlation across two stocks must come from the joint effect of the market.

We decompose the return on stock i, R_i, into a constant; a component due to the market, R_M, through a "beta" coefficient; and some residual risk:

$$R_i = \alpha_i + \beta_i \times R_M + \epsilon_i \tag{13.9}$$

where β_i is called systematic risk of stock i. It is also the regression slope ratio:

$$\beta_i = \frac{\text{Cov}[R_i, R_M]}{V[R_M]} = \rho_{iM}\frac{\sigma(R_i)}{\sigma(R_M)} \tag{13.10}$$

Note that the residual is uncorrelated with R_M by assumption.

The contribution of William Sharpe was to show that equilibrium in capital markets imposes restrictions on the α_i. If we redefine returns in excess of the risk-free rate, R_f, we have

$$E(R_i) - R_f = 0 + \beta_i[E(R_M) - R_f] \tag{13.11}$$

This relationship is also known as the **Capital Asset Pricing Model** (CAPM). So, αs should be zero in equilibrium.

The CAPM is based on equilibrium in capital markets, which requires that the demand for securities from risk-averse investors matches the available supply. It also assumes that asset returns have a normal distribution. When these conditions are satisfied, the CAPM predicts a relationship between α_i and the factor exposure β_i: $\alpha_i = R_f(1 - \beta_i)$.

A major problem with this theory is that it may not be testable unless the "market" is exactly identified. For risk managers, who primarily focus on risk instead of expected returns, however, this is of little importance. What matters is the simplification bought by the diagonal model.

Consider a portfolio that consists of positions w_i on the various assets. We have

$$R_p = \sum_{i=1}^{N} w_i R_i \tag{13.12}$$

Using Equation (13.9), the portfolio return is also

$$R_p = \sum_{i=1}^{N}(w_i\alpha_i + w_i\beta_i R_M + w_i\epsilon_i) = \alpha_p + \beta_p R_M + \sum_{i=1}^{N}(w_i\epsilon_i) \tag{13.13}$$

Such decomposition is useful for **performance attribution**. Suppose a stock portfolio returns 10% over the last year. How can we tell if the portfolio manager is doing a good job? We need to know the performance of the overall stock market, as well as the portfolio beta. Suppose the market went up by 8%, and the portfolio beta is 1.1. portfolio alpha. Taking expected values, we find

$$E(R_p) = \alpha_p + \beta_p E(R_M) \tag{13.14}$$

The portfolio "alpha" is $\alpha_p = 10\% - 1.1 \times 8\% = 1.2\%$. In this case, the active manager provided value added. More generally, we could have additional risk factors. Performance attribution is the process of decomposing the total return on various sources of risk, with the objective of identifying the value added of active management.[3]

We now turn to the use of the diagonal model for risk simplification, and ignore the intercept in what follows. The portfolio variance is

$$V[R_p] = \beta_p^2 V[R_M] + \sum_{i=1}^{N} \left(w_i^2 V[\epsilon_i]\right) \tag{13.15}$$

since all the residual terms are uncorrelated. Suppose that, for simplicity, the portfolio is equally weighted and that the residual variances are all the same $V[\epsilon_i] = V$. This implies $w_i = w = 1/N$. As the number of assets, N, increases, the second term will tend to

$$\sum_{i=1}^{N} \left(w_i^2 V[\epsilon_i]\right) \to N \times [(1/N)^2 V] = (V/N) \tag{13.16}$$

which should vanish as N increases. In this situation, the only remaining risk is the general market risk, consisting of the beta squared times the variance of the market.

Next, we can derive the covariance between any two stocks

$$\text{Cov}[R_i, R_j] = \text{Cov}[\beta_i R_M + \epsilon_i, \beta_j R_M + \epsilon_j] = \beta_i \beta_j \sigma_M^2 \tag{13.17}$$

using the assumption that the residual components are uncorrelated with each other and with the market. Also, the variance of a stock is

$$\text{Cov}[R_i, R_i] = \beta_i^2 \sigma_M^2 + \sigma_{\epsilon,i}^2 \tag{13.18}$$

The covariance matrix is then

$$\Sigma = \begin{bmatrix} \beta_1^2 \sigma_M^2 + \sigma_{\epsilon,1}^2 & \beta_1 \beta_2 \sigma_M^2 & \dots & \beta_1 \beta_N \sigma_M^2 \\ \vdots & & & \\ \beta_N \beta_1 \sigma_M^2 & \beta_N \beta_2 \sigma_M^2 & \dots & \beta_N^2 \sigma_M^2 + \sigma_{\epsilon,N}^2 \end{bmatrix}$$

[3]This process can also be used to detect **timing ability**, which consists of adding value by changing exposure on risk factors and **security selection ability**, which adds value beyond exposures on major risk factors.

which can also be written as

$$\Sigma = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} [\beta_1 \dots \beta_N] \sigma_M^2 + \begin{bmatrix} \sigma_{\epsilon,1}^2 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & \sigma_{\epsilon,N}^2 \end{bmatrix}$$

Using matrix notation, we have

$$\Sigma = \beta\beta'\sigma_M^2 + D_\epsilon \tag{13.19}$$

This consists of N elements in the vector β, of N elements on the diagonal of the matrix D_ϵ, plus the variance of the market itself. The diagonal model reduces the number of parameters from $N \times (N+1)/2$ to $2N + 1$, a considerable improvement. For example, with 100 assets the number is reduced from 5,050 to 201.

In summary, this diagonal model substantially simplifies the risk structure of an equity portfolio. Risk managers can proceed in two steps: first, managing the overall market risk of the portfolios, and second, managing the concentration risk of individual securities.

13.5.2 Factor Models

Still, this one-factor model could miss common effects among groups of stocks, such as industry effects. To account for these, Equation (13.9) can be generalized to K factors

$$R_i = \alpha_i + \beta_{i1} y_1 + \cdots + \beta_{iK} y_K + \epsilon_i \tag{13.20}$$

where $y_1, \dots, y_K$ are the factors, which are assumed independent of each other for simplification. The covariance matrix generalizes Equation (13.19) to

$$\Sigma = \beta_1\beta_1'\sigma_1^2 + \cdots + \beta_K\beta_K'\sigma_K^2 + D_\epsilon \tag{13.21}$$

The number of parameters is now $(N \times K + K + N)$. For example, with 100 assets and five factors, this number is 605, which is still much lower than 5,050 for the unrestricted model.

As in the case of the CAPM, the **Arbitrage Pricing Theory** (APT), developed by Professor Stephen Ross, shows that there is a relationship between α_i and the factor exposures. The theory does not rely on equilibrium but simply on the assumption that there should be no arbitrage opportunities in capital markets, a much weaker requirement. It does not even need the factor model to hold strictly; instead, it requires only that the residual risk is very small. This must be the case if a sufficient number of common factors is identified and in a well-diversified portfolio.

The APT model does not require the market to be identified, which is an advantage. Like the CAPM, however, tests of this model are ambiguous since the theory provides no guidance as to what the factors should be.

Example 13-13: FRM Exam 1998—Question 62/Capital Markets

13-13. In comparing CAPM and APT, which of the following advantages does APT have over CAPM:

I. APT makes less restrictive assumptions about investor preferences toward risk and return.

II. APT makes no assumption about the distribution of security returns.

III. APT does not rely on the identification of the true market portfolio, and so the theory is potentially testable.

a) I only
b) II and III only
c) I and III only
d) I, II, and III

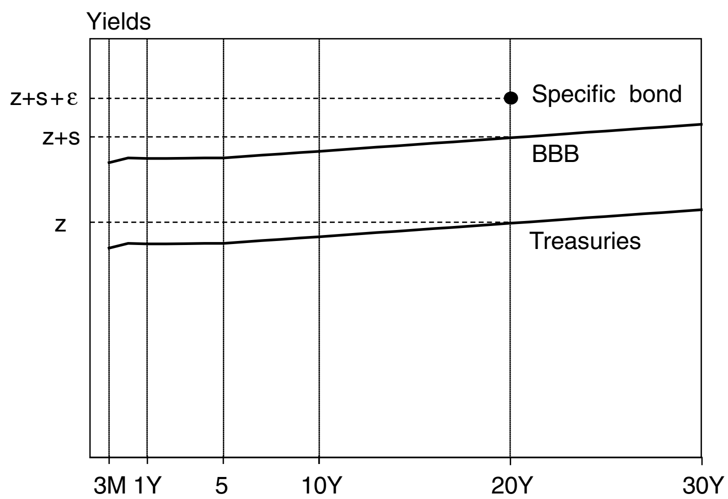
13.5.3 Fixed-Income Portfolio Risk

As an example of portfolio simplification, we turn to the analysis of a corporate bond portfolio with N individual bonds. Each "name" is potentially a source of risk. Instead of modelling all securities, the risk manager should attempt to simplify the risk profile of the portfolio. Potential major risk factors are movements in a set of J Treasury zero-coupon rates, z_j, and in K credit spreads, s_k, sorted by credit rating. The goal is to provide a good approximation to the risk of the portfolio.

In addition, it is not practical to model the risk of all bonds. The bonds may not have a sufficient history. Even if they do, the history may not be relevant if it does not account for the probability of default. In all cases, risk is best modelled by focusing on yields instead of prices.

We model the movement in each corporate bond yield y_i by a movement in the Treasury factor z_j at the closest maturity and in the credit rating s_k class to which it belongs. The remaining component is ϵ_i, which is assumed to be independent across i. We have $y_i = z_j + s_k + \epsilon_i$. This decomposition is illustrated in Figure 13-8 for a corporate bond rated BBB with a 20-year maturity.

FIGURE 13-8 Yield Decomposition



The movement in the bond price is

$$\Delta P_i = -\text{DVBP}_i \Delta y_i = -\text{DVBP}_i \, \Delta z_j - \text{DVBP}_i \, \Delta s_k - \text{DVBP}_i \, \Delta \epsilon_i$$

where DVBP = DV01 is the total dollar value of a basis point for the associated risk factor. We hold n_i units of this bond.

Summing across the portfolio and collecting terms across the common risk factors, the portfolio price movement is

$$\Delta V = -\sum_{i=1}^{N} n_i \text{DVBP}_i \Delta y_i = -\sum_{j=1}^{J} \text{DVBP}_j^z \, \Delta z_j - \sum_{k=1}^{K} \text{DVBP}_k^s \, \Delta s_k - \sum_{i=1}^{N} n_i \text{DVBP}_i \, \Delta \epsilon_i \quad (13.22)$$

where DVBP_j^z results from the summation of $n_i\text{DVBP}_i$ for all bonds that are exposed to the jth maturity. The total variance can be decomposed into

$$V(\Delta V) = \text{General Risk} + \sum_{i=1}^{N} n_i^2 \text{DVBP}_i^2 \, V(\Delta \epsilon_i) \quad (13.23)$$

If the portfolio is well diversified, the general risk term should dominate. So, we could simply ignore the second term.

Ignoring specific risk, a portfolio composed of thousands of securities can be characterized by its exposure to just a few government maturities and credit spreads. This is a considerable simplification.

13.6 Answers to Chapter Examples

Example 13-1: FRM Exam 1997—Question 10/Market Risk

b) From the table. Among floating exchange rates, the USD/CAD has low volatility.

Example 13-2: FRM Exam 1997–Question 14/Market Risk

d) The logs of JPY/EUR and EUR/USD add up to that of JPY/USD: ln[JPY/USD] = ln[JPY/eur] + ln[eur/USD]. So, σ^2(JPY/USD) = σ^2(JPY/EUR) + σ^2(EUR/USDD) + $2\rho\sigma$(JPY/EUR)σ(EUR/USDD), or $8^2 = 10^2 + 6^2 + 2\rho 10 \times 6$, or $2\rho 10 \times 6 = -72$, or $\rho = -0.60$.

Example 13-3: FRM Exam 1999–Question 86/Market Risk

d) Historical yield volatility is more stable than price risk for a specific bond.

Example 13-4: FRM Exam 1999–Question 80/Market Risk

c) *(Lengthy.)* Assuming normally distributed returns, the 95% worst loss for the bond can be found from the yield volatility and Equation (13.4). First, we compute the gross market value of the position, which is $P = \$20,000,000 \times (98 + 8/32 + 1.43)/100 = \$19,936,000$. Next, we compute the daily yield volatility, which is $\sigma(\Delta y) = y\sigma^{ANNUAL}(\Delta y/y)/\sqrt{250} = 0.06509 \times 0.12/\sqrt{250} = 0.000494$. The bond's VAR is then VAR $= D^* \times P \times 1.64485 \times \sigma(\Delta y)$, or VAR $= 12.719 \times \$19,936,000 \times 1.64485 \times 0.000494 = \$206,036$. Note that it is important to use an accurate value for the normal deviate. Using an approximation, such as $\alpha = 1.645$, will give a wrong answer, (d) in this case.

Example 13-5: FRM Exam 2000–Question 96/Market Risk

d) Most of the movements in yields can be explained by a single-factor model, or parallel moves. Once this effect is taken into account, short-term yields move more than long-term yields, so that (a) and (b) are wrong.

Example 13-6: FRM Exam 1997–Question 42/Market Risk

d) The yield on the inflation-protected bond is a real yield, or nominal yield minus expected inflation.

Example 13-7: FRM Exam 1999–Question 71/Market Risk
b) If most of the term structure is unaffected, the hedge will not change in value given that it is driven by 10-year yields. Also, there will be little change in refinancing. For the IO, the slight decrease in the short-term discount rate will increase the present value of short-term cash flows, but the effect is small.

Example 13-8: FRM Exam 1999–Question 73/Market Risk
c) The TR swap will eliminate all market risk; shorting Treasuries protects against interest rate risk; since the ARM is already short options, the manager should be buying caps, not selling them.

Example 13-9: FRM Exam 1997–Question 43/Market Risk
a) The “smile” effect represents different implied vols for the same maturity, so that (II) is correct. Otherwise, the index is computed using market values, number of shares times price, so that (I) is wrong. The stocks are selected by Standard and Poor’s but are not always the largest ones. Finally, the stocks in the index are regularly changed.

Example 13-10 FRM Exam 1997–Question 44/Market Risk
c) The futures price is a function of the spot price, interest rate, and dividend yield.

Example 13-11: FRM Exam 1997–Question 12/Market Risk
a) From comparing Tables 13-1, 13-6, 13-7. The volatility of crude oil, at around 35% per annum, is the highest.

Example 13-12: FRM Exam 1997–Question 23/Market Risk
c) There is no spot risk since the two contracts have offsetting exposure to the spot rate. There is, however, basis risk (lease rate and interest rate) and liquidity risk.

Example 13-13: FRM Exam 1998–Question 62/Capital Markets
d) The CAPM assumes that returns are normally distributed and that markets are in equilibrium. In other words, the demand from mean-variance optimizers must be equal to the supply. In contrast, the APT simply assumes that returns are driven by a factor model with a small number of factors, whose risk can be eliminated through arbitrage. So, the APT is less restrictive, does not assume that returns are normally distributed, and does not rely on the identification of the true market portfolio.

Chapter 14

Hedging Linear Risk

Risk that has been measured can be managed. This chapter turns to the active management of market risks.

The traditional approach to market risk management is **hedging**. Hedging consists of taking positions that lower the risk profile of the portfolio. The techniques for hedging have been developed in the futures markets, where farmers, for instance, use financial instruments to hedge the price risk of their products.

This implementation of hedging is quite narrow, however. Its objective is to find the optimal position in a futures contract that minimizes the variance of the total position. This is a special case of minimizing the VAR of a portfolio with two assets, an inventory and a "hedging" instrument. Here, the hedging position is fixed and the value of the hedging instrument is linearly related to the underlying asset.

More generally, we can distinguish between

- **Static hedging**, which consists of putting on, and leaving, a position until the hedging horizon. This is appropriate if the hedge instrument is linearly related to the underlying asset price.
- **Dynamic hedging**, which consists of continuously rebalancing the portfolio to the horizon. This can create a risk profile similar to positions in options.

Dynamic hedging is associated with options, which will be examined in the next chapter. Since options have nonlinear payoffs in the underlying asset, the hedge ratio, which can be viewed as the slope of the tangent to the payoff function, must be readjusted as the price moves.

In general, hedging will create **hedge slippage**, or **basis risk**. This can be measured by unexpected changes in the value of the hedged portfolio. Basis risk arises when changes in payoffs on the hedging instrument do not perfectly offset changes in the value of the underlying position.

Obviously, if the objective of hedging is to lower volatility, hedging will eliminate downside risk but also any upside in the position. the objective of hedging is to lower risk, not to make profits. Whether hedging is beneficial should be examined in the context of the trade-off between risk and return.

This chapter discusses linear hedging. A particularly important application is hedging with futures. Section 14.1 presents an introduction to futures hedging with a unit hedge ratio. Section 14.2 then turns to a general method for finding the optimal hedge ratio. This method is applied in Section 14.3 for hedging bonds and equities.

14.1 Introduction to Futures Hedging

14.1.1 Unitary Hedging

Consider the situation of a U.S. exporter who has been promised a payment of 125 million Japanese yen in seven months. The perfect hedge would be to enter a 7-month forward contract over-the-counter (OTC). This OTC contract, however, may not be very liquid. Instead, the exporter decides to turn to an exchange-traded futures contract, which can be bought or sold more easily.

The Chicago Mercantile Exchange (CME) lists yen contracts with face amount of Y12,500,000 that expire in 9 months. The exporter places an order to sell 10 contracts, with the intention of reversing the position in 7 months, when the contract will still have 2 months to maturity.[1] Because the amount sold is the same as the underlying, this is called a **unitary hedge**.

Table 14-1 describes the initial and final conditions for the contract. At each date, the futures price is determined by interest parity. Suppose that the yen depreciates sharply, leading to a loss on the anticipated cash position of $Y125,000,000 \times 0.006667 - 0.00800) = -\$166{,}667$. This loss, however, is offset by a gain on the futures, which is $(-10) \times Y12.5,000,000 \times 0.006711 - 0.00806) = \$168{,}621$. This creates a very small gain of $1,954. Overall, the exporter has been hedged.

This example shows that futures hedging can be quite effective, removing the effect of fluctuations in the risk factor. Define Q as the amount of yen transacted and

[1]In practice, if the liquidity of long-dated contracts is not adequate, the exporter could use nearby contracts and roll them over prior to expiration into the next contracts. When there are multiple exposures, this practice is known as a **stack hedge**. Another type of hedge is the **strip hedge**, which involves hedging the exposures with a number of different contracts. While a stack hedge has superior liquidity, it also entails greater basis risk than a strip hedge. Hedgers must decide whether the greater liquidity of a stack hedge is worth the additional basis risk.

TABLE 14-1 A Futures Hedge

Item	Initial Time	Exit Time	Gain or Loss
Market Data:			
Maturity (months)	9	2	
US interest rate	6%	6%	
Yen interest rate	5%	2%	
Spot (Y/$)	Y125.00	Y150.00	
Futures (Y/$)	Y124.07	Y149.00	
Contract Data:			
Spot ($/Y)	0.008000	0.006667	−$166,667
Futures ($/Y)	0.008060	0.006711	$168,621
Basis ($/Y)	−0.000060	−0.000045	$1,954

S and F as the spot and futures rates, indexed by 1 at the initial time and by 2 at the exit time. The P&L on the unhedged transaction is

$$Q[S_2 - S_1] \tag{14.1}$$

Instead, the hedged profit is

$$Q[(S_2 - S_1) - (F_2 - F_1)] = Q[(S_2 - F_2) - (S_1 - F_1)] = Q[b_2 - b_1] \tag{14.2}$$

where $b = S - F$ is the **basis**. The hedged profit only depends on the movement in the basis. Hence the effect of hedging is to transform price risk into basis risk. A short hedge position is said to be *long the basis*, since it benefits from an increase in the basis.

In this case, the basis risk is minimal for a number of reasons. First, the cash and futures correspond to the same asset. Second, the cash-and-carry relationship holds very well for currencies. Third, the remaining maturity at exit is rather short.

14.1.2 Basis Risk

Basis risk arises when the characteristics of the futures contract differ from those of the underlying position. Futures contracts are standardized to a particular grade, say West Texas Intermediate (WTI) for oil futures traded on the NYMEX. This defines the grade of crude oil deliverable against the contract. A hedger, however, may have a position in a different grade, which may not be perfectly correlated with WTI.

Thus basis risk is the uncertainty whether the cash-futures spread will widen or narrow during the hedging period. Hedging can be effective, however, if movements in the basis are dominated by movements in cash markets.

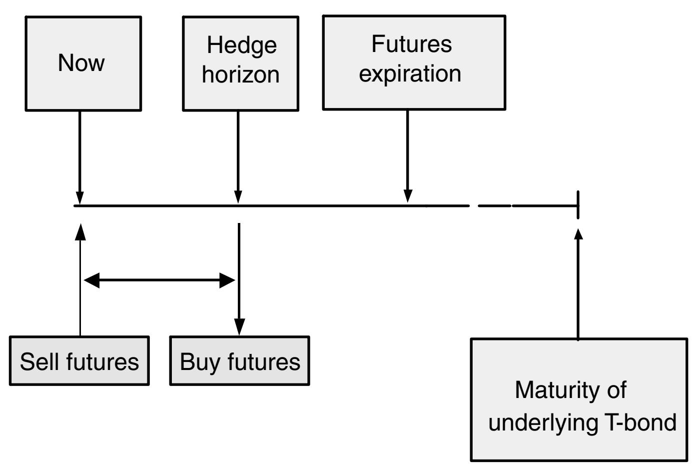
For most commodities, basis risk is inevitable. Organized exchanges strive to create enough trading and liquidity in their listed contracts, which requires standardization. Speculators also help to increase trading volumes and provide market liquidity. Thus there is a trade-off between liquidity and basis risk.

Basis risk is higher with **cross-hedging**, which involves using a futures on a totally different asset or commodity than the cash position. For instance, a U.S. exporter who is due to receive a payment in Norwegian Kroner (NK) could hedge using a futures contract on the $/euro exchange rate. Relative to the dollar, the euro and the NK should behave similarly, but there is still some basis risk.

Basis risk is lowest when the underlying position and the futures correspond to the same asset. Even so, some basis risk remains because of differing maturities. As we have seen in the yen hedging example, the maturity of the futures contract is 9 instead of 7 months. As a result, the liquidation price of the futures is uncertain.

Figure 14-1 describes the various time components for a hedge using T-bond futures. The first component is the *maturity of the underlying bond*, say 20 years. The second component is the *time to futures expiration*, say 9 months. The third component is the *hedge horizon*, say 7 months. Basis risk occurs when the hedge horizon does not match the time to futures expiration.

FIGURE 14-1 Hedging Horizon and Contract Maturity



Example 14-1: FRM Exam 2000—Question 78/Market Risk

14-1. What feature of cash and futures prices tends to make hedging possible?

a) They always move together in the same direction and by the same amount.

b) They move in opposite directions by the same amount.

c) They tend to move together generally in the same direction and by the same amount.

d) They move in the same direction by different amounts.

Example 14-2: FRM Exam 2000—Question 17/Capital Markets

14-2. Which one of the following statements is *most* correct?

a) When holding a portfolio of stocks, the portfolio's value can be fully hedged by purchasing a stock index futures contract.

b) Speculators play an important role in the futures market by providing the liquidity that makes hedging possible and assuming the risk that hedgers are trying to eliminate.

c) Someone generally using futures contracts for hedging does not bear the basis risk.

d) Cross hedging involves an additional source of basis risk because the asset being hedged is exactly the same as the asset underlying the futures.

Example 14-3: FRM Exam 2000—Question 79/Market Risk

14-3. Under which scenario is basis risk likely to exist?

a) A hedge (which was initially matched to the maturity of the underlying) is lifted before expiration.

b) The correlation of the underlying and the hedge vehicle is less than one and their volatilities are unequal.

c) The underlying instrument and the hedge vehicle are dissimilar.

d) All of the above are correct.

14.2 Optimal Hedging

The previous section gave an example of a unit hedge, where the amounts transacted are identical in the two markets. In general, this is not appropriate. We have to decide how much of the hedging instrument to transact.

Consider a situation where a portfolio manager has an inventory of carefully selected corporate bonds that should do better than their benchmark. The manager wants to guard against interest rate increases, however, over the next three months. In this situation, it would be too costly to sell the entire portfolio only to buy it back later. Instead, the manager can implement a temporary hedge using derivative contracts, for instance T-Bond futures.

Here, we note that the only risk is **price risk**, as the quantity of the inventory is known. This may not always be the case, however. Farmers, for instance, have uncertainty over both prices and the size of their crop. If so, the hedging problem is substantially more complex as it involves hedging *revenues*, which involves analyzing demand and supply conditions.

14.2.1 The Optimal Hedge Ratio

Define ΔS as the change in the dollar value of the inventory and ΔF as the change in the dollar value of the one futures contract. In other markets, other reference currencies would be used. The inventory, or position to be hedged, can be existing or **anticipatory**, that is, to be received in the future with a great degree of certainty. The manager is worried about potential movements in the value of the inventory ΔS.

If the manager goes long N futures contracts, the total change in the value of the portfolio is

$$\Delta V = \Delta S + N\Delta F \tag{14.3}$$

One should try to find the hedge that reduces risk to the minimum level. The variance of profits is equal to

$$\sigma^2_{\Delta V} = \sigma^2_{\Delta S} + N^2\sigma^2_{\Delta F} + 2N\sigma_{\Delta S,\Delta F} \tag{14.4}$$

Note that volatilities are initially expressed in dollars, not in rates of return, as we attempt to stabilize dollar values.

Taking the derivative with respect to N

$$\frac{\partial \sigma^2_{\Delta V}}{\partial N} = 2N\sigma^2_{\Delta F} + 2\sigma_{\Delta S,\Delta F} \tag{14.5}$$

For simplicity, drop the Δ in the subscripts. Setting Equation (14.5) equal to zero and solving for N, we get

$$N^* = -\frac{\boldsymbol{\sigma}_{\Delta S,\Delta F}}{\boldsymbol{\sigma}^2_{\Delta F}} = -\frac{\boldsymbol{\sigma}_{SF}}{\boldsymbol{\sigma}^2_{F}} = -\boldsymbol{\rho}_{SF}\frac{\boldsymbol{\sigma}_{S}}{\boldsymbol{\sigma}_{F}} \tag{14.6}$$

where σ_{SF} is the covariance between futures and spot price changes. Here, N^* is the **minimum variance hedge ratio**.

We can do more than this, though. At the optimum, we can find the variance of profits by replacing N in Equation (14.4) by N^*, which gives

$$\sigma_V^{*2} = \sigma_S^2 + \left(\frac{\sigma_{SF}}{\sigma_F^2}\right)^2 \sigma_F^2 + 2\left(\frac{-\sigma_{SF}}{\sigma_F^2}\right)\sigma_{SF} = \sigma_S^2 + \frac{\sigma_{SF}^2}{\sigma_F^2} + 2\frac{-\sigma_{SF}^2}{\sigma_F^2} = \sigma_S^2 - \frac{\sigma_{SF}^2}{\sigma_F^2} \quad (14.7)$$

In practice, there is often confusion about the definition of the portfolio value and unit prices. Here S consists of the number of units (shares, bonds, bushels, gallons) times the unit price (stock price, bond price, wheat price, fuel price).

It is sometimes easier to deal with unit prices and to express volatilities in terms of *rates of changes in unit prices*, which are unitless. Defining quantities Q and unit prices s, we have $S = Qs$. Similarly, the notional amount of one futures contract is $F = Q_f f$. We can then write

$$\sigma_{\Delta S} = Q\sigma(\Delta s) = Qs\sigma(\Delta s/s)$$
$$\sigma_{\Delta F} = Q_f\sigma(\Delta f) = Q_f f\sigma(\Delta f/f)$$
$$\sigma_{\Delta S,\Delta F} = \rho_{sf}[Qs\sigma(\Delta s/s)][Q_f f\sigma(\Delta f/f)]$$

Using Equation (14.6), the optimal hedge ratio N^* can also be expressed as

$$N^* = -\rho_{SF}\frac{Qs\sigma(\Delta s/s)}{Q_f f\sigma(\Delta f/f)} = -\rho_{SF}\frac{\sigma(\Delta s/s)}{\sigma(\Delta f/f)}\frac{Qs}{Q_f f} = -\beta_{sf}\frac{Q \times s}{Q_f \times f} \quad (14.8)$$

where β_{sf} is the coefficient in the regression of $\Delta s/s$ over $\Delta f/f$. The second term represents an adjustment factor for the size of the cash position and of the futures contract.

14.2.2 The Hedge Ratio as Regression Coefficient

The optimal amount N^* can be derived from the slope coefficient of a regression of $\Delta s/s$ on $\Delta f/f$:

$$\frac{\Delta s}{s} = \alpha + \beta_{sf}\frac{\Delta f}{f} + \epsilon \quad (14.9)$$

As seen in Chapter 3, standard regression theory shows that

$$\beta_{sf} = \frac{\sigma_{sf}}{\sigma_f^2} = \rho_{sf}\frac{\sigma_s}{\sigma_f} \quad (14.10)$$

Thus the **best hedge** is obtained from a regression of the (change in the) value of the inventory on the value of the hedge instrument.

> **Key concept:**
> The optimal hedge is given by the negative of the beta coefficient of a regression of changes in the cash value on changes in the payoff on the hedging instrument.

Further, we can measure the quality of the optimal hedge ratio in terms of the amount by which we decreased the variance of the original portfolio:

$$R^2 = \frac{(\sigma_S^2 - \sigma_V^{*2})}{\sigma_S^2} \tag{14.11}$$

After substitution of Equation (14.7), we find that $R^2 = (\sigma_S^2 - \sigma_S^2 + \sigma_{SF}^2/\sigma_F^2)/\sigma_S^2 = \sigma_{SF}^2/(\sigma_F^2\sigma_S^2) = \rho_{SF}^2$. This unitless number is also the coefficient of determination, or the percentage of variance in $\Delta s/s$ explained by the independent variable $\Delta f/f$. Thus this regression also gives us the **effectiveness** of the hedge, which is measured by the proportion of variance eliminated.

We can also express the volatility of the hedged position from Equation (14.7) using the R^2 as

$$\sigma_V^* = \sigma_S \sqrt{(1 - R^2)} \tag{14.12}$$

This shows that if $R^2 = 1$, the regression fit is perfect, and the resulting portfolio has zero risk. In this situation, the portfolio has no basis risk. However, if the R^2 is very low, the hedge is not effective.

> **Example 14-4: FRM Exam 2001—Question 86**
> 14-4. If two securities have the same volatility and a correlation equal to -0.5, their minimum variance hedge ratio is
> a) 1:1
> b) 2:1
> c) 4:1
> d) 16:1

Example 14-5: FRM Exam 1999—Question 66/Market Risk
14-5. The hedge ratio is the ratio of the size of the position taken in the futures contract to the size of the exposure. Assuming the standard deviation of change of spot price is σ_1 and the standard deviation of change of future price is σ_2, the correlation between the changes of spot price and future price is ρ. What is the optimal hedge ratio?
a) $1/\rho \times \sigma_1/\sigma_2$
b) $1/\rho \times \sigma_2/\sigma_1$
c) $\rho \times \sigma_1/\sigma_2$
d) $\rho \times \sigma_2/\sigma_1$

Example 14-6: FRM Exam 2000—Question 92/Market Risk
14-6. The hedge ratio is the ratio of derivatives to a spot position (or vice versa) that achieves an objective, such as minimizing or eliminating risk. Suppose that the standard deviation of quarterly changes in the price of a commodity is 0.57, the standard deviation of quarterly changes in the price of a futures contract on the commodity is 0.85, and the correlation between the two changes is 0.3876. What is the optimal hedge ratio for a three-month contract?
a) 0.1893
b) 0.2135
c) 0.2381
d) 0.2599

14.2.3 Example

An airline knows that it will need to purchase 10,000 metric tons of jet fuel in three months. It wants some protection against an upturn in prices using futures contracts.

The company can hedge using heating oil futures contracts traded on NYMEX. The notional for one contract is 42,000 gallons. As there is no futures contract on jet fuel, the risk manager wants to check if heating oil could provide an efficient hedge instead. The current price of jet fuel is \$277/metric ton. The futures price of heating oil is \$0.6903/gallon. The standard deviation of the rate of change in jet fuel prices over three months is 21.17%, that of futures is 18.59%, and the correlation is 0.8243.

Compute

a) The notional and standard deviation of the unhedged fuel cost in dollars
b) The optimal number of futures contract to buy/sell, rounded to the closest integer
c) The standard deviation of the hedged fuel cost in dollars

Answer

a) The position notional is Qs = \$2,770,000. The standard deviation in dollars is

$$\sigma(\Delta s/s)sQ = 0.2117 \times \$277 \times 10{,}000 = \$586{,}409$$

For reference, that of one futures contract is

$$\sigma(\Delta f/f)fQ_f = 0.1859 \times \$0.6903 \times 42{,}000 = \$5{,}389.72$$

with a futures notional of $fQ_f = \$0.6903 \times 42{,}000 = \$28{,}992.60$.

b) The cash position corresponds to a payment, or liability. Hence, the company will have to *buy* futures as protection. First, we compute beta, which is $\beta_{sf} = 0.8243(0.2117/0.1859) = 0.9387$. The corresponding covariance term is $\sigma_{sf} = 0.8243 \times 0.2117 \times 0.1859 = 0.03244$. Adjusting for the notionals, this is $\sigma_{SF} = 0.03244 \times \$2{,}770{,}000 \times \$28{,}993 = 2{,}605{,}268{,}452$. The optimal hedge ratio is, using Equation (14.8)

$$N^* = \beta_{sf}\frac{Q \times s}{Q_f \times f} = 0.9387\frac{10,000 \times \$277}{42,000 \times \$0.69} = 89.7$$

or 90 contracts after rounding (which we ignore in what follows).

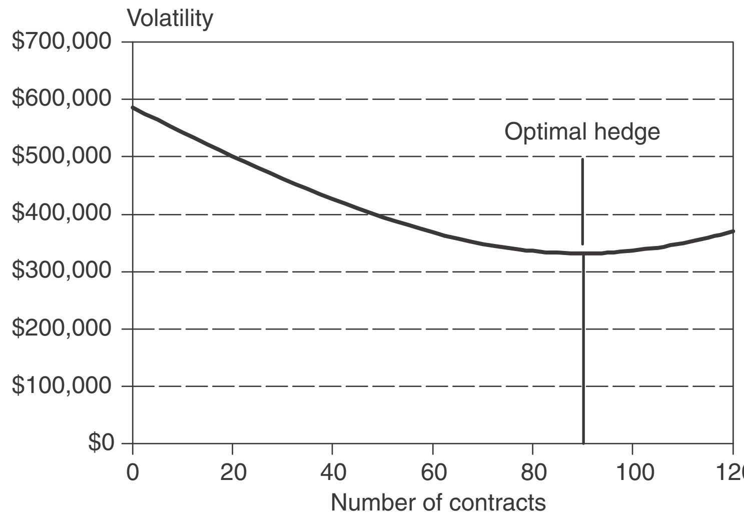
c) To find the risk of the hedged position, we use Equation (14.8). The volatility of the unhedged position is $\sigma_S = \$586,409$. The variance of the hedged position is

$$\begin{aligned}
\sigma_S^2 &= (\$586{,}409)^2 && = +343{,}875{,}515{,}281 \\
-\sigma_{SF}^2/\sigma_F^2 &= -(2{,}605{,}268{,}452/5{,}390)^2 && = -233{,}653{,}264{,}867 \\
\text{V}(\mathit{hedged}) & && = +110{,}222{,}250{,}414
\end{aligned}$$

The volatility of the hedged position is $\sigma_V^* = \$331,997$. Thus the hedge has reduced the risk from \$586,409 to \$331,997. that one minus the ratio of the hedged and unhedged variances is $(1 - 110{,}222{,}250{,}414/343{,}875{,}515{,}281) = 67.95\%$. This is exactly the square of the correlation coefficient, $0.8243^2 = 0.6795$. Thus the effectiveness of the hedge can be judged from the correlation coefficient.

Figure 14-2 displays the relationship between the risk of the hedged position and the number of contracts. As N increases, the risk decreases, reaching a minimum for $N^* = 90$ contracts. The graph also shows that the quadratic relationship is relatively flat for a range of values around the minimum. Choosing anywhere between 80 and 100 contracts will have little effect on the total risk.

FIGURE 14-2 Risk of Hedged Position and Number of Contracts



14.2.4 Liquidity Issues

Although futures hedging can be successful at mitigating market risk, it can create other risks. Futures contracts are marked to market daily. Hence they can involve large cash inflows or outflows. Cash outflows, in particular, can create liquidity problems, especially when they are not offset by cash inflows from the underlying position.

Example 14-7: FRM Exam 1999—Question 67/Market Risk

14-7. In the early 1990s, Metallgesellschaft, a German oil company, suffered a loss of $1.33 billion in their hedging program. They rolled over short-dated futures to hedge long term exposure created through their long-term fixed-price contracts to sell heating oil and gasoline to their customers. After a time, they abandoned the hedge because of large negative cash flow. The cash-flow pressure was due to the fact that MG had to hedge its exposure by
a) Short futures and there was a decline in oil price
b) Long futures and there was a decline in oil price
c) Short futures and there was an increase in oil price
d) Long futures and there was an increase in oil price

14.3 Applications of Optimal Hedging

The linear framework presented here is completely general. We now specialize it to two important cases, duration and beta hedging. The first applies to the bond market, the second to the stock market.

14.3.1 Duration Hedging

Modified duration can be viewed as a measure of the exposure of relative changes in prices to movements in yields. Using the definitions in Chapter 1, we can write

$$\Delta P = (-D^* P)\Delta y \tag{14.13}$$

where D^* is the modified duration. The **dollar duration** is defined as $(D^* P)$.

Assuming the duration model holds, which implies that the change in yield Δy does not depend on maturity, we can rewrite this expression for the cash and futures positions

$$\Delta S = (-D_S^* S)\Delta y \qquad \Delta F = (-D_F^* F)\Delta y$$

where D_S^* and D_F^* are the modified durations of S and F, respectively. Note that these relationships are supposed to be perfect, without an error term. The variances and covariance are then

$$\sigma_S^2 = (D_S^* S)^2 \sigma^2(\Delta y) \qquad \sigma_F^2 = (D_F^* F)^2 \sigma^2(\Delta y) \qquad \sigma_{SF} = (D_F^* F)(D_S^* S)\sigma^2(\Delta y)$$

We can replace these in Equation (14.6)

$$N^* = -\frac{\sigma_{SF}}{\sigma_F^2} = -\frac{(D_F^* F)(D_S^* S)}{(D_F^* F)^2} = -\frac{(D_S^* S)}{(D_F^* F)} \tag{14.14}$$

Alternatively, this can be derived as follows. Write the total portfolio payoff as

$$\begin{aligned} \Delta V &= \Delta S + N \Delta F \\ &= (-D_S^* S)\Delta y + N(-D_F^* F)\Delta y \\ &= -[(D_S^* S) + N(D_F^* F)] \times \Delta y \end{aligned}$$

which is zero when the net exposure, represented by the term between brackets, is zero. In other words, the optimal hedge ratio is simply minus the ratio of the dollar duration of cash relative to the dollar duration of the hedge. This ratio can also be expressed in dollar value of a basis point (DVBP).

More generally, we can use N as a tool to modify the total duration of the portfolio. If we have a target duration of D_V, this can be achieved by setting $[(D_S^* S) + N(D_F^* F)] = D_V^* V$, or

$$N = \frac{(D_V^* V - D_S^* S)}{(D_F^* F)} \tag{14.15}$$

of which Equation (14.14) is a special case.

> **Key concept:**
> The optimal duration hedge is given by the ratio of the dollar duration of the position to that of the hedging instrument.

Example 1

A portfolio manager holds a bond portfolio worth \$10 million with a modified duration of 6.8 years, to be hedged for 3 months. The current futures price is 93-02, with a notional of \$100,000. We assume that its duration can be measured by that of the cheapest-to-deliver, which is 9.2 years.

Compute

a) The notional of the futures contract

b) The number of contracts to buy/sell for optimal protection

Answer

a) The notional is $[93 + (2/32)]/100 \times \$100{,}000 = \$93{,}062.5$.

b) The optimal number to *sell* is from Equation (14.14)

$$N^* = -\frac{(D_S^* S)}{(D_F^* F)} = -\frac{6.8 \times \$10{,}000{,}000}{9.2 \times \$93{,}062.5} = -79.4$$

or 79 contracts after rounding. Note that the DVBP of the futures is about $9.2 \times \$93{,}000 \times 0.01\% = \$85$.

Example 2

On February 2, a corporate Treasurer wants to hedge a July 17 issue of \$5 million of commercial paper with a maturity of 180 days, leading to anticipated proceeds of \$4.52 million. The September Eurodollar futures trades at 92, and has a notional amount of \$1 million.

Compute

a) The current dollar value of the futures contract

b) The number of contracts to buy/sell for optimal protection

Answer

a) The current dollar price is given by $\$10{,}000[100 - 0.25(100 - 92)] = \$980{,}000$. Note that the duration of the futures is always 3 months (90 days), since the contract refers to 3-month LIBOR.

b) If rates increase, the cost of borrowing will be higher. We need to offset this by a gain, or a short position in the futures. The optimal number is from Equation (14.14)

$$N^* = -\frac{(D_S^* S)}{(D_F^* F)} = -\frac{180 \times \$4{,}520{,}000}{90 \times \$980{,}000} = -9.2$$

or 9 contracts after rounding. Note that the DVBP of the futures is about $0.25 \times \$1,000,000 \times 0.01\% = \$25$.

Example 14-8: FRM Exam 2000—Question 73/Market Risk

14-8. What assumptions does a duration-based hedging scheme make about the way in which interest rates move?

a) All interest rates change by the same amount.
b) A small parallel shift occurs in the yield curve.
c) Any parallel shift occurs in the term structure.
d) Interest rates movements are highly correlated.

Example 14-9: FRM Exam 1999—Question 61/Market Risk

14-9. If all spot interest rates are increased by one basis point, a value of a portfolio of swaps will increase by $1,100. How many Eurodollar futures contracts are needed to hedge the portfolio?

a) 44
b) 22
c) 11
d) 1,100

Example 14-10: FRM Exam 1999—Question 109/Market Risk

14-10. Roughly how many 3-month LIBOR Eurodollar futures contracts are needed to hedge a position in a $200 million, 5-year receive-fixed swap?

a) Short 250
b) Short 3,200
c) Short 40,000
d) Long 250

14.3.2 Beta Hedging

We now turn to equity hedging using stock index futures. **Beta**, or **systematic risk** can be viewed as a measure of the exposure of the rate of return on a portfolio i to movements in the "market" m

$$R_{it} = \alpha_i + \beta_i R_{mt} + \epsilon_{it} \tag{14.16}$$

where β represents the systematic risk, α the intercept (which is not a source of risk and therefore ignored for risk management purposes), and ϵ the residual component, which is uncorrelated with the market. We can also write, in line with the previous sections and ignoring the residual and intercept

$$(\Delta S/S) \approx \beta(\Delta M/M) \tag{14.17}$$

Now, assume that we have at our disposal a stock-index futures contract, which has a beta of unity $(\Delta F/F) = 1(\Delta M/M)$. For options, the beta is replaced by the net delta, $(\Delta C) = \delta(\Delta M)$.

As in the case of bond duration, we can write the total portfolio payoff as

$$\begin{aligned}\Delta V &= \Delta S + N\Delta F \\ &= (\beta S)(\Delta M/M) + NF(\Delta M/M) \\ &= [(\beta S) + NF] \times (\Delta M/M)\end{aligned}$$

which is set to zero when the net exposure, represented by the term between brackets is zero. The optimal number of contracts to short is

$$N^* = -\frac{\beta S}{F} \tag{14.18}$$

Key concept:
The optimal hedge with stock index futures is given by the the beta of the cash position times its value divided by the notional of the futures contract.

Example

A portfolio manager holds a stock portfolio worth \$10 million with a beta of 1.5 relative to the S&P 500. The current futures price is 1,400, with a multiplier of \$250.

Compute

a) The notional of the futures contract

b) The number of contracts to sell short for optimal protection

Answer

a) The notional amount of the futures contract is $\$250 \times 1400 = \$350{,}000$.

b) The optimal number of contract to short is, from Equation (14.18)

$$N^* = -\frac{\beta S}{F} = -\frac{1.5 \times \$10{,}000{,}000}{1 \times \$350{,}000} = -42.9$$

or 43 contracts after rounding.

The quality of the hedge will depend on the size of the residual risk in the market model of Equation (14.16). For large portfolios, the approximation may be good. In contrast, hedging an individual stock with stock index futures may give poor results.

For instance, the correlation of a typical U.S. stock with the S&P 500 is 0.50. For an industry index, it is typically 0.75. Using the regression effectiveness in Equation (14.12), we find that the volatility of the hedged portfolio is still about $\sqrt{1-0.5^2} =$ 87% of the unhedged volatility for a typical stock and about 66% of the unhedged volatility for a typical industry. The lower number shows that hedging with general stock index futures is more effective for large portfolios. To obtain finer coverage of equity risks, hedgers could use futures contracts on industrial sectors, or even single stock futures.

Example 14-11: FRM Exam 2000—Question 93/Market Risk

14-11. Assume Global Funds manages an equity portfolio worth \$50,000,000 with a beta of 1.8. Further, assume that there exists an index call option contract with a delta of 0.623 and a value of \$500,000. How many options contracts are needed to hedge the portfolio?

a) 169
b) 289
c) 306
d) 321

14.4 Answers to Chapter Examples

Example 14-1: FRM Exam 2000—Question 78/Market Risk

c) Hedging is made possible by the fact that cash and futures prices usually move in the same direction and by the same amount.

Example 14-2: FRM Exam 2000—Question 17/Capital Markets

b) Answer (a) is wrong because we need to hedge by *selling* futures. Answer (c) is wrong because futures hedging creates some basis risk. Answer (d) is wrong because cross-hedging involves *different* assets. Speculators do serve some social function, which is to create liquidity for others.

Example 14-3: FRM Exam 2000—Question 79/Market Risk

d) Basis risk occurs if movements in the value of the cash and hedged positions do not offset each other perfectly. This can happen if the instruments are dissimilar or if the correlation is not unity. Even with similar instruments, if the hedge is lifted before the maturity of the underlying, there is some basis risk.

Example 14-4: FRM Exam 2001—Question 86

b) Set x as the amount to invest in the second security, relative to that in the first (or the hedge ratio). The variance is then proportional to $1 + x^2 + 2x\rho$. Taking the derivative and setting to zero, we have $x = -rho = 0.5$. Thus one security must have twice the amount in the other. Alternatively, the hedge ratio is given by $N^* = -\rho\frac{\sigma_S}{\sigma_F}$, which gives 0.5. Answer (b) is the only one which is consistent with this number or its inverse.

Example 14-5: FRM Exam 1999—Question 66/Market Risk

c) See Equation (14.6).

Example 14-6: FRM Exam 2000—Question 92/Market Risk

d) The hedge ratio is $\rho_{fs}\sigma_s/\sigma_f = 0.3876 \times 0.57/0.85 = 0.2599$.

Example 14-7: FRM Exam 1999—Question 67/Market Risk

b) MG was long futures to offset the promised forward sales to clients. It lost money as oil futures prices fell.

Example 14-8: FRM Exam 2000—Question 73/Market Risk

b) The assumption is that of (1) parallel and (2) small moves in the yield curve. Answers (a) and (c) are the same, and omit the size of the move. Answer (d) would require perfect, not high, correlation plus small moves.

Example 14-9: FRM Exam 1999—Question 61/Market Risk

a) The DVBP of the portfolio is \$1100. That of the futures is \$25. Hence the ratio is 1100/25 = 44.

Example 14-10: FRM Exam 1999—Question 109/Market Risk

b) The dollar duration of a 5-year 6% par bond is about 4.3 years. Hence the DVBP of the position is about $\$200,000,000 \times 4.3 \times 0.0001 = \$86,000$. That of the futures is \$25. Hence the ratio is 86000/25 = 3,440.

Example 14-11: FRM Exam 2000—Question 93/Market Risk

b) The hedging instrument has a market beta that is not unity, but instead 0.623. The optimal hedge ratio is $N = -(1.8 \times \$50{,}000{,}000)/(0.623 \times \$500{,}000) = 288.9$.

Chapter 15

Nonlinear Risk: Options

The previous chapter focused on "linear" hedging, using contracts such as forwards and futures whose values are linearly related to the underlying risk factors. Positions in these contracts are fixed over the hedge horizon. Because linear combinations of normal random variables are also normally distributed, linear hedging maintains normal distributions, albeit with lower variances.

Hedging nonlinear risks, however, is much more complex. Because options have nonlinear payoffs, the distribution of option values can be sharply asymmetrical. Since options are ubiquitous instruments, it is important to develop tools to evaluate the risk of positions with options. Since options can be replicated by dynamic trading of the underlying instruments, this also provides insights into the risks of active trading strategies.

In Chapter 12, we have seen that market losses can be ascribed to the combination of two factors: exposure and adverse movements in the risk factor. Thus a large loss could occur because of the risk factor, which is bad luck. Too often, however, losses occur because the exposure profile is similar to a short option position. This is less forgivable, because exposure is under the control of the risk manager.

The challenge is to develop measures that provide an intuitive understanding of the exposure profile. Section 15.1 introduces option pricing and the Taylor approximation.[1] It also briefly reviews the Black-Scholes formula that was presented in Chapter 6. Partial derivatives, also known as "Greeks," are analyzed in Section 15.2. Section 15.3 then turns to the interpretation of dynamic hedging and discusses the distribution profile of option positions.

---

[1]The reader should be forewarned that this chapter is more technical than others. It presupposes some exposure to option pricing and hedging.

15.1 Evaluating Options

15.1.1 Definitions

We consider a **derivative** instrument whose value depends on an underlying asset, which can be a price, an index, or a rate. As an example, consider a call option where the underlying asset is a foreign currency. We use these definitions:

$$
\begin{aligned}
S_t &= \text{current spot price of the asset in dollars} \\
F_t &= \text{current forward price of the asset} \\
K &= \text{exercise price of option contract} \\
f_t &= \text{current value of derivative instrument} \\
r_t &= \text{domestic risk-free rate} \\
r_t^* &= \text{foreign risk-free rate (also written as } y) \\
\sigma_t &= \text{annual volatility of the rate of change in } S \\
\tau &= \text{time to maturity.}
\end{aligned}
$$

More generally, r^* represents the income payment y on the asset, which represents the *annual rate* of dividend or coupon payments on a stock index or bond.

For most options, we can write the value of the derivative as the function

$$f_t = f(S_t, r_t, r_t^*, \sigma_t, K, \tau) \tag{15.1}$$

The contract specifications are represented by K and the time to maturity τ. The other factors are affected by market movements, creating volatility in the value of the derivative. For simplicity, we drop the time subscripts in what follows.

Derivatives pricing is all about finding the value of f, given the characteristics of the option at expiration and some assumptions about the behavior of markets. For a forward contract, for instance, the expression is very simple. It reduces to

$$f = Se^{-r^*\tau} - Ke^{-r\tau} \tag{15.2}$$

More generally, we may not be able to derive an analytical expression for the functional form of the derivative, requiring numerical methods.

15.1.2 Taylor Expansion

We are interested in describing the movements in f. The exposure profile of the derivative can be described *locally* by taking a Taylor expansion,

$$df = \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial r^*}dr^* + \frac{\partial f}{\partial \sigma}d\sigma + \frac{\partial f}{\partial \tau}d\tau + \cdots \quad (15.3)$$

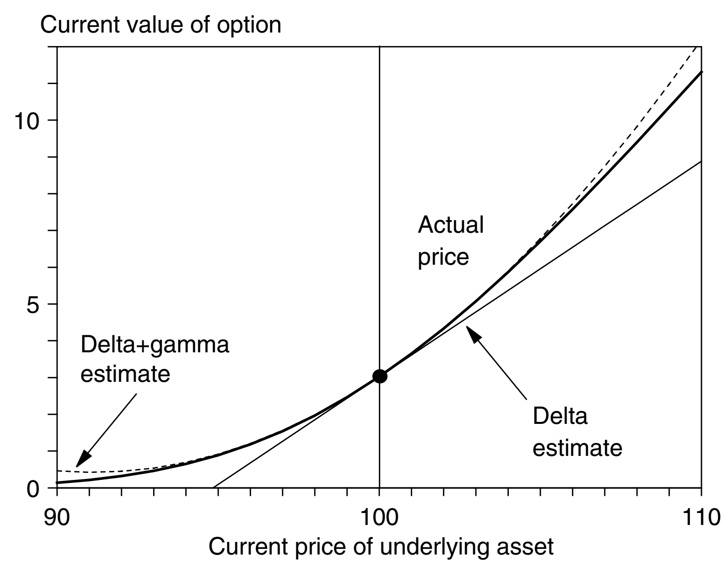
Because the value depends on S in a nonlinear fashion, we added a quadratic term for S. The terms in Equation (15.3) approximate a nonlinear function by linear and quadratic polynomials.

Option pricing is about finding f. **Option hedging** uses the partial derivatives. **Risk management** is about combining those with the movements in the risk factors.

Figure 15-1 describes the relationship between the value of a European call an the underlying asset. The actual price is the solid line. The thin line is the linear (delta) estimate, which is the tangent at the initial point. The dotted line is the quadratic (delta plus gamma) estimates, which gives a much better fit because it has more parameters.

Note that, because we are dealing with sums of local price movements, we can aggregate the sensitivities at the portfolio level. This is similar to computing the portfolio duration from the sum of durations of individual securities, appropriately weighted.

FIGURE 15-1 Delta-Gamma Approximation for a Long Call



Defining $\Delta = \frac{\partial f}{\partial S}$, for example, we can summarize the portfolio, or "book" Δ_P in terms of the total sensitivity,

$$\Delta_P = \sum_{i=1}^{N} x_i \Delta_i \tag{15.4}$$

where x_i is the number of options of type i in the portfolio. To hedge against first-order price risk, it is sufficient to hedge the *net* portfolio delta. This is more efficient than trying to hedge every single instrument individually.

The Taylor approximation may fail for a number of reasons:

- *Large movements in the underlying risk factor*
- *Highly nonlinear exposures*, such as options near expiry or exotic options
- *Cross-partials effect*, such as σ changing in relation with S

If this is the case, we need to turn to a **full revaluation** of the instrument. Using the subscripts 0 and 1 as the initial and final values, the change in the option value is

$$f_1 - f_0 = f(S_1, r_1, r_1^*, \sigma_1,\ K, \tau_1) - f(S_0, r_0, r_0^*, \sigma_0,\ K, \tau_0) \tag{15.5}$$

15.1.3 Option Pricing

We now present the various partial derivatives for conventional European call and put options. As we have seen in Chapter 6, the **Black-Scholes** (BS) model provides a closed-form solution, from which these derivatives can be computed analytically.

The key point of the BS derivation is that a position in the option can be replicated by a "delta" position in the underlying asset. Hence, a portfolio combining the asset and the option in appropriate proportions is risk-free "locally", that is, for small movements in prices. To avoid arbitrage, this portfolio must return the risk-free rate. The option value is the discounted expected payoff,

$$f_t = E_{\text{RN}}[e^{-r\tau} F(S_T)] \tag{15.6}$$

where E_{RN} represents the expectation of the future payoff in a "risk-neutral" world, that is, assuming the underlying asset grows at the risk-free rate and the discounting also employs the risk-free rate.

In the case of a European call, the final payoff is $F(S_T) = \text{Max}(S_T - K, 0)$, and the current value of the call is given by:

$$c = S e^{-r_t^* \tau} N(d_1) - K e^{-r\tau} N(d_2) \tag{15.7}$$

where $N(d)$ is the cumulative distribution function for the standard normal distribution:

$$N(d) = \int_{-\infty}^{d} \Phi(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}x^2} dx$$

with Φ defined as the standard normal distribution function. $N(d)$ is also the area to the left of a standard normal variable with value equal to d. The values of d_1 and d_2 are:

$$d_1 = \frac{ln(Se^{-r_t^*\tau}/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

By put-call parity, the European put option value is:

$$p = Se^{-r_t^*\tau}[N(d_1) - 1] - Ke^{-r\tau}[N(d_2) - 1] \tag{15.8}$$

Example 15-1: FRM Exam 1999—Question 65/Market Risk

15-1. It is often possible to estimate the value at risk of a vanilla European options portfolio by using a delta-gamma methodology rather than exact valuation formulas because

a) Delta and gamma are the first two terms in the Taylor series expansion of the change in an option price as a function of the change in the underlying and the remaining terms are often insignificant.
b) It is only delta and gamma risk that can be hedged.
c) Unlike the price, delta and gamma for a European option can be computed in closed form.
d) Both a and c, but not b, are correct.

Example 15-2: FRM Exam 1999—Question 88/Market Risk

15-2. Why is the delta normal approach not suitable for measuring options portfolio risk?

a) There is a lack of data to compute the variance/covariance matrix.
b) Options are generally short-dated instruments.
c) There are nonlinearities in option payoff.
d) Black-Scholes pricing assumptions are violated in the real world.

15.2 Option "Greeks"

15.2.1 Option Sensitivities: Delta and Gamma

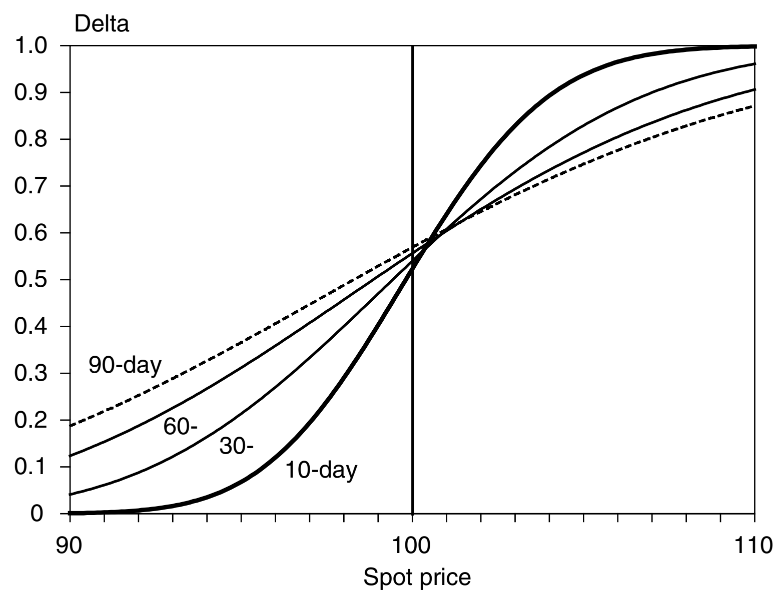
Given these closed-form solutions for European options, we can derive all partial derivatives. The most important sensitivity is the **delta**, which is the first partial derivative with respect to the price. For a call option, this can be written explicitly as:

$$\Delta_c = \frac{\partial c}{\partial S} = e^{-r_t^* \tau} N(d_1) \tag{15.9}$$

which is always positive and below unity.

Figure 15-2 relates delta to the current value of S, for various maturities. The essential feature of this figure is that Δ varies substantially with the spot price and with time. As the spot price increases, d_1 and d_2 become very large, and Δ tends toward $e^{-r_t^* \tau}$, close to one. in this situation, the option behaves like an outright position in the asset. Indeed the limit of Equation (15.7) is $c = Se^{-r_t^* \tau} - Ke^{-r\tau}$, which is exactly the value of our forward contract, Equation (15.2).

FIGURE 15-2 Option Delta



At the other extreme, if S is very low, Δ is close to zero and the option is not very sensitive to S. When S is close to the strike price K, Δ is close to 0.5, and the option behaves like a position of 0.5 in the underlying asset.

> **Key concept:**
> The delta of an at-the-money call option is close to 0.5. Delta moves to one as the call goes deep in the money. It moves to zero as the call goes deep out of the money.

The delta of a put option is:

$$\Delta_p = \frac{\partial p}{\partial S} = e^{-r_t^* \tau}[N(d_1) - 1] \tag{15.10}$$

which is always negative. It behaves similarly to the call Δ, except for the sign. The delta of an at-the-money put is about -0.5.

> **Key concept:**
> The delta of an at-the-money put option is close to -0.5. Delta moves to one as the put goes deep in the money. It moves to zero as the put goes deep out of the money.

The figure also shows that, as the option nears maturity, the Δ function becomes more curved. The function converges to a step function, 0 when $S < K$, and 1 otherwise. Close-to-maturity options have unstable deltas.

For a European call or put, gamma (Γ) is the second order term,

$$\Gamma = \frac{\partial^2 c}{\partial S^2} = \frac{e^{-r_t^* \tau} \Phi(d_1)}{S\sigma\sqrt{\tau}} \tag{15.11}$$

which is driven by the "bell shape" of the normal density function Φ. This is also the derivative of Δ with respect to S. Thus Γ measures the "instability" in Δ. Note that gamma is identical for a call and put with identical characteristics.

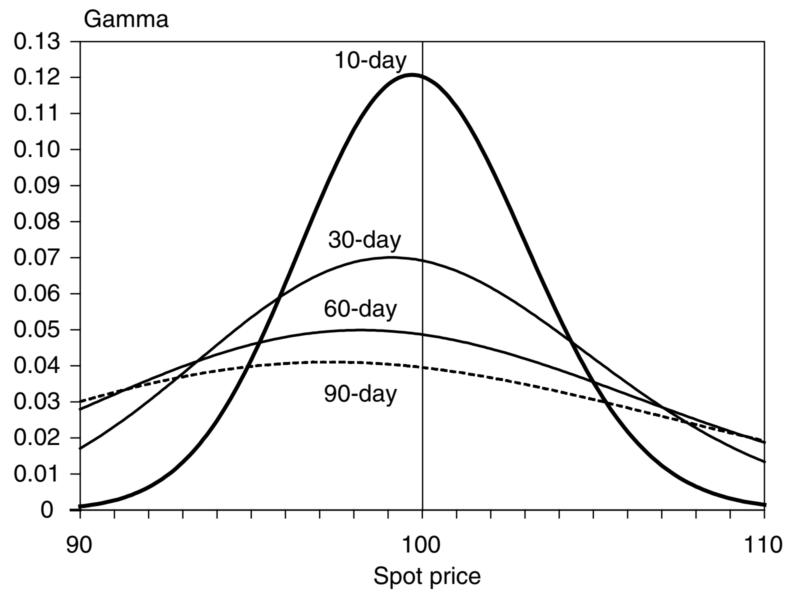
Figure 15-3 plots the call option gamma. At-the-money options have the highest gamma, which indicates that Δ changes very fast as S changes. In contrast, both in-the-money options and out-of-the-money options have low gammas because their delta is constant, close to one or zero, respectively.

The figure also shows that as the maturity nears, the option gamma increases. This leads to a useful rule:

> **Key concept:**
> For vanilla options, nonlinearities are most pronounced for short-term at-the-money options.

Thus, gamma is similar to the concept of convexity developed for bonds. Fixed-coupon bonds, however, always have positive convexity, whereas options can create positive or negative convexity. Positive convexity or gamma is beneficial, as it implies that the value of the asset drops more slowly and increases more quickly than

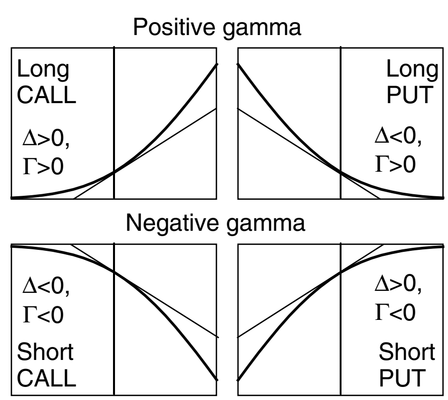
FIGURE 15-3 Option Gamma



otherwise. In contrast, negative convexity can be dangerous because it implies faster price falls and slower price increases.

Figure 15-4 summarizes the delta and gamma exposures of positions in options. Long positions in options, whether calls or puts, create positive convexity. Short positions create negative convexity. In exchange for assuming the harmful effect of this negative convexity, option sellers receive the premium.

FIGURE 15-4 Delta and Gamma of Option Positions



Example 15-3: FRM Exam 2001—Question 79
15-3. A bank has sold USD 300,000 of call options on 100,000 equities. The equities trade at 50, the option strike price is 49, the maturity is in 3 months, volatility is 20%, and the interest rate is 5%. How does the bank delta hedge? (Round to the nearest thousand share)
a) Buy 65,000 shares
b) Buy 100,000 shares
c) Buy 21,000 shares
d) Sell 100,000 shares

Example 15-4: FRM Exam 1999—Question 69/Market Risk
15-4. A portfolio is long a call that is delta hedged by trading in the underlying security. Assuming that the call is fairly valued and the market is in equilibrium, which of the following formulas indicates the standard deviation of the expected profit or loss from holding the hedged position until option expiry? In the following N is the frequency of hedging (52 = weekly), T is the time to expiry and σ is the annualized volatility. K is a constant.
a) $K\sigma/\sqrt{N}$
b) $K\sqrt{N}/\sigma^2$
c) $K\sigma^2/N$
d) KN/σ

15.2.2 Option Sensitivities: Vega

Unlike linear contracts, options are exposed not only to movements in the direction of the spot price, but also in its volatility. Options therefore can be viewed as "volatility bets."

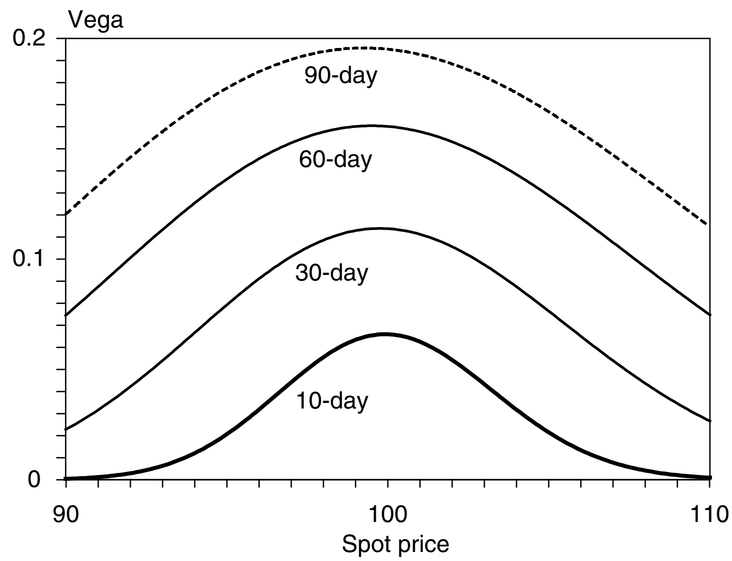
The sensitivity of an option to volatility is called the option **vega** (sometimes also called lambda, or kappa). For European calls and puts, this is

$$\Lambda = \frac{\partial c}{\partial \sigma} = Se^{-r_t^* \tau} \sqrt{\tau}\, \Phi(d_1) \tag{15.12}$$

which also has the "bell shape" of the normal density function Φ. As with gamma, vega is identical for similar call and put positions. Λ must be positive for long option positions.

Figure 15-5 plots the call option vega. The graph shows that at-the-money options are the most sensitive to volatility. time effect, however, is different from that

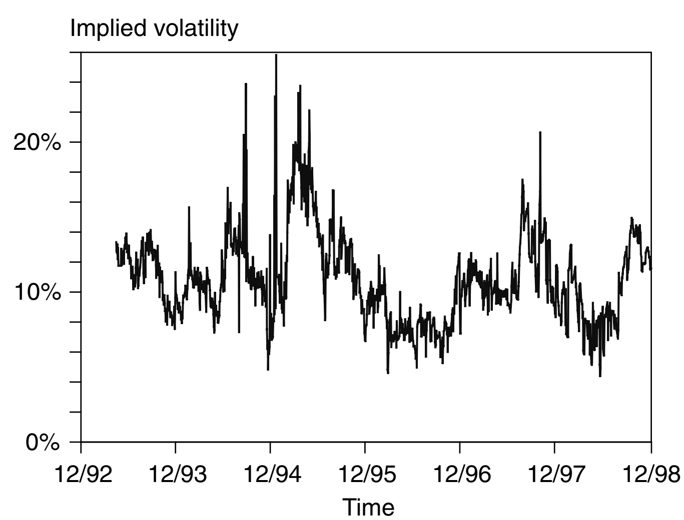
FIGURE 15-5 Option Vega



for gamma, because the term $\sqrt{\tau}$ appears in the numerator instead of denominator. This implies that vega decreases with maturity, unlike gamma, which increases with maturity.

Changes in the volatility parameter can be a substantial source of risk. Figure 15-6 illustrates the time-variation in the option-σ for options on the dollar/mark exchange rate. Here, the average value is about 11%, with a typical daily volatility in σ of 1.5%.[2]

FIGURE 15-6 Movements in Implied Volatility



[2]There is strong mean reversion in these data, so that daily volatilities cannot be extrapolated to annual data.

15.2.3 Option Sensitivities: Rho

The sensitivity to the domestic interest rate, also called **rho**, is

$$\rho_c = \frac{\partial c}{\partial r} = Ke^{-r\tau}\tau N(d_2) \tag{15.13}$$

For a put,

$$\rho_p = \frac{\partial p}{\partial r} = -Ke^{-r\tau}\tau N(-d_2) \tag{15.14}$$

An increase in the rate of interest increases the value of the call, as the underlying asset grows at a higher rate, which increases the probability of exercising the call, with a fixed strike price K. In the limit, for an infinite interest rate, the probability of exercise is one and the call option is equivalent to the stock itself. The reasoning is opposite for a put option.

The exposure to the yield on the asset is, for calls and puts, respectively,

$$\rho^*_C = \frac{\partial c}{\partial r^*} = -Se^{-r^*_t\tau}\tau N(d_1) \tag{15.15}$$

$$\rho^*_P = \frac{\partial p}{\partial r^*} = Se^{-r^*_t\tau}\tau N(-d_1) \tag{15.16}$$

An increase in the dividend yield decreases the growth rate of the underlying asset, which is harmful to the value of the call. Again, the reasoning is opposite for a put option.

15.2.4 Option Sensitivities: Theta

Finally, the variation in option value due to the passage of time is also called **theta**. This is also the **time decay**. Unlike other factors, however, movements in remaining maturity are perfectly predictable; time is not a risk factor.

For a European call, this is

$$\Theta_c = \frac{\partial c}{\partial t} = -\frac{\partial c}{\partial \tau} = -\frac{Se^{-r^*_t\tau}\sigma\Phi(d_1)}{2\sqrt{\tau}} + r^*Se^{-r^*_t\tau}N(d_1) - rKe^{-r\tau}N(d_2) \tag{15.17}$$

For a European put, this is

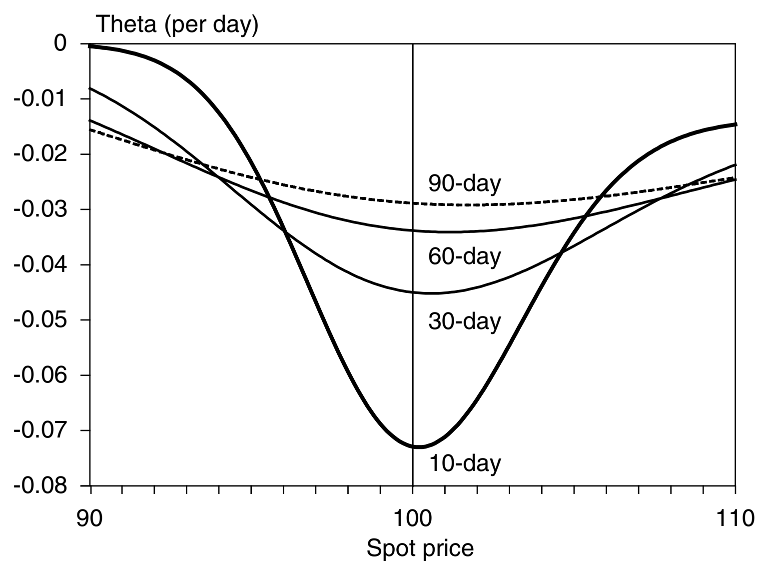
$$\Theta_p = \frac{\partial p}{\partial t} = -\frac{\partial p}{\partial \tau} = -\frac{Se^{-r^*_t\tau}\sigma\Phi(d_1)}{2\sqrt{\tau}} - r^*Se^{-r^*_t\tau}N(-d_1) + rKe^{-r\tau}N(-d_2) \tag{15.18}$$

Θ is generally negative for long positions in both calls and puts. This means that the option loses value as time goes by.

For American options, however, Θ is *always* negative. Because they give their holder the choice to exercise early, shorter-term American options are unambiguously less valuable than longer-term options.

Figure 15-7 displays the behavior of a call option theta for various prices of the underlying asset and maturities. For long positions in options, theta is negative, which reflects the fact that the option is a wasting asset. Like gamma, theta is greatest for short-term at-the-money options, when measured in absolute value. At-the-money options lose a lot of value when the maturity is near.

FIGURE 15-7 Option Theta



15.2.5 Option Pricing and the "Greeks"

Having defined the option sensitivities, we can illustrate an alternative approach to the derivation of the Black-Scholes formula. Recall that the underlying process for the asset follows a stochastic process known as a **geometric Brownian motion** (GBM),

$$dS = \mu S dt + \sigma S dz \tag{15.19}$$

where dz has a normal distribution with mean zero and variance dt.

Considering only this *single* source of risk, we can return to the Taylor expansion in Equation (15.3). The value of the derivative is a function of S and time, which we can write as $f(S, t)$. The question is, How does f evolve over time?

We can relate the stochastic process of f to that of S using **Ito's lemma**, so named after its creator. This can be viewed as an extension of the Taylor approximation to a stochastic environment. Applied to the GBM, this gives

$$df = \left(\frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 + \frac{\partial f}{\partial \tau}\right)dt + \left(\frac{\partial f}{\partial S}\sigma S\right)dz \tag{15.20}$$

This is also

$$df = (\Delta\mu S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt + (\Delta\sigma S)dz \tag{15.21}$$

The first term, including dt, is the trend. The second, including dz, is the stochastic component.

Next, we construct a portfolio delicately balanced between S and f that has no exposure to dz. Define this portfolio as

$$\Pi = f - \Delta S \tag{15.22}$$

Using (15.19) and (15.21), its stochastic process is

$$\begin{aligned} d\Pi &= [\Delta\mu S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt + (\Delta\sigma S)dz - \Delta[\mu S dt + \sigma S)dz] \\ &= (\Delta\mu S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt + (\Delta\sigma S)dz - (\Delta\mu S)dt - (\Delta\sigma S)dz \\ &= (\frac{1}{2}\Gamma\sigma^2 S^2 + \Theta)dt \end{aligned} \tag{15.23}$$

This simplification is extremely important. Note how the terms involving dz cancel out each other: the portfolio has been immunized against this source of risk. At the same time, the terms in μS also happened to cancel out each other. The fact that μ disappears from the trend in the portfolio is important, as it explains why the trend of the underlying asset does not appear in the Black-Scholes formula.

Continuing, we note that the portfolio Π has no risk. To avoid arbitrage, it must return the risk-free rate:

$$d\Pi = [r\Pi]dt = r(f - \Delta S)dt \tag{15.24}$$

If the underlying asset has a dividend yield of y, this must be adjusted to

$$d\Pi = (r\Pi)dt + y\Delta S dt = r(f - \Delta S)dt + y\Delta S dt \tag{15.25}$$

Setting the trends in Equations (15.23) and (15.25) equal to each other, we must have

$$(r - y)\Delta S + \frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = rf \tag{15.26}$$

This is the Black-Scholes **partial differential equation (PDE)**, which applies to any contract, or portfolio, that derives its value from S. The solution of this equation, with appropriate boundary conditions, leads to the BS formula for a European call, Equation (15.7).

We can use this relationship to understand how the sensitivities relate to each other. Consider a portfolio of derivatives, all on the same underlying asset, that is delta-hedged. Setting $\Delta = 0$ in Equation (15.26), we have

$$\frac{1}{2}\Gamma\sigma^2 S^2 + \Theta = rf$$

This shows that, for such portfolio, when Γ is large and positive, Θ must be negative if rf is small. In other words, a delta-hedged position with positive gamma, which is beneficial in terms of price risk, must have negative theta, or time decay. An example is the long straddle examined in Chapter 6. Such position is delta-neutral and has large gamma or convexity. It would benefit from a large move in S, whether up or down. This portfolio, however, involves buying options whose value decay very quickly with time. Thus, there is an intrinsic trade-off between Γ and Θ.

Key concept:
For delta-hedged portfolios, Γ and Θ must have opposite signs. Portfolios with positive convexity, for example, must experience time decay.

15.2.6 Option Sensitivities: Summary

We now summarize the sensitivities of option positions with some illustrative data in Table 15-1. Three strike prices are considered, $K = 90$, 100, and 110. We verify that the Γ, Λ, Θ measures are all highest when the option is at-the-money ($K = 100$). Such options have the most nonlinear patterns.

The table also shows the loss for the worst daily movement in each risk factor at the 95 percent confidence level. For S, this is $dS = -1.645 \times 20\% \times \$100/\sqrt{252} = -\$2.08$. We combine this with delta, which gives a potential loss of $\Delta \times dS = -\$1.114$, or about a fourth of the option value.

Next, we examine the second order term, S^2. The worst squared daily movement is $dS^2 = 2.08^2 = 4.33$ in the risk factor at the 95 percent confidence level. We combine this with gamma, which gives a potential gain of $\frac{1}{2}\Gamma \times dS^2 = 0.5 \times 0.039 \times 4.33 = \$0.084$. Note that this is a gain because gamma is positive, but much smaller than the

TABLE 15-1 Derivatives for a European Call
Parameters: $S = \$100, \sigma = 20\%, r = 5\%, y = 3\%, \tau = 3$ month

Variable		Unit	Strike			Worst Loss	
			K = 90	K = 100	K = 110	Variable	Loss
c		Dollars	\$11.02	\$4.22	\$1.05		
		Change per:					
Δ	spot price	dollar	0.868	0.536	0.197	−\$2.08	−\$1.114
Γ	spot price	dollar	0.020	0.039	0.028	4.33	\$0.084
Λ	volatility	(% pa)	0.103	0.198	0.139	−0.025	−\$0.005
ρ	interest rate	(% pa)	0.191	0.124	0.047	−0.10	−\$0.013
ρ^*	asset yield	(% pa)	−0.220	−0.135	−0.049	0.10	−\$0.014
Θ	time	day	−0.014	−0.024	−0.016		

first-order effect. Thus the worst loss due to S would be $-\$1.114 + \$0.084 = -\$1.030$ using the linear and quadratic effects.

For σ, we observe a volatility of volatility on the order of 1.5%. The worst daily move is therefore $-1.645 \times 1.5 = -2.5$, expressed in percent, which gives a worst loss of $-\$0.0049$. Finally, for r, we assuming an annual volatility of changes in rates of 1%. The worst daily move is then $-1.645 \times 1/\sqrt{252} = -0.10$, in percent, which gives a worst loss of $-\$0.013$. So, most of the risk originates from S. In this case, a linear approximation using Δ only would capture most of the downside risk. For near-term at-the-money options, however, the quadratic effect will be more important.

Example 15-5: FRM Exam 2001—Question 123

15-5. Which of the following “Greeks” contributes most to the risk of an option that is close to expiration and deep in the money?

a) Vega
b) Rho
c) Gamma
d) Delta

Example 15-6: FRM Exam 1998—Question 43/Capital Markets

15-6. If risk is defined as a potential for unexpected loss, which factors contribute to the risk of a long put option position?

a) Delta, vega, rho
b) Vega, rho
c) Delta, vega, gamma, rho
d) Delta, vega, gamma, theta, rho

Example 15-7: FRM Exam 1998—Question 44/Capital Markets

15-7. Same as above for a short call position.

Example 15-8: FRM Exam 1998—Question 45/Capital Markets

15-8. Same as above for a long straddle position.

Example 15-9: FRM Exam 1999—Question 38/Capital Markets

15-9. Which of the following statements about option time value is *true*?

a) Deeply out-of-the-money options have more time value than at-the-money options with the same remaining time to expiration.

b) Deeply in-the-money options have more time value than at-the-money options with the same amount of time to expiration.

c) At-the-money options have higher time value than either out-of-the money or in-the-money options with the same remaining time to expiration.

d) At-the-money options have no time value.

Example 15-10: FRM Exam 1999—Question 39/Capital Markets

15-10. Which type of option experiences accelerating time decay as expiration approaches in an unchanged market?

a) In-the-money

b) Out-of-the-money

c) At-the-money

d) None of the above

Example 15-11: FRM Exam 1999—Question 56/Capital Markets

15-11. According to the Black-Scholes model for evaluating European options on non dividend-paying stock, which option sensitivity (Greek) would be identical for both a call and a put option, given that the implied volatility, time to maturity, strike price, and risk free interest rate were the same?

I) Gamma

II) Vega

III) Theta

IV) Rho

a) II only

b) I and II

c) All the above

d) III and IV

Example 15-12: FRM Exam 1998—Question 36/Capital Markets

15-12. An investor bought a short-term at-the-money swaption straddle from a derivative dealer two days ago. Which of the following risk factors could lead to a loss to the investor?

I. Interest rate delta risk
II. Gamma risk
III. Vega risk
IV. Theta (time decay) risk
V. Counterparty credit risk

a) I and II only
b) I, II and III only
c) I, III, IV, and V
d) I, II, III, IV, and V

Example 15-13: FRM Exam 1998—Question 37/Capital Markets

15-13. An investor sold a short-term at-the-money swaption straddle to a derivative dealer two days ago. The option premium was paid up-front. Which of the following risk factors could lead to a loss to the investor?

I. Interest rate delta risk
II. Gamma risk
III. Vega risk
IV. Theta (time decay) risk
V. Counterparty credit risk

a) I and II only
b) I, II and III only
c) I, III, IV, and V only
d) I, II, III, IV, and V

Example 15-14: FRM Exam 2000—Question 76/Market Risk

15-14. How can a trader produce a short vega, long gamma position?

a) Buy short-maturity options, sell long-maturity options.
b) Buy long-maturity options, sell short-maturity options.
c) Buy and sell options of long maturity.
d) Buy and sell options of short maturity.

Example 15-15: FRM Exam 2001—Question 113

15-15. An option portfolio exhibits high unfavorable sensitivity to increases in implied volatility and while experiencing significant daily losses with the passage of time. Which strategy would the trader most likely employ to hedge his portfolio?

a) Sell short dated options and buy long dated options
b) Buy short dated options and sell long dated options
c) Sell short dated options and sell long dated options
d) Buy short dated options and buy long dated options

15.3 Dynamic Hedging

The BS derivation taught us how to price and hedge options. Perhaps even more importantly, it showed that holding a call option is equivalent to holding a fraction of the underlying asset, where the fraction dynamically changes over time.

15.3.1 Delta and Dynamic Hedging

This equivalence is illustrated in Figure 15-8, which displays the current value of a call as a function of the current spot price. The long position in one call is replicated by a partial position in the underlying asset. For an at-the-money position, the initial delta is about 0.5.

FIGURE 15-8 Dynamic Replication of a Call Option

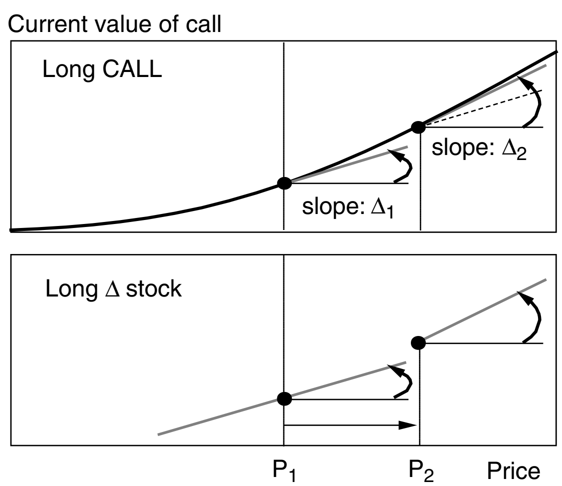
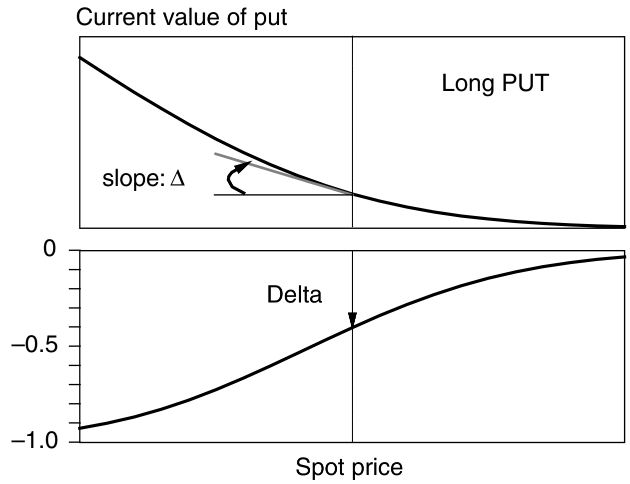


FIGURE 15-9 Dynamic Replication of a Put Option



As the stock price increases from P_1 to P_2, the slope of the option curve, or delta, increases from Δ_1 to Δ_2 . As a result, the option can be replicated by a larger position in the underlying asset. Conversely, when the stock price decreases, the size of the position is cut, as in a graduated stop-loss order. Thus the dynamic adjustment buys more of the asset as its price goes up, and conversely, sells it after a fall.

Figure 15-9 shows the dynamic replication of a put. We start at-the-money with Δ close to -0.5. As the price S goes up, Δ increases toward 0. Note that this is an increase since the initial delta was negative. As with the long call position, we *buy* more of the asset *after* its price has gone up. In contrast, short positions in calls and puts imply opposite patterns. Dynamic hedging implies selling more of the asset after its price has gone up.

15.3.2 Implications

These patterns are important to understand for a number of reasons. First, a dynamic replication of a long option position is bound to lose money. This is because it buys the asset *after* the price has gone up; in other words, too late. Each transaction loses a small amount of money, which will accumulate precisely to the option premium.

A second point is that these automatic trading systems, if applied on large scale, have the potential to be destabilizing. Selling on a downturn in price can only exacerbate the downside move. Some have argued that the crash of 1987 was due to the large-scale selling of portfolio insurers in a falling market. These portfolio insurers were in effect replicating a long position in puts, blindly selling when the market was falling.[3]

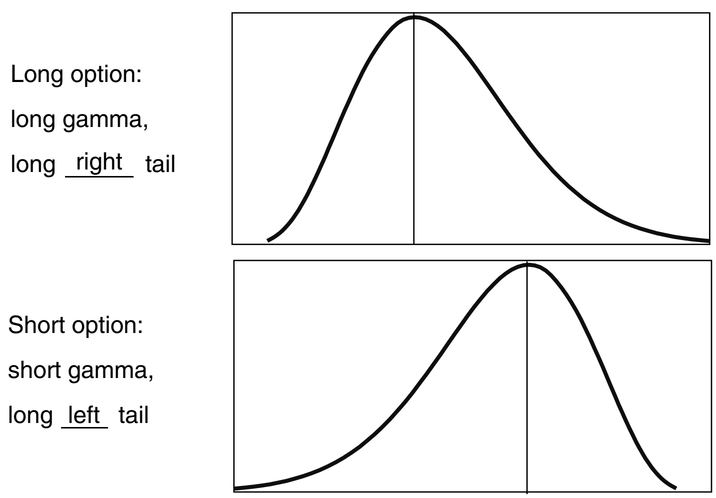
A third point is that this pattern of selling an asset after its price went down is similar to prudent risk-management practices. Typically, traders must cut down their positions after they incur large losses. This is similar to decreasing Δ when S drops. Thus, loss-limit policies bear some resemblance to a long position in an option.

Finally, the success of this replication strategy critically hinges on the assumption of a continuous GBM price process. This makes it theoretically possible to rebalance the portfolio as often as needed. In practice, the replication may fail if prices experience drastic jumps.

15.3.3 Distribution of Option Payoffs

Unlike linear derivatives like forwards and futures, payoffs on options are intrinsically asymmetric. This is not necessarily because of the distribution of the underlying factor, which is often symmetric, but rather is due to the exposure profile. Long positions in options, whether calls or puts, have positive gamma, positive skewness, or long right tails. In contrast, short positions in options are short gamma and hence have negative skewness or long left tails. This is illustrated in Figure 15-10.

FIGURE 15-10 Distributions of Payoffs on Long and Short Options



[3]The exact role of portfolio insurance, however, is still hotly debated. Others have argued that the crease was aggravated by a breakdown in market structures, i.e. the additional uncertainty due to the inability of the stock exchanges to handle abnormal trading volumes.

We now summarize VAR formulas for simple option positions. Assuming a normal distribution, the VAR of the underlying asset is

$$\text{VAR}(dS) = \alpha S\sigma(dS/S) \tag{15.27}$$

where α corresponds to the desired confidence level, e.g. $\alpha = 1.645$ for a 95% confidence level. The linear VAR for an option is

$$\text{VAR}_1(dc) = \Delta\text{VAR}(dS) \tag{15.28}$$

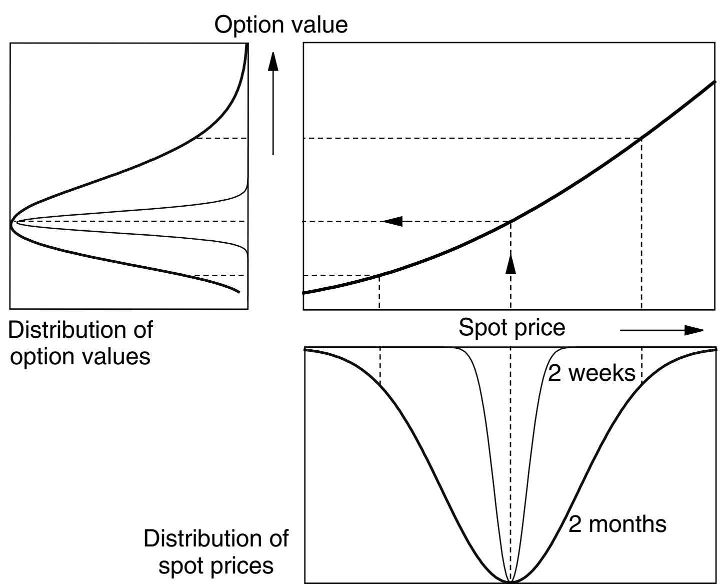
The quadratic VAR for an option is

$$\text{VAR}_2(dc) = \Delta\text{VAR}(dS) - \frac{1}{2}\Gamma\text{VAR}(dS)^2 \tag{15.29}$$

Long option positions have positive gammas and hence slightly lower risk than using a linear model. Conversely, negative gammas translate into quadratic VARs that exceed linear VARs.

Lest we think that such options require sophisticated risk management methods, what matters is the *extent* of nonlinearity. Figure 15-11 illustrates the risk of a call option with a maturity of 3 months. It shows that the degree of nonlinearity also depends on the horizon. With a VAR horizon of 2 weeks, the range of possible values for S is quite narrow. If S follows a normal distribution, the option value will be approximately normal. However, if the VAR horizon is set at 2 months, the nonlinearities in

FIGURE 15-11 Skewness and VAR Horizon



the exposure combine with the greater range of price movements to create a heavily skewed distribution.

So, for plain-vanilla options, the linear approximation may be adequate as long as the VAR horizon is kept short. For more exotic options, or longer VAR horizons, the risk manager needs to account for nonlinearities.

Example 15-16: FRM Exam 2001—Question 80

15-16. Which position is most risky?

a) Gamma-negative, delta-neutral
b) Gamma-positive, delta-positive
c) Gamma-negative, delta-positive
d) Gamma-positive, delta-neutral

Example 15-17: FRM Exam 1997—Question 28/Market Risk

15-17. Consider the risk of a long call on an asset with a notional amount of $1 million. The VAR of the underlying asset is 7.8%. If the option is a short-term at-the-money option, the VAR of the option position is slightly:

a) Less than $39,000 when second-order terms are considered
b) More than $39,000 when second-order terms are considered
c) Less than $78,000 when second-order terms are considered
d) More than $78,000 when second-order terms are considered

Example 15-18: FRM Exam 1998—Question 27/Risk Measurement

15-18. A trader has an option position in crude oil with a delta of 100,000 barrels and gamma of minus 50,000 barrels per dollar move in price. Using the delta-gamma methodology, compute the VAR on this position, assuming the extreme move on crude oil is $2.00 per barrel.

a) $100,000
b) $200,000
c) $300,000
d) $400,000

Example 15-19: FRM Exam 1999—Question 94/Market Risk

15-19. A commodities trading firm has an options portfolio with a two-day VAR of $1.6 million. What would be an appropriate translation of this VAR to a ten-day horizon?

a) $8.0 million
b) $3.2 million
c) $5.6 million
d) Cannot be determined from the information provided

Example 15-20: FRM Exam 1997—Question 51/Market Risk

15-20. A risk manager would like to measure VAR for a bond. He notices that the bond has a putable feature. What affect on the VAR will this puttable feature have?

a) The VAR will increase.
b) The VAR will decrease.
c) The VAR will remain the same.
d) The affect on the VAR will depend on the volatility of the bond.

Example 15-21: FRM Exam 2000—Question 97/Market Risk

15-21. A trader buys an at-the-money call option with the intention of delta-hedging it to maturity. Which one of the following is likely to be the most profitable over the life of the option?

a) An increase in implied volatility
b) The underlying price steadily rising over the life of the option
c) The underlying price steadily decreasing over the life of the option
d) The underlying price drifting back and forth around the strike over the life of the option

15.4 Answers to Chapter Examples

Example 15-1: FRM Exam 1999—Question 65/Market Risk

a) The delta-gamma approximation is reasonably good for vanilla options (especially not too close to maturity).

Example 15-2: FRM Exam 1999—Question 88/Market Risk

c) Nonlinearities cause distributions to be non-normal. Note that for long-term vanilla options, the delta-normal method may be appropriate.

Example 15-3: FRM Exam 2001—Question 79

a) This is an at-the-money option with a delta of about 0.5. Since the bank sold calls, it needs to delta-hedge by buying the shares. With a delta of 0.54, it would need to buy approximately 50,000 shares. Answer (a) is the closest. Note that most other information is superfluous.

Example 15-4: FRM Exam 1999—Question 69/Market Risk

a) The volatility of the hedged portfolio must be proportional to the volatility of the underlying asset, σ. The volatility of the hedged position increases as the rebalancing horizon increases. If we have continuous rebalancing (N very large), there should be no risk. Otherwise, it must be inversely related to the number of rebalancings N.

Example 15-5: FRM Exam 2001—Question 123

d) A short-dated in-the-money option behaves essentially like a position of delta in the underlying asset. The gamma and vega are low.

Example 15-6: FRM Exam 1998—Question 43/Capital Markets

a) Theta is nor a risk factor since time movements are deterministic. Gamma is positive for a long position and therefore lowers risk. The remaining exposures are delta, vega, and rho.

Example 15-7: FRM Exam 1998—Question 44/Capital Markets

c) Gamma now creates risk.

Example 15-8: FRM Exam 1998—Question 45/Capital Markets

b) The position is now delta-neutral and has positive gamma. The remaining exposures are vega, and rho.

Example 15-9: FRM Exam 1999—Question 38/Capital Markets

c) See Figure 15-7 describing the option theta.

Example 15-10: FRM Exam 1999—Question 39/Capital Markets

c) Time decay describes the loss of option value, which is greatest for at-the-money option with short maturities.

Example 15-11: FRM Exam 1999—Question 56/Capital Markets

b) An otherwise identical call and put have the same gamma and vega. Theta is different, even though the formula contains the same first term, due to the differential effect of time on r and y. Rho is totally different, positive for the call and negative for the put.

Example 15-12: FRM Exam 1998—Question 36/Capital Markets
c) The investor is long the option and has already paid the premium. Therefore, there is credit risk as counterparty could default when the contracts have positive value. The position is also exposed to decreases in volatility (vega risk) and the passage of time (theta risk). There is no gamma risk as the position has positive gamma.

Example 15-13: FRM Exam 1998—Question 37/Capital Markets
b) This is the reverse of the previous position. There is no credit risk as only the investor can lose money, not the dealer. Now there is gamma risk. The position is also exposed to increases in volatility (vega risk).

Example 15-14: FRM Exam 2000—Question 76/Market Risk
a) Long positions in options have positive gamma and vega. Gamma (or instability in delta) increases near maturity; vega decreases near maturity. So, to obtain positive gamma and negative vega, we need to buy short-maturity options and sell long-maturity options.

Example 15-15: FRM Exam 2001—Question 113
b) Such a portfolio is short vega (volatility) and short theta (time). We need to implement a hedge that is delta-neutral and involves buying and selling options with different maturities. Long positions in short-dated options have high negative theta and low positive vega. Hedging can be achieved by selling short-term options and buying long-term options.

Example 15-16: FRM Exam 2001—Question 80
c) The worst combination involves some directional risk plus some negative gamma. Directional risk, delta-positive, could lead to a large loss if the underlying price falls.

Example 15-17: FRM Exam 1997—Question 28/Market Risk
a) An ATM option has a delta of about 50% delta and is long gamma. Its linear VAR is $0.50 \times 0.078 \times \$1,000,000 = \$39,000$. Because the gamma is positive, the risk is slightly lower than the linear VAR.

Example 15-18: FRM Exam 1998—Question 27/Risk Measurement

c) Note that Gamma is negative. Using the Taylor approximation, the worst loss is obtained as the price move of $df = \Delta(-dS) + \frac{1}{2}\Gamma(dS)^2 = 100{,}000 \times \; -\$2 + \frac{1}{2}(-50{,}000)(\$2)^2 = -\$200{,}000 - \$100{,}000 = -\$300{,}000$.

Example 15-19: FRM Exam 1999—Question 94/Market Risk

d) As Figure 15-11 shows, the distribution profile of an option changes as the horizon changes. This makes it difficult to extrapolate short-horizon VAR to longer-horizons without knowing more information on gamma, for instance.

Example 15-20: FRM Exam 1997—Question 51/Market Risk

b) Relative to a bullet bond, the investor is long an option, because he or she can "put" back the bond to the issuer. This will create positive gamma, or lower VAR than otherwise.

Example 15-21: FRM Exam 2000—Question 97/Market Risk

d) An important aspect of the question is the fact that the option is held to maturity. Answer (a) is incorrect because changes in the implied volatility would change the value of the option, but this has no effect when holding to maturity. The profit from the dynamic portfolio will depend on whether the actual volatility differs from the initial implied volatility. It does not depend on whether the option ends up in-the-money or not, so answers (b) and (c) are incorrect. The portfolio will be profitable if the actual volatility is small, which implies small moves around the strike price.

Chapter 16

Modeling Risk Factors

We now turn to a description of the risk factors used in the value-at-risk (VAR) analysis. Such analysis requires various levels of assumptions. A starting point is historical data. Typically, the following assumptions are made: (1) the recent history is a good guide to future movements of risk factors, (2) the risk factors are jointly distributed as normal variables, (3) the distributions have fixed parameters, mean and standard deviation.

As with all models, these assumptions are simple representations of a complex world. The question is how well they allow the risk manager to model and measure portfolio risk.

Section 16.1 starts by describing the normal distribution. We compare the normal and lognormal distributions and explain why this choice is so popular. A major failing of this distribution is its inability to represent the frequency of large observations found in financial data.

Section 16.2 discusses other distributions that have fatter tails than the normal. Section 16.3 then turns to an alternative class of explanation, which is time-variation in risk, summarizing the main approaches, generalized autoregressive conditional heteroskedastic (GARCH), RiskMetrics' exponentially weighted moving averge (EWMA).

16.1 The Normal Distribution

16.1.1 Why the Normal?

The normal, or Gaussian, distribution is usually the first choice when modeling asset returns. This distribution plays a special role in statistics, as it is easy to handle, is stable under addition, and provides the limiting distribution of the average of *independent* random variables (through the central limit theorem).

Empirically, the normal distribution provides a rough, first-order approximation to the distribution of many random variables: rates of changes in currency prices, rates of changes in stock prices, rates of changes in bond prices, changes in yields, and rates of changes in commodity prices. All of these are characterized by greater frequencies of small moves than large moves, thus having a greater weight in the center of the distribution.

16.1.2 Computing Returns

In what follows, the random variable is the new price P_1, given the current price P_0. Defining $r = (P_1 - P_0)/P_0$ as the rate of return in the price, the assumption is that this random variable is drawn from a normal distribution

$$r \sim \Phi(\mu, \sigma) \tag{16.1}$$

with some mean μ and standard deviation σ. Turning to prices, we have $P_1 = P_0(1+r)$ and

$$P_1 \sim P_0 + \Phi(P_0\mu, P_0\sigma) \tag{16.2}$$

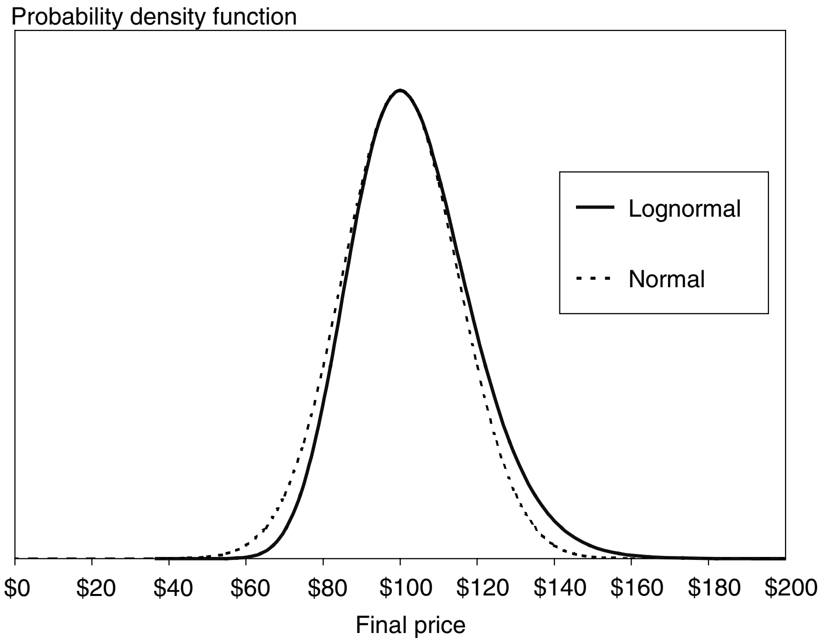
For instance, starting from a stock price of \$100, if $\mu = 0\%$ and $\sigma = 15\%$, we have $P_1 \sim \$100 + \Phi(\$0, \$15)$. Over short horizons, the mean is not too important relative to the volatility.

For many of these variables, however, the normal distribution cannot even be theoretically correct. Because of limited liability, stock prices cannot go below zero. Similarly, commodity prices and yields cannot turn negative. This is why another popular distribution is the **lognormal distribution**, which is such that

$$R = \ln(P_1/P_0) \sim \Phi(\mu, \sigma) \tag{16.3}$$

By taking the logarithm, the price is given by $P_1 = P_0 \exp(R)$, which precludes prices from turning negative as the exponential function is always positive. Figure 16-1 compares the normal and lognormal distributions over a 1-year horizon with $\sigma = 15\%$ annually. The distributions are very similar, except for the tails. The lognormal is skewed to the right.

FIGURE 16-1 Normal and Lognormal Distributions–Annual Horizon



Over a shorter horizon such as a week, the distributions are virtually identical, as are the distributions for assets with low volatilities. The intuition is that with either a low volatility or a short horizon, there is very little chance of prices turning negative. The limited liability constraint is not important.

Key concept:
The normal and lognormal distributions are very similar for short horizons or low volatilities.

As an example, Table 16-1 compares the computation of returns over a one-day and one-year horizon. The one-day returns are 1.000% and 0.995% for discrete and log-returns, respectively, which translates into a relative difference of 0.5%, which is minor. In contrast, the difference is more significant over longer horizons, or when the initial and ending prices are quite different.

The advantage of using log-returns is that they aggregate easily from one period to multiple periods. Indeed, if daily log-returns are normally distributed, so is the multiple-period return. Discrete returns aggregate easily across the portfolio. If discrete asset returns are normally distributed, so is the portfolio return.

TABLE 16-1 Comparison between Discrete and Log Returns

	Daily	Annual
Initial Price	100	100
Ending Price	101	115
Discrete Return	1.0000	15.0000
Log Return	0.9950	13.9762
Relative Difference	0.50%	7.33%

16.1.3 Time Aggregation

Longer horizons can be accommodated assuming a constant lognormal distribution across horizons. Over two periods, for instance, the price movement can be described as the sum of the price movements over each day

$$R_{t,2} = \ln(P_t/P_{t-2}) = \ln(P_t/P_{t-1}) + \ln(P_{t-1}/P_{t-2}) = R_{t-1} + R_t \tag{16.4}$$

If returns are identically and independently distributed (i.i.d.), the variance of multiple-period returns is, defining T as the number of steps,

$$V[R(0,T)] = V[R(0,1)] + V[R(1,2)] + \cdots + V[R(T-1,T)] = V[R(0,1)]T \tag{16.5}$$

since the variances are all the same. Similarly, the mean of multiple-period returns is

$$E[R(0,T)] = E[R(0,1)] + E[R(1,2)] + \cdots + E[R(T-1,T)] = E[R(0,1)]T \tag{16.6}$$

assuming expected returns are the same for each day.

Thus the multiple-period volatility is

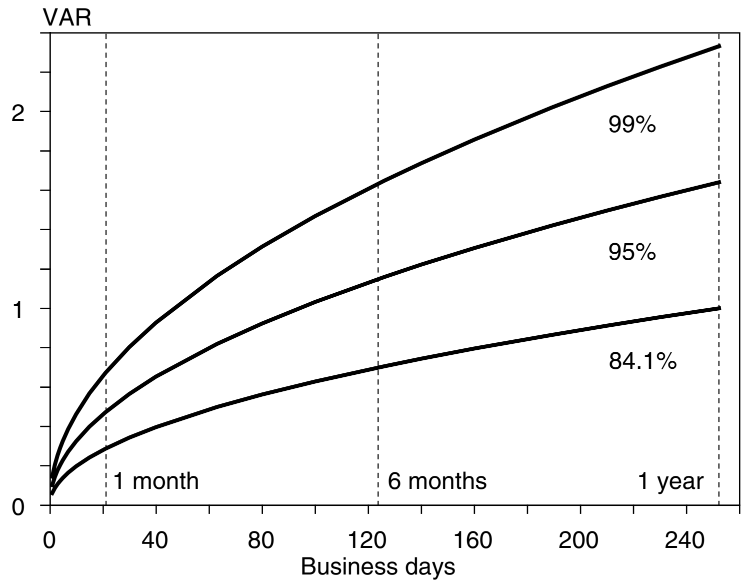
$$\sigma_T = \sigma\sqrt{T} \tag{16.7}$$

If the distribution is stable under addition, i.e. we can use the same multiplier for a 1-period and T-period return, we have a multiple-period VAR of

$$\text{VAR} = \alpha\sigma\sqrt{T}W \tag{16.8}$$

In other words, extension to a multiple period follows a square root of time rule. Figure 16-2 shows how VAR grows with the length of the horizon and for various confidence

FIGURE 16-2 VAR at Increasing Horizons



levels. This is scaled to an annual standard deviation of 1, which is a 84.1% VAR. The figure shows that VAR increases more slowly than time. The 1-month 99% VAR is 0.67, but increases only to 2.33 at a 1-year horizon.

In summary, the square root of time rule applies under the following conditions:

1. The distribution is the same at each period, i.e. there is no predictable time variation in expected return nor in risk.
2. Returns are uncorrelated/independent across each period, so that all covariances terms disappear.
3. The distribution is the same for 1- or T-period, or is stable under addition, such as the normal.

If returns are not independent, we may be able to characterize the risk in some cases. For instance, returns follow a first-order autoregressive process,

$$R_t = \rho R_{t-1} + u_t \tag{16.9}$$

we can write the variance of two-day returns as

$$V[R_t + R_{t-1}] = \sigma^2 + \sigma^2 + 2\rho\sigma^2 = \sigma^2[2 + 2\rho] \tag{16.10}$$

which is greater than the independent and identically distributed (i.i.d.) case when ρ is positive, in other words when markets are trending.

To illustrate the lack of importance of the mean at short horizons, consider Table 16-2. Take a market distribution with an annual expected return of 6 percent with volatility of 15 percent. The last column reports the ratio of the computed volatility to the mean. For a one-year horizon, this ratio is low, at 2.5. For short horizons, such as one day, this ratio is much higher, at 39.7. Thus a small mistake in the measurement of the mean, or even ignoring the mean altogether, is of no consequence at short horizons.

TABLE 16-2 Risk and Returns for Different Horizons

Horizon	Year	Mean	S.D.	Ratio
Annual	1	0.0600	0.1500	2.5
Quarterly	1/4	0.0150	0.0750	5.0
Monthly	1/12	0.0050	0.0433	8.7
Daily	1/252	0.0002	0.0094	39.7

Example 16-1: FRM Exam 1999—Question 64/Market Risk

16-1. Under what circumstances is it appropriate to scale up a VAR estimate from a shorter holding period to a longer holding period using the square root of time?

a) It is never appropriate.
b) It is always appropriate.
c) When either mean reversion or trend are present in the historical data series.
d) When neither mean reversion nor trend are present in the historical data series.

Example 16-2: FRM Exam 1998—Question 5/Risk Measurement

16-2. Consider a portfolio with a 1-day VAR of $1 million. Assume that the market is trending with an autocorrelation of 0.1. Under this scenario, what would you expect the 2-day VAR to be?

a) $2 million
b) $1.414 million
c) $1.483 million
d) $1.449 million

16.2 Fat Tails

Perhaps the most serious problem with the normal distribution is the fact that its tails “disappear” too fast, at least faster than what is empirically observed in financial data. We typically observe that every market experiences one or more daily moves of 4 standard deviations or more per year. Such frequency is incompatible with a normal distribution. With a normal distribution, the probability of this happening is 0.0032% for one day, which implies a frequency of once every 125 years.

Key concept:
Every financial market experiences one or more daily price moves of 4 standard deviations or more each year. And in any year, there is usually at least one market that has a daily move greater than 10 standard deviations.

This empirical observation can be explained in a number of ways: (1) the true distribution has fatter tails (e.g., the Student’s *t*), (2) the observations are drawn from a mix of distributions (e.g. a mix of two normals, one with low risk, the other with high risk), or (3) the distribution is non-stationary.

The first explanation is certainly a possibility. Figure 16-3 displays the density function of the normal and Student’s *t* distribution, with 4 and 6 degrees of freedom (df). The student density has fatter tails, which better reflect the occurrences of extreme observations in empirical financial data.

FIGURE 16-3 Normal and Student Distributions

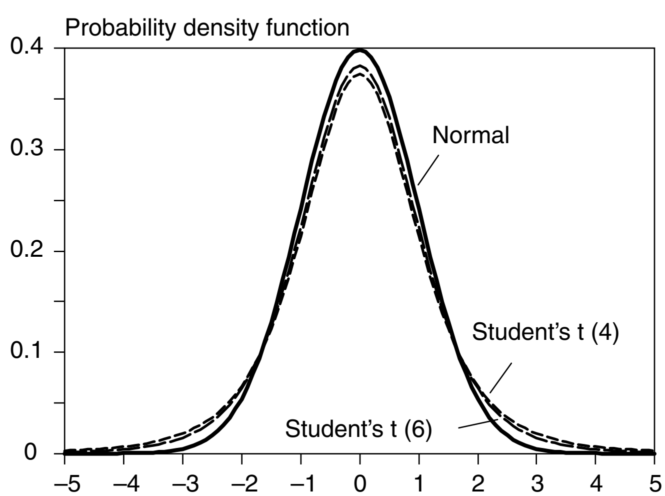


TABLE 16-3 Comparison of the Normal and Student's *t* Distributions

	Tail probability			Expected Number in 250 days		
Deviate	Normal	*t* df = 6	*t* df = 4	Normal	*t* df = 6	*t* df = 4
−5	0.00000	0.00123	0.00375	0.00	0.31	0.94
−4	0.00003	0.00356	0.00807	0.01	0.89	2.02
−3	0.00135	0.01200	0.01997	0.34	3.00	4.99
−2	0.02275	0.04621	0.05806	5.69	11.55	14.51
−1	0.15866	0.17796	0.18695	39.66	44.49	46.74
				Deviate (alpha)		
Probability = 1%				2.33	3.14	3.75
Ratio to normal				1.00	1.35	1.61

This information is further detailed in Table 16-3. The left-side panel reports the tail probability of an observation lower than the deviate. For instance, the probability of observing a draw less than −3 is 0.001, or 0.1% for the normal, 0.012 for the Student's *t* with 6 degrees of freedom, and 0.020 for the Student's *t* with 4 degrees of freedom.

We can transform these into an expected number of occurrences in one year, or 250 business days. The right-side panel shows that the corresponding numbers are 0.34, 3.00 and 4.99 for the respective distributions. In other words, the normal distribution projects only 0.3 days of movements below $z = -3$. With a Student's *t* with df=4, the expected number is 5 in a year, which is closer to reality.

The bottom panel reports the deviate that corresponds to a 99 percent right-tail confidence level, or 1 percent left tail. For the normal distribution, this is the usual 2.33. For the Student's *t* with df=4, α is 3.75, much higher. The ratio of the two is 1.61. Thus a rule of thumb would be to correct the VAR measure from a normal distribution by a ratio of 1.61 to achieve the desired coverage in the presence of fat tails. More generally, this explains why "safety factors" are used to multiply VAR measures, such as the Basel multiplicative factor of three.

Example 16-3: FRM Exam 1999—Question 83/Market Risk

16-3. In the presence of fat tails in the distribution of returns, VAR based on the delta-normal method would (for a linear portfolio)

a) Underestimate the true VAR
b) Be the same as the true VAR
c) Overestimate the true VAR
d) Cannot be determined from the information provided

16.3 Time-Variation in Risk

An alternative class of explanation is that empirical data can be viewed as drawn from a normal distribution with time-varying parameters. This is only useful if this time variation has some structure, or predictability.

16.3.1 GARCH

A specification that has proved quite successful in practice is the **generalized autoregressive conditional heteroskedastic (GARCH)** model developed by Engle (1982) and Bollerslev (1986).

This class of models assumes that the return at time t has a normal distribution conditional on parameters μ_t and σ_t.

$$r_t \sim \Phi(\mu_t, \sigma_t) \qquad (16.11)$$

The important point is that σ is indexed by time. In this context, we define the **conditional variance** as that conditional on current information h_t. This may differ from the **unconditional variance**, which is the same for the whole sample. Thus the average variance is unconditional, whereas a time-varying variance is conditional.

There is substantial empirical evidence that conditional volatility models successfully forecast risk. in modeling slowly changing changes. The general assumption is that the conditional returns have a normal distribution, although this could be extended to other distributions such as the Student's t.

The GARCH model assumes that the conditional variance depends on the latest innovation, and on the previous conditional variance. Define h_t as the conditional variance, using information up to time $t-1$, and r_{t-1} as the previous day's return. The simplest such model is the GARCH(1,1) process

$$h_t = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta h_{t-1} \qquad (16.12)$$

A particular specification of this is the ARCH(1) model, which sets $\beta = 0$, but has been generally found as inadequate as it allows no persistence in the shocks.

The average, unconditional variance is found by setting $E[r_{t-1}^2] = h_t = h_{t-1} = h$. Solving for h, we find

$$h = \frac{\alpha_0}{1 - \alpha_1 - \beta} \qquad (16.13)$$

TABLE 16-4 Building a GARCH Forecast

time	return	conditional variance	conditional risk	conditional 95% limit
$t-1$	r_{t-1}	h_t	$\sqrt{h_t}$	$2\sqrt{h_t}$
0	0.0	1.10	1.05	±2.10
1	3.0	1.32	1.15	±2.30
2	0.0	1.27	1.13	±2.25
3	0.0	1.22	1.10	±2.20

This model will be stationary when the sum of parameters $\alpha_1 + \beta$ are less than unity. This sum is also called the **persistence**, as it defines the speed at which shocks to the variance revert to their long run values.

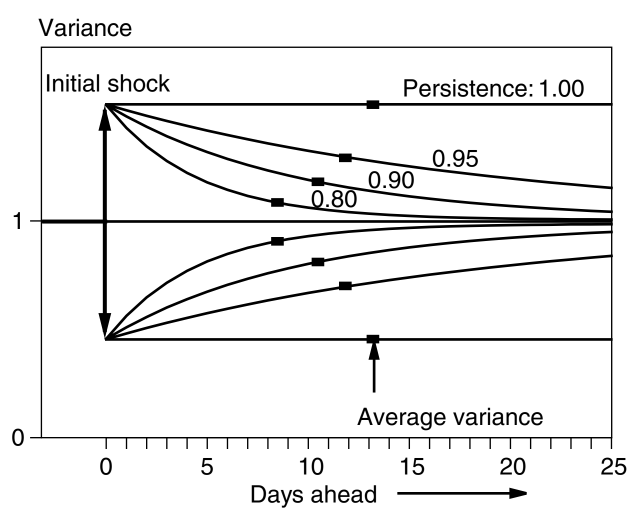
To understand how the process works, consider Table 16-4. The parameters are $\alpha_0 = 0.01$, $\alpha_1 = 0.03$, $\beta = 0.95$. The unconditional variance is $0.01/(1-0.03-0.95) = 0.7$ daily, which is typical of a currency series. The process is stable since $\alpha_1 + \beta = 0.98 < 1$.

At time 0, we start with the variance at $h_0 = 1.1$ (expressed in percent squared). The conditional volatility is $\sqrt{h_0} = 1.05\%$. The next day, there is a large return of 3%. The new variance forecast is then $h_1 = 0.01 + 0.03 \times 3^2 + 0.95 \times 1.1 = 1.32$. The conditional volatility just went up to 1.15%.

If nothing happens the following days, the next variance forecast is $h_2 = 0.01 + 0.03 \times 0^2 + 0.95 \times 1.32 = 1.27$. And so on.

Figure 16-4 illustrates the dynamics of shocks to a GARCH process for various values of the persistence parameter. As the conditional variance deviates from the

FIGURE 16-4 Shocks to a GARCH Process



starting value, it slowly reverts to the long-run value at a speed determined by $\alpha_1 + \beta$. Note that these are forecasts of one-day variances. From the viewpoint of risk management, what matters is the average variance over the horizon, which is marked on the graph.

The graph also shows why the square root of time rule for extrapolating returns does not apply when risk is time-varying. Starting from an initial value of the variance greater than the long-run average, simply extrapolating the 1-day variance to a longer horizon will overstate the average variance. Conversely, starting from a lower value and applying the square root of time rule will understate risk.

Key concept:
The square root of time rule used to scale 1-day returns into longer horizons is generally inappropriate when risk is time-varying.

16.3.2 EWMA

The RiskMetrics approach is a particular, convenient case of the GARCH process. Variances are modeled using an **exponentially weighted moving average (EWMA)** forecast. The forecast is a weighted average of the previous forecast, with weight λ, and of the latest squared innovation, with weight $(1 - \lambda)$

$$h_t = \lambda h_{t-1} + (1 - \lambda) r_{t-1}^2 \tag{16.14}$$

The λ parameter, also called the **decay factor**, determines the relative weights placed on previous observations. The EWMA model places geometrically declining weights on past observations, assigning greater importance to recent observations. By recursively replacing h_{t-1} in Equation (16.14), we have

$$h_t = (1 - \lambda)[r_{t-1}^2 + \lambda r_{t-2}^2 + \lambda^2 r_{t-3}^2 + \cdots] \tag{16.15}$$

The weights therefore decrease at a geometric rate. The lower λ, the more quickly older observations are forgotten. RiskMetrics has chosen $\lambda = 0.94$ for daily data and $\lambda = 0.97$ for monthly data.

Table 16-5 shows how to build the EWMA forecast using a parameter of $\lambda = 0.95$, which is consistent with the previous GARCH example. At time 0, we start with the

TABLE 16-5 Building a EWMA Forecast

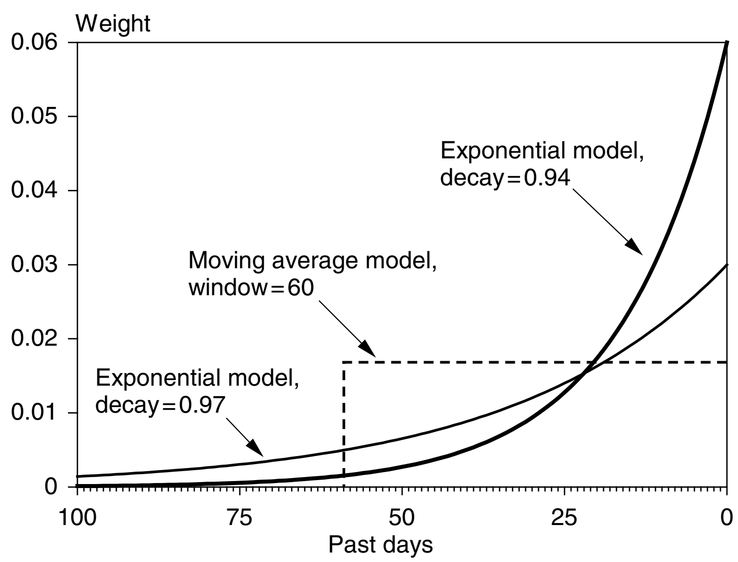
Time	Return	Conditional Variance	Conditional Risk	Conditional 95% Limit
$t-1$	r_{t-1}	h_t	$\sqrt{h_t}$	$2\sqrt{h_t}$
0	0.0	1.10	1.05	±2.1
1	3.0	1.50	1.22	±2.4
2	0.0	1.42	1.19	±2.4
3	0.0	1.35	1.16	±2.3

variance at $h_0 = 1.1$, as before. The next day, we have a return of 3%. The new variance forecast is then $h_1 = 0.05 \times 3^2 + 0.95 \times 1.1 = 1.50$. The next day, this moves to $h_2 = 0.05 \times 0^2 + 0.95 \times 1.50 = 1.42$. And so on.

This model is a special case of the GARCH process, where α_0 is set to 0, and α_1 and β sum to unity. The model therefore has permanent persistence. Shocks to the volatility do not decay, as shown in Figure 16-4 when the persistence is 1.00. Thus longer-term extrapolation from the GARCH and EWMA models may give quite different forecasts. Over a one-day horizon, however, the two models are quite similar and often indistinguishable from each other.

Figure 16-5 displays the pattern of weights for previous observations. With $\lambda = 0.94$, the weights decay rather quickly, dropping below 0.00012 for data more than

FIGURE 16-5 Weights on Past Observations



100 days old. With $\lambda = 0.97$, the weights decay more slowly. In comparison, moving average models have a fixed window, with equal weights within the window but otherwise zero.

Example 16-4: FRM Exam 1999—Question 103/Market Risk

16-4. The current estimate of daily volatility is 1.5 percent. The closing price of an asset yesterday was \$30.00. The closing price of the asset today is \$30.50. Using the EWMA model with $\lambda = 0.94$, the updated estimate of volatility is

a) 1.5096
b) 1.5085
c) 1.5092
d) 1.5083

Example 16-5: FRM Exam 1999—Question 72/Market Risk

16-5. Until January 1999 the historical volatility for the Brazilian real versus the U.S. dollar had been very small for several years. On January 13, 1999, Brazil abandoned the defense of the currency peg. Using the data from the close of business on January 13th, which of the following methods for calculating volatility would have shown the greatest jump in measured historical volatility?

a) 250 day equal weight
b) Exponentially weighted with a daily decay factor of 0.94
c) 60 day equal weight
d) All of the above

16.3.3 Option Data

All the previous forecasts were based on historical data. While conditional volatility models are a substantial improvement over models that assume constant risk, they are always, by definition, one step too late.

These models start to react *after* a big shock has occurred. In many situations, this may be too late. Hence, the quest for forward-looking risk measures.

Such forward-looking measures are contained in option implied standard deviations (ISD). ISD are obtained by, first, assuming an option pricing model and, next, inverting the model, that is, solving for the parameter that will make the model price equal to the observed market price.

Define $f()$ as an option pricing function, such as the Black-Scholes model for European options. Normally, we input σ into f along with other parameters and then solve for the option price. However, if the market trades these options and if all the other inputs are observable, we can recover σ_{ISD} by setting the model price equal to the market price

$$c_{\text{MARKET}} = f(\sigma_{\text{ISD}}) \tag{16.16}$$

This assumes that the model fits the data perfectly, which may not be the case for out-of-the-money options. Hence, this method works best for short-term (2 weeks to 3 months) at-the-money options.

This approach can even be generalized to implied correlations. For this, we need triplets of options, e.g. $/yen, $/euro, yen/euro. The first one will imply σ_1, the second σ_2, and the third the covariance σ_{12}, from which the implied correlation ρ_{12} can be recovered.

There is much empirical evidence that ISD provide superior forecasts of future risk. This was expected, as the essence of option trading is to place volatility bets.

> **Key concept:**
> Whenever possible, use option ISD to forecast risk.

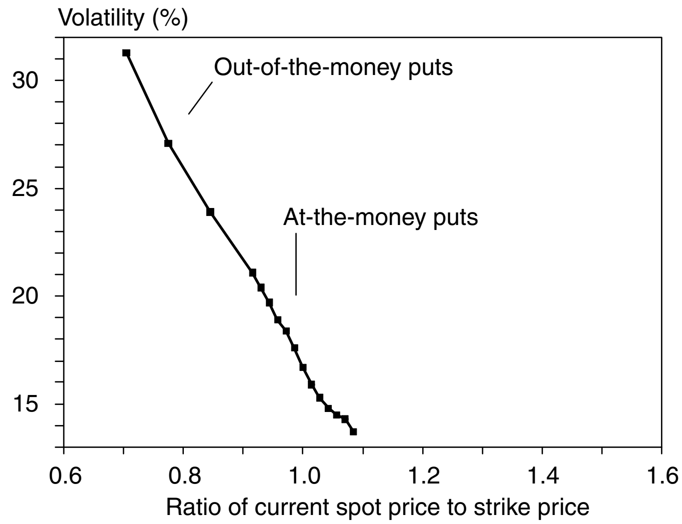
The main drawback of this method is that, while historical time-series models can be applied systematically to all series for which we have data, we do not have actively traded options for all risk factors. In addition, we have even fewer combinations of options that permit us to compute implied correlations. This makes it difficult to integrate ISD with time-series models.

16.3.4 Implied Distributions

Options can be used to derive much more than the volatility. Recently, option watchers have observed some inconsistencies in the pricing of options, especially for stock index options. In particular, options that differ only by their strike prices are characterized by different ISDs. Options that are out-of-the-money have higher ISDs than at-the-money options. This has become known as the **smile effect** in ISDs, which is shown in Figure 16-6, where equity ISDs are plotted against the ratio of the strike price over the current spot price.

Low values of the ratio, describing out-of-the-money puts, are associated with high ISDs. In other words, out-of-the-money puts appear overpriced relative to others. Here the effect is asymmetric, or most pronounced for the left side.

FIGURE 16-6 Smile Effect



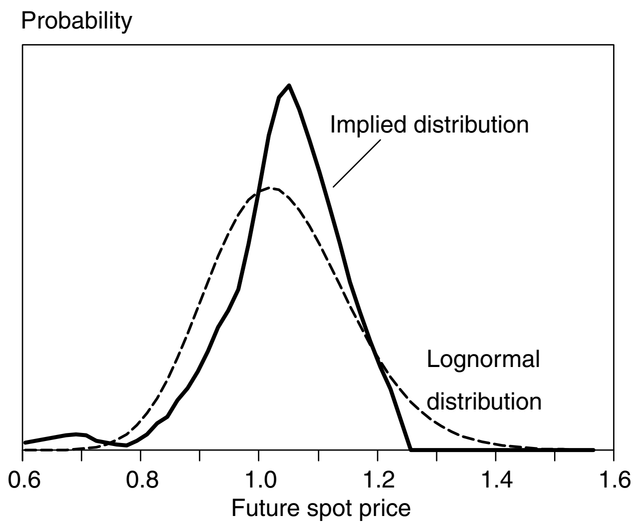
Different ISDs are clearly inconsistent with the joint assumption of a lognormal distribution for prices and efficient markets. Perhaps the data are trying to tell a story. This effect became most pronounced after the stock market crash of 1987, raising the possibility that the market expected another crash, although with low probability.

Recently, Rubinstein (1994) has extended the concept of ISD to the whole **implied distribution** of future prices. By judiciously choosing options with sufficiently spaced strike prices, one can recover the entire implied distribution that is consistent with option prices. This distribution, shown in Figure 16-7, displays a hump for values of the future price 30% below the current price. This hump is nowhere apparent from the usual log-normal distribution.

This puzzling result can be given two interpretations. The first is that the market indeed predicts a small probability of a future crash. The second has to do with the fact that this distribution derived from option prices assumes risk-neutrality, since the Black-Scholes approach values options assuming investors are risk neutral. Thus this distribution may differ from the true, objective distribution due to a **risk premium**. Intuitively, investors may be very averse to a situation where they have to suffer a large fall in the value of their stock portfolios. As a result, they will bid up the price of put options, which is reflected in a higher than otherwise implied volatility.

This is currently an area of active research. The consensus, however, is that options should contain valuable information about future distributions since, after all, option traders bet good money on their forecasts.

FIGURE 16-7 Implied Distribution



16.4 Answers to Chapter Examples

Example 16-1: FRM Exam 1999—Question 64/Market Risk

d) The presence of either mean reversion or trend (or time variation in risk) implies a different distribution of returns for different holding periods.

Example 16-2: FRM Exam 1998–Question 5/Risk Measurement

c) Knowing that the variance is $V(2 - \text{day}) = V(1 - \text{day})[2+2\rho]$, we find $\text{VAR}(2 - \text{day}) = \text{VAR}(1 - \text{day})\sqrt{2 + 2\rho} = \$1\sqrt{2 + 0.2} = \$1.483$, assuming the same distribution for the different horizons.

Example 16-3: FRM Exam 1999–Question 83/Market Risk

a) With fat tails, the normal VAR would underestimate the true VAR.

Example 16-4: FRM Exam 1999–Question 103/Market Risk

a) The updated volatility is from Equation (16.14) the square root of

$$h_t = \lambda(\text{current vol.})^2 + (1 - \lambda)(\text{current return})^2$$

Using log-returns, we find $R = 1.653\%$ and $\sigma_t = 1.5096\%$. With discrete-returns, we find $R = 1.667\%$ and $\sigma_t = 1.5105\%$.

Example 16-5: FRM Exam 1999–Question 72/Market Risk

b) The EWMA puts a weight of 0.06 on the latest observation, which is higher than the weight of 0.0167 for the 60-day MA and 0.004 for the 250-day MA.

Chapter 17

VAR Methods

So far, we have considered sources of risk in isolation. This approach reflects the state of the art up to the beginning of the 1990s. Until then, risk was measured and managed at the level of a desk or business unit. Similarly, university courses in finance dealt separately with equity risk, interest-rate risk, and currency risk. Textbooks on derivatives did not mention aggregate risk. The profession of finance was basically compartmentalized.

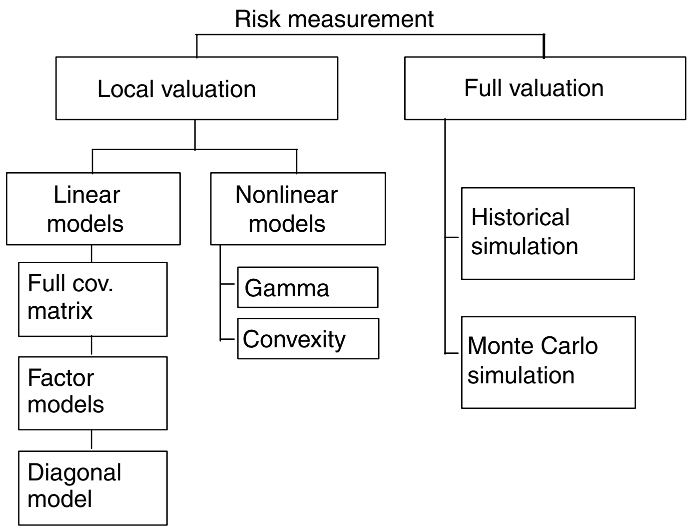
This approach, however, totally fails to take advantage of portfolio theory, which has taught us that risk should be measured at the level of the portfolio. The revolution in risk management has finally made this possible. Indeed, the purpose of VAR is to measure firm-wide risk.

At the most basic level, VAR methods can be separated into local valuation and full valuation methods. **Local valuation methods** make use of the valuation of the instrument at the current point, along with the first and perhaps the second partial derivatives. **Full valuation methods**, in contrast, reprice the instrument over a broad range of values for the risk factors. These methods are discussed in Section 17.1 and described in Figure 17-1.

The left branch describes local valuation methods, also known as **analytical methods**. These include linear models and nonlinear models. Linear models are based on the covariance matrix approach. This can be simplified using factor models, or even a diagonal model. The right branch describes full valuation methods and include historical or Monte Carlo simulations. Section 17.2 presents an overview of the three main VAR methods.

Turning now to individual positions, we start with one of the fundamental principles behind risk management: Divide to conquer. It would be infeasible to model all financial instruments as having their individual source of risk, simply because there are too many. The art of risk management consists of choosing a set of limited risk factors that hopefully will span or cover the whole spectrum of risks. Instruments are then decomposed into these elemental risk factors by a process called **mapping**,

FIGURE 17-1 VAR Methods



which consists of replacing each instrument by its exposures on the selected risk factors. Thus, risk management is truly the art of the approximation.

Section 17.3 works through a detailed example, a forward currency contract. Movements in the value of this contract depend on three risk factors, the spot exchange rate, and the local and foreign interest rates. We first mark-to-market the contract, then we show how to implement the delta-normal and simulation methods. The **delta normal** approach maps all instruments on their risk factors, using their deltas, and assumes that all risk factors have a jointly normal distribution.

Finally, Section 17.4 illustrates how VAR methods are changing the portfolio management process. Risk budgeting is increasingly used to allocate risk across units and is only made feasible by firm-wide measures of risk. Ultimately, portfolio decisions should reflect the best trade-off between expected return and risk. VAR methods provide tools to measure an essential component of this choice, which is downside risk.

17.1 Local vs. Full Valuation

17.1.1 Local Valuation

VAR was born from the recognition that we need an estimate that accounts for various sources of risk and expresses loss in terms of probability. Extending the duration equation to the worst change in yield at some confidence level dy, we have

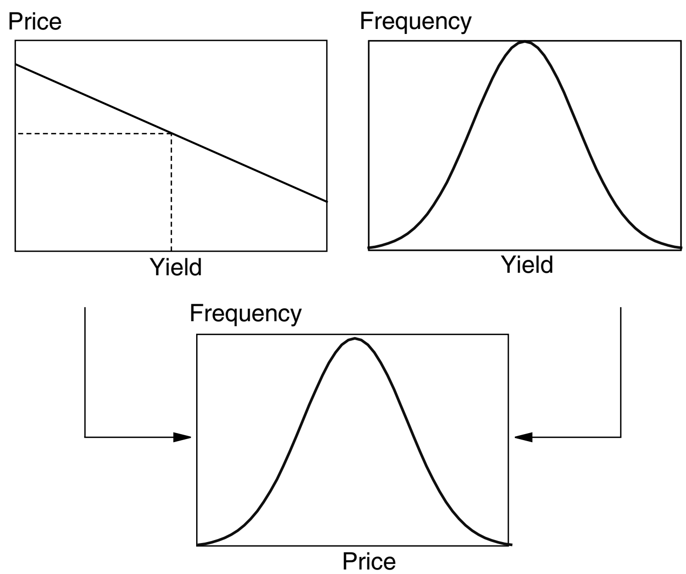
$$(\text{Worst } dP) = (-D^*P) \times (\text{Worst } dy) \tag{17.1}$$

where D^* is modified duration. For a long position in the bond, the worst movement in yield is an increase at say, the 95% confidence level. This will lead to a fall in the bond value at the same confidence level. We call this approach **local valuation**, because it uses information about the initial price and the exposure at the initial point. As a result, the VAR for the bond is given by

$$\text{VAR}(dP) = (D^*P) \times \text{VAR}(dy) \tag{17.2}$$

The main advantage of this approach is its simplicity: The distribution of the price is the same as that of the change in yield. This is particularly convenient for portfolios with numerous sources of risks, because linear combinations of normal distributions are normally distributed. Figure 17-2, for example, shows how the linear exposure combined with the normal density (in the right panel) combines to create a normal density.

FIGURE 17-2 Distribution with Linear Exposures



17.1.2 Full Valuation

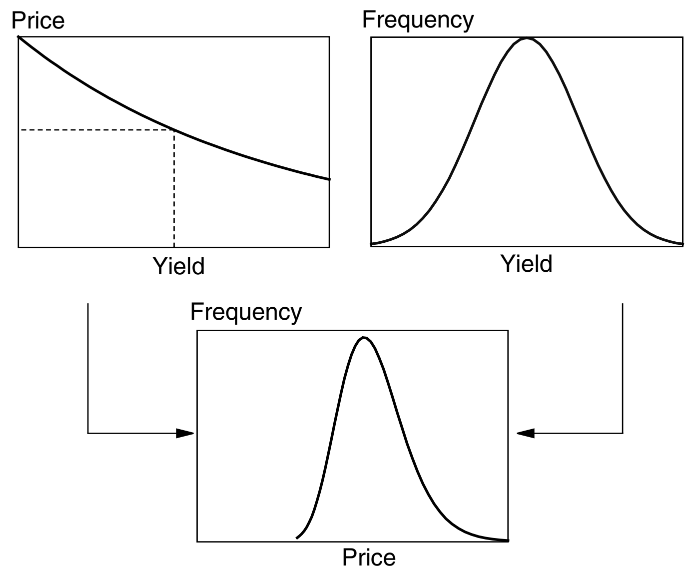
More generally, to take into account nonlinear relationships, one would have to reprice the bond under different scenarios for the yield. Defining y_0 as the initial yield,

$$(\text{Worst } dP) = P[y_0 + (\text{Worst } dy)] - P[y_0] \tag{17.3}$$

We call this approach **full valuation**, because it requires repricing the asset.

This approach is illustrated in Figure 17-3, where the nonlinear exposure combined with the normal density creates a distribution that is not symmetrical any more, but skewed to the right. Unfortunately, full valuation methods are a quantum leap in difficulty relative to simple, linear valuation methods.

FIGURE 17-3 Distribution with Nonlinear Exposures



17.1.3 Delta-Gamma Method

Ideally, we would like to keep the simplicity of the local valuation while accounting for nonlinearities in the payoffs patterns. Using the Taylor expansion,

$$dP \approx \frac{\partial P}{\partial y} dy + (1/2)\frac{\partial^2 P}{\partial y^2}(dy)^2 = -D^* P dy + (1/2) C P (dy)^2 \tag{17.4}$$

where the second-order term involves convexity C. Note that the valuation is still local because we only value the bond once, at the original point. The first and second derivatives are also evaluated at the local point.

Because the price is a monotonous function of the underlying yield, we can use the Taylor expansion to find the worst downmove in the bond price from the worst move in the yield. Calling this $dy^* = \text{VAR}(dy)$

$$(\text{Worst } dP) = P(y_0 + dy^*) - P(y_0) \approx (-D^* P)(dy^*) + (1/2)(C\ P)(dy^*)^2 \tag{17.5}$$

This leads to a simple adjustment for VAR

$$\mathrm{VAR}(dP) = (D^*P) \times \mathrm{VAR}(dy) - (1/2)(C\ P) \times \mathrm{VAR}(dy)^2 \tag{17.6}$$

More generally, this method can be applied to derivatives, for which we write the Taylor approximation as

$$df \approx \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} dS^2 = \Delta dS + \frac{1}{2} \Gamma dS^2 \tag{17.7}$$

where Γ is now the second derivative, or gamma, like convexity.

For a long call option, the worst value is achieved as the underlying price moves down by $\mathrm{VAR}(dS)$. With $\Delta > 0$ and $\Gamma > 0$, the VAR for the derivative is now

$$\mathrm{VAR}(df) = \Delta \times \mathrm{VAR}(dS) - \frac{1}{2} \Gamma \times \mathrm{VAR}(dS)^2 \tag{17.8}$$

This method is called **delta-gamma** because it provides an analytical, second-order correction to the delta-normal VAR. This explains why long positions in options, with positive gamma, have less risk than with a linear model. Conversely, short positions in options have greater risk than implied by a linear model.

This simple adjustment, unfortunately, only works when the payoff function is monotonous, that is, involves a one-to-one relationship between the option value f and S. More generally, the **delta-gamma-delta** VAR method involves, first, computing the moments of df using Equation (17.7) and, second, choosing the normal distribution that provides the best fit to these moments.

The improvement brought about by this method depends on the size of the second-order coefficient, as well as the size of the worst move in the risk factor. For forward contracts, for instance, $\Gamma = 0$, and there is no point in adding second-order terms. Similarly, for most fixed-income instruments over a short horizon, the convexity effect is relatively small and can be ignored.

Example 17-1: FRM Exam 1997—Question 13/Regulatory

17-1. An institution has a fixed-income desk and an exotic-options desk. Four risk reports were produced, each with a different methodology. With all four methodologies readily available, which of the following would you use to allocate economic capital?

a) Simulation applied to both desks
b) Delta-normal applied to both desks
c) Delta-gamma for the exotic-options desk and the delta-normal for the fixed-income desk
d) Delta-gamma applied to both desks

17.2 VAR Methods: Overview

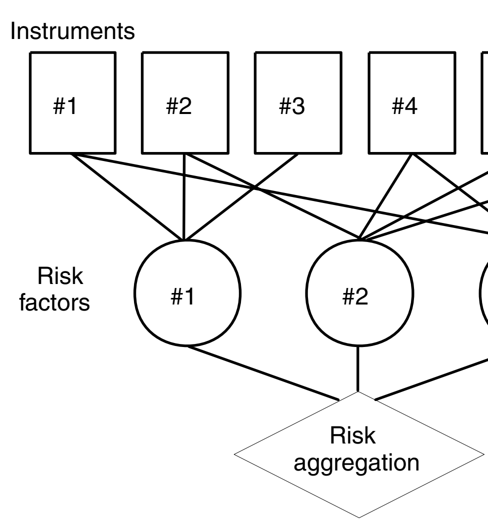
17.2.1 Mapping

This section provides an introduction to the three VAR methods. The portfolio could consist of a large number of instruments, say M. Because it would be too complex to model each instrument separately, the first step is **mapping**, which consists of replacing the instruments by positions on a limited number of risk factors. Say we have N risk factors. The positions are then aggregated across instruments, which yields dollar exposures x_i.

The distribution of the portfolio return $R_{p,t+1}$ is then derived from the exposures and movements in risk factors, Δf. Some care has to be taken defining the risk factors (in gross return, change in yield, rate of return, and so on); the exposures x have to be consistently defined. Here, R_p must be measured as the change in *dollar* value of the portfolio (or whichever base currency is used).

Figure 17-4 displays the mapping process. For instance, we could reduce the large spectrum of maturities in the U.S. fixed-income market by 14 maturities. In the next section, we provide a fully worked-out example.

FIGURE 17-4 Mapping Approach



17.2.2 Delta-Normal Method

The **delta-normal method** is the simplest VAR approach. It assumes that the portfolio exposures are linear and that the risk factors are jointly normally distributed. As such, it is a local valuation method.

Because the portfolio return is a linear combination of normal variables, it is normally distributed. Using matrix notations, the portfolio variance is given by

$$\sigma^2(R_{p,t+1}) = x_t' \Sigma_{t+1} x_t \tag{17.9}$$

where Σ_{t+1} is the forecast of the covariance matrix over the horizon.

If the portfolio volatility is measured in dollars, VAR is directly obtained from the standard normal deviate α that corresponds to the confidence level c:

$$\text{VAR} = \alpha\sigma(R_{p,t+1}) \tag{17.10}$$

This is called the **diversified VAR**, because it accounts for diversification effects. In contrast, the **undiversified VAR** is simply the sum of the individual VARs for each risk factor. It assumes that all prices will move in the worst direction simultaneously, which is unrealistic.

The RiskMetrics approach is basically similar to the delta-normal approach. The only difference is that the risk factor returns are measured as logarithms of the price ratios, instead of rates of returns.

The main benefit of this approach is its appealing simplicity. This is also its drawback. The delta-normal method cannot account for nonlinear effects such as encountered with options. It may also underestimate the occurrence of large observations because of its reliance on a normal distribution.

17.2.3 Historical Simulation Method

The **historical-simulation** (HS) method is a full valuation method. It consists of going back in time, e.g. over the last 250 days, and applying current weights to a time-series of historical asset returns. It replays a "tape" of history with current weights.

Define the current time as t; we observe data from 1 to t. The current portfolio value is P_t, which is a function of the current risk factors

$$P_t = P[f_{1,t}, f_{2,t}, \ldots, f_{N,t}] \tag{17.11}$$

We sample the factor movements from the historical distribution, without replacement

$$\Delta f_i^k = \{\Delta f_{i,1}, \Delta f_{i,2}, \ldots, \Delta f_{i,t}\} \tag{17.12}$$

From this we can construct hypothetical factor values, starting from the current one

$$f_i^k = f_{i,t} + \Delta f_i^k \tag{17.13}$$

which are used to construct a hypothetical value of the current portfolio under the new scenario, using Equation (17.11)

$$P^k = P\left[f_1^k, f_2^k, \ldots, f_N^k\right] \tag{17.14}$$

We can now compute changes in portfolio values from the current position $R^k = (P^k - P_t)/P_t$.

We sort the t returns and pick the one that corresponds to the cth quantile, $R_p(c)$. VAR is obtained from the difference between the average and the quantile,

$$\text{VAR} = \text{AVE}[R_p] - R_p(c) \tag{17.15}$$

The advantage of this method is that it makes no distributional assumption about return distribution, which may include fat tails. The main drawback of the method is its reliance on a short historical moving window to infer movements in market prices. If this window does not contain some market moves that are likely, it may miss some risks.

17.2.4 Monte Carlo Simulation Method

The **Monte Carlo simulation method** is basically similar to the historical simulation, except that the movements in risk factors are generated by drawings from some distribution. Instead of Equation (17.12), we have

$$\Delta f^k \sim g(\theta), \quad k = 1, \ldots K \tag{17.16}$$

where g is the joint distribution (e.g. a normal or Student's t) and θ the required parameters. be the *joint* distribution of all risk factors. The risk manager samples **pseudo-random numbers** from this distribution and then generates pseudo-dollar returns as before. Finally, the returns are sorted to produce the desired VAR.

This method is the most flexible, but also carries an enormous computational burden. It requires users to make assumptions about the stochastic process and to understand the sensitivity of the results to these assumptions. Thus, it is subject to **model risk**.

Monte Carlo methods also create inherent sampling variability because of the randomization. Different random numbers will lead to different results. It may take a large number of iterations to converge to a stable VAR measure. It should be noted that when all risk factors have a normal distribution and exposures are linear, the method should converge to the VAR produced by the delta-normal VAR.

17.2.5 Comparison of Methods

Table 17-1 provides a summary comparison of the three mainstream VAR methods. Among these methods, the delta-normal is by far the easiest to implement and communicate. For simple portfolios with little optionality, this may be perfectly appropriate. In contrast, the presence of options may require a full valuation method.

TABLE 17-1 Comparison of Approaches to VAR

Features	Delta-normal	Historical simulation	Monte Carlo simulation
Valuation	Linear	Full	Full
Distribution			
Shape	Normal	Actual	General
Extreme events	Low probability	In recent data	Possible
Implementation			
Ease of computation	Yes	Intermediate	No
Communicability	Easy	Easy	Difficult
VAR precision	Excellent	Poor with short window	Good with many iterations
Major pitfalls	Nonlinearities, fat tails	Time variation in risk, unusual events	Model risk

Example 17-2: FRM Exam 2001 Question 92

17-2. Under usually accepted rules of market behavior, the relationship between parametric delta-normal VAR and historical VAR will tend to be:

a) Parametric VaR will be higher.
b) Parametric VaR will be lower.
c) It depends on the correlations.
d) None of the above are correct.

Example 17-3: FRM Exam 1997—Question 12/Risk Measurement

17-3. Delta-normal, historical simulation, and Monte Carlo are various methods available to compute VAR. If underlying returns are normally distributed, then the

a) Delta-normal method VAR will be identical to the historical-simulation VAR.
b) Delta-normal method VAR will be identical to the Monte-Carlo VAR.
c) Monte-Carlo VAR will approach the delta-normal VAR as the number of replications ("draws") increases.
d) Monte-Carlo VAR will be identical to the historical-simulation VAR.

Example 17-4: FRM Exam 1998—Question 6/Regulatory

17-4. Which VAR methodology is least effective for measuring options risks?

a) Variance/covariance approach
b) Delta/gamma
c) Historical simulation
d) Monte Carlo

Example 17-5: FRM Exam 1999—Question 82/Market Risk

17-5. BankLondon with substantial position in 5-year AA-grade Eurobonds has recently launched an initiative to calculate 10 day spread VAR. As a risk manager for the Eurobond trading desk you have been asked to provide an estimate for the AA-spread VAR. The extreme move used for the gilts yield is 40bp, and for the Eurobond yield is 50bp. These are based on the standard deviation of absolute (not proportional) changes in yields. The correlation between changes in the two is 89%. What is the extreme move for the spread?

a) 19.35bp
b) 14.95bp
c) 10bp
d) 23.24bp

Example 17-6: FRM Exam 1999—Questions 15 and 90/Market Risk

17-6. The VAR of one asset is 300 and the VAR of another one is 500. If the correlation between changes in asset prices is 1/15, what is the combined VAR?

a) 525
b) 775
c) 600
d) 700

17.3 Example

17.3.1 Mark-to-Market

We now illustrate the computation of VAR for a simple example. The problem at hand is to evaluate the 1-day downside risk of a currency forward contract. We will show that to compute VAR we need first to value the portfolio, mapping the value of the portfolio on fundamental risk factors, then to generate movements in these risk factors, and finally to combine the risk factors with the valuation model to simulate movements in the contract value.

Assume that on December 31, 1998, we have a forward contract to buy £10 million in exchange for delivering $16.5 million in 3 months.

As before, we use these definitions:

S_t = current spot price of the pound in dollars
F_t = current forward price
K = purchase price set in contract
f_t = current value of contract
r_t = domestic risk-free rate
r_t^* = foreign risk-free rate
τ = time to maturity

To be consistent with conventions in the foreign exchange market, we define the present value factors using discrete compounding

$$P_t = \text{PV}(\$1) = \frac{1}{1 + r_t\tau} \qquad P_t^* = \text{PV}(£1) = \frac{1}{1 + r_t^*\tau} \tag{17.17}$$

The current market value of a forward contract to buy one pound is given by

$$f_t = S_t \frac{1}{1 + r_t^*\tau} - K \frac{1}{1 + r_t\tau} = S_t P_t^* - K P_t \tag{17.18}$$

which is exposed to 3 risk factors, the spot rate and the two interest rates. In addition, we can use this equation to derive the exposures on the risk factors. After differentiation, we have

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial P} dP + \frac{\partial f}{\partial P^*} dP^* = P^* dS + S dP^* - K dP \tag{17.19}$$

Alternatively,

$$df = (SP^*)\frac{dS}{S} + (SP^*)\frac{dP^*}{P^*} - (KP)\frac{dP}{P} \tag{17.20}$$

Intuitively, the forward contract is equivalent to (1) A long position of (SP^*) on the spot rate (2) A long position of (SP^*) in the foreign bill (3) A short position of (KP) in the domestic bill (borrowing)

We can now mark-to-market our contract. If Q represents our quantity, £10 million, the current market value of our contract is

$$V_t = Qf_t = £10,000,000 S_t \frac{1}{1 + r_t^* \tau} - \$16,500,000 \frac{1}{1 + r_t \tau} \tag{17.21}$$

On the valuation date, we have $S = 1.6637, r = 4.9375\%$, and $r^* = 5.9688\%$. Hence

$$P_0 = \frac{1}{1 + r_t \tau} = \frac{1}{(1 + 4.9375\% \times 90/360)} = 0.9879$$

and similarly, $P_0^* = 0.9854$. The current market value of our contract is

$$V_t = £10,000,000 \times 1.6637 \times 0.9854 - \$16,500,000 \times 0.9879 = \$93{,}581$$

which is slightly in the money. We are going to use this formula to derive the distribution of contract values under different scenarios for the risk factors.

17.3.2 Risk Factors

Assume now that we only consider the last 100 days to be representative of movements in market prices. Table 17-2 displays quotations on the spot and 3-month rates for the last 100 business days, starting on August 10.

We first need to convert these quotes into true random variables, that is, with zero mean and constant dispersion. Table 17-3 displays the one-day changes in interest rates dr, as well as the relative changes in the associated present value factors dP/P and in spot rates dS/S. For instance, for the first day,

$$dr_1 = 5.5625 - 5.5938 = -0.0313$$

and

$$dS/S_1 = (1.6315 - 1.6341)/1.6341 = -0.0016$$

This information is now used to construct the distribution of risk factors.

TABLE 17-2 Historical Market Factors

Date	$ Eurorate (3mo-%pa)	£ Eurorate (3mo-%pa)	Spot Rate S($/£)	Number
	Market Factors			
8/10/98	5.5938	7.4375	1.6341	
8/11/98	5.5625	7.5938	1.6315	1
8/12/98	6.0000	7.5625	1.6287	2
8/13/98	5.5625	7.4688	1.6267	3
8/14/98	5.5625	7.6562	1.6191	4
8/17/98	5.5625	7.6562	1.6177	5
8/18/98	5.5625	7.6562	1.6165	6
8/19/98	5.5625	7.5625	1.6239	7
8/20/98	5.5625	7.6562	1.6277	8
8/21/98	5.5625	7.6562	1.6387	9
8/24/98	5.5625	7.6562	1.6407	10
Ö				
12/15/98	5.1875	6.3125	1.6849	90
12/16/98	5.1250	6.2188	1.6759	91
12/17/98	5.0938	6.3438	1.6755	92
12/18/98	5.1250	6.1250	1.6801	93
12/21/98	5.1250	6.2812	1.6807	94
12/22/98	5.2500	6.1875	1.6789	95
12/23/98	5.2500	6.1875	1.6769	96
12/24/98	5.1562	6.1875	1.6737	97
12/29/98	5.1875	6.1250	1.6835	98
12/30/98	4.9688	6.0000	1.6667	99
12/31/98	**4.9375**	**5.9688**	**1.6637**	100

TABLE 17-3 Movements in Market Factors

Number	*dr* ($1)	*dr* (£1)	*dP*/*P* ($1)	*dP*/*P* (£1)	*dS* ($/£)/*S*
	Movements in Market Factors				
1	-0.0313	0.1563	0.00000	-0.00046	-0.0016
2	0.4375	-0.0313	-0.00116	0.00000	-0.0017
3	-0.4375	-0.0937	0.00100	0.00015	-0.0012
4	0.0000	0.1874	-0.00008	-0.00054	-0.0047
5	0.0000	0.0000	-0.00008	-0.00008	-0.0009
6	0.0000	0.0000	-0.00008	-0.00008	-0.0007
7	0.0000	-0.0937	-0.00008	0.00015	0.0046
8	0.0000	0.0937	-0.00008	-0.00031	0.0023
9	0.0000	0.0000	-0.00008	-0.00008	0.0068
10	0.0000	0.0000	-0.00008	-0.00008	0.0012
90	0.0937	0.0625	-0.00031	-0.00023	-0.0044
91	-0.0625	-0.0937	0.00008	0.00015	-0.0053
92	-0.0312	0.1250	0.00000	-0.00038	-0.0002
93	0.0312	-0.2188	-0.00015	0.00046	0.0027
94	0.0000	0.1562	-0.00008	-0.00046	0.0004
95	0.1250	-0.0937	-0.00039	0.00015	-0.0011
96	0.0000	0.0000	-0.00008	-0.00008	-0.0012
97	-0.0938	0.0000	0.00015	-0.00008	-0.0019
98	0.0313	-0.0625	-0.00015	0.00008	0.0059
99	-0.2187	-0.1250	0.00046	0.00023	-0.0100
100	-0.0313	-0.0312	0.00000	0.00000	-0.0018

17.3.3 VAR: Historical Simulation

The **historical-simulation method** takes historical movements in the risk factors to simulate potential future movements. For instance, one possible scenario for the U.S. interest rate is that, starting from the current value $r_0 = 4.9375$, the movement the next day could be similar to that observed on August 11, which is a decrease of $dr_1 = -0.0313$. The new value is $r(1) = 4.9062$. the simulated values of other variables as

$$r^*(1) = 5.9688 + 0.1563 = 6.1251$$

and

$$S(1) = 1.6637 \times (1 - 0.0016) = 1.6611.$$

Armed with these new values, we can reprice the forward contract, now worth

$$V_t = £10,000,000 \times 1.6611 \times 0.9849 - \$16,500,000 \times 0.9879 = \$59{,}941.$$

Note that, because the contract is long the pound that fell in value, the current value of the contract has decreased relative to the initial value of $93,581.

We record the new contract value and repeat this process for all the movements from day 1 to day 100. This creates a distribution of contract values, which is reported in the last column of Table 17-4.

The final step consists of sorting the contract values, as shown in Table 17-5. Suppose we want to report VAR relative to the initial value (instead of relative to the average on the target date.) The last column in the table reports the *change* in the portfolio value, i.e. $V(k) - V_0$. These range from a loss of $200,752 to a gain of $280,074.

We can now characterize the risk of the forward contract by its entire distribution, which is shown in Figure 17-5. The purpose of VAR is to report a single number as a downside risk measure. Let us take, for instance, the 95 percent lower quantile. From Table 17-5, we identify the fifth lowest value out of a hundred, which is $127,232. Ignoring the mean, the 95 percent VAR is $\text{VAR}_{HS} = \$127{,}232$.

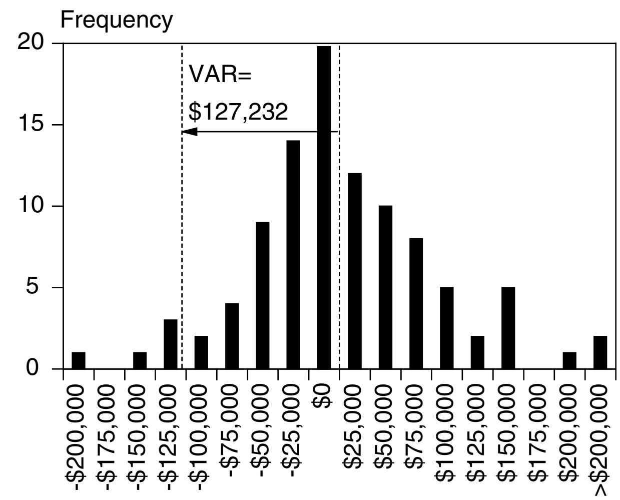
TABLE 17-4 Simulated Market Factors

	Simulated Market Factors					Hypothetical MTM
Number	*r* ($1)	*r* (£1)	*S* ($/£)	PV($1)	PV(£1)	Contract
1	4.9062	6.1251	1.6611	0.9879	0.9849	$59,941
2	5.3750	5.9375	1.6608	0.9867	0.9854	$84,301
3	4.5000	5.8751	1.6617	0.9889	0.9855	$59,603
4	4.9375	6.1562	1.6559	0.9878	0.9848	$9,467
5	4.9375	5.9688	1.6623	0.9878	0.9853	$79,407
6	4.9375	5.9688	1.6625	0.9878	0.9853	$81,421
7	4.9375	5.8751	1.6713	0.9878	0.9855	$172,424
8	4.9375	6.0625	1.6676	0.9878	0.9851	$128,149
9	4.9375	5.9688	1.6749	0.9878	0.9853	$204,361
10	4.9375	5.9688	1.6657	0.9878	0.9853	$113,588
90	5.0312	6.0313	1.6564	0.9876	0.9851	$23,160
91	4.8750	5.8751	1.6548	0.9880	0.9855	$7,268
92	4.9063	6.0938	1.6633	0.9879	0.9850	$83,368
93	4.9687	5.7500	1.6683	0.9877	0.9858	$148,705
94	4.9375	6.1250	1.6643	0.9878	0.9849	$93,128
95	5.0625	5.8751	1.6619	0.9875	0.9855	$84,835
96	4.9375	5.9688	1.6617	0.9878	0.9853	$74,054
97	4.8437	5.9688	1.6605	0.9880	0.9853	$58,524
98	4.9688	5.9063	1.6734	0.9877	0.9854	$193,362
99	4.7188	5.8438	1.6471	0.9883	0.9856	±$73,811
100	4.9062	5.9376	1.6607	0.9879	0.9854	$64,073
	4.9375	**5.9688**	**1.6637**	**0.9879**	**0.9854**	**$93,581**

17-5 Distribution of Portfolio Values

	Sorted Values	
Number	Hypothetical MTM	Change in MTM
1	-$107,171	-$200,752
2	-$73,811	-$167,392
3	-$46,294	-$139,875
4	-$37,357	-$130,938
5	-$33,651	**-$127,232**
6	-$22,304	-$115,885
7	-$11,694	-$105,275
8	$7,268	-$86,313
9	$9,467	-$84,114
10	$13,744	-$79,837
90	$193,362	$99,781
91	$194,405	$100,824
92	$204,361	$110,780
93	$221,097	$127,515
94	$225,101	$131,520
95	$228,272	$134,691
96	$233,479	$139,897
97	$241,007	$147,426
98	$279,672	$186,091
99	$297,028	$203,447
100	$373,655	$280,074

17-5 Empirical Distribution of Value Changes



17.3.4 VAR: Delta-Normal Method

The **delta-normal** approach takes a different approach to constructing the distribution of the portfolio value. We assume that the three risk factors (dS/S), (dP/P), (dP^*/P^*) are jointly normally distributed. We can write Equation (17.20) as

$$df = (SP^*)\frac{dS}{S} + (SP^*)\frac{dP^*}{P^*} - (KP)\frac{dP}{P} = x_1 dz_1 + x_2 dz_2 + x_3 dz_3 \quad (17.22)$$

where the dz are normal variables and x are exposures.

Define Σ as the (3 by 3) covariance matrix of the dz, and x as the vector of exposures. We compute VAR from $\sigma^2(df) = x'\Sigma x$. Table 17-6 details the steps. First, we compute the covariance matrix of the 3 risk factors. The top of the table shows the standard deviation of daily returns as well as correlations. From these, we construct the covariance matrix.

Next, the table shows the vector of exposures, x'. The matrix multiplication Σx is shown on the following lines. After that, we compute $x'(\Sigma x)$, which yields the variance. Taking the square root, we have $\sigma(df) = \$77{,}306$. Finally, we transform into a 95 percent quantile by multiplying by 1.645, which gives $\text{VAR}_{\text{DN}} = \$127{,}169$.

Note how close this number is to the VAR_{HS} of $127,232 we found previously. This suggests that the distribution of these variables is close to a normal distribution. Indeed, the empirical distribution in Figure 17-5 roughly looks like a normal. The fitted distribution is shown in Figure 17-6.

TABLE 17-6 Covariance Matrix Approach

Covariance Matrix of Market Factors

	dP/P($1)	*dP/P(£1)*	*dS($/£)/S*
Standard Deviation:	0.022%	0.026%	0.473%

Correlation Matrix:	*dP/P($1)*	*dP/P(£1)*	*dS($/£)/S*
dP/P($1)	1.000	0.137	0.040
dP/P(£1)	0.137	1.000	–0.063
dS($/£)/S	0.040	–0.063	1.000

Covariance Matrix:	*dP/P($1)*	*dP/P(£1)*	*dS($/£)/S*
Σ *dP/P($1)*	4.839E-08	7.809E-09	4.155E-08
dP/P(£1)	7.809E-09	6.720E-08	–7.688E-08
dS($/£)/S	4.155E-08	–7.688E-08	2.237E-05
Exposures:			
x'	–$16,300,071	$16,393,653	$16,393,653

Σx

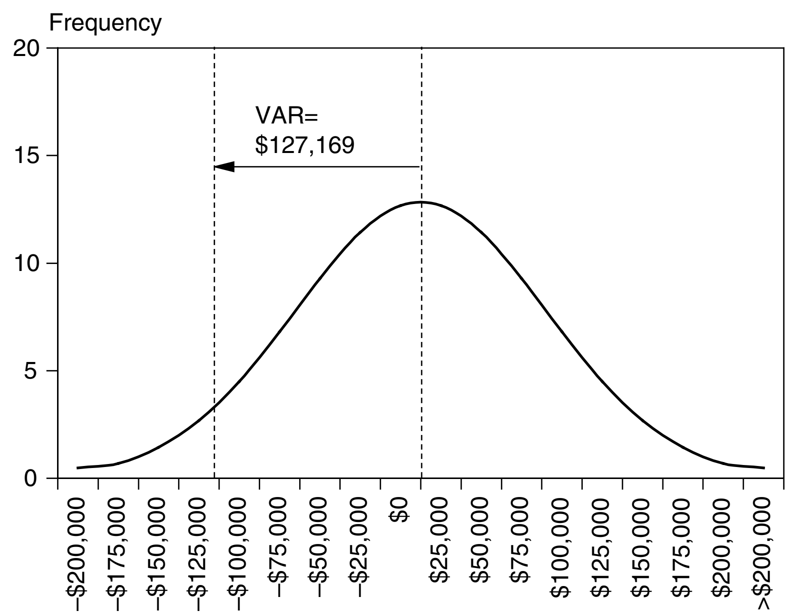
4.839E-08	7.809E-09	4.155E-08		–$16,300,071		$0.020
7.809E-09	6.720E-08	–7.688E-08	×	$16,393,653	=	–$0.286
4.155E-08	–7.688E-08	2.237E-05		$16,393,653		$364.852

$s^2 = x'(\Sigma x)$ Variance:

				$0.020		
–$16,300,071	$16,393,653	$16,393,653	×	–$0.286	=	$5,976,242,188
				$364.852		

s Standard deviation... **$77,306**

FIGURE 17-6 Normal Distribution of Value Changes



17.4 Risk Budgeting

The revolution is risk management reflects the recognition that risk should be measured at the highest level, that is, firm wide or portfolio wide. This ability to measure total risk has led to a top-down allocation of risk, called **risk budgeting**.

This concept is being implemented in pension plans as a follow-up to their **asset allocation process**. Asset allocation consists of finding the optimal allocation into major asset classes that provides the best risk/return trade-off for the investor. This defines the risk profile of the portfolio. For instance, assume that the asset allocation led to a choice of annual volatility of 10.41%. With a portfolio of \$100 million, this translates into a 95% annual VAR of \$17.1 million, assuming normal distributions. More generally, VAR can be computed using any of the three methods presented in this chapter.

This VAR budget can then be parcelled out to various asset classes and active managers within asset classes. Table 17-7 illustrates the risk budgeting process for three major asset classes, U.S. stocks, U.S. bonds, and non-U.S. bonds. Data are based on dollar returns over the period 1978 to 2002.

TABLE 17-7 Risk Budgeting

Asset		Expected Return	Volatility	Correlations 1	2	3	Percentage Allocation	VAR (per \$100)
U.S. stocks	1	13.27	15.62	1.000			60.3	\$15.5
U.S. bonds	2	8.60	7.46	0.207	1.000		7.4	\$0.9
Non-U.S. bonds	3	9.28	11.19	0.036	0.385	1.000	32.3	\$6.0
Portfolio			10.41				100.0	\$17.1

The table shows a portfolio allocation of 60.3%, 7.4%, and 32.3% to U.S. stocks, U.S. bonds, and non-U.S. bonds, respectively. Risk budgeting is the process by which these efficient portfolio allocations are transformed into VAR assignments. This translates into individual VARs of \$15.5, \$0.9, and \$6.0 million respectively. For instance, the VAR budget for U.S. stocks is 60.3% $\times$ (\$100 $\times$ 1.645 $\times$ 15.62%) = \$15.5 million. Note that the sum of individual VARs is \$22.4 million, which is more than the portfolio VAR of \$17.1 million due to diversification effects.

This risk budgeting approach is spreading rapidly to the management of pension plans. Such an approach has all the benefits of VAR. It provides a consistent measure of risk across all subportfolios. It forces managers and investors to confront squarely the amount of risk they are willing to assume. It gives them tools to monitor their risk in real time.

17.5 Answers to Chapter Examples

Example 17-1: FRM Exam 1997—Question 13/Regulatory

c) Delta-normal is appropriate for the fixed-income desk, unless it contains many MBSs. For the option desk, at least the second derivatives should be considered; so, the delta-gamma method is adequate.

Example 17-2: FRM Exam 2001—Question 92

b) Parametric VAR usually assumes a normal distribution. Given that actual distributions of financial variables have fatter tails than the normal distribution, parametric VAR at high confidence levels will generally underestimate VAR.

Example 17-3: FRM Exam 1997—Question 12/Risk Measurement

c) In finite samples, the simulation methods will be in general different from the delta-normal method, and from each other. As the sample size increases, however, the Monte-Carlo VAR should converge to the delta-normal VAR when returns are normally distributed.

Example 17-4: FRM Exam 1998—Question 6/Regulatory

a) The variance/covariance approach does not take into account second-order curvature effects.

Example 17-5: FRM Exam 1999—Questions 82/Market Risk

d) $\text{VAR} = \sqrt{40^2 + 50^2 - 2 \times 40 \times 50 \times 0.89} = 23.24$.

Example 17-6: FRM Exam 1999—Questions 15 and 90/Market Risk

c) $\text{VAR} = \sqrt{300^2 + 500^2 + 2 \times 300 \times 500 \times 1/15} = \$600$.

PART four

Credit Risk Management

Chapter 18

Introduction to Credit Risk

Credit risk is the risk of an economic loss from the failure of a counterparty to fulfill its contractual obligations. Its effect is measured by the cost of replacing cash flows if the other party defaults.

This chapter provides an introduction to the measurement of credit risk. Credit risk has undergone tremendous developments in the last few years. Fuelled by advances in the measurement of market risk, institutions are now, for the first time, attempting to quantify credit risk on a portfolio basis.

Credit risk, however, offers unique challenges. It requires constructing the distribution of default probabilities, of loss given default, and of credit exposures, all of which contribute to credit losses and should be measured in a portfolio context. In comparison, the measurement of market risk using value at risk (VAR) is a simple affair.

For most institutions, however, market risk pales in significance compared with credit risk. Indeed, the amount of risk-based capital for the banking system reserved for credit risk is vastly greater than that for market risk. The history of financial institutions has also shown that the biggest banking failures were due to credit risk.

Credit risk involves the possibility of non-payment, either on a future obligation or during a transaction. Section 18.1 introduces **settlement risk**, which arises from the exchange of principals in different currencies during a short window. We discuss exposure to settlement risk and methods to deal with it.

Traditionally, however, credit risk is viewed as presettlement risk. Section 18.2 analyzes the components of a credit risk system and the evolution of credit risk measurement systems.

Section 18.3 then shows how to construct the distribution of credit losses for a portfolio given default probabilities for the various credits in the portfolio.

The key drivers of portfolio credit risk are the correlations between defaults. Section 18.4 takes a fixed $100 million portfolio with an increasing number of obligors and shows how the distribution of losses is dramatically affected by correlations.

18.1 Settlement Risk

18.1.1 Presettlement vs. Settlement Risk

Counterparty credit risk consists of both presettlement and settlement risk. **Presettlement risk** is the risk of loss due to the counterparty's failure to perform on an obligation during the life of the transaction. This includes default on a loan or bond or failure to make the required payment on a derivative transaction. Presettlement risk can exist over long periods, often years, starting from the time it is contracted until settlement.

In contrast, **settlement risk** is due to the exchange of cash flows and is of a much shorter-term nature. This risk arises as soon as an institution makes the required payment until the offsetting payment is received. This risk is greatest when payments occur in different time zones, especially for foreign exchange transactions where notionals are exchanged in different currencies. Failure to perform on settlement can be caused by counterparty default, liquidity constraints, or operational problems.

Most of the time, settlement failure due to operational problems leads to minor economic losses, such as additional interest payments. In some cases, however, the loss can be quite large, extending to the full amount of the transferred payment. An example of major settlement risk is the 1974 failure of Herstatt Bank. The day it went bankrupt, it had received payments from a number of counterparties but defaulted before payments were made on the other legs of the transactions.

18.1.2 Handling Settlement Risk

In March 1996, the Bank for International Settlements (BIS) issued a report warning that the private sector should find ways to reduce settlement risk in the $1.2 trillion-a-day global foreign exchange market.[1] The report noted that central banks had "significant concerns regarding the risk stemming from the current arrangements for settling FX trades." It explained that "the amount at risk to even a single counterparty could exceed a bank's capital," which creates **systemic risk**. The threat of regulatory action led to a reexamination of settlement risk.

[1]Committee on Payment and Settlement Systems (1996). *Settlement Risk in Foreign Exchange Transactions*, BIS [On-line]. Available: http://www.bis.org/publ/cpss17.pdf

The status of a trade can be classified into five categories:

- *Revocable:* when the institution can still cancel the transfer without the consent of the counterparty
- *Irrevocable:* after the payment has been sent and before payment from the other party is due
- *Uncertain:* after the payment from the other party is due but before it is actually received
- *Settled:* after the counterparty payment has been received
- *Failed:* after it has been established that the counterparty has not made the payment

Settlement risk occurs during the periods of irrevocable and uncertain status, which can take from one to three days.

While this type of credit risk can lead to substantial economic losses, the short nature of settlement risk makes it fundamentally different from presettlement risk. Managing settlement risk requires unique tools, such as **real-time gross settlement** (RTGS) systems. These systems aim at reducing the time interval between the time an institution can no longer stop a payment and the receipt of the funds from the counterparty.

Settlement risk can be further managed with netting agreements. One such form is **bilateral netting**, which involves two banks. Instead of making payments of gross amounts to each other, the banks would tot up the balance and settle only the net balance outstanding in each currency. At the level of instruments, netting also occurs with **contracts for differences** (CFD). Instead of exchanging principals in different currencies, the contracts are settled in dollars at the end of the contract term.[2]

The next step up is a **multilateral netting system**, also called **continuous-linked settlements**, where payments are netted for a group of banks that belong to the system. This idea became reality when the **CLS Bank**, established in 1998 with 60 bank participants, became operational on September 9, 2002. Every evening, CLS Bank provides a schedule of payments for the member banks to follow during the next day. Payments are not released until funds are received and all transaction confirmed.

[2]These are similar to **nondeliverable forwards**, which are used to trade emerging market currencies outside the jurisdiction of the emerging-market regime and are also settled in dollars.

The risk now has been reduced to that of the netting institution. In addition to reducing settlement risk, the netting system has the advantage of reducing the number of trades between participants, by up to 90%, which lowers transaction costs.

Example 18-1: FRM Exam 2000—Question 36/Credit Risk

18-1. Settlement risk in foreign exchange is generally due to
a) Notionals being exchanged
b) Net value being exchanged
c) Multiple currencies and countries involved
d) High volatility of exchange rates

Example 18-2: FRM Exam 2000—Question 85/Market Risk

18-2. Which one of the following statements about multilateral netting systems is *not* accurate?
a) Systemic risks can actually increase because they concentrate risks on the central counterparty, the failure of which exposes all participants to risk.
b) The concentration of risks on the central counterparty eliminates risk because of the high quality of the central counterparty.
c) By altering settlement costs and credit exposures, multilateral netting systems for foreign exchange contracts could alter the structure of credit relations and affect competition in the foreign exchange markets.
d) In payment netting systems, participants with net-debit positions will be obligated to make a net settlement payment to the central counterparty that, in turn, is obligated to pay those participants with net credit positions.

18.2 Overview of Credit Risk

18.2.1 Drivers of Credit Risk

We now examine the drivers of credit risk, traditionally defined as presettlement risk. Credit risk measurement systems attempts to quantify the risk of losses due to counterparty default. The distribution of credit risk can be viewed as a compound process driven by these variables

- **Default**, which is a discrete state for the counterparty—either the counterparty is in default or not. This occurs with some **probability of default** (PD).
- **Credit exposure** (CE), also known as **exposure at default** (EAD), which is the economic value of the claim on the counterparty at the time of default.
- **Loss given default** (LGD), which represents the fractional loss due to default. As an example, take a situation where default results in a **fractional recovery rate** of 30% only. LGD is then 70% of the exposure.

Traditionally, credit risk has been measured in the context of loans or bonds for which the exposure, or economic value, of the asset is close to its notional, or face value. This is an acceptable approximation for bonds but certainly not for derivatives, which can have positive or negative value. Credit exposure is defined as the positive value of the asset:

$$\text{Credit Exposure}_t = \text{Max}(V_t, 0) \tag{18.1}$$

This is so because if the counterparty defaults with money owed to it, the full amount has to be paid.[3] In contrast, if it owes money, only a fraction may be recovered. Thus, presettlement risk only arises when the contract's replacement cost has a positive value to the institution (i.e., is "in-the-money").

18.2.2 Measurement of Credit Risk

The evolution of credit risk management tools has gone through these steps:

- Notional amounts
- Risk-weighted amounts
- External/internal credit ratings
- Internal portfolio credit models

Initially, risk was measured by the total notional amount. A multiplier, say 8 percent, was applied to this amount to establish the amount of required capital to hold as a reserve against credit risk.

The problem with this approach is that it ignores variations in the probability of default. In 1988, the Basel Committee instituted a very rough categorization of credit risk by *risk-class*, providing risk weights to scale each notional amount. This was the first attempt to force banks to carry enough capital in relation to the risks they were taking.

These risk weights proved to be too simplistic, however, creating incentives for banks to alter their portfolio in order to maximize their shareholder returns subject to the Basel capital requirements. This had the perverse effect of creating more risk into the balance sheets of commercial banks, which was certainly not the intended purpose of the 1988 rules. As an example, there was no differentiation between AAA-rated and C-rated corporate credits. Since loans to C-credits are more profitable than

[3]This is due to *no walk-away* clauses, explained in Chapter 28.

those to AAA-credits, given the same amount of regulatory capital, the banking sector responded by shifting its loan mix toward lower-rated credits.

This led to the 2001 proposal by the Basel Committee to allow banks to use their own internal or external credit ratings. These credit ratings provide a better representation of credit risk, where *better* is defined as more in line with economic measures. The new proposals will be described in more detail in a following chapter.

Even with these improvements, credit risk is still measured on a stand-alone basis. This harks back to the ages of finance before the benefits of diversification were formalized by Markowitz. One would have to hope that eventually the banking system will be given proper incentives to diversify its credit risk.

18.2.3 Credit Risk vs. Market Risk

The tools recently developed to measure market risk have proved invaluable to assess credit risk. Even so, there are a number of major differences between market and credit risks, which are listed in Table 18-1.

TABLE 18-1 Comparison of Market Risk and Credit Risk

Item	Market Risk	Credit Risk
Sources of risk	Market risk only	Default risk, recovery risk, market risk
Distributions	Mainly symmetric, perhaps fat tails	Skewed to the left
Time horizon	Short term (days)	Long term (years)
Aggregation	Business/trading unit	Whole firm vs. counterparty
Legal issues	Not applicable	Very important

As mentioned previously, credit risk results from a compound process with three sources of risk. The nature of this risk creates a distribution that is strongly skewed to the left, unlike most market risk factors. This is because credit risk is akin to short positions in options. At best, the counterparty makes the required payment and there is no loss. At worst, the entire amount due is lost.

The time horizon is also different. Whereas the time required for corrective action is relatively short in the case of market risk, it is much longer for credit risk. Positions also turn over much more slowly for credit risk than for market risk, although the advent of credit derivatives now makes it easier to hedge credit risk.

Finally, the level of aggregation is different. Limits on market risk may apply at the level of a trading desk, business units, and eventually the whole firm. In contrast, limits on credit risk must be defined at the counterparty level, for all positions taken by the institution.

Credit risk can also mix with market risk. Movements in corporate bond prices indeed reflect changing expectations of credit losses. In this case, it is not so clear whether this volatility should be classified into market risk or credit risk.

18.3 Measuring Credit Risk

18.3.1 Credit Losses

To simplify, consider only credit risk due to the effect of defaults. This is what is called **default mode**. The distribution of losses due to credit risk from a portfolio of N instruments can be described as

$$\text{Credit Loss} = \sum_{i=1}^{N} b_i \times \text{CE}_\text{i} \times (1 - f_i) \tag{18.2}$$

where:

- b_i is a (Bernoulli) random variable that takes the value of 1 if default occurs and 0 otherwise, with probability p_i, such that $E[b_i] = p_i$
- CE_i is the credit exposure at the time of default
- f_i is the recovery rate, or $(1 - f)$ the loss given default In theory, all of these could be random variables. For what follows, we will assume that the only random variable is the event of default b.

18.3.2 Joint Events

Assuming that the only random variable is default, Equation (18.2) shows that the expected credit loss is

$$E[\text{CL}] = \sum_{i=1}^{N} E[b_i] \times \text{CE}_\text{i} \times (1 - f_i) = \sum_{i=1}^{N} p_i \times \text{CE}_\text{i} \times (1 - f_i) \tag{18.3}$$

The dispersion in credit losses, however, critically depends on the correlations between the default events.

It is often convenient, although not necessarily accurate, to assume that the events are statistically independent. This simplifies the analysis considerably, as the probability of any joint event is then simply the product of the individual event probabilities

$$p(A \text{ and } B) = p(A)p(B) \tag{18.4}$$

At the other extreme, if the two events are perfectly correlated, that is, if B always default when A defaults, we have

$$p(A \text{ and } B) = p(B \mid A) \times p(A) = 1 \times p(A) = p(A) \tag{18.5}$$

when the marginal probabilities are equal, $p(A) = p(B)$.

Suppose for instance that the marginal probabilities are each $p(A) = p(B) = 1\%$. Then the probability of the joint event is 0.01% in the independence case and still 1% in the perfect correlation case.

More generally, one can show that the probability of a joint default depends on the marginal probabilities and the correlations. As we have seen in Chapter 2, the expectation of the product is

$$E[b_A \times b_B] = \text{C}[b_A, b_B] + E[b_A]E[b_B] = \rho\sigma_A\sigma_B + p(A)p(B) \tag{18.6}$$

Given that b_A is a Bernoulli variable, its standard deviation is $\sigma_A = \sqrt{p(A)[1 - p(A)]}$ and similarly for b_B. We then have

$$p(A \text{ and } B) = \text{Corr}(A, B)\sqrt{p(A)[1 - p(A)]}\sqrt{p(B)[1 - p(B)]} + p(A)p(B) \tag{18.7}$$

For example, if the correlation is unity and $p(A) = p(B) = p$, we have

$$p(A \text{ and } B) = 1 \times [p(1 - p)]^{1/2} \times [p(1 - p)]^{1/2} + p^2 = [p(1 - p)] + p^2 = p,$$

as shown in Equation (18.5).

If the correlation is 0.5 and $p(A) = p(B) = 0.01$, however, we have $p(A \text{ and } B) = 0.00505$, which is only half of the marginal probabilities. This example is illustrated in Table 18-2, which lays out the full joint distribution. Note how the probabilities in each row and column sum to the marginal probability. From this information, we can infer all missing probabilities.

TABLE 18-2 Joint Probabilities

B A	Default	No def.	Marginal
Default	0.00505	0.00495	0.01
No def.	0.00495	0.98505	0.99
Marginal	0.01	0.99	

18.3.3 An Example

Consider for instance a portfolio of $100 million with 3 bonds A, B, and C, with various probabilities of default. To simplify, we assume (1) that the exposures are constant, (2) that the recovery in case of default is zero, and (3) that default events are independent across issuers.

Table 18-3 displays the exposures and default probabilities. The second panel lists all possible states. In state one, there is no default, which has a probability of $(1-p_1)(1-p_2)(1-p_3) = (1-0.05)(1-0.10)(1-0.20) = 0.684$, given independence. In state two, bond A defaults and the others do not, with probability $p_1(1-p_2)(1-p_3) = 0.05(1-0.10)(1-0.20) = 0.036$. And so on for the other states.

TABLE 18-3 Portfolio Exposures, Default Risk, and Credit Losses

Issuer	Exposure	Probability
A	$25	0.05
B	$30	0.10
C	$45	0.20

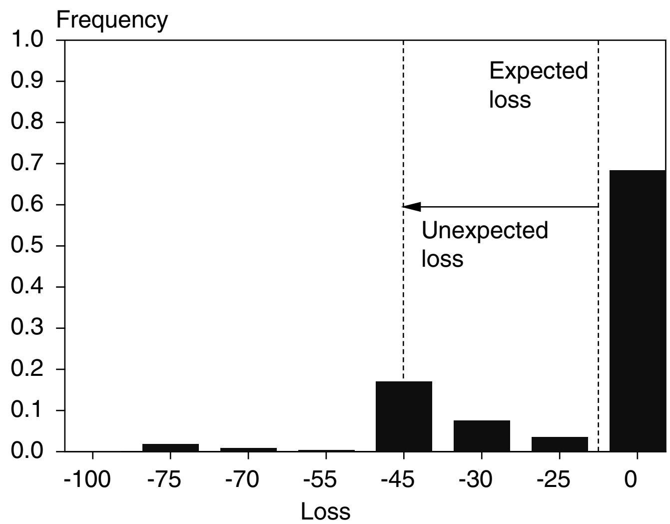
Default i	Loss L_i	Probability $p(L_i)$	Cumulative Prob.	Expected $L_i p(L_i)$	Variance $(L_i - EL_i)^2 p(L_i)$
None	$0	0.6840	0.6840	0.000	120.08
A	$25	0.0360	0.7200	0.900	4.97
B	$30	0.0760	0.7960	2.280	21.32
C	$45	0.1710	0.9670	7.695	172.38
A,B	$55	0.0040	0.9710	0.220	6.97
A,C	$70	0.0090	0.9800	0.630	28.99
B,C	$75	0.0190	0.9990	1.425	72.45
A,B,C	$100	0.0010	1.0000	0.100	7.53
Sum				$13.25	434.7

Figure 18-1 graphs the frequency distribution of credit losses. From the table, we can compute an expected loss of $13.25 million, which is also $E[\text{CL}] = \sum p_i \times \text{CE}_i = 0.05 \times 25 + 0.10 \times 30 + 0.20 \times 45$. This is the average credit loss over many repeated, hypothetical "samples." The table also shows how to compute the variance as

$$V[\text{CL}] = \sum_{i=1}^{N} (L_i - E[\text{CL}_i])^2 p(L_i),$$

which yields a standard deviation of $\sigma(\text{CL}) = \sqrt{434.7} = \$20.9$ million.

FIGURE 18-1 Distribution of Credit Losses



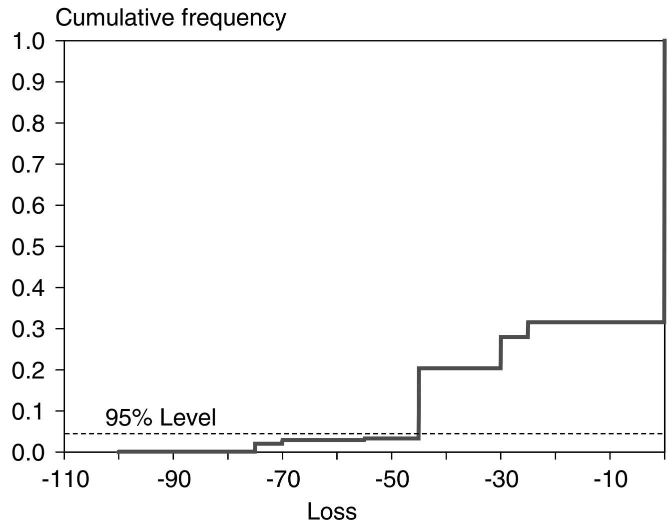
Alternatively, we can express the range of losses with a 95 percent quantile, which is the lowest number CL_i such that

$$P(CL \leq CL_i) \geq 95\% \tag{18.8}$$

From Table 18-3, this is \$45 million. Figure 18-2 plots the cumulative distribution function and shows that the 95% quantile is \$45 million. In other words, a loss up to \$45 million will not be exceeded in at least 95% of the time. In terms of deviations from the mean, this gives an unexpected loss of $45 - 13.2 = \$32$ million. This is a measure of **credit VAR**.

This very simple 3-bond portfolio provides a useful example of the measurement of the distribution of credit risk. It shows that the distribution is skewed to the left. In addition, the distribution has irregular “bumps” that correspond to the default events. The chapter on managing credit risk will further elaborate this point.

FIGURE 18-2 Cumulative Distribution of Credit Losses



Example 18-3: FRM Exam 2000—Question 46/Credit Risk

18-3. An investor holds a portfolio of \$50 million. This portfolio consists of A-rated bonds (\$20 million) and BBB-rated bonds (\$30 million). Assume that the one-year probabilities of default for A-rated and BBB-rated bonds are 2 and 4 percent, respectively, and that they are independent. If the recovery value for A-rated bonds in the event of default is 60 percent and the recovery value for BBB-rated bonds is 40 percent, what is the one-year expected credit loss from this portfolio?

a) \$672,000
b) \$742,000
c) \$880,000
d) \$923,000

Example 18-4: FRM Exam 1998—Question 38/Credit Risk

18-4. Calculate the probability of a subsidiary and parent company both defaulting over the next year. Assume that the subsidiary will default if the parent defaults, but the parent will not necessarily default if the subsidiary defaults. Also assume that the parent has a 1-year probability of default of 0.50% and the subsidiary has a 1-year probability of default of 0.90%.

a) 0.450%
b) 0.500%
c) 0.545%
d) 0.550%

Example 18-5: FRM Exam 1998—Question 16/Credit Risk

18-5. A portfolio manager has been asked to take the risk related to the default of two securities A and B. She has to make a large payment if, and only if, both A and B default. For taking this risk, she will be compensated by receiving a fee. What can be said about this fee?

a) The fee will be larger if the default of A and of B are highly correlated.
b) The fee will be smaller if the default of A and of B are highly correlated.
c) The fee is independent of the correlation between the default of A and of B.
d) None of the above are correct.

Example 18-6: FRM Exam 1998—Question 42/Credit Risk

18-6. A German Bank lends DEM 100 million to a Russian Bank for one year and receives DEM 120 million worth of Russian government securities as collateral. Assuming that the 1-year 99% VAR on the Russian government securities is DEM 20 million and the Russian bank's 1-year probability of default is 5%, what is the German bank's probability of losing money on this trade over the next year?

a) Less than 0.05%
b) Approximately 0.05%
c) Between 0.05% – 5%
d) Greater than 5%

Example 18-7: FRM Exam 2000—Question 51/Credit Risk

18-7. A portfolio consists of two (long) assets £100 million each. The probability of default over the next year is 10% for the first asset, 20% for the second asset, and the joint probability of default is 3%. Estimate the expected loss on this portfolio due to credit defaults over the next year assuming 40% recovery rate for both assets.

a) £18 million
b) £22 million
c) £30 million
d) None of the above

18.4 Credit Risk Diversification

Modern banking was built on the sensible notion that a portfolio of loans is less risky than single loans. As with market risk, the most important feature of credit risk management is the ability to diversify across defaults.

To illustrate this point, Figure 18-3 presents the distribution of losses for a $100 million loan portfolio. The probability of default is fixed at 1 percent. If default occurs, recovery is zero.

In the first panel, we have one loan only. We can either have no default, with probability 99%, or a loss of $100 million with probability 1%. The expected loss is

$$\text{EL} = 0.01 \times \$100 + 0.99 \times 0 = \$1 \text{ million.}$$

The problem, of course, is that, if default occurs, it will be a big hit to the bottom line, possibly bankrupting the lending bank.

Basically, this is what happened to Peregrine Investments Holdings, one of Hong Kong's leading investment banks that failed due to the Asian crisis of 1997. The bank failed in large part from a single loan to PT Steady Safe, an Indonesian taxi-cab operator, that amounted to $235 million, a quarter of the bank's equity capital.

In the case of our single loan, the spread of the distribution is quite large, with a variance of 99, which implies a standard deviation (SD) of about $10 million. Simply focusing on the standard deviation, however, is not fully informative given the severe skewness in the distribution.

In the second panel, we consider ten loans, each for $10 million. The total notional is the same as before. We assume that defaults are independent. The expected loss is still $1 million, or 10 × 0.01 × $10 million. The SD, however, is now $3 million, much less than before.

Next, the third panel considers a hundred loans of $1 million each. The expected loss is still $1 million, but the SD is now $1 million, even lower. Finally, the fourth panel considers a thousand loans of $100,000, which create a SD of $0.3 million.

For comparability, all these graphs use the same vertical and horizontal scale. This, however, does not reveal the distributions fully. This is why the fifth panel expands the distribution with 1000 counterparties, which looks similar to a normal distribution. This reflects the **central limit theorem**, which states that the distribution of the sum of *independent* variables tends to a normal distribution. Remarkably, even starting from a highly skewed distribution, we end up with a normal distribution due to diversification effects. This explains why portfolios of consumer loans, which are spread over a large number of credits, are less risky than typical portfolios of corporate loans.

With N events that occur with the same probability p, define the variable $X = \sum_{i=1}^{N} b_i$ as the number of defaults (where $b_i = 1$ when default occurs). The expected credit loss on our portfolio is then

$$E[\text{CL}] = E[X] \times \$100/N = pN \times \$100/N = p \times \$100 \tag{18.9}$$

which does not depend on N but rather on the average probability of default and total exposure, \$100 million. When the events are independent, the variance of this variable is, using the results from a binomial distribution,

$$V[\mathrm{CL}] = V[X] \times (\$100/N)^2 = p(1-p)N \times (\$100/N)^2 \tag{18.10}$$

which gives a standard deviation of

$$\mathrm{SD}[\mathrm{CL}] = \sqrt{p(1-p)} \times \$100/\sqrt{N} \tag{18.11}$$

For a constant total notional, this shrinks to zero as N increases.

We should note the crucial assumption that the credits are independent. When this is not the case, the distribution will lose its asymmetry more slowly. Even with a very large number of consumer loans, the dispersion may not tend to zero because the general state of the economy is a common factor behind consumer credits. Indeed, many more defaults occur in a recession than in an expansion.

Institutions loosely attempt to achieve diversification by **concentration limits.** In other words, they limit the extent of exposure, say loans, to a particular industrial or geographical sector. The rationale behind this is that defaults are more highly correlated within sectors than across sectors. Conversely, **concentration risk** is the risk that too many defaults could occur at the same time.

Example 18-8: FRM Exam 1997—Question 11/Credit Risk

18-8. A commercial loan department lends to two different BB-rated obligors for one year. Assume the one-year probability of default for a BB-rated obligor is 10% and there is zero correlation (independence) between the obligor's probability of defaulting. What is the probability that both obligors will default in the same year?

a) 1%
b) 2%
c) 10%
d) 20%

FIGURE 18-3 Distribution of Credit Losses

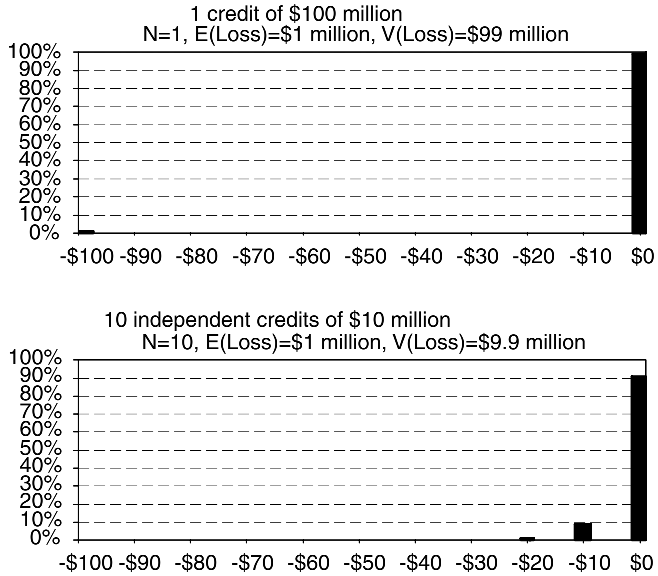


FIGURE 18-3 Distribution of Credit Losses (Continued)

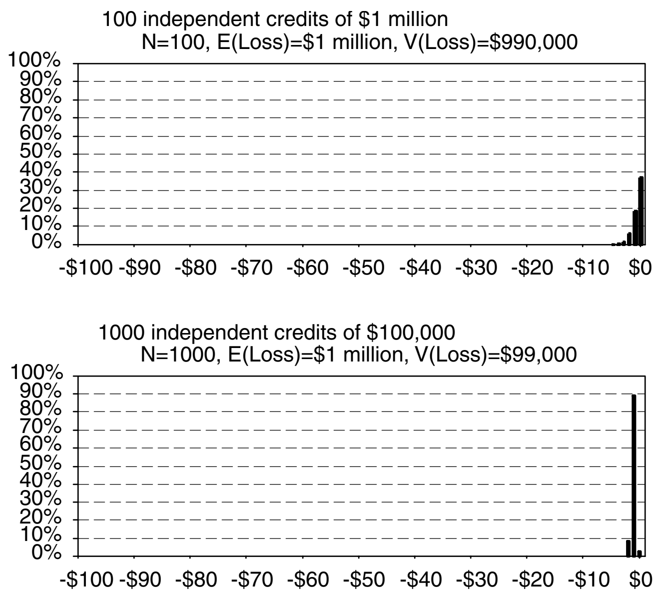
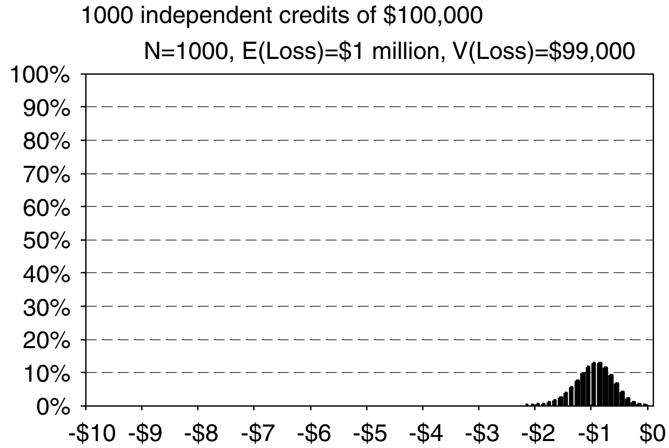


FIGURE 18-3b Distribution of Credit Losses (Continued)



Example 18-9: FRM Exam 1997—Question 12/Credit Risk

18-9. What is the probability of no defaults over the next year from a portfolio of 10 BBB-rated obligors? Assume the one-year probability of default for a BBB-rated counterparty is 5% and assumes zero correlation (independence) between the obligor's probability of default.

a) 5.0%
b) 50.0%
c) 60.0%
d) 95.0%

Example 18-10: FRM Exam 2001—Question 5

18-10. What is the approximate probability of one particular bond defaulting, and none of the others, over the next year from a portfolio of 20 BBB-rated obligors? Assume the 1-year probability of default for a BBB-rated counterparty to be 4% and obligor defaults to be independent from one another.

a) 2%
b) 4%
c) 45%
d) 96%

18.5 Answers to Chapter Examples

Example 18-1: FRM Exam 2000—Question 36/Credit Risk

a) Settlement risk is due to the exchange of notional principal in different currencies at different points in time, which exposes one counterparty to default after it has made payment. There would be less risk with netted payments.

Example 18-2: FRM Exam 2000—Question 85/Market Risk

b) Answers (c) and (d) are both correct. Answers (a) and (b) are contradictory. A multilateral netting system concentrates the credit risk into one institution. This could potentially create much damage if this institution fails.

Example 18-3: FRM Exam 2000—Question 46/Credit Risk

c) The expected loss is $\sum_i p_i \times \text{CE}_i \times (1 - f_i) = \$20{,}000{,}000 \times 0.02(1 - 0.60) + \$30{,}000{,}000 \times 0.04(1 - 0.40) = \$880{,}000$.

Example 18-4: FRM Exam 1998—Question 38/Credit Risk

b) Since the subsidiary defaults when the parent defaults, the joint probability is simply that of the parent defaulting.

Example 18-5: FRM Exam 1998—Question 16/Credit Risk

a) The fee must reflect the joint probability of default. As described in Equation (18.7), if defaults of A and B are highly correlated, the default of one implies a greater probability of a second default. Hence the fee must be higher.

Example 18-6: FRM Exam 1998—Question 42/Credit Risk

c) The probability of losing money is driven by (i) a fall in the value of the collateral and (ii) default by the Russian bank. If the two events are independent, the joint probability is $5\% \times 1\% = 0.05\%$. In contrast, if the value of securities always drops at the same time the Russian bank defaults, the probability is simply that of the Russian bank's default, or 5%.

Example 18-7: FRM Exam 2000—Question 51/Credit Risk

a) The three loss events are

(i) Default by the first alone, with probability $0.10 - 0.03 = 0.07$

(ii) Default by the second, with probability $0.20 - 0.03 = 0.17$

(iii) Default by both, with probability 0.03

The respective losses are £$100 \times (1-0.4) \times 0.07 = 4.2$, £$100 \times (1-0.4) \times 0.17 = 10.2$, £$200 \times (1 - 0.4) \times 0.03 = 3.6$, for a total expected loss of £18 million.

Example 18-8: FRM Exam 1997—Question 11/Credit Risk

a) With independence, this probability is $10\% \times 10\% = 1\%$.

Example 18-9: FRM Exam 1997—Question 12/Credit Risk

c) Since the probability of one default is 5%, that on a bond no defaulting is $100 - 5 = 95\%$. With independence, the joint probability of 10 no defaults is $(1 - 5\%)^{10} = 60\%$.

Example 18-10: FRM Exam 2001—Question 5

a) This question asks the probability that one particular bond will default and 19 others will not. Assuming independence, this is $0.04(1 - 0.04)^{19} = 1.84\%$. Note that the probability that any bond will default and none others is 20 times this, or 36.8%.

Chapter 19

Measuring Actuarial Default Risk

Default risk is the primary component of credit risk. It represents the **probability of default** (PD), as well as the **loss given default** (LGD). When default occurs, the actual loss is the combination of exposure at default and loss given default.

Default risk can be measured using two approaches: (1) **Actuarial methods**, which provide "objective" (as opposed to risk-neutral) measures of default rates, usually based on historical default data, and (2) **Market-price methods**, which infer from traded prices the market's assessment of default risk, along with a possible risk premium. The market prices of debt, equity, or credit derivatives can be used to derive risk-neutral measures of default risk.

Risk-neutral measures provide a useful shortcut to price assets, such as options. For risk management purposes, however, they are contaminated by the effect of risk premiums and therefore do not exactly measure default probabilities. In contrast, objective measures describe the "actual" or "natural" probability of default. On the other hand, since risk-neutral measures are derived directly from market data, they should incorporate all the news about a creditor's prospects.

Actuarial measures of default probabilities are provided by **credit rating agencies**, which classify borrowers by credit ratings that are supposed to quantify default risk. Such ratings are **external** to the firm. Similar techniques can be used to develop **internal** ratings.

Such measures can also be derived from **accounting variables models**. These models relate the occurrence of default to a list of firm characteristics, such as accounting variables. Statistical techniques such as discriminant analysis then examine how these variables are related to the occurrence or nonoccurrence of default. Presumably, rating agencies use similar procedures, augmented by additional data.

This chapter focuses on actuarial measures of default risk. Market-based measures of default risk will be examined in the next chapter. Section 19.1 examines first the definition of a credit event. Section 19.2 then examines credit ratings, describing how historical default rates can be used to infer default probabilities. Recovery rates are discussed in Section 19.3. Section 19.4 then presents an application to the construction and rating of a collateralized bond obligation. Finally, Section 19.5 broadly discusses the evaluation of corporate and sovereign credit risk.

19.1 Credit Event

A credit event is a discrete state. Either it happens or not. The issue is the definition of the event, which must be framed in legal terms.

One could say, for instance, that the definition of default for a bond obligation is quite narrow. Default on the bond occurs when payment on that same bond is missed.

Default on a bond, however, reflects the creditor's financial distress and is almost always accompanied by default on other obligations. This is why rating agencies give a credit rating for the issuer.[1] Likewise, the state of **default** is defined by **Standard & Poor's** (S&P), a credit rating agency, as

> *The first occurrence of a payment default on any financial obligation, rated or unrated, other than a financial obligation subject to a bona fide commercial dispute; an exception occurs when an interest payment missed on the due date is made within the grace period.*

This definition, however, needs to be defined more precisely for credit derivatives, whose payoffs are directly related to credit events. We will cover credit derivatives in Chapter 22. The definition of a **credit event** has been formalized by the **International Swaps and Derivatives Association** (ISDA), an industry group, which lists these events:

- **Bankruptcy**, which is a situation involving (1) The *dissolution* of the obligor (other than merger) (2) The *insolvency*, or inability to pay its debt, (3) The *assignment* of claims (4) The *institution of bankruptcy proceeding* (5) The *appointment of receivership* (6) The *attachment of substantially all assets by a third party*
- **Failure to pay**, which means failure of the creditor to make due payment; this is usually triggered after an agreed-upon grace period and above a certain amount

[1]Specific bonds can be as higher as or lower than this issuer rating, depending on their relative priority.

- **Obligation/cross default**, which means the occurrence of a default (other than failure to make a payment) on any other similar obligation
- **Obligation/cross acceleration**, which means the occurrence of a default (other than failure to make a payment) on any other similar obligation that results in that obligation becoming due immediately
- **Repudiation/moratorium**, which means that the counterparty is rejecting, or challenges, the validity of the obligation
- **Restructuring**, which means a waiver, deferral, or rescheduling of the obligation with the effect that the terms are less favorable than before.

In addition, other events sometimes included are

- **Downgrade**, which means the credit rating is lower than previously, or withdrawn
- **Currency inconvertibility**, which means the imposition of exchange controls or other currency restrictions imposed by a governmental or associated authority
- **Governmental action**, which means either (1) declarations or actions by a government or regulatory authority that impair the validity of the obligation, or (2) the occurrence of war or other armed conflict that impairs the functioning of the government or banking activities

The ISDA definitions are designed to minimize legal risks, by precisely wording the definition of credit event. Sometimes unforeseen situations develop. Even now, it is sometimes not clear whether a bank debt restructuring constitutes a credit event, as in the recent cases of Conseco, Xerox, and Marconi.

Another notable default is that of Argentina, which represents the largest sovereign default recorded so far, in terms of external debt. Argentina announced in November 2001 a restructuring of its local debt that was more favorable to itself. Some holders of credit default swaps argued that this was a "credit event," since the exchange was coerced, and that they were entitled to payment. Swap sellers disagreed. This became an unambiguous default, however, when Argentina announced in December it would stop paying interest on its $135 billion foreign debt. Nonetheless, the situation was unresolved for holders of credit swaps that expired just before the official default. In such situations, the ISDA tries to clarify the language of its agreement.

Example 19-1: FRM Exam 1998—Question 5/Credit Risk

19-1. Which of the following events is not a "credit event"?

a) Bankruptcy

b) Calling back a bond

c) Downgrading

d) Default on payments

Example 19-2: FRM Exam 1999—Question 128/Credit Risk

19-2. Which of the following losses can be considered as resulting from an "event risk"?

I) Losses on a diversified portfolio of stocks during the stock market decline and hedge fund crisis in the Autumn/Fall of 1998.

II) A U.S. investor bought a bond whose payments are in Japanese yen. The investor made a loss as Japanese Yen depreciated relative to the dollar.

III) A holding in RJR Nabisco corporate bonds sustained a loss in 1988 when RJR Nabisco was taken over for $25 billion via a leveraged buyout which resulted in a reduction of its debt rating to noninvestment grade.

IV) A municipal bond portfolio suffers a loss when municipal bonds are declared as no longer tax exempt by the tax authority, with no compensation being paid to investors.

a) III only

b) All the above

c) I and IV

d) III and IV

19.2 Default Rates

19.2.1 Credit Ratings

A **credit rating** is an "evaluation of creditworthiness" issued by a rating agency. More technically, it has been defined by **Moody's**, a ratings agency, as an "opinion of the future ability, legal obligation, and willingness of a bond issuer or other obligor to make full and timely payments on principal and interest due to investors."

Table 19-1 presents the interpretation of various credit ratings issued by the two major rating agencies, Moody's and Standard and Poor's. These ratings correspond to long-term debt; other ratings apply to short-term debt. Generally, the two agencies provide similar ratings for the same issuer.

Table 19-1. Classification by Credit Ratings

Explanation	Standard & Poor's	Moody's Services
Investment grade:		
Highest grade	AAA	Aaa
High grade	AA	Aa
Upper medium grade	A	A
Medium grade	BBB	Baa
Speculative grade:		
Lower medium grade	BB	Ba
Speculative	B	B
Poor standing	CCC	Caa
Highly speculative	CC	Ca
Lowest quality, no interest	C	C
In default	D	

Modifiers: Example A+, A, A−, A1, A2, A3

Ratings are broadly divided into

- **Investment grade**, that is, at and above BBB for S&P and Baa for Moody's
- **Speculative grade**, or **below investment grade**, for the rest This classification is sometimes used to define classes of investments allowable to some investors, such as pension funds.

These ratings represent objective (or actuarial) probabilities of default.[2] Indeed, the agencies have published studies that track the frequency of bond default in the United States, classified by initial ratings for different horizons. These frequencies can be used to convert ratings to default probabilities.

The agencies use a number of criteria to decide on the credit rating, among other accounting ratios. Table 19-2 presents median value for selected accounting ratios for industrial corporations. The first column (under "leverage") shows that the ratio of total debt to total capital (debt plus book equity) varies systematically across ratings. Highly rated companies have low ratios, 23% for AAA firms. In contrast, BB-rated (just below investment grade) companies have a debt-to-capital ratio of 63%. This implies a capital-to-equity leverage ratio of 2.7 to 1.[3]

[2]In fact, the ratings measure the probability of default (PD) for S&P and the joint effect of PD × LGD for Moody's, where LGD is the proportional loss given default.

[3]Defining D, E as debt and equity, this is obtained as $(D+E)/E = D/E+1 = 63\%/(1-63\%)+1 = 2.7$

The right-hand-side panel (under "cash flow") also shows systematic variations in a measure of free cash flow divided by interest payments. This represents the number of times the cash flow can cover interest payments. Focusing on earnings before interest and taxes (EBIT), AAA-rated companies have a safe cushion of 21.4, whereas BB-rated companies have coverage of 2.1 only.

Table 19-2. S&P's Financial Ratios Across Ratings

Rating	Leverage: (Percent)		Cash Flow Coverage: (Multiplier)	
	Total Debt /Capital	LT Debt /Capital	EBITDA /Interest	EBIT /Interest
AAA	23	13	26.5	21.4
AA	38	28	12.9	10.1
A	43	34	9.1	6.1
BBB	48	43	5.8	3.7
BB	63	57	3.4	2.1
B	75	70	1.8	0.8
CCC	88	69	1.3	0.1

Note: From S&P's *Corporate Ratings Criteria* (2002), based on median financial ratios over 1998 to 2000 for industrial corporations. EBITDA is defined as earnings before interest, taxes, depreciation and amortization.

Example 19-3: FRM Exam 1997—Question 8/Credit Risk

19-3. Which of the following is Moody's lowest credit rating?
a) Aaa2
b) Baa1
c) Baa3
d) Ba2

Example 19-4: FRM Exam 1998—Question 37/Credit Risk

19-4. A credit-risk analyst has calculated two significant financial figures for Company X; a pretax interest coverage ratio of 3.75 and long-term debt/equity of 35%. Given this information, what is the most likely rating grade that the analyst will assign to Company X?
a) Investment grade
b) Speculative grade
c) Noninvestment grade
d) Junk grade

19.2.2 Historical Default Rates

Tables 19-3 and 19-4 display historical default rates as reported by Moody's and Standard and Poor's, respectively. These describe the proportion of firms that default, $\bar{X}$, which is a statistical estimate of the true default probability:

$$E(\bar{X}) = p \tag{19.1}$$

For example, borrowers with an initial Moody's rating of Baa experienced an average 0.34% default rate over the next year, and 7.99% over the following ten years. Similar rates are obtained for S&P's BBB-rated credits, who experienced an average 0.36% default rate over the next year, and 7.60% over the following ten years.

Thus, higher ratings are associated with lower default rates. As a result, this information could be used as estimates of default probability for an initial rating class. In addition, the tables show that the default rate increases with the horizon, for a given initial credit rating. Credit risk increases with the horizon.

TABLE 19-3: Moody's Cumulative Default Rates (Percent), 1920–2002

Rating	Year									
	1	2	3	4	5	6	7	8	9	10
Aaa	0.00	0.00	0.02	0.09	0.19	0.29	0.41	0.59	0.78	1.02
Aa	0.07	0.22	0.36	0.54	0.85	1.21	1.60	2.01	2.37	2.78
A	0.08	0.27	0.57	0.92	1.28	1.67	2.09	2.48	2.93	3.42
Baa	0.34	0.99	1.79	2.69	3.59	4.51	5.39	6.25	7.16	7.99
Ba	1.42	3.43	5.60	7.89	10.16	12.28	14.14	15.99	17.63	19.42
B	4.79	10.31	15.59	20.14	23.99	27.12	30.00	32.36	34.37	36.10
Caa-C	14.74	23.95	30.57	35.32	38.83	41.94	44.23	46.44	48.42	50.19
Inv.	0.17	0.50	0.93	1.41	1.93	2.48	3.03	3.57	4.14	4.71
Spec.	3.83	7.75	11.41	14.69	17.58	20.09	22.28	24.30	26.05	27.80
All	1.50	3.09	4.62	6.02	7.28	8.41	9.43	10.38	11.27	12.14

Rating	Year									
	11	12	13	14	15	16	17	18	19	20
Aaa	1.24	1.40	1.61	1.70	1.75	1.85	1.96	2.02	2.14	2.20
Aa	3.24	3.77	4.29	4.82	5.23	5.51	5.75	5.98	6.30	6.54
A	3.95	4.47	4.94	5.40	5.88	6.35	6.63	6.94	7.23	7.54
Baa	8.81	9.62	10.41	11.12	11.74	12.33	12.95	13.49	13.93	14.39
Ba	21.06	22.65	24.23	25.61	26.83	27.96	29.13	30.24	31.14	32.05
B	37.79	39.37	40.85	42.33	43.62	44.94	45.91	46.68	47.32	47.60
Caa-C	52.30	54.4	56.24	58.22	60.08	61.78	63.27	64.81	66.25	67.59
Inv.	5.30	5.90	6.46	7.00	7.48	7.92	8.30	8.65	8.99	9.32
Spec.	29.47	31.08	32.64	34.07	35.36	36.58	37.72	38.78	39.67	40.46
All	13.01	13.85	14.66	15.40	16.07	16.69	17.24	17.75	18.21	18.64

TABLE 19-4: S&P's Cumulative Default Rates, 1981–2002 (Percent)

Rating	Year														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
AAA	0.00	0.00	0.03	0.07	0.11	0.20	0.30	0.47	0.54	0.61	0.61	0.61	0.61	0.75	0.92
AA	0.01	0.03	0.08	0.17	0.28	0.42	0.61	0.77	0.90	1.06	1.20	1.37	1.51	1.63	1.77
A	0.05	0.15	0.30	0.48	0.71	0.94	1.19	1.46	1.78	2.10	2.37	2.60	2.84	3.08	3.46
BBB	0.36	0.96	1.61	2.58	3.53	4.49	5.33	6.10	6.77	7.60	8.48	9.34	10.22	11.28	12.44
BB	1.47	4.49	8.18	11.69	14.77	17.99	20.43	22.63	24.85	26.61	28.47	29.76	30.98	31.70	32.56
B	6.72	14.99	22.19	27.83	31.99	35.37	38.56	41.25	42.90	44.59	45.84	46.92	47.71	48.68	49.57
CCC	30.95	40.35	46.43	51.25	56.77	58.74	59.46	59.85	61.57	62.92	63.41	63.41	63.41	64.25	64.25
Inv.	0.13	0.34	0.59	0.93	1.29	1.65	1.99	2.33	2.64	2.99	3.32	3.63	3.95	4.30	4.75
Spec.	5.56	11.39	16.86	21.43	25.12	28.35	31.02	33.32	35.24	36.94	38.40	39.48	40.40	41.24	42.05
All	1.73	3.51	5.12	6.48	7.57	8.52	9.33	10.04	10.66	11.27	11.81	12.28	12.71	13.17	13.69

Note: Static pool average cumulative default rates (adjusted for "not rated" borrowers).

One problem with such historical information, however, is the relative paucity of data. There are simply not many instances of highly rated borrowers that default over long horizons. For instance, S&P reports default rates up to 15 years using data from 1981 to 2002. The one-year default rates represent 23 years of data, that is, 1981, 1982, and so on to 2002. There are, however, only eight years of data for the 15-year default rates, that is, 1981-1995 to 1988-2002. Thus the sample size is much shorter (and also overlapping and therefore not independent). If so, omitting or adding a few borrowers can drastically alter the reported default rates.

This can lead to inconsistencies in the tables. For instance, the default rates for CCC-borrowers is the same, at 63.41 percent, from year 11 to 13. This would imply that there is no further risk of default after 11 years, which is unrealistic. Also, when the categories are further broken down into modifiers (e.g., Aaa1, Aaa2, Aaa3), default rates sometimes do not decrease monotonically with the ratings, which is a small sample effect.

We can try to assess the accuracy of these default rates by computing their standard error. Consider for instance the default rate over the first year for AA-rated credits, which averaged out to $\bar{X} = 0.01\%$ in this S&P sample. This was taken out of a total of about $N = 8,000$ observations, which we assume to be independent. The variance of the average is, from the distribution of a binomial process,

$$V(\bar{X}) = \frac{p(1-p)}{N} \tag{19.2}$$

which gives a standard error of about 0.011%. This is on the same order as the average of 0.01%, indicating that there is substantial imprecision in this average default rate. So, we could not really distinguish an AA credit from an AAA credit.

The problem is made worse with lower sample sizes, which is the case in non-U.S. markets or when the true p is changing over time. For instance, if we observe a 5% default rate among 100 observations, the standard error becomes 2.2%, which is very large. Therefore, a major issue with credit risk is that estimation of default rates for low-probability events can be very imprecise.

Example 19-5: FRM Exam 1997—Question 28/Credit Risk

19-5. Based on historical data from S&P, what is the approximate historical 1-year probability of default for a BB-rated obligor?
a) 0.05%
b) 0.20%
c) 1.0%
d) 5.0%

Example 19-6: FRM Exam 1998—Question 29/Credit Risk

19-6. Based on historical evidence, a B-rated counterparty is approximately 16 times more likely to default over a 1-year time period than a BBB-rated counterparty. Over a 10-year time period, a B-rated counterparty is how many more times likely to default than a BBB-rated counterparty?
a) 5
b) 9
c) 16
d) 24

19.2.3 Cumulative and Marginal Default Rates

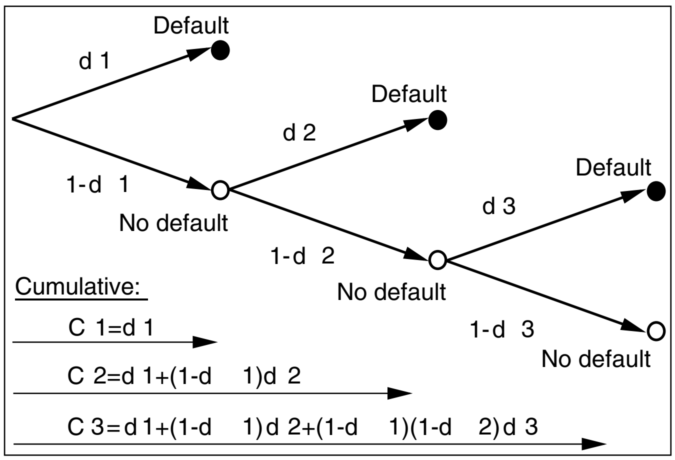
The default rates reported in Tables 19-3 and 19-4 are **cumulative default rates** for an initial credit rating, that is, measure the total frequency of default *at any time* between the starting date and year T. It is also informative to measure the **marginal default rate**, which is the frequency of default *during* year T.

The default process is illustrated in Figure 19-1. Here, d_1 is the marginal default rate during year 1. Next, d_2 is the marginal default rate during year 2. In order to default during the second year, the firm must have survived the first year and defaulted in the second. Thus, the probability of defaulting in year 2 is given by $(1-d_1)d_2$. The cumulative probability of defaulting up to year 2 is then $C_2 = d_1 + (1-d_1)d_2$. Subtracting and adding one, this is also $C_2 = 1 - (1 - d_1)(1 - d_2)$, which perhaps has a more intuitive interpretation, as this is one minus the probability of surviving the whole period.

More formally, we define

- $m[t + N \mid R(t)]$ as the number of issuers rated R at the end of year t that default in year $T = t + N$
- $n[t + N \mid R(t)]$ as the number of issuers rated R at the end of year t that have not defaulted by the beginning of year $t + N$

FIGURE 19-1 Sequential Default Process



Marginal Default Rate during Year T This is the proportion of issuers initially rated R at initial time t that default in year T, relative to the remaining number at the beginning of the same year T:

$$d_N(R) = \frac{m[t + N \mid R(t)]}{n[t + N \mid R(t)]}$$

Survival Rate This is the proportion of issuers initially rated R that will not have defaulted by T:

$$S_N(R) = \Pi_{i=1}^{N}(1 - d_i(R)) \qquad (19.3)$$

Marginal Default Rate from Start to Year T This is the proportion of issuers initially rated R that defaulted in year T, relative to the initial number in year t. For this to happen, the issuer will have survived until year $t + N - 1$, then default the next year

$$k_N(R) = S_{N-1}(R)d_N(R) \qquad (19.4)$$

Cumulative Default Rate This is the proportion of issuers initially rated R that defaulted at any point until year T

$$C_N(R) = k_1(R) + k_2(R) + \cdots + k_N(R) = 1 - S_N(R) \qquad (19.5)$$

Average Default Rate We can express the total cumulative default rate into an average, per period default rate d, by setting

$$C_N = 1 - \Pi_{i=1}^{N}(1 - d_i) = 1 - (1 - d)^N \tag{19.6}$$

As we move from annual to semiannual and ultimately continuous compounding, the average default rate becomes

$$C_N = 1 - (1 - d^a)^N = 1 - (1 - d^s/2)^{2N} \rightarrow 1 - e^{-d^c N} \tag{19.7}$$

where d_a, d_s, d_c are default rates using annual, semiannual, and continuous compounding. This is exactly equivalent to various definitions for the compounding of interest.

Example: Computing marginal and cumulative default probabilities

Consider a "B" rated firm that has default rates of $d_1 = 5\%$, $d_2 = 7\%$.

- In the first year, $k_1 = d_1 = 5\%$.
- After 1 year, the survival rate is $S_1 = 0.95$.
- The probability of defaulting in year 2 is then $k_2 = S_1 \times d_2 = 0.95 \times 0.07 \times = 6.65\%$.
- After 2 years, the survival rate is $(1 - d_1)(1 - d_2) = 0.95 \times 0.93 = 0.8835$.
- The cumulative probability of defaulting in years 1 and 2 is $5\% + 6.65\% = 11.65\%$.

Based on this information, we can map these "forward", or marginal, default rates from cumulative default rates for various credit ratings. Figure 19-2, for instance, displays cumulative default rates reported by Moody's in Table 19-3. The marginal default rates are derived from these and plotted in Figure 19-3.

It is interesting to see that the marginal probability of default increases with maturity for initial high credit ratings, but decreases for initial low credit ratings. The increase is due to a mean reversion effect. The fortunes of an Aaa-rated firm can only stay the same, at best, and often will deteriorate over time. In contrast, a B-rated firm that has survived the first few years must have a decreasing probability of defaulting as time goes by. This is a survival effect.

The analysis of default probabilities is similar to that of mortality rates for mortgage-backed securities. If the annual default rate is d, the monthly default rate, assuming it is constant, is implicitly given by

$$(1 - d_M)^{12} = (1 - d) \tag{19.8}$$

FIGURE 19-2 Moody's Cumulative Default Rates, 1920–2002

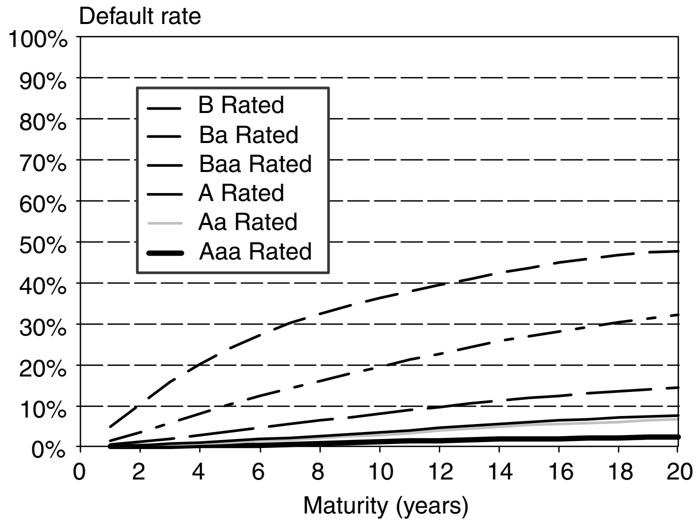
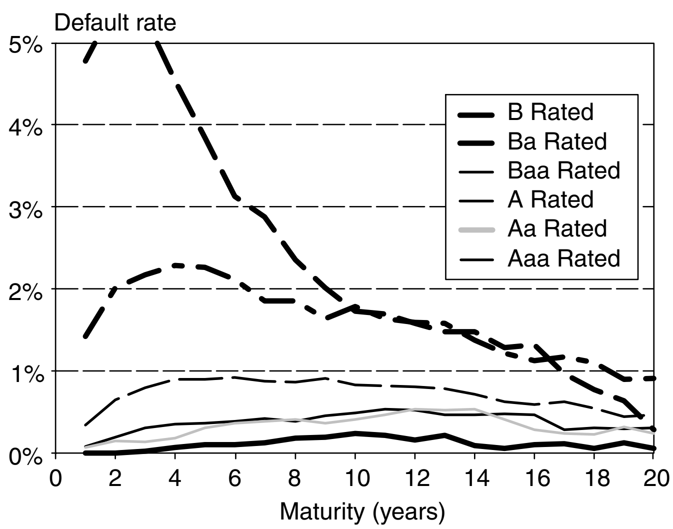


FIGURE 19-3 Moody's Marginal Default Rates, 1920–2002



which says that the firm must survive all 12 months sequentially to survive the year. But, as we have seen, the marginal probability of default increases with time for high credits.

Example 19-7: FRM Exam 1997—Question 2/Credit Risk

19-7. The probability of an AA-rated counterparty defaulting over the next year is 0.06%. Therefore, one would expect the probability of it defaulting over the next 3 months to be

a) Between 0% – 0.015%
b) Exactly 0.015%
c) Between 0.015% – 0.030%
d) Greater than 0.030%

Example 19-8: FRM Exam 2000—Question 37/Credit Risk

19-8. A company has a constant 30% per year probability of default. What is the probability the company will be in default after three years?

a) 34%
b) 48%
c) 66%
d) 90%

Example 19-9: FRM Exam 2000—Question 31/Credit Risk

19-9. According to Standard and Poor's, the 5-year cumulative probability default for BB-rated debt is 15%. If the marginal probability of default for BB debt from year 5 to year 6 (conditional on no prior default) is 10%, then what is the 6-year cumulative probability default for BB-rated debt?

a) 25%
b) 16.55%
c) 15%
d) 23.50%

Example 19-10: FRM Exam 1997—Question 10/Credit Risk

19-10. The ratio of the default probability of an AA-rated issuer over the default probability of a B-rated issuer

a) Generally increases with time to maturity
b) Generally decreases with time to maturity
c) Remains roughly the same with time to maturity
d) Depends on the industry sector

Example 19-11: FRM Exam 2000—Question 43/Credit Risk

19-11. The marginal default rates (conditional on no previous default) for a BB-rated firm during the first, second, and third years are 3, 4, and 5 percent, respectively. What is the cumulative probability of defaulting over the next three years?

a) 10.78 percent
b) 11.54 percent
c) 12.00 percent
d) 12.78 percent

Example 19-12: FRM Exam 2000—Question 34/Credit Risk

19-12. What is the difference between the marginal default probability and the cumulative default probability?

a) Marginal default probability is the probability that a borrower will default in any given year, whereas the cumulative default probability is over a specified multi-year period.
b) Marginal default probability is the probability that a borrower will default due to a particular credit event, whereas the cumulative default probability is for all possible credit events.
c) Marginal default probability is the minimum probability that a borrower will default, whereas the cumulative default probability is the maximum probability.
d) Both a and c are correct.

19.2.4 Transition Probabilities

As we have seen, the measurement of long-term default rates can be problematic with small sample sizes. The computation of these default rates can be simplified by assuming a Markov process for the ratings migration, described by a transition matrix. **Migration** is a discrete process that consists of credit ratings changing from one period to the next.

The **transition matrix** gives the probability of moving to one rating conditional on the rating at the beginning of the period. The usual assumption is that these moves follow a **Markov process**, or that migrations across states are independent from one period to the next.[4] This type of process exhibits *no carry-over effect*. More formally, a **Markov chain** describes a stochastic process in discrete time where the conditional distribution, given today's value, is constant over time. Only present values are relevant.

[4] There is some empirical evidence, however, that credit downgrades are not independent but instead display a momentum effect.

Table 19-5 gives an example of a simplified transition matrix for 4 states, A, B, C, D, where the last represents default. Consider a company in year 0 in the B category. The company could default:

- In year 1, with probability $D[t_1 \mid B(t_0)] = P(D_1 \mid B_0) = 3\%$

- In year 2, after going from B to A in the first year, then A to D in the second, or from B to B, then to D, or from B to C, then to D. The total probability is

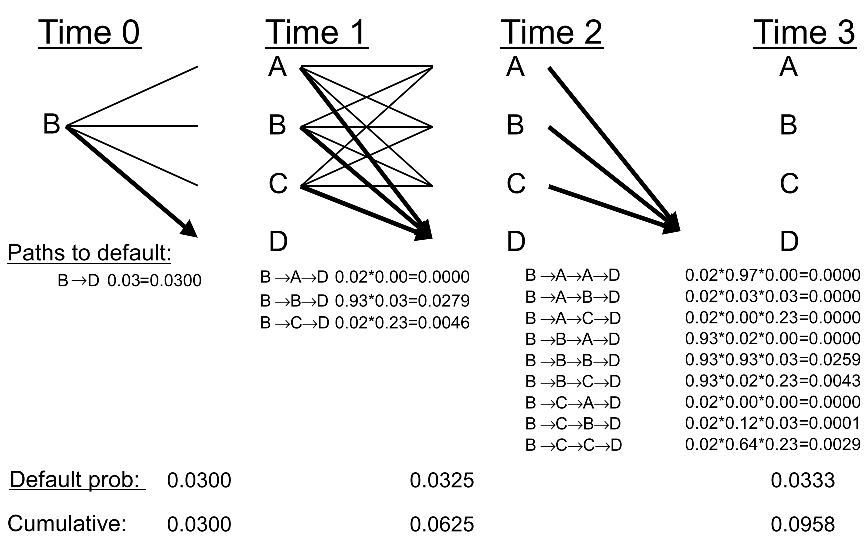
$$P(D_2 \mid A_1)P(A_1) + P(D_2 \mid B_1)P(B_1) + P(D_2 \mid C_1)P(C_1)$$
$$= 0.00 \times 0.02 + 0.03 \times 0.93 + 0.23 \times 0.02 = 3.25\%$$

Table 19-5 Credit Ratings Transition Probabilities

State	Ending				Total
Starting	A	B	C	D	Prob.
A	0.97	0.03	0.00	0.00	1.00
B	0.02	0.93	0.02	0.03	1.00
C	0.01	0.12	0.64	0.23	1.00
D	0	0	0	1.00	1.00

The cumulative probability of default over the two years is then 3% + 3.25% = 6.25%. Figure 19-4 illustrates the various paths to default in years 1, 2, and 3.

FIGURE 19-4 Paths to Default



The advantage of using this approach is that the resulting data are more robust and consistent. For instance, the 15-year cumulative default rate obtained this way will always be greater than the 14-year default rate. much greater precision.

Example 19-13: FRM Exam 2000—Question 50/Credit Risk

19-13. The transition matrix in credit risk measurement generally represents

a) Probabilities of migrating from one rating quality to another over the lifetime of the loan

b) Correlations among the transitions for the various rating quality assets within one year

c) Correlations of various market movements that impact rating quality for a 10-day holding period

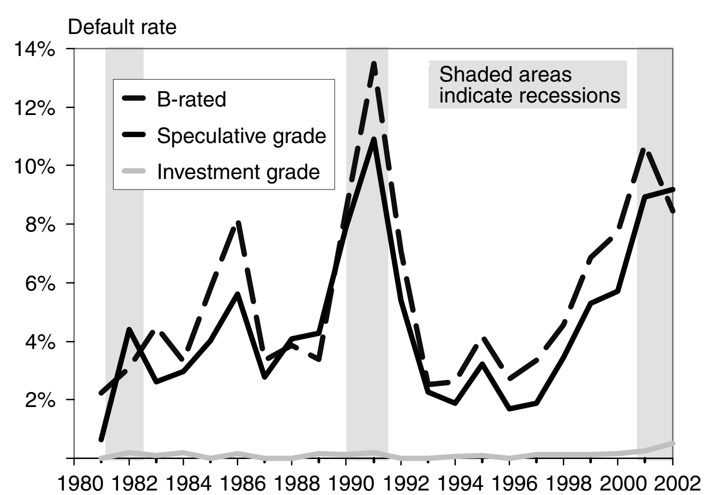
d) Probabilities of migrating from one rating quality to another within one year

19.2.5 Predicting Default Probabilities

Defaults are also correlated with economic activity. Moody's, for example, has compared the annual default rate to the level of industrial production since 1920. Moody's reports a marked increase in the default rate in the 1930s at the time of the great depression. Similarly, the slowdown in economic activity around the 1990 and 2001 recessions was associated with an increase in defaults.

These default rates, however, do not control for structural shifts in the credit quality. In recent years, many issuers came to the market with a lower initial credit rating than in the past. This should lead to more defaults even with a stable economic environment.

FIGURE 19-5 Time Variation in Defaults (from S&P)



To control for this effect, Figure 19-5 plots the default rate for B credits as well as for investment-grade and speculative credits over the years 1981 to 2002. As expected, the default rate of investment-grade bonds is very low. More interestingly, however, it displays minimal variation through time. We do observe, however, significant variation in the default rate of B credits, which peaks during the recessions that started in 1981, 1990, and 2001. Thus, economic activity significantly affects credit risk and the effect is most marked for speculative grade bonds.

19.3 Recovery Rates

Credit risk also depends on the **loss given default** (LGD). This can be measured as one minus the **recovery rate**, or fraction recovered after default.

19.3.1 The Bankruptcy Process

Normally, default is a state that affects all obligations of an issuer equally, especially when accompanied by a bankruptcy filing. In most countries, a formal bankruptcy process provides a centralized forum for resolving all the claims against the corporation. The bankruptcy process creates a **pecking order** for a company's creditors. This spells out the order in which creditors are paid, thereby creating differences in the recovery rate across creditors. Within each class, however, creditors should be treated equally.

In the United States, firms that are unable to make required payments can file for either **Chapter 7** bankruptcy, which leads to the liquidation of the firm's assets, or **Chapter 11** bankruptcy, which leads to a reorganization of the firm during which the firm continues to operate under court supervision.

Under Chapter 7, the proceeds from liquidation should be divided according to the **absolute priority rule**, which states that payments should be made first to claimants with the highest priority.

Table 19-6 describes the pecking order in bankruptcy proceedings. At the top of the list come **secured creditors**, who because of their property right are paid to the fullest extent of the value of the collateral. Then come **priority creditors**, which consist mainly of post-bankruptcy creditors. Finally, **general creditors** can be paid if funds remain after distribution to others.

TABLE 19-6: Pecking Order in U.S. Federal Bankruptcy Law

Seniority	Type of Creditor
Highest (paid first)	(1) Secured creditors (up to the extent of secured collateral) (2) Priority creditors: - Firms that lend money during bankruptcy period - Providers of goods and services during bankruptcy period (e.g., employees, laywers, vendors) - Taxes (3) General creditors: - Unsecured creditors before bankruptcy
Lowest (paid last)	- Shareholders

Similar rules apply under Chapter 11. In this situation, the firm must submit a **reorganization plan**, which specifies new financial claims to the firm's assets. The absolute priority rule, however, is often violated in Chapter 11 settlements. Junior debt holders and stockholders often receive some proceeds even though senior shareholders are not paid in full. This is allowed to facilitate timely resolution of the bankruptcy and to avoid future lawsuits. Even so, there remain sharp differences in the recovery across seniority.

19.3.2 Estimates of Recovery Rates

Credit rating agencies measure recovery rates using the value of the debt right after default. This is viewed as the market's best estimate of the future recovery and takes into account the value of the firm's assets, the estimated cost of the bankruptcy process, and various means of payment (e.g., using equity to pay bondholders), discounted into the present.

The recovery rate has been shown to depend on a number of factors.

- *The status or seniority of the debtor:* claims with lower seniority have lower recovery rates.
- *The state of the economy:* recovery rates tend to be lower when the economy is in a recession.

Ratings can also include the loss given default. The same borrower may have various classes of debt, which may have different credit ratings due to the different level of protection. If so, debt with lower seniority should carry a lower rating.

Tables 19-7 and 19-8 display recovery rates for corporate debt. Moody's, for instance, estimates the average recovery rate for senior unsecured debt at $f = 49\%$. S&P estimates this number at around $f = 47\%$, which is quite close. Generally, agencies conservatively estimate recovery rates to be in the range of 25 to 44 percent for senior unsecured bonds. Derivative instruments rank as senior unsecured creditors and would be expected to have the same recovery rates as senior unsecured debt. Bank loans are usually secured and therefore have higher recovery rates, typically assumed to be in the range of 50 to 60 percent. As expected, subordinated bonds and preferred stocks have the lowest recovery rates, typically assumed to be in the range of 15 to 28 percent.

TABLE 19-7: Moody's Recovery Rates for U.S. Corporate Debt

Seniority/Security	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Std.Dev.
Senior/Secured bank loans	15.00	60.00	75.00	69.91	88.00	98.00	23.47
Equipment trust bonds	8.00	26.25	70.63	59.96	85.00	103.00	31.08
Senior/Secured bonds	7.50	31.00	53.00	52.31	65.25	125.00	25.15
Senior/Unsecured bonds	0.50	30.75	48.00	48.84	67.00	122.60	25.01
Senior/Subordinated bonds	0.50	21.34	35.50	39.46	53.47	123.00	24.59
Subordinated bonds	1.00	19.62	30.00	33.17	42.94	99.13	20.78
Junior/Subordinated bonds	3.63	11.38	16.25	19.69	24.00	50.00	13.85
Preferred stocks	0.05	5.03	9.13	11.06	12.91	49.50	9.09
All	0.05	21.00	38.00	42.11	61.22	125.00	26.53

TABLE 19-8: S&P's Historical Recovery Rates for Corporate Debt

Seniority ranking	Number of observations	Average issue size ($ million)	Simple average Price	Standard deviation of Price	Weighted average Price
Senior secured	91	117.8	54.28	24.25	49.32
Senior unsecured	237	97.5	46.57	25.24	47.09
Subordinated	177	145.5	35.20	24.67	32.46
Junior subordinated	144	81.9	34.98	22.32	35.51
Total	649	110.0	41.98	25.23	40.23

Source: S&P, from 649 defaulted bond prices over 1981–1999.

There is, however, much variation around the average recovery rates, as Table 19-7 shows. The table reports not only the average value but also the standard deviation, minimum, maximum, and first and third quartile. Recovery rates vary widely. In addition, recovery rates are negatively related to default rates. During years with more bond defaults, prices after default are more depressed than usual. This correlation creates bigger losses, which extends the left tail of the credit loss distribution.

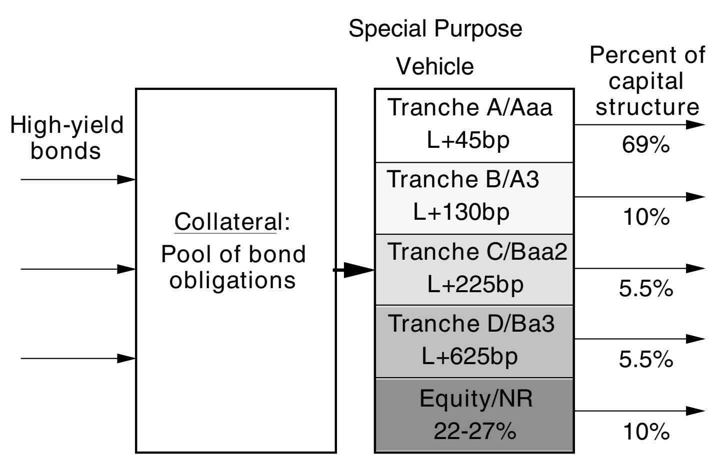
Another difficulty is that these recovery rates are mainly drawn from a sample of U.S. firms, which fall under the jurisdiction of U.S. bankruptcy laws. Differences across national jurisdictions will create additional differences among recovery rates. So, these numbers can only serve as a guide to non-U.S. recovery rates.

Example 19-14: FRM Exam 2000—Question 58/Credit Risk
19-14. When measuring credit risk, for the same counterparty
a) A loan obligation is generally rated higher than a bond obligation.
b) A bond obligation is generally rated higher than a loan obligation.
c) A bond obligation is generally rated the same as a loan obligation.
d) Loans are never rated so it's impossible to compare.

19.4 Application to Portfolio Rating

Much of financial engineering is about repackaging financial instruments to make them more palatable to investors, creating value in the process. In the 1980s, **collateralized mortgage obligations** (CMOs) brought mortgage-backed securities to the masses by repackaging their cash flows into **tranches** with different characteristics. The same magic is performed with **collateralized debt obligations** (CDOs), which are securities backed by a diversified pool of corporate bonds and loans. **Collateralized bond obligations** (CBOs) and **collateralized loan obligations** (CLOs) are backed by bonds and loans, respectively. Figure 19-6 illustrates a typical CDO structure.

FIGURE 19-6 Collateralized Debt Obligation Structure



The first step is to place a package of high-yield bonds in a **special-purpose vehicle** (SPV). The second step is to specify the **waterfall**, or priority of payments to the various tranches. Here, 69% of the capital structure is apportioned to tranche A, which has the highest credit rating of Aaa; it pays LIBOR + 45bp. Other tranches have lower priority and rating; intermediate tranches are typically called *mezzanine.* At the bottom comes the equity tranche, which is not rated. After payment to the other tranches and costs, the excess spread can be around 2.5 to 3%, which with a 10-to-1 leverage gives a yield of 25 to 30% to equity investors. In exchange, the equity is exposed to the first dollar loss in the portfolio. Thus, the rating enhancement for the senior classes is achieved through prioritizing the cash flows. Rating agencies have developed internal models to rate the senior tranches based on the probability of shortfalls due to defaults.

Whatever transformation is brought about, the resulting package must obey some basic laws of conservation. For the underlying and resulting securities, we must have the same cash flows at each point in time. As a result, this implies (1) The same total market value (2) The same risk profile, both for interest rate and default risk The weighted duration of the final package must equal that of the underlying securities. The expected default rate, averaged by market values, must be the same. So, if some tranches are less risky, others must bear more risk. Like CMOs, CDOs are often structured so that most of the tranches have less risk. Inevitably, the remaining **residual** tranche is more risky. This is sometimes called "toxic waste." If this residual is cheap enough, however, some investors should be willing to buy it.

CDO transactions are typically classified as balance sheet or arbitrage. The primary goal of **balance sheet CDOs** is to move loans off the balance sheet of commercial banks to lower regulatory capital requirements. In contrast, **arbitrage CDOs** are designed to capture the spread between the portfolio of underlying securities and that of highly rated, overlying, tranches. Such CDOs exploit differences in the funding costs of assets and liabilities. The spreads on high-yield debt have historically more than compensated investors for their credit risk, which reflects a liquidity effect, or risk premium. Because CDO senior tranches create more liquid assets with automatic diversification, investors require a lower risk premium for these. The arbitrage profit then goes into the equity tranche (but also into management and investment banking fees.)

The credit risk transfer can be achieved by cash flow or synthetic structures. The example in Figure 19-6 is typical of traditional, or *funded*, **cash-flow CDOs**. The physical assets are sold to a SPV and the underlying cash flows used to back payments to the issued notes. In contrast, the credit risk exposure of **synthetic CDOs** is achieved with credit derivatives, which will be covered in a later chapter.

Finally, CDOs differ in the management of the asset pool. In **static CDOs**, the asset pool is basically fixed. In contrast, with **managed CDOs**, a portfolio manager is allowed to trade actively the underlying assets. This allows him or her to unwind assets with decreasing credit quality or to reinvest redeemed issues.

Example 19-15: FRM Exam 2001—Question 12

19-15. A pool of high yield bonds is placed in a SPV and three tranches (including the equity tranche) of bonds are issued collateralized by the bonds to create a Collateralized Bond Obligation (CBO). Which of the following is true?

a) At fair value the value of the issued bonds should be less than the collateral.

b) At fair value the total default probability, weighted by size of issue, of the issued bonds should equal the default probability of the collateral pool.

c) The equity tranche of the CBO has the least risk of default. d) The yield on the low risk tranche must be greater than the yield on the collateral pool.

Example 19-16: FRM Exam 1998—Question 8/Credit Risk

19-16. In a typical collateralized bond obligation (CBO), a pool of high-yield bonds is posted as collateral and the cash flows from the collateral are structured as several classes of securities (the offered securities) with different credit ratings and a residual piece (the equity), which absorbs most of the default risk. When comparing the market value weighted average rating of the collateral and that of the offered securities, which of the following is *true*?

a) The market value weighted average rating of the collateral is about the same as the offered securities.

b) The market value weighted average rating of the collateral is higher than the offered securities.

c) The market value weighted average rating of the collateral is lower than the offered securities.

d) The market value weighted average rating of the collateral may be lower or higher than the offered securities.

19.5 Assessing Corporate and Sovereign Rating

19.5.1 Corporate Default

One issue is whether these ratings are the best forecasts of default probabilities based on public information. A substantial academic literature has examined this question and has generally concluded that ratings can be reasonably predicted from accounting information. provide important information about a firm's viability.

Analysts focus on the balance sheet **leverage** often defined in terms of the debt-to-equity ratio, and the **debt coverage**, defined in terms of the ratio of income over debt payment. All else equal, companies with higher leverage and lower debt coverage are more likely to default. By nature, however, accounting information is backward-looking.

The economic prospects of a company are even more important for assessing credit risk. These include growth potential, market competition, and exposure to financial risk factors. Because they are forward-looking, market-based variables such as bond credit spreads and equity prices contain better forecasts of default probabilities than ratings.

The data presented so far described default rates for U.S. industrial corporations. The next question is whether this historical experience applies to other countries. We would expect some difference in ratings transition because of a number of factors:

- *Differences in financial stability across countries:* countries have different financial market structures, such as the strength of the banking system, and different government policies. The mishandling of economic policy can turn, for instance, what should be a minor devaluation into a major problem leading to a recession.
- *Differences in legal systems:* the protection accorded to creditors can vary widely across countries, some of which have not yet established a bankruptcy process.
- *Differences in industrial structure:* there may be differences in default rates across countries simply due to different industrial structure. There is evidence that default rates vary across U.S. industries even with identical credit ratings.

19.5.2 Sovereign Default

Rating agencies have only recently started to rate sovereign bonds. In 1975, S&P only rated seven countries, all of which were investment grade. By 1990, the pool had expanded to thirty-one countries, of which only nine were from emerging markets.

In now, S&P rates approximately 90 countries. The history of default is even more sparse, making it difficult to generalize from a very small sample.

Assessing sovereign credit risk is significantly more difficult than for corporates. When a corporate borrower defaults, legal action can be taken by the creditors. For instance, an unsecured creditor can file an action against a debtor and have the defendant's assets seized under a "writ of attachment." This creates a **lien** on its assets, or a claim on the assets as security for the payment of the debt. In contrast, it is impossible to attach the domestic assets of a sovereign nation. This implies that recovery rates on sovereign debt are usually lower than on corporate debt. Thus, sovereign credit evaluation involves not only **economic risk** (the ability to repay debts when due), but also **political risk** (the willingness to pay).

Sovereign credit ratings also differ depending on whether the debt is **local currency debt** or **foreign currency debt**. Table 19-9 displays the factors entering local and foreign currency ratings.

TABLE 19-9: Credit Ratings Factors

Categories	Local Currency	Foreign Currency
Political risk	x	x
Price stability	x	x
Income and economic structure	x	x
Economic growth prospects	x	x
Fiscal flexibility	x	x
Public debt burden	x	x
Balance of payment flexibility		x
External debt and liquidity		x

Political risk factors (e.g., degree of political consensus, integration in global trade and financial system, and internal or external security risk) play an important part in sovereign credit risk. Factors affecting *local currency debt* include economic, fiscal, and especially monetary risks. High rates of inflation typically reflect economic mismanagement and are associated with political instability. Countries rated AAA, for instance, have inflation rates from 0 to at most 10%; BB-rated countries have inflation rates ranging from 25% to 100%.

Important factors affecting *foreign currency debt* include the international investment position of a country (that is, public and private external debt), the stock of foreign currency reserves, and patterns in the balance of payment. In particular, the ratio of external interest payments to exports is closely watched.

In the case of the Asian crisis, agencies seem to have overlooked other important aspects of creditworthiness, such as the currency and maturity structure of national debt. Too many Asian creditors had borrowed short-term in dollars to invest in the local currency, which created a severe liquidity problem. Admittedly, the credit valuation process can be hindered by the reluctance of foreign nations to provide timely information. In the case of Argentina, on the other hand, most observers had anticipated a default. This was due to a combination of high external debt, slow economic growth, unwillingness to make the necessary spending adjustments, and ultimately was a political decision.

Because local currency debt is backed by the taxation power of the government, local currency debt is considered to have less credit risk than foreign currency debt. Table 19-10 displays local and foreign currency debt ratings for a sample of countries. Ratings for foreign currency debt are the same, or one notch below, those of local currency debt. Similarly, sovereign debt is typically rated higher than corporate debt in the same country. Governments can repay foreign currency debt, for instance, by controlling capital flows or seizing foreign currency reserves.

TABLE 19-10: Standard & Poor's Sovereign Long-Term Credit Ratings Selected Countries, March 2003

Issuer	Local Currency	Foreign Currency
Argentina	SD	SD
Australia	AAA	AAA
Belgium	AA+	AA+
Brazil	BB	B+
Canada	AAA	AAA
China		BBB
France	AAA	AAA
Germany	AAA	AAA
Hong Kong	AA−	A+
Japan	AA−	AA−
Korea	A+	A−
Mexico	A−	BBB−
Netherlands	AAA	AAA
Russia	BB+	BB
South Africa	A−	BBB−
Spain	AA+	AA+
Switzerland	AAA	AAA
Taiwan	AA−	AA−
Thailand	A−	BBB−
Turkey	B−	B−
United Kingdom	AAA	AAA
United States	AAA	AAA

Note : Argentina is rated selective default (SD).

Overall, sovereign debt ratings are considered less reliable than corporate ratings. Indeed, corporate bond spreads are greater for sovereigns than corporate issuers. In 1999, for example, the average spread on dollar-denominated sovereign bonds rated BB was about 160bp higher than for identically-rated corporates. There are also greater differences in sovereign ratings across agencies than for corporates. The evaluation of sovereign credit risk seems to be a much more subjective process than for corporates.

Example 19-17: FRM Exam 1997—Question 27/Credit Risk

19-17. Which of the following credit events usually takes place first?

a) A bond is downgraded by a rating agency.

b) A bond's credit spread widens.

Example 19-18: FRM Exam 2001—Question 2

19-18. *(Requires knowledge of markets)* Which of the following is the best rated country according to the most important ratings agencies?

a) Argentina

b) Brazil

c) Mexico

d) Peru

Example 19-19: FRM Exam 1999—Question 121/Credit Risk

19-19. In assessing the sovereign credit, a number of criteria are considered. Which of the following is the more critical one?

a) Fiscal position of the government

b) Prospect for domestic output and demand

c) International asset position

d) Structure of the government's debt and debt service (external and internal)

Example 19-20: FRM Exam 1998—Question 36/Credit Risk

19-20. What is the most significant difference to consider when assessing the credit worthiness of a country rather than a company?

a) The country's willingness and its ability to pay must be analyzed.

b) Financial data on a country is often available only with long lags.

c) It is more costly to do due diligence on a country rather than on a company.

d) A country is often unwilling to disclose sensitive financial information.

19.6 Answers to Chapter Examples

Example 19-1: FRM Exam 1998—Question 5/Credit Risk

b) Calling back a bond occurs when the borrower wants to refinance its debt at a lower cost, which is not a credit event.

Example 19-2: FRM Exam 1999—Question 128/Credit Risk

d) Losses I and II are due to market risk. Loss III is a credit event, due to restructuring. Loss IV is a tax event deriving from governmental action.. So, III and IV qualify as event risks.

Example 19-3: FRM Exam 1997—Question 8/Credit Risk

d) Ba2 is the lowest rating among the list.

Example 19-4: FRM Exam 1998—Question 37/Credit Risk

a) The cutoff point for pretax interest coverage ratio in Table 19-4 is 3.7 for BBB credits, which is similar to the ratio of 3.75 for company X. More importantly, the LT debt/equity ratio of 35% for company X translates into a LT debt/capital ratio of 26% (obtained as 35%/(1 + 35%) = 26%). Because this is well below the cutoff point of 43% for BBB-credits in Table 19-4, the category must be investment grade.

Example 19-5: FRM Exam 1997—Question 28/Credit Risk

c) This default rate is 1.47% from Table 19-4. Similarly, the Moody's default rate for Ba credits is 1.42%.

Example 19-6: FRM Exam 1998—Question 29/Credit Risk

a) From Table 19-4, the ratio of B to BBB defaults for a 1-year horizon is $6.72/0.36 = 19$, which is slightly higher than the 16 ratio in the first part of the question. The numbers are different because of variances in sample periods. The ratio at 10-year horizon is $44.59/7.60 = 6$, which is close to 5. Intuitively, the default rate on B credits should increase at a lower rate than that on BBB credits. The cumulative default rate on B credits starts with a high value but cannot go above one.

Example 19-7: FRM Exam 1997—Question 2/Credit Risk

a) Using $(1 - d_M)^4 = (1 - 0.06\%)$, we find an average rate of $d_M = 0.015\%$. For the next quarter, however, the marginal default rate will be lower because d increases with maturity for high credit ratings.

Example 19-8: FRM Exam 2000—Question 37/Credit Risk

c) The probability of surviving is $(1 - d)^3 = 0.343$; hence the probability of default at any point during the next three years is 66%.

Example 19-9: FRM Exam 2000—Question 31/Credit Risk

d) The cumulative 6-year default rate is given by $C_6(R) = C_5(R) + k_6 = C_5(R) + S_5 \times d_6 = 0.15 + (1 - 0.15) \times 0.10 = 0.235$.

Example 19-10: FRM Exam 1997—Question 10/Credit Risk

a) The question could refer to the cumulative or marginal probabilities. Intuitively, the probability is low for AA credit for short maturities but increases more, relative to the starting value, than for lower credits. Using the cumulative probabilities for AA and B credits in Table 19-4, we have, for 1 year, a ratio of 0.01/6.72 = 0.001 and, for 10 years, a ratio of 2.10/44.59 = 0.05. This increases with maturity. Similarly, the marginal default probability increases with time for high credits and decreases for low credits.

Example 19-11: FRM Exam 2000—Question 43/Credit Risk

b) This is one minus the survival rate over 3 years: $S_3(R) = (1 - d_1)(1 - d_2)(1 - d_3) = (1 - 0.03)(1 - 0.04)(1 - 0.05) = 0.8856$. Hence, the cumulative default rate is 0.1154.

Example 19-12: FRM Exam 2000—Question 34/Credit Risk

a) The marginal default rate is the probability of defaulting over the next year, conditional on having survived to the beginning of the year.

Example 19-13: FRM Exam 2000—Question 50/Credit Risk

d) The transition matrix represents the conditional probability of moving from one rating to another over a fixed period, typically a year.

Example 19-14: FRM Exam 2000—Question 58/Credit Risk

a) The recovery rate on loans is typically higher than that on bonds. Hence the credit rating, if it involves both probability of default and recovery, should be higher for loans than for bonds.

Example 19-15: FRM Exam 2001-15

b) The market values and weighted probability of default should be equal for the collateral and various tranches. So, (a) is wrong. The equity tranche has the highest risk of default, so (c) is wrong. The yield on the low risk tranche must be the lowest, so (d) is wrong.

Example 19-16: FRM Exam 1998—Question 8/Credit Risk

c) The rating of the collateral must be between that of the offered securities and the residual. Say that the collateral is rated B, with 5% probability of default (PD); the offered securities represent 80% of the total market value. These are more highly rated than the collateral because the equity absorbs the default risk. If the offered securities are rated BB (with 1% PD), the equity must be such that $80\% \times 0.01 + 20\% \times x = 0.05$, which yields an PD of 21% for the equity, close to a CCC rating.

Example 19-17: FRM Exam 1997—Question 27/Credit Risk

b) The empirical evidence is that bond prices lead changes in credit ratings, because they are forward-looking instead of ratings.

Example 19-18: FRM Exam 2001—Question 2

c) Mexico is the most highly rated country of this group, according to the table of S&P ratings. Argentina is in Selective Default (SD) since 2001. As of early 2003, Mexico is rated BBB−, Peru is rated BB−, Brazil is rated B+.

Example 19-19: FRM Exam 1999—Question 121/Credit Risk

d) Empirically, the ratio of debt to exports seems to be the most important factor driving sovereign ratings (see the Handbook of Emerging Markets, pp. 10–11).

Example 19-20: FRM Exam 1998—Question 36/Credit Risk

a) Countries cannot be forced into bankruptcy. There is no enforcement mechanism for payment to creditors such as for private companies. Recent history has shown that a country can simply decide to renege on its debt. So, willingness to pay is a major factor.

Chapter 20

Measuring Default Risk from Market Prices

The previous chapter discussed how to quantify credit risk from categorization into credit risk ratings. Based on these external ratings, we can forecast credit losses from historical default rates and recovery rates.

Credit risk can also be assessed from market prices of securities whose values are affected by default. This includes corporate bonds, equities, and credit derivatives. In principle, these should provide more up-to-date and accurate measures of credit risk because financial markets have access to a large amount of information. This chapter shows how to infer default risk from market prices.

Section 20.1 will show how to use information about the market prices of credit-sensitive bonds to infer default risk. In this chapter, we will call defaultable debt interchangeably credit-sensitive, corporate, and risky debt. Here, *risky* refers to credit risk and not market risk. We show how to break down the yield on a corporate bond into a default probability, a recovery rate, and a risk-free yield.

Section 20.2 then turns to equity prices. The advantage of using equity prices is that they are much more widely available and of much better quality than corporate bond prices. We show how equity can be viewed as a call option on the value of the firm and how a default probability can be inferred from the value of this option. This approach also explains why credit positions are akin to short positions in options and are characterized by distributions that are skewed to the left. Chapter 22 will discuss credit derivatives, which can also be used to infer default risk.

20.1 Corporate Bond Prices

To assess the credit risk of a transaction with a counterparty, consider **credit-sensitive** bonds issued by the same counterparty. We assume that default is a state that affects all obligations equally.

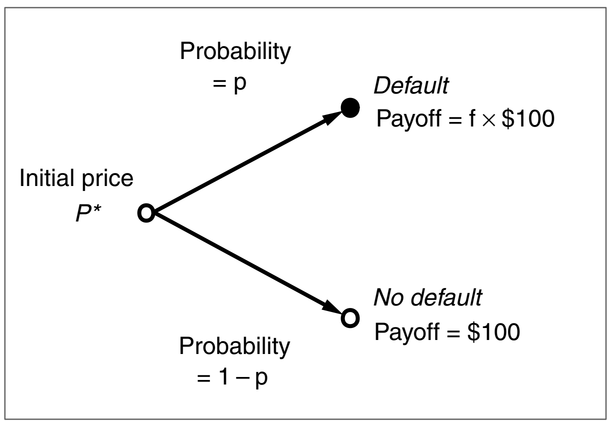
20.1.1 Spreads and Default Risk

Assume for simplicity that the bond makes only one payment of \$100 in one period. We can compute a market-determined yield y^* from the price P^* as

$$P^* = \frac{\$100}{(1 + y^*)} \tag{20.1}$$

This can be compared with the risk-free yield over the same period y. The payoffs on the bond can be described by a simplified default process, which is illustrated in Figure 20-1. At maturity, the bond can be in default or not. Its value is \$100 if there is no default and $f \times \$100$ if default occurs, where f is the fractional recovery. We define π as the default rate over the period. How can we value this bond?

FIGURE 20-1 A Simplified Bond Default Process



Using **risk-neutral pricing**, the current price must be the mathematical expectation of the values in the two states, discounting the payoffs at the risk-free rate. Hence,

$$P^* = \frac{\$100}{(1 + y^*)} = \left[\frac{\$100}{(1 + y)}\right] \times (1 - \pi) + \left[\frac{f \times \$100}{(1 + y)}\right] \times \pi \tag{20.2}$$

Note that the discounting uses the risk-free rate y because there is no risk premium with risk-neutral valuation. After rearranging terms,

$$(1 + y) = (1 + y^*)[1 - \pi(1 - f)] \tag{20.3}$$

which implies a default probability of

$$\pi = \frac{1}{(1 - f)}\left[1 - \frac{(1 + y)}{(1 + y^*)}\right] \tag{20.4}$$

Dropping second-order terms, this simplifies into

$$y^* \approx y + \pi(1 - f) \tag{20.5}$$

This equation shows that the credit spread $y^* - y$ measures credit risk, more specifically the probability of default, π, times the loss given default, $(1 - f)$.

Let us now consider multiple periods, which number T. We compound interest rates and default rates over each period. In other words, π is now the *average* annual default rate. Assuming one payment only, the present value is

$$P^* = \frac{\$100}{(1 + y^*)^T} = \left[\frac{\$100}{(1 + y)^T}\right] \times (1 - \pi)^T + \left[\frac{f \times \$100}{(1 + y)^T}\right] \times [1 - (1 - \pi)^T] \tag{20.6}$$

which can be written as

$$(1 + y)^T = (1 + y^*)^T\{(1 - \pi)^T + f[1 - (1 - \pi)^T]\} \tag{20.7}$$

Unfortunately, this does not simplify further.

When we have risky bonds of various maturities, this can be used to compute default probabilities for different horizons. If we have two periods, for example, we could use Equation (20.3) to find the probability of defaulting over the first period π_1, and Equation (20.7) to find the annualized, or average, probability of defaulting over the first two periods, or π_2. As we have seen in the previous chapter, the marginal probability of defaulting d_2 in the second period is given by solving

$$(1 - \pi_2)^2 = (1 - \pi_1)(1 - d_2) \tag{20.8}$$

This enables us to recover a term structure of forward default probabilities from a sequence of zero-coupon bonds. In practice, if we only have access to coupon-paying bonds, the computation becomes more complicated because we need to consider the payments in each period with and without default.

20.1.2 Risk Premium

It is worth emphasizing that this approach assumed risk-neutrality. As in the methodology for pricing options, we assumed both that the value of any asset grows at the risk-free rate and can be discounted at the same risk-free rate. Thus the probability measure π is a risk-neutral measure, which is not necessarily equal to the objective, physical, probability of default.

Defining this objective probability as π' and the discount rate as y', the current price can be also expressed in terms of the true expected value discounted at the risky rate y':

$$P^* = \frac{\$100}{(1+y^*)} = \left[\frac{\$100}{(1+y')}\right] \times (1-\pi') + \left[\frac{f \times \$100}{(1+y')}\right] \times \pi' \qquad (20.9)$$

Equation (20.4) allows us to recover a risk-neutral default probability only. More generally, if investors require some compensation for bearing credit risk, the credit spread will include a risk premium rp

$$y^* \approx y + \pi'(1-f) + \mathrm{rp} \qquad (20.10)$$

To be meaningful, this risk premium must be tied to some measure of bond riskiness as well as investor risk aversion. In addition, this premium may incorporate a **liquidity premium**, because the corporate issue may not be as easily traded as the corresponding Treasury issue and tax effects.[1]

> **Key concept:**
> The yield spread betwen a corporate bond and an otherwise identical bond with no credit risk reflects the expected actuarial loss, or annual default rate times the loss given default, plus a risk premium.

Example: Deriving default probabilities

We wish to compare a 10-year U.S. Treasury strip and a 10-year zero issued by International Business Machines (IBM), which is rated A by S&P and Moody's. The respective yields are 6% and 7%, using semiannual compounding. Assuming that the recovery rate is 45% of the face value, what does the credit spread imply for the probability of default?

Using Equation (20.1), we find that $\pi(1-f) = 1 - (1+y/200)^{20}/(1+y^*/200)^{20} = 0.0923$. Hence, $\pi = 9.23\%/(1-45\%) = 16.8\%$. Therefore, the cumulative (risk-neutral) probability of defaulting during next ten years is 16.8%. This number is rather high

[1] For a decomposition of the yield spread into risk premium effects, see Elton, E., Gruber M., Agrawal D., & Mann C. (2001). Explaining the rate spread on corporate bonds. *Journal of Finance*, 56(1), 247–277. The authors find a large risk premium, which they relate to common risk factors from the stock market. Part of the risk premium is also due to tax effects. Because Treasury coupon payments are no taxable at the state level, for example New York state, investors are willing to accept a lower yield on Treasury bonds, which artificially increases the corporate yield spread.

compared with the historical record for this risk class. Table 19-3, Moody's reports a historical 10-year default rate for A-credits of 3.4% only.

If these historical default rates are used as the future probability of default, the implication is that a large part of the credit spread reflects a risk premium. For instance, assume that 80 basis points out of the 100 basis points credit spread reflects a risk premium. We change the 7% yield to 6.2% and find a probability of default of 3.5%. This is more in line with the actual default experience of such issuers.

Example 20-1: FRM Exam 1998—Question 3/Credit Risk

20-1. When comparing the zero curve (semiannual compounding) of riskless bonds and risky bonds, one can estimate the implied default probabilities by examining the spread between the two. Assuming the 1-year riskless zero rate is 5%, the risky zero rate is 5.5%, and the recovery rate is zero, what is the implied 1-year default probability?

a) 0.24%
b) 0.48%
c) 0.97%
d) 1.92%

Example 20-2: FRM Exam 1997—Question 23/Credit Risk

20-2. Assume the 3-month U.S. Treasury yield is 5.5% and the Eurodollar deposit rate is 6% (both on simple interest basis). What is the approximate probability of the Eurodollar deposit defaulting over its life (assuming a zero recovery rate)?

a) 0.01%
b) 0.1%
c) 0.5%
d) 1.0%

Example 20-3: FRM Exam 1997—Question 24/Credit Risk

20-3. Assume the 1-year U.S. Treasury yield is 5.5% (on simple interest basis) and a default probability of 1% for 1-year Commercial Paper. What should the yield of 1-year Commercial Paper be (on simple interest basis) assuming 50% recovery rate?

a) 6.0%
b) 6.5%
c) 7.0%
d) 7.5%

20.1.3 The Cross-Section of Yield Spreads

We now turn to actual market data. Figure 20-2 illustrates a set of par yield curves for various credits as of December 1998. For reference, the spreads are listed in Table 20-1. The curves are sorted by credit rating, from AAA to B, using S&P's ratings. cumulative default rates reported in the previous chapter. They increase with maturity and with lower credit quality.

FIGURE 20-2 Yield Curves For Different Credits

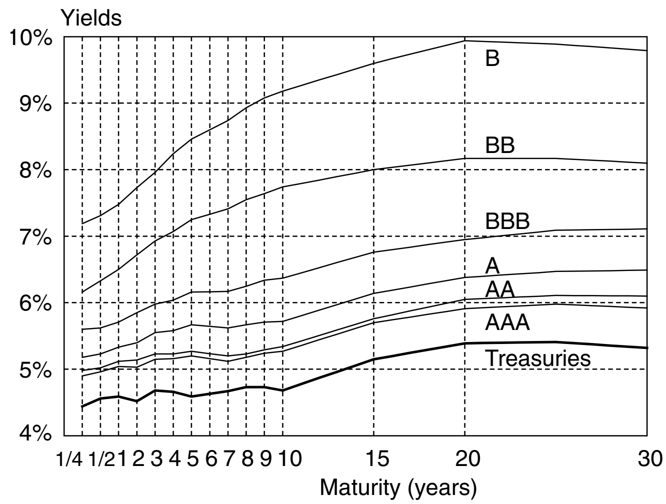


TABLE 20-1 Credit Spreads

Maturity (Years)	Credit Rating AAA	AA	A	BBB	BB	B
3M	46	54	74	116	172	275
6M	40	46	67	106	177	275
1	45	53	74	112	191	289
2	51	62	88	133	220	321
3	47	55	87	130	225	328
4	50	57	92	138	241	358
5	61	68	108	157	266	387
6	53	61	102	154	270	397
7	45	53	95	150	274	407
8	45	50	94	152	282	420
9	51	56	98	161	291	435
10	59	66	104	169	306	450
15	55	61	99	161	285	445
20	52	66	99	156	278	455
30	60	78	117	179	278	447

The lowest curve is the Treasury curve, which represents risk-free bonds. Spreads for AAA-credits are low, starting at 46bp at short maturities and increasing to 60bp at longer maturities Spreads for B-credits are much wider; they also increase faster, from 275 to 450. Finally, note how close together the AAA to AA spreads are, in spite of the fact that default probabilities approximately double from AAA to AA. The transition from Treasuries to AAA credits most likely reflects other factors, such as liquidity and tax effects, rather than credit risk.

The previous sections have shown that we could use information in corporate bond yields to make inferences about credit risk. Indeed, bond prices represent the best assessment of traders, or real "bets," on credit risk. As such, we would expect bond prices to be the best predictors of credit risk and to outperform credit ratings. To the extent that agencies use public information to form their credit rating, this information should be subsumed into market prices. Bond prices are also revised more frequently than credit ratings. As a result, movements in corporate bond prices tend to *lead* changes in credit ratings.

Example 20-4: FRM Exam 1998—Question 11/Credit Risk

20-4. What can be said about the spread (S1) between AAA and A credits, and the spread between BBB and B credits (S2) in general?

a) S1 is equal to S2.
b) S1 $\geq$ S2.
c) S1 $\leq$ S2.
d) S1 may be less or more than S2.

Example 20-5: FRM Exam 1999—Question 136/Credit Risk

20-5. Suppose XYZ Corp. has two bonds paying semiannually according to the table:

Remaining maturity	Coupon (sa 30/360)	T-bill Price	rate (bank discount)
6 months	8.0%	99	5.5%
1 year	9.0%	100	6.0%

The recovery rate for each in the event of default is 50%. For simplicity, assume that each bond will default only at the end of a coupon period.

The market-implied risk-neutral probability of default for XYZ Corp. is

a) Greater in the first six-month period than the second
b) Equal for both coupon periods
c) Greater in the second six-month period than the first
d) Cannot be determined from the information provided

20.1.4 The Time-Series of Yield Spreads

Credit spreads reflect potential losses caused by default risk, and perhaps a risk premium. Some of this default risk is specific to the issuer and requires a detailed analysis of its prospective financial condition. Part of this risk, however, can be attributed to common credit factors. These common factors are particularly important as they cannot be diversified away in a large portfolio of credit-sensitive bonds.

First among these factors are general economic conditions. Economic growth is negatively correlated with credit spreads. When the economy slows down, more companies are likely to have cash-flow problems and to default on their bonds. Indeed, Figure 13-6 shows that spreads widen during recessions. An inverting term structure, which indicates monetary tightening and lower expectations of growth, is similarly associated with a widening credit spread.

Volatility is also a factor. In a more volatile environment, investors may require larger risk premiums, thus increasing credit spreads. When this happens, liquidity may also dry up. Investors may then require a greater credit spread in order to hold increasingly illiquid securities.

In addition, volatility can have another effect. Corporate bond indices include many callable bonds, unlike Treasury indices. As a result, credit spreads also reflect this option component. The buyer of a callable bond requires a higher yield in exchange for granting the call option. Because the value of this option increases with volatility, greater volatility should also increase the credit spread.

20.2 Equity Prices

The credit spread approach, unfortunately, is only useful when there is good bond market data. The problem is that this is rarely the case, for a number of reasons.

1. Many countries do not have a well-developed corporate bond market. As Table 7-1 has shown, the United States has by far the largest corporate bond market. This means that other countries have much fewer outstanding bonds and a much less active market.
2. The counterparty may not have an outstanding publicly traded bond or if so, the bond may contain other features such as a call.
3. The bond may not trade actively and instead reported prices may simply be **matrix prices**, that is, interpolated from other, current yields.

20.2.1 The Merton Model

An alternative is to turn to default risk models based on stock prices, because equity prices are available for a larger number of companies and are more actively traded than corporate bonds. The Merton (1974) model views equity as akin to a call option on the assets of the firm, with an exercise price given by the face value of debt.

To simplify to the extreme, consider a firm with total value V that has one bond due in one period with face value K. If the value of the firm exceeds the promised payment, the bond is repaid in full and stockholders receive the remainder. However, if V is less than K, the firm is in default and the bondholders receive V only. The value of equity goes to zero. Throughout, we assume that there are no transaction costs. Hence, the value of the stock at expiration is

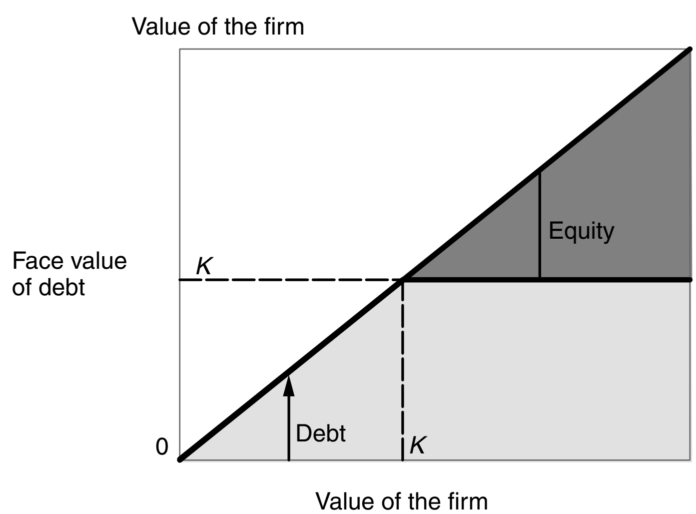
$$S_T = \text{Max}(V_T - K, 0) \tag{20.11}$$

Because the bond and equity add up to the firm value, the value of the bond must be

$$B_T = V_T - S_T = V_T - \text{Max}(V_T - K, 0) = \text{Min}(V_T, K) \tag{20.12}$$

The current stock price, therefore, embodies a forecast of default probability, in the same way that an option embodies a forecast of being exercised. Figures 20-3 and 20-4 describe how the value of the firm can be split up into the bond and stock values.

FIGURE 20-3 Equity as an Option on the Value of the Firm

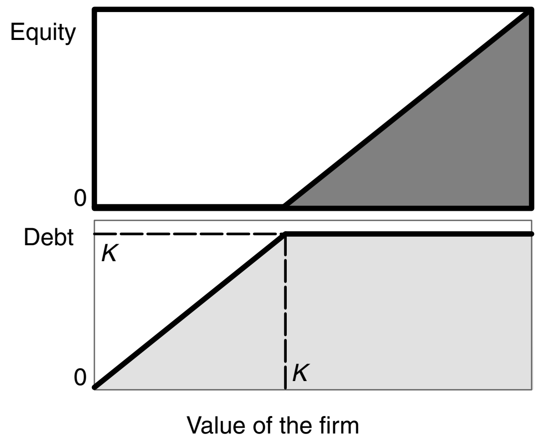


Note that the bond value can also be described as

$$B_T = K - \text{Max}(K - V_T, 0) \quad (20.13)$$

In other words, a long position in a risky bond is equivalent to a long position in a risk-free bond plus a short put option, which is really a credit derivative.

FIGURE 20-4 Components of the Value of the Firm



Key concept:
Equity can be viewed as a call option on the firm value with strike price equal to the face value of debt. Corporate debt can be viewed as risk-free debt minus a put option on the firm value.

This approach is particularly illuminating because it demonstrates that corporate debt has a payoff akin to a short position in an option, explaining the left skewness that is so characteristic of credit losses. In contrast, equity is equivalent to a long position in an option due to its **limited liability feature**, that is, investors can lose no more than their equity investment.

20.2.2 Pricing Equity and Debt

To illustrate, we proceed along the lines of the usual Black-Scholes (BS) framework, assuming the firm value follows the geometric Brownian motion process

$$dV = \mu V dt + \sigma V dz \quad (20.14)$$

If we assume that markets are frictionless and that there are no bankruptcy costs, the value of the firms is simply the sum of the firm's equity and debt: $V = B + S$.

To price a claim on the value of the firm, we need to solve a partial differential equation with appropriate boundary conditions. The corporate bond price is obtained as

$$B = F(V, t), \quad F(V, T) = \text{Min}[V, B_F] \tag{20.15}$$

where $B_F = K$ is the face value of the bond to be repaid at expiration, or strike price.

Similarly, the equity value is

$$S = f(V, t), \quad f(V, T) = \text{Max}[V - B_F, 0] \tag{20.16}$$

Stock Valuation

The value of the stock is given by the BS formula

$$S = \text{Call} = VN(d_1) - Ke^{-r\tau}N(d_2) \tag{20.17}$$

where $N(d)$ is the cumulative distribution function for the standard normal distribution, and

$$d_1 = \frac{\ln(V/Ke^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, \quad d_2 = d_1 - \sigma\sqrt{\tau}$$

where $\tau = T - t$ is the time to expiration. If we define $x = Ke^{-r\tau}/V$ as the debt/value ratio, this shows that the option value depends solely on x and $\sigma\sqrt{\tau}$.

Note that, in practice, this application is different from the BS model where we plug in the value of V, of its volatility $\sigma = \sigma_V$, and solve for the value of the call. Here, we observe the market value of the firm S and the equity volatility σ_S and must infer the values of V and its volatility so that Equation (20.17) is satisfied. This can only be done iteratively. Defining Δ as the hedge ratio, we have

$$dS = \frac{\partial S}{\partial V}dV = \Delta dV \tag{20.18}$$

Defining σ_S as the volatility of (dS/S), we have $(\sigma_S S) = \Delta(\sigma_V V)$ and

$$\sigma_V = \Delta\sigma_S(S/V) \tag{20.19}$$

Bond Valuation

Next, the value of the bond is given by $B = V - S$, or

$$B = Ke^{-r\tau}N(d_2) + V[1 - N(d_1)] \tag{20.20}$$

$$B/Ke^{-r\tau} = [N(d_2) + (V/Ke^{-r\tau})N(-d_1)] \tag{20.21}$$

Risk-Neutral Dynamics of Default

In the Black-Scholes model, $N(d_2)$ is also the probability of exercising the call, or that the bond will not default. Conversely, $1 - N(d_2) = N(-d_2)$ is the risk-neutral probability of default.

Pricing Credit Risk

At maturity, the credit loss is the value of the risk-free bond minus the corporate bond, $\text{CL} = B_F - B_T$. At initiation, the expected credit loss (ECL) is

$$\begin{aligned} B_F e^{-r\tau} - B &= Ke^{-r\tau} - \{Ke^{-r\tau}N(d_2) + V[1 - N(d_1)]\} \\ &= Ke^{-r\tau}[1 - N(d_2)] - V[1 - N(d_1)] \\ &= Ke^{-r\tau}N(-d_2) - VN(-d_1) \end{aligned}$$

This decomposition is quite informative. Multiplying by the future value factor $e^{r\tau}$, it shows that the ECL at maturity is

$$\text{ECL}_T = N(-d_2)[K - Ve^{r\tau}N(-d_1)/N(-d_2)] = p \times [\text{Exposure} \times \text{LGD}] \tag{20.22}$$

This involves two terms. The first is the probability of default, $N(-d_2)$. The second is the loss when there is default. This is obtained as the face value of the bond K minus the recovery value of the loan when in default, $Ve^{r\tau}N(-d_1)/N(-d_2)$, which is also the expected value of the firm in the state of default. Note that the recovery rate is endogenous here, as it depends on the value of the firm, time, and debt ratio.

Credit Option Valuation This approach can also be used to value the put option component of the credit-sensitive bond. This option pays $K - B_T$ in case of default. A portfolio with the bond plus the put is equivalent to a risk-free bond $Ke^{-r\tau} = B + \text{Put}$. Hence, using Equation (20.20), the credit put should be worth

$$\text{Put} = Ke^{-r\tau} - \{Ke^{-r\tau}N(d_2) + V[1 - N(d_1)]\} = -V[N(-d_1)] + Ke^{-r\tau}[N(-d_2)] \tag{20.23}$$

This will be used later in the chapter on credit derivatives.

Example 20-6: FRM Exam 2001—Question 14

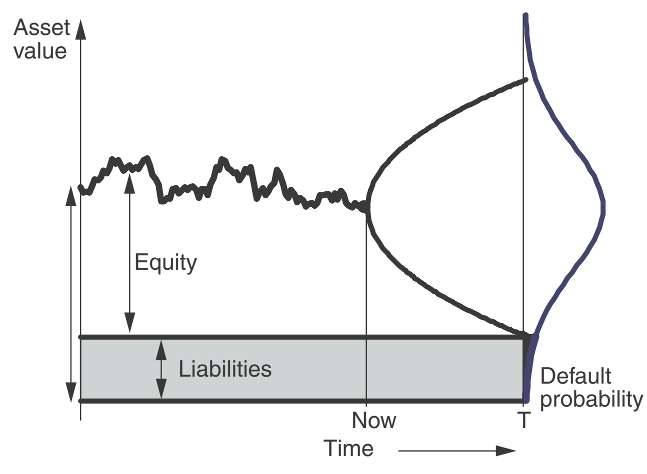
20-6. To what sort of option on the counterparty's assets can the current exposure of a credit-risky position better be compared?

a) A short call
b) A short put
c) A short knock-in call
d) A binary option

20.2.3 Applying the Merton Model

These valuation formulas can be used to recover, given the current value of equity and of nominal liabilities, the value of the firm and its probability of default. Figure 20-5 illustrates the evolution of the value of the firm. The firm defaults if this value falls below the liabilities at the horizon. We measure this risk-neutral probability by $N(-d_2)$.

FIGURE 20-5 Default in the Merton Model



In practice, default is much more complex than pictured here. We would have to collect information about all the nominal, fixed liabilities of the company, as well as their maturities. Default can also happen at any intermediate point. also more complex than this one-period framework. So, instead of default on the target date, we could measure default probability as a function of the distance relative to a moving floor that represents liabilities. This was essentially the approach undertaken by

KMV Corporation, which sells **estimated default frequencies** (EDF) for firms all over the world.[2]

The Merton approach has many advantages. First, it relies on equity prices rather than bond prices. There are many more firms with an actively traded stock price than with bonds. Second, correlations between equity prices can generate correlations between defaults, which would be otherwise difficult to measure. Perhaps the most important advantage of this model is that it generates movements in EDFs that seem to *lead* changes in credit ratings.

FIGURE 20-6 KMV's EDF and Credit Rating

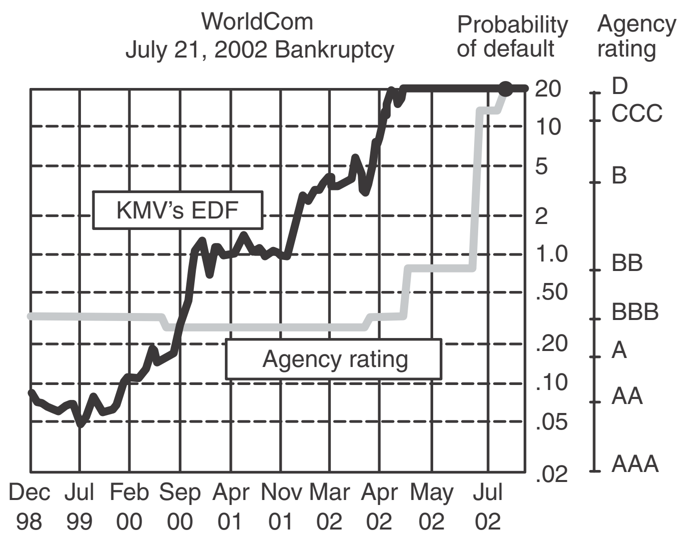


Figure 20-6 displays movements in EDFs and credit rating for Worldcom, using the same vertical scale. Worldcom went bankrupt on July 21, 2002. With $104 billion in assets, this was America's largest bankruptcy ever. The agency rating was BBB until April 2002. It gives no warning of the impending default. In contrast, starting one year before the default, the EDF starts to move up. In April, it reached 20%, presaging bankruptcy.

These models have disadvantages as well. The first limitation of the model is that it cannot be used to price sovereign credit risk, as countries obviously do not have a stock price. This is a problem for credit derivatives, where a large share of the market consists of sovereign risks.

[2]KMV is now part of Moody's.

A more fundamental drawback is that it relies on a static model of the firm's capital and risk structure. The debt level is assumed to be constant over the horizon. Also, the model needs to be expanded to a more realistic setting where debt matures at various points in time, which is not an obvious extension.

Another problem is that management could undertake new projects that increase not only the value of equity but also its volatility, thereby increasing the credit spread. This runs counter to the fundamental intuition of the Merton model, which is that, all else equal, a higher stock price reflects a lower probability of default and hence should be associated with a smaller credit spread.

Finally, this class of models also fails to explain the magnitude of credit spreads we observe on credit-sensitive bonds. Recent work attempts to add other sources of risk such as interest rate risk, but it still falls short of explaining these spreads. Thus these models are most useful in tracking *changes* in EDFs over time.

20.2.4 Example

It is instructive to work through a simplified example. Consider a firm with assets worth $V = \$100$, with volatility of $\sigma_V = 20\%$. In practice, one would have to start from the observed stock price and volatility and iterate to find σ_V.

The horizon is $\tau = 1$ year. The risk-free rate is $r = 10\%$ using continuous compounding. We assume a leverage $x = 0.9$, which implies a face value of $K = \$99.46$ and a risk-free current value of $Ke^{-r\tau} = \$90$.

Working through the Merton analysis, one finds that the current stock price should be $S = \$13.59$. Hence the current bond price is

$$B = V - S = \$100 - \$13.59 = \$86.41$$

which implies a yield of $\ln(K/B)/\tau = \ln(99.46/86.41) = 14.07\%$ or yield spread of 4.07%. The current value of the credit put is then

$$P = Ke^{-r\tau} - B = \$90 - \$86.41 = \$3.59$$

The analysis also generates values for $N(d_2) = 0.6653$ and $N(d_1) = 0.7347$. Thus the *risk-neutral* probability of default is $\text{EDF} = N(-d_2) = 1 - N(d_2) = 33.47\%$. Note that this could differ from the *actual*, or *objective* probability of default since the stock could very well grow at a rate which is greater than the risk-free rate of 10%.

Finally, let us decompose the expected loss at expiration from Equation (20.22), which gives

$$N(-d_2)[K - Ve^{r\tau}N(-d_1)/N(-d_2)]$$
$$= 0.3347 \times [\$99.46 - \$110.56 \times 0.2653/0.3347]$$
$$= 0.3347 \times [\$11.85] = \$3.96$$

This combines the probability of default with the expected loss upon default, which is \$11.85. This future expected credit loss of \$3.96 must also be the future value of the credit put, or $\$3.59e^{r\tau} = \$3.96$.

Note that the model needs very high leverage, here $x = 90\%$, to generate a sizeable credit spread, here 4.07%. This implies a debt-to-equity ratio of 0.9/0.1 = 900%, which is unrealistically high.

With lower leverage, say $x = 0.7$, the credit spread shrinks rapidly, to 0.36%. At $x =$ 50% or below, the predicted spread goes to zero. As this leverage would be considered normal, the model fails to reproduce the size of observed credit spreads. Perhaps it is most useful for tracking time variation in estimated default frequencies.

Example 20-7: FRM Exam 1998—Question 22/Credit Risk

20-7. Which of the following is used to estimate the probability of default in the KMV Model?

a) Vector analysis
b) Total return analysis
c) Equity price volatility
d) None of the above

Example 20-8: FRM Exam 1999—Question 155/Credit Risk

20-8. Having equity in a firm's capital structure adds to the creditworthiness of the firm. Which of the following statements support(s) this argument?

I. Equity does not require payments that could lead to default.
II. Equity capital does not mature, so it represents a permanent capital base.
III. Equity provides a cushion for debt holders in case of bankruptcy.
IV. The cost of equity is lower than the cost of debt.

a) I, II, and III
b) All of the above
c) I, II, and IV
d) III only

20.3 Answers to Chapter Examples

Example 20-1: FRM Exam 1998—Question 3/Credit Risk

b) Using Equation (20.3), we have

$$(1 - \pi) = \frac{(1 + y/200)^2}{(1 + y^*/200)^2}$$

which gives

$$\pi = 1 - \frac{(1 + 5/200)^2}{(1 + 5.5/200)^2} = 0.49\%$$

Example 20-2: FRM Exam 1997—Question 23/Credit Risk

b) Using Equation (20.3), the annual probability of default is $\pi = 1 - \frac{(1+0.055)}{(1+0.06)} = 0.47\%$, which gives 0.1% quarterly.

Example 20-3: FRM Exam 1997—Question 24/Credit Risk

a) We add 50% of 1% to the risk-free rate, which gives 6.0%.

Example 20-4: FRM Exam 1998—Question 11/Credit Risk

c) Credit spreads widen considerably for lower rated credits.

Example 20-5: FRM Exam 1999—Question 136/Credit Risk

a) First, we compute the current yield on the 6-month bond, which is selling at a discount. We solve for y^* such that $99 = 104/(1 + y^*/200)$ and find $y^* = 10.10\%$. Thus the yield spread for the first bond is $10.1 - 5.5 = 4.6\%$. The second bond is at par, so the yield is $y^* = 9\%$. The spread for the second bond is $9 - 6 = 3\%$. The default rate for the first period must be greater. The recovery rate is the same for the two periods, so does not matter for this problem.

Example 20-6: FRM Exam 2001—Question 14

b) The lender is short a put option, since exposure only exists if the value of assets falls below the amount lent.

Example 20-7: FRM Exam 1998—Question 22/Credit Risk

c) The KMV model is based on the value of the equity and liabilities, the risk-free rate, and equity price volatility.

Example 20-8: FRM Exam 1999—Question 155/Credit Risk

a) The cost of equity is generally higher than that of debt because it is riskier. Otherwise, all of the other arguments (a), (b), (c) are true. Equity will not cause default. It does not mature and provides a cushion for debtholders, as stockholders should lose money before debtholders.

Chapter 21

Credit Exposure

Credit exposure is the amount at risk when default occurs. It is also called **exposure at default** (EAD). When banking simply consisted of making loans, exposure was essentially the face value of the loan or other obligation. This value could be taken as being roughly constant.

Since the development of the swap markets, however, the measurement of credit exposure has become much more sophisticated. This is because swaps, like most derivatives, have an up-front value that is much smaller than the notional amount. Indeed, the initial value of a swap is typically zero, which means that at the outset, there is no credit risk because there is nothing to lose.

As the swap contract matures, however, it can turn into a positive or negative value. The asymmetry of bankruptcy treatment is such that a credit loss can only occur if the asset owed by the defaulted counterparty has positive value. Thus, the credit exposure is the value of the asset if positive, like an option.

This chapter turns to the quantitative measurement of credit exposure. Section 21.1 describes the general features of credit exposure for various types of financial instruments, including loans or bonds, guarantees, credit commitments, repos, and derivatives. Section 21.2 shows how to compute the distribution of credit exposure and gives detailed examples of exposures of interest rate and currency swaps. Section 21.3 discusses exposure modifiers, or techniques that have been developed to reduce credit exposure further. It shows how credit risk can be controlled by marking to market, margins, position limits, recouponing, and netting agreements. For completeness, Section 21.4 includes credit risk modifiers such as credit triggers and time puts, which also control default risk instead of exposure only.

21.1 Credit Exposure by Instrument

The credit exposure is the positive part of the value of the asset. In particular, the **current exposure** is equal to the value of the asset at the current time V_t if positive:

$$\text{Exposure}_{\text{t}} = \text{Max}(V_t, 0). \tag{21.1}$$

The **potential exposure** represents the exposure on some future date, or sets of dates. Based on this definition, we can characterize the exposure of a variety of financial instruments. The measurement of current and potential exposure also motivates regulatory capital charges for credit risk, which are explained in Chapter 31.

Loans or Bonds

Loans or **bonds** are balance-sheet assets whose current and potential exposure is the notional, which is the amount loaned or invested. In fact, this should be the market value of the asset given current interest rates, but, as we will show, this is not very far from the notional on a percentage basis. The exposure is also the notional for **receivables, trade credits** as the potential loss is the amount due.

Guarantees

These are off-balance-sheet contracts whereby the bank has underwritten, or agrees to assume, the obligations of a third party. The exposure is the notional amount, because this will be fully drawn when default occurs. By nature, guarantees are **irrevocable**, that is, unconditional and binding whatever happens.

An example of a **guarantee** is a contract whereby Bank A makes a loan to Client C only if guaranteed by Bank B. Should C default, B is exposed to the full amount of the loan. Another example is an **acceptance**, whereby a bank agrees to pay the face value of the bill at maturity. Alternatively, **standby facilities**, or **financial letters of credit**, provide a guarantee to a third party to make a payment should the obligor default.

Commitments

These are off-balance-sheet contracts whereby the bank commits to a future transaction that may result in creating a credit exposure at a *future* date. For instance, a bank may provide a **note issuance facility** whereby it promises a minimum price for notes regularly issued by a borrower. If the notes cannot be placed at the market at the minimum price, the bank commits to buy them at a fixed price. Such commitments have less risk than a guarantee because it is not certain that the bank will have to provide backup support.

It is also useful to distinguish between **irrevocable commitments**, which are unconditional and binding on the bank, and **revocable commitments**, where the bank has the option to revoke the contract should the counterparty's credit quality deteriorate. This option substantially decreases the credit exposure.

Swaps or Forwards

These are off-balance-sheet items that can be viewed as irrevocable commitments to purchase or sell some asset on prearranged terms. The current and potential exposure will vary from zero to a large amount depending on the movement in the driving risk factors. Similar arrangements are **sale-repurchase agreements** (repos), whereby a bank sells an asset to another in exchange for a promise to buy it back later.

Long Options

Options are also off-balance-sheet items that may create credit exposure. The current and potential exposure also depends on movements in the driving risk factors. Here, there is no possibility of negative values V_t because options always have positive value, or zero at worst.

Short Options

Unlike long options, the current and potential exposure is zero because the bank writing the option can only incur a negative cash flow, assuming the option premium has been fully paid.

Exposure also depends on the features of any embedded option. With an American option, for instance, the holder of an in-the-money swap may want to exercise early if the credit rating of its counterparty starts to deteriorate. This decreases the exposure relative to an equivalent European option.

Example 21-1: FRM Exam 1999—Question –-130/Credit Risk

21-1. By selling a call option on the S&P 500 futures contract, which is cash settled, an organization is subject to

a) Market risk, but not credit risk
b) Credit risk, but not market risk
c) Both market risk and credit risk
d) Neither market risk nor credit risk

Example 21-2: FRM Exam 1999—Question 151/Credit Risk

21-2. Trader A purchased an at-the-money 1-year OTC put option on the DAX index for a cost of EUR 10,000. What does trader A consider his maximum potential credit exposure to the counterparty over the term of the trade?

a) 0

b) Less than EUR 8,000

c) Between EUR 8,000 and EUR 12,000

d) Greater than EUR 12,000

Example 21-3: FRM Exam 2001—Question 84

21-3. If a counterparty defaults before maturity, which of the following situations will cause a credit loss?

a) You are short EUR in a 1-year EUR/USD forward FX contract and the EUR has appreciated.

b) You are short EUR in a 1-year EUR/USD forward FX contract and the EUR has depreciated.

c) You sold a 1-year OTC EUR call option and the EUR has appreciated.

d) You sold a 1-year OTC EUR call option and the EUR has depreciated.

Example 21-4: FRM Exam 2000—Question 35/Credit Risk

21-4. Contracts such as interest-rate swaps that are private arrangements between two parties entail credit risks. Consider a financial institution that has entered into offsetting interest-rate swap contracts with two manufacturing companies, General Equipment and Universal Tools. In which one of the following situations is the financial institution exposed to credit risk from the swap position? The most likely possibility is

a) A default by General Equipment when the value of the swap to the financial institution is positive

b) A default by Universal Tools when the value of the swap to the financial institution is negative

c) That the interest rates will move so that the value of the swap to Universal Tools becomes negative

d) That the interest rates will move so that the value of the swap to General Equipment becomes positive

21.2 Distribution of Credit Exposure

The credit exposure consists of the **current exposure**, which is readily observable, and the **potential exposure**, or future exposure, which is random. Define x as the potential value of the asset on the target date. We describe this variable by its probability density function $f(x)$. This is where market risk mingles with credit risk.

21.2.1 Expected and Worst Exposure

The **expected credit exposure** (ECE) is the expected value of the asset replacement value x, if positive, on a target date:

$$\text{Expected Credit Exposure} = \int_{-\infty}^{+\infty} \text{Max}(x, 0) f(x) dx \tag{21.2}$$

The **worst credit exposure** (WCE) is the largest (worst) credit exposure at some level of confidence. defined as **Credit at Risk** (CAR). It is implicitly defined as the value such that it is not exceeded at the given confidence level p:

$$1 - p = \int_{\text{WCE}}^{\infty} f(x) dx \tag{21.3}$$

To model the potential credit exposure, we need to (i) model the distribution of risk factors, and (ii) evaluate the instrument given these risk factors. This process is identical to a market value-at-risk (VAR) computation except that the aggregation takes place first at the counterparty level and second at the portfolio level.

To simplify to the extreme, suppose that the payoff x is normally distributed with mean zero and volatility σ. The expected credit exposure is then

$$\text{ECE} = \frac{1}{2} E(x \mid x > 0) = \frac{1}{2} \sigma \sqrt{\frac{2}{\pi}} = \frac{\sigma}{\sqrt{2\pi}} \tag{21.4}$$

Note that we divided by 2 because there is a 50 percent probability that the value will be positive. The worst credit exposure at the 95 percent level is given by

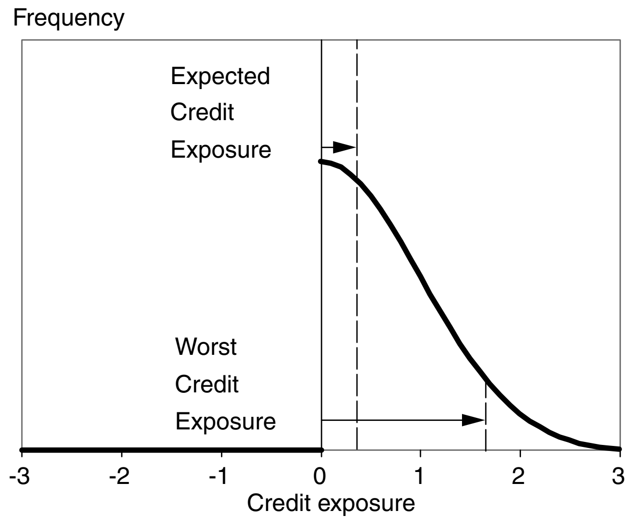
$$\text{WCE} = 1.645\sigma \tag{21.5}$$

Figure 21-1 illustrates the measurement of ECE and WCE for a normal distribution. Note that negative values of x are not considered.

21.2.2 Time Profile

The distribution can be summarized by the expected and worst credit exposures at each point in time. To summarize even further, we can express the average credit exposure by taking a simple arithmetic average over the life of the instrument.

FIGURE 21-1 Expected and Worst Credit Exposures–Normal Distribution



The **average expected credit exposure** (AECE) is the average of the expected credit exposure over time, from now to maturity T:

$$\text{AECE} = (1/T)\int_{t=0}^{T} \text{ECE}_t \, dt \tag{21.6}$$

The **average worst credit exposure** (AWCE) is defined similarly:

$$\text{AWCE} = (1/T)\int_{t=0}^{T} \text{WCE}_t \, dt \tag{21.7}$$

21.2.3 Exposure Profile for Interest-Rate Swaps

We now consider the computation of the exposure profile for an interest-rate swap. In general, we need to define (1) The number of market factor variables (2) The function and parameters for the joint stochastic processes (3) The pricing model for the swap

We start with a one-factor stochastic process for the interest rate, defining the movement in the rate r_t at time t as

$$dr_t = \kappa(\theta - r_t)dt + \sigma {r_t}^{\gamma} dz_t \tag{21.8}$$

as seen in Chapter 4. The first term imposes **mean reversion**. When the current value of r_t is higher than the long-run value, the term between parentheses is negative, which creates a downward trend. More generally, the mean term could reflect the path implied in forward interest rates.

The second term defines the innovation, which can be given a normal distribution. An important issue is whether the volatility of the innovation should be constant or proportional to some power γ of the current value of the interest rate r_t. If the horizon is short, this issue is not so important because the current rate will be close to the initial rate anyway.

When $\gamma = 0$, changes in yields are normally distributed, which is the Vasicek model (1977). As seen in Chapter 13, a typical volatility of *absolute* changes in yields is 1% per annum. A potential problem with this is that the volatility is the same whether the yields starts at 20% or 1%. As a result, the yield could turn negative, depending on the initial starting point and the strength of the mean reversion.

Another class of models is the lognormal model, which takes $\gamma = 1$. The model can then be rewritten in terms of $dr_t/r_t = d\ln(r_t)$. This specification ensures that the volatility shrinks as r gets close to zero, avoiding negative values. A typical volatility of *relative* changes in yields is 15% per annum, which is also the 1% for changes in the level of rates divided by an initial rate of 6.7%.

For illustration purposes, we choose the normal process $\gamma = 0$ with mean reversion $\kappa = 0.02$ and volatility $\sigma = 0.25\%$ per month, which are realistic parameters based on recent U.S. data. The initial and long-run values of r are 6%. Typical simulation values are shown in Figure 21-2. Note how rates can deviate from their initial value but are pulled back to the long-term value of 6%. We need, however, the whole distribution of values at each point in time.

FIGURE 21-2 Simulation Paths for the Interest Rate

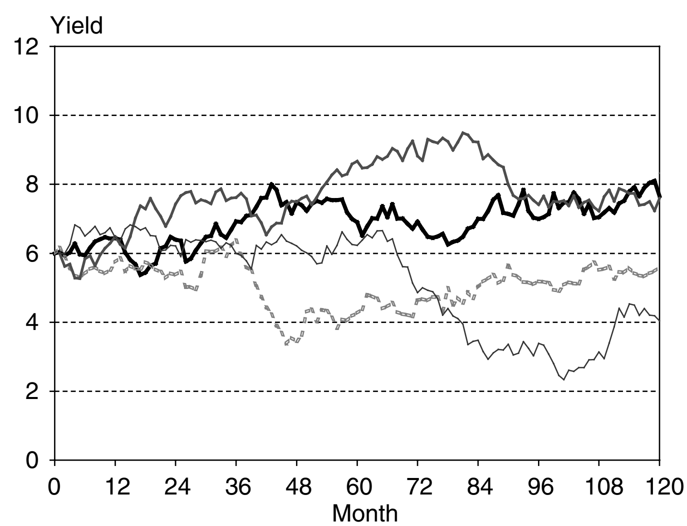
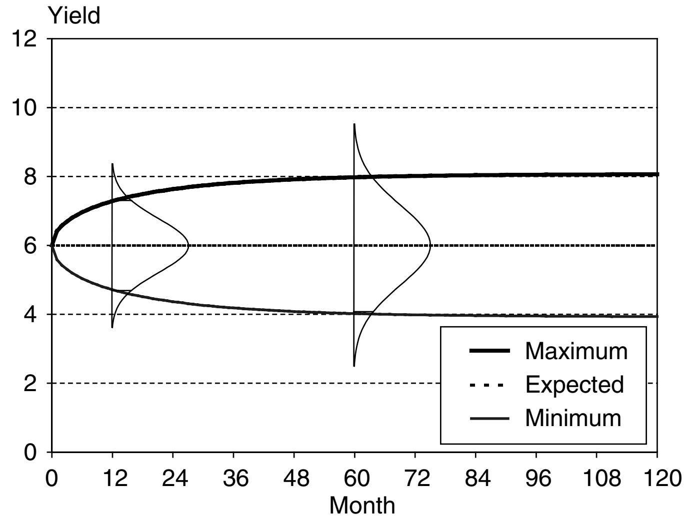


FIGURE 21-3 Distribution Profile for the Interest Rate



This model is convenient because it leads to closed-form solutions. The distribution of future values for r is summarized in Figure 21-3 by its mean and two-tailed 90 percent confidence band (called maximum and minimum values). The graph shows that the mean is 6%, which is also the long-run value. The confidence bands initially widen due to the increasing horizon, then converge to a fixed value due to the mean reversion effect.

The next step is to value the swap. At each point in time, the current market value of the swap is the difference between the value of a fixed-coupon bond and a floating-rate note

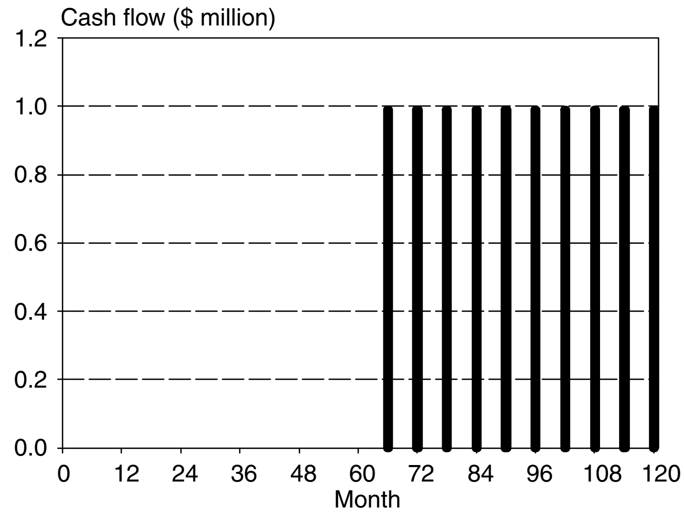
$$V_t = B(\$100, t, T, c, r_t) - B(\$100, \mathrm{FRN}) \tag{21.9}$$

Here, c is the annualized coupon rate, and T is the maturity date. The risk to the swap comes from the fact that the fixed leg has a coupon c that could differ from prevailing market rates. The principals are not exchanged.

Figure 21-4 illustrates the changes in cash flows that could arise from a drop in rates from 6% to 4% after 5 years. The receive-fixed party would then be owed every six months, for a semiannual pay swap, $\$100 \times (6 - 4)\% \times 0.5 = \$1$ million until the maturity of the swap. With 10 payments remaining, this adds up to a positive credit exposure of \$10 million. Discounting over the life of the remaining payments gives \$8.1 million as of the valuation date.

In what follows, we assume that the swap receives fixed payments that are paid at a continuous rate instead of semiannually, which simplifies the example. Otherwise, there would be discontinuities in cash-flow patterns and we would have to consider

FIGURE 21-4 Net Cash Flows When Rates Fall to 4% after 5 Years



the risk of the floating leg as well. We also use continuous compounding. Defining N as the number of remaining years, the coupon bond value is

$$B(\$100, N, c, r) = \$100\frac{c}{r}[1 - e^{-rN}] + \$100e^{-rN} \tag{21.10}$$

as we have seen in the Appendix to Chapter 1. The first term is the present value of the fixed-coupon cash flows discounted at the current rate r. The second term is the repayment of principal. For the special case where the coupon rate is equal to the current market rate, $c = r$, and the market value is indeed \$100 for this par bond. If $c > r$, the market value must be above par.

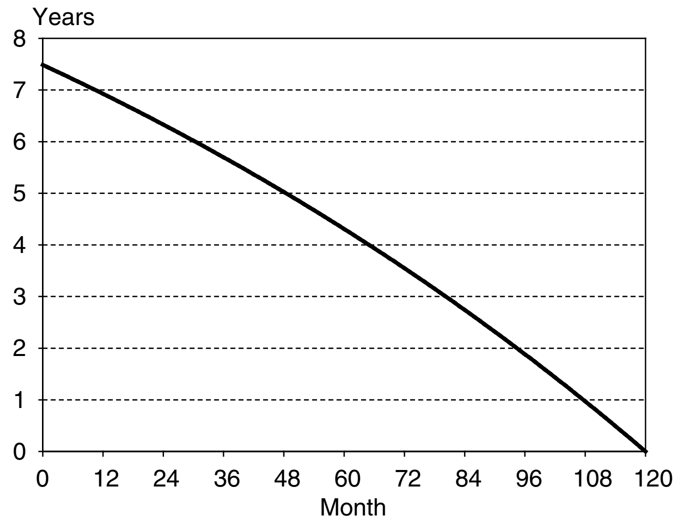
The floating-rate note can be priced in the same way, but with a coupon rate that is always equal to the current rate. Hence, its value is always at par.

To understand the exposure profile of the coupon bond, we need to consider two opposing effects. (1) The **diffusion effect:** As time goes by, the uncertainty in the interest rate increases. (2) The **amortization effect:** As maturity draws near, the bond's duration decreases to zero.

This second effect is described in Figure 21-5, which shows the bond's duration converging to zero. This explains why the bond's market value converges to the face value upon maturity whatever happens to the current interest rate.

Because the bond is a strictly monotonous function of the current yield, we can compute the 90 percent confidence bands by valuing the bond using the extreme interest rates range at each point in time. We use Equation (21.10) at each point in time in Figure 21-3. This exposure profile is shown in Figure 21-6.

FIGURE 21-5 Duration Profile for a 10-Year Bond



Initially, the market value of the bond is $100. After two or three years, the range of values is the greatest, from $87 to $115. Thereafter, the range converges to the face value of $100. But overall, the fluctuations as a *proportion* of the face value are relatively small. When considering other approximations in the measurement of credit risk, such as the imprecision in default probability and recovery rate, assuming a constant exposure for the bond is not a bad approximation.

FIGURE 21-6 Exposure Profile for a 10-Year Bond

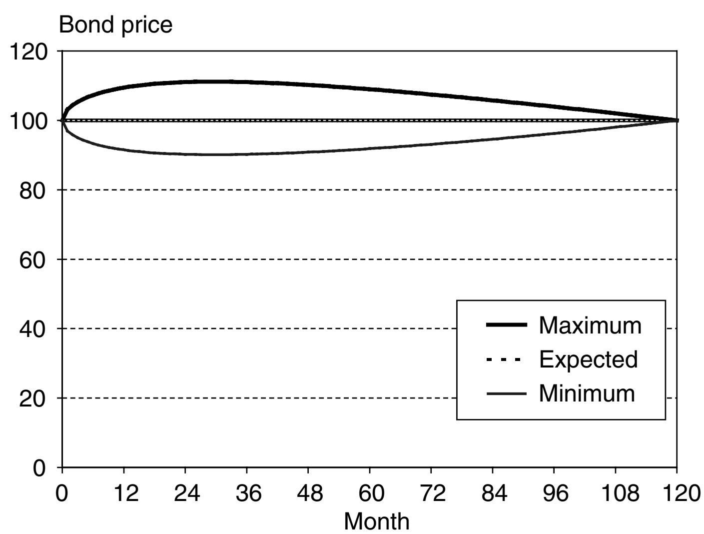
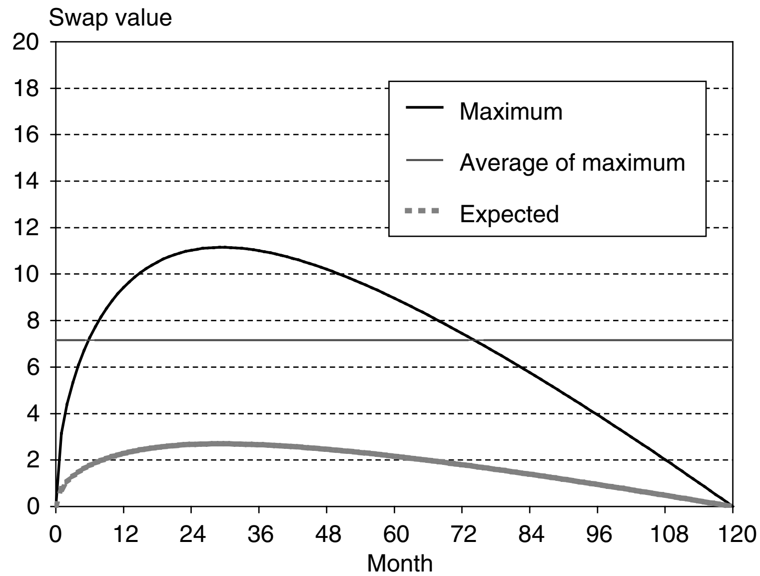


FIGURE 21-7 Exposure Profile for a 10-Year Interest-Rate Swap



This is not the case, however, for the interest-rate swap. Its value can be found from subtracting \$100 (the value of the floating-rate note) from that of the coupon bond. Initially, its value is zero. Thereafter, it can take on positive or negative values. Credit exposure is the positive value only. Figure 21-7 presents the profile of the expected exposure and of the maximum (worst) exposure at the one-sided 95 percent level. It also shows the average maximum exposure over the whole life of the swap.

Intuitively, the value of the swap is derived from the difference between the fixed and floating cash flows. Consider a swap with two remaining payments and a notional amount of \$100. Its value is

$$V_t = \$100\left[\frac{c}{(1+r)} + \frac{c}{(1+r)^2} + \frac{1}{(1+r)^2}\right] - \$100\left[\frac{r}{(1+r)} + \frac{r}{(1+r)^2} + \frac{1}{(1+r)^2}\right] \tag{21.11}$$

$$= \$100\left[\frac{(c-r)}{(1+r)} + \frac{(c-r)}{(1+r)^2}\right]$$

Note how the principal payments cancel out and we are left with the discounted *net* difference between the fixed coupon and the prevailing rate $(c - r)$.

This information can be used to assess the expected exposure and worst exposure on a target date. The peak exposure occurs around the second year into the swap, or about a fourth of the swap's life. At that point, the expected exposure is about

3 to 4 percent of the notional, which is much less than that of the bond. The worst exposure peaks at about 10 to 15 percent of notional. In practice, these values depend on the particular stochastic process used, but the exposure profiles will be qualitatively similar.

To assess the potential variation in swap values, we can make some approximations based on duration. Consider first the very short-term exposure, for which mean reversion and changes in durations are not important. The volatility of changes in rates then simply increases with the square root of time. Given a 0.25% per month volatility and 7.5-year initial duration, we can approximate the volatility of the swap value over the next year as

$$\sigma(V) = \$100 \times 7.5 \times [0.25\% \sqrt{12}] = \$6.5 \text{ million}$$

Multiplying by 1.645, we get \$10.7 million, which is close to the \$9.4 million actual 95% worst exposure in a year reported in Figure 21-7. a fixed duration and

The trade-off between declining duration and increasing risk can be formalized with a slightly more realistic example. Assume that the bond's (modified) duration is proportional to the remaining life, or $D = k(T - t)$ at any date t. The volatility from 0 to time t can be written as $\sigma(r_t - r_0) = \sigma \sqrt{t}$. Hence, the swap volatility is

$$\sigma(V) = [k(T - t)]\sigma \sqrt{t} \tag{21.12}$$

To see where it reaches a maximum, we differentiate with respect to t, and get

$$\frac{d\sigma(V)}{dt} = [k(-1)]\sigma \sqrt{t} + [k(T - t)]\sigma \frac{1}{2\sqrt{t}}$$

Setting this to zero, we have

$$\sqrt{t} = (T - t)\frac{1}{2\sqrt{t}}$$

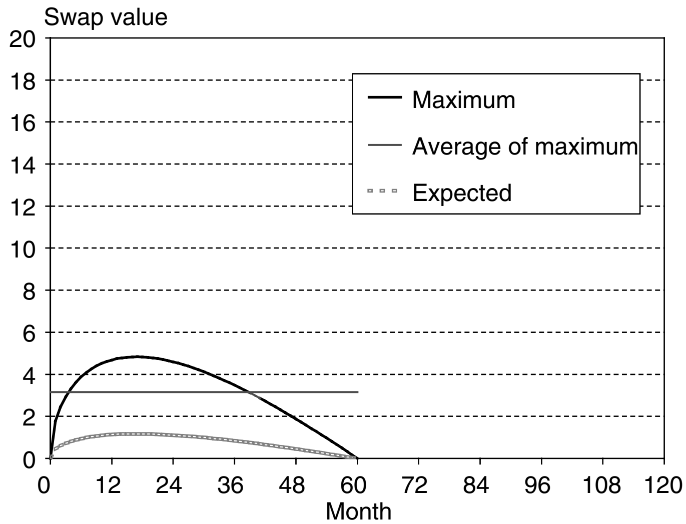
or

$$2t = (T - t)$$

or

$$t_{\text{MAX}} = (1/3)T \tag{21.13}$$

FIGURE 21-8 Exposure Profile for a 5-Year Interest-Rate Swap



The maximum exposure occurs at one-third of the life of the swap. This occurs later than the one-fourth reported previously because we assumed no mean reversion.

At that point, the worst credit exposure will be

$$1.645\ \sigma(V_{\text{MAX}}) = 1.645k(2/3)T\sigma\sqrt{T/3} = 1.645k(2/3)\sigma\sqrt{1/3}\ T^{3/2} \tag{21.14}$$

which shows that the WCE increases as $T^{3/2}$, which is faster than the maturity.

Figure 21-8 shows the exposure profile of a 5-year swap. Here again, the peak exposure occurs at a third of the swap's life. As expected, the magnitude is lower, with The peak expected exposure is only about 1 percent of notional.

Finally, Figure 21-9 displays the exposure profile when the initial interest rate is at 5% with a coupon of 6%. As a result, the swap is already in-the-money, with a mark-to-market value of $7.9 million. If we assume a long-run rate of 6%, the total exposure profile starts from a positive value, reaches a maximum after about two years, then converges to zero.

Example 21-5: FRM Exam 1999—Question 111/Credit Risk

21-5. What is the primary difference between the default implications of loans versus those of interest-rate swaps?

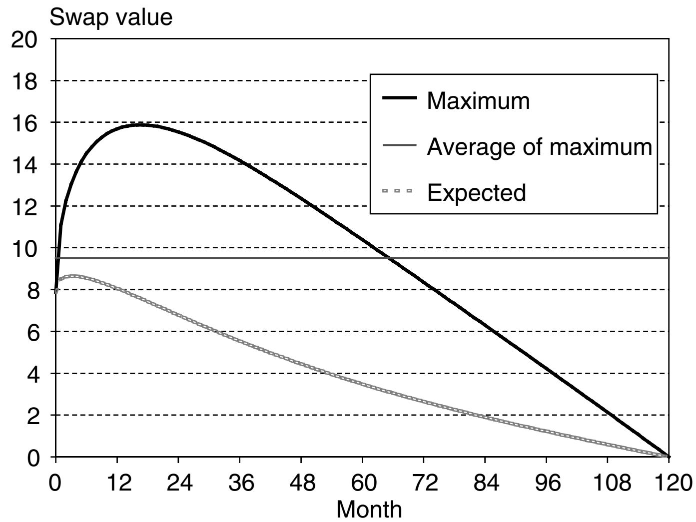
a) The principal in a swap is not at risk.

b) The cash flows in the loans are determined by the level of rates, not the difference in rates.

c) Default on a loan requires only that the firm be in financial distress, a swap also requires that the remaining value be positive to the dealer.

d) All of the above.

FIGURE 21-9 Exposure Profile for a 10-Year In-the-Money Swap



Example 21-6: FRM Exam 1999—Question 133/Credit Risk

21-6. Which criteria would result in the best measure of loan equivalent exposure for risk management and capital allocation purposes?

a) Current mark-to-market value of a contract

b) Current mark-to-market value of a contract plus an add-on factor for future potential exposure

c) A factor of 3 percent multiplied by the notional amount multiplied by the number of years, or fraction thereof, until maturity, i.e. $3\% \times NT$, where N is notional, and T is time to maturity in years d) Sum of the net notional amount of all transactions with the same counterparty

Example 21-7: FRM Exam 1999—Question 118/Credit Risk

21-7. Assume that swap rates are identical for all swap tenors. A swap dealer entered into a plain vanilla swap one year ago as the receive-fixed party, when the price of the swap was 7%. Today, this swap dealer will face credit risk exposure from this swap only if the value of the swap for the dealer is

a) Negative, which will occur if new swaps are being priced at 6%

b) Negative, which will occur if new swaps are being priced at 8%

c) Positive, which will occur if new swaps are being priced at 6%

d) Positive, which will occur if new swaps are being priced at 8%

Example 21-8: FRM Exam 1999—Question 148/Credit Risk

21-8. Assume that the DV01 of an interest-rate swap is proportional to its time to maturity (which at the initiation of the swap is equal to T). Also, assume that the interest-rate curve moves are parallel, stochastic with constant volatility, normally distributed, and independent. At what time will the maximum potential exposure be reached?

a) T/4
b) T/3
c) T/2
d) 3T/4

Example 21-9: FRM Exam 2000—Question 29/Credit Risk

21-9. Determine at what point in the future a derivatives portfolio will reach its maximum potential exposure. All the derivatives are on one underlying, which is assumed to move in a stochastic fashion (variance in the underlying's value increases linearly with time passage). The derivatives portfolio sensitivity to the underlying is expected to drop off as $(T - t)^2$ (square of the time left to maturity), where T is the time from today the last contract in the portfolio rolls off, and t is the time from today.

a) T/5
b) T/3
c) T/2
d) None of the above

Example 21-10: FRM Exam 1999—Question 149/Credit Risk

21-10. *(Complex)* Assume that the DV01 of an interest-rate swap is equal to 4,000 times its time left to maturity in years. At initiation, the swap tenor is three years and the swap is at par. Assume that the interest-rate curve moves are parallel, stochastic with constant volatility, and normally distributed and independent with 1 day standard deviation of 5 bp. Assume 250 business days per year. The swap's maximum potential exposure at the 99% confidence level is approximately

a) 700,000
b) 1,000,000
c) 1,500,000
d) 2,000,000

21.2.4 Exposure Profile for Currency Swaps

Exposure profiles are substantially different for other swaps. Consider, for instance, a currency swap where the notionals are \$100 million against £50 million, set at an initial exchange rate of $S(\$/£) = 2$.

The market value of a currency swap that receives foreign currency is

$$V_t = S_t(\$/£)B^*(£50, t, T, c^*, r^*) - B(\$100, t, T, c, r) \tag{21.15}$$

Following usual conventions, asterisks refer to foreign currency values.

In general, this swap is exposed to domestic as well as foreign interest-rate risk. When we just have two remaining coupons, the value of the swap evolves according to

$$S£50\left[\frac{c^*}{(1+r^*)} + \frac{c^*}{(1+r^*)^2} + \frac{1}{(1+r^*)^2}\right] - \$100\left[\frac{c}{(1+r)} + \frac{c}{(1+r)^2} + \frac{1}{(1+r)^2}\right] \tag{21.16}$$

Note that, relative to Equation (21.11), the principals do not cancel each other since they are paid in different currencies.

In what follows, we will assume for simplicity that there is no interest-rate risk, or that the value of the swap is dominated by currency risk. Further, we assume that the coupons are the same in the two currencies, otherwise there would be an asymmetrical accumulation of payments. As before, we have to choose a stochastic process for the spot rate. Say this is a lognormal process with constant variance and no trend:

$$dS_t = \sigma S_t dz_t \tag{21.17}$$

We choose $\sigma = 12\%$ annually, which is realistic as seen in Chapter 13. This process ensures that the rate never turns negative.

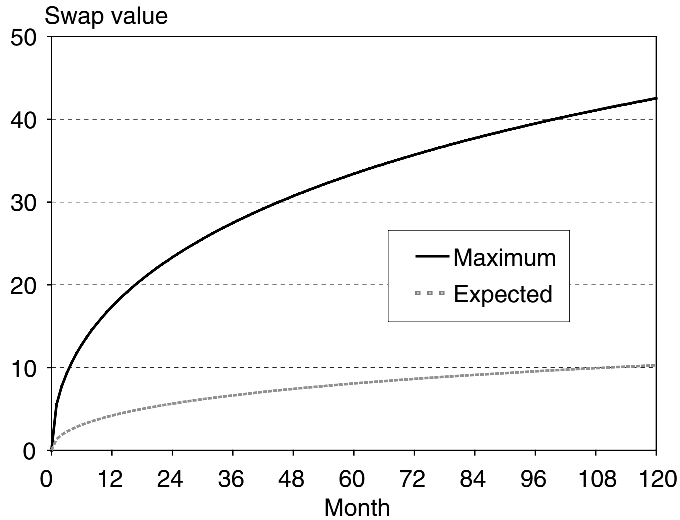
Figure 21-10 presents the exposure profile of a 10-year currency swap. Here, there is no amortization effect, and exposure increases continuously over time. The peak exposure occurs at the end of the life of the swap. At that point, the expected exposure is about 10 percent of the notional, which is much higher than for the interest-rate swap. The worst exposure is commensurately high, at about 45 percent of notional.

Although these values depend on the particular stochastic process and parameters used, this example demonstrates that credit exposures for currency swaps is far greater than for interest-rate swaps, even with identical maturities.

21.2.5 Exposure Profile for Different Coupons

So far, we have assumed a flat term structure and equal coupon payments in different currencies, which create a symmetric situation for the exposure for the long and

FIGURE 21-10 Exposure Profile for a 10-Year Currency Swap



short party. In reality, these conditions will not hold, creating asymmetric exposure patterns.

Consider, for instance, the interest-rate swap in Equation (21.11). If the term structure slopes upward, the coupon rate is greater than the floating rate, $c > r$, and the fixed receiver receives a net payment in the near term. The value of the swap can be analyzed projecting floating payments at the forward rate

$$V_t = \frac{(c - s_1)}{(1 + s_1)} + \frac{(c - f_{12})}{(1 + s_2)^2}$$

where s_1, s_2 are the 1- and 2-year spot rates, and f_{12} is the 1- to 2-year forward rate.

Example:

Consider a \$100 million interest-rate swap with two remaining payments. We have $s_1 = 5\%$, $s_2 = 6.03\%$ and hence using $(1 + s_2)^2 = (1 + s_1)(1 + f_{12})$, we have $f_{12} = 7.07\%$. The coupon yield of $c = 6\%$ is such that the swap has zero initial value. The table below shows that the present value of the first payment (to the party who receives fixed) is positive and equal to \$0.9524. The second payment then must be negative, and is equal to $-\$0.9524$. The two payments exactly offset each other because the swap has zero value.

Time	Expected Spot	Expected Payment	Discounted
1	5%	6.00 − 5.00 = +1.00	+0.9524
2	7.07%	6.00 − 7.07 = −1.07	−0.9524
Total			0.0000

This pattern of payments, however, creates more credit exposure to the fixed payer because it involves a payment in the first period offset by a receipt in the second. If the counterparty defaults shortly after the first payment is made, there could be a credit loss even if interest rates have not changed.

Key concept:
With a positively sloped term structure, the receiver of floating rate (payer of the fixed rate) has greater credit exposure than the counterparty.

A similar issue arises with currency swaps when the two coupon rates differ. Low nominal interest rates imply a higher forward exchange rate. The party that receives payments in a low-coupon currency is expected to receive greater payments later during the exchange of principal. If the counterparty defaults, there could be a credit loss even if rates have not changed.

Table 21-1 gives the example of a fixed-rate swap where one party receives 6% in dollars against paying 9% in pounds. We assume a flat term structure in both currencies and an initial spot rate of \$2/£. The first panel describes the present-value factors as well as the forward rates. Because of the higher pound interest rate, the forward exchange value of the pound drops from \$2.0000 to \$1.5129 after 10 years. The two rightmost columns in the first panel report the present value of the stream of payments, each discounted in its own currency. They sum to \$100 million and −£50 million respectively, which at the current spot rate of \$2/£ adds up to zero. The initial value of the swap is zero.

The second panel lays out the cash flows in each currency. The three columns on the right describe the credit exposure. First, the pound cash flow is translated into dollars at the forward rate. For instance, the first payment of £4.50 is also $4.5 \times 1.9449 =$ \$8.75. The sum of the receipt of \$6 million and payment of \$8.75 million gives a net outflow of \$2.75 million. The table shows that the first 9 years involve an outflow, which is eventually offset by an inflow of \$23.54 million in year 10. The last column converts these expected credit exposures at different point in time into a current

TABLE 21-1 Credit Exposure Profile for a Currency Swap
$100 million at 6% against £50 million at 9%

Time	Market Data			Swap Valuation	
	PV-$	PV-£	FX($/£)	NPV($)	NPV(£)
			2.0000		
1	0.9434	0.9174	1.9449	$5.66	−£4.13
2	0.8900	0.8417	1.8914	$5.34	−£3.79
3	0.8396	0.7722	1.8394	$5.04	−£3.47
4	0.7921	0.7084	1.7887	$4.75	−£3.19
5	0.7473	0.6499	1.7395	$4.48	−£2.92
6	0.7050	0.5963	1.6916	$4.23	−£2.68
7	0.6651	0.5470	1.6451	$3.99	−£2.46
8	0.6274	0.5019	1.5998	$3.76	−£2.26
9	0.5919	0.4604	1.5558	$3.55	−£2.07
10	0.5584	0.4224	1.5129	$59.19	−£23.02
Total				$100.00	−£50.00

Time	Cash Flows		Exposure		
	Receive	Pay	Pay in $	Difference	NPV(Diff.)
1	$6.00	−£4.50	−$8.75	−$2.75	−$2.60
2	$6.00	−£4.50	−$8.51	−$2.51	−$2.24
3	$6.00	−£4.50	−$8.28	−$2.28	−$1.91
4	$6.00	−£4.50	−$8.05	−$2.05	−$1.62
5	$6.00	−£4.50	−$7.83	−$1.83	−$1.37
6	$6.00	−£4.50	−$7.61	−$1.61	−$1.14
7	$6.00	−£4.50	−$7.40	−$1.40	−$0.93
8	$6.00	−£4.50	−$7.20	−$1.20	−$0.75
9	$6.00	−£4.50	−$7.00	−$1.00	−$0.59
10	$106.00	−£54.50	−$82.46	$23.54	$13.15
Total					$0.00

value, discounting at the 6% dollar rate. The net present values (NPVs) of the differences sum to zero, as required. The table, however, shows that if the counterparty defaults in year 9, the remaining credit exposure is quite high.

> **Key concept:**
> The receiver of a low-coupon currency has greater credit exposure than the counterparty.

Example 21-11: FRM Exam 2001—Question 8

21-11. Which of the following 10-year swaps has the highest potential credit exposure?

a) A cross-currency swap *after* 2 years
b) A cross-currency swap *after* 9 years
c) An interest rate swap *after* 2 years
d) An interest rate swap *after* 9 years

Example 21-12: FRM Exam 2000—Question 47/Credit Risk

21-12. Which one of the following deals would have the largest credit exposure for a $1,000,000 deal size (assume the counterparty in each deal is a AAA-rated bank and has no settlement risk)?

a) Pay fixed in an AUD interest-rate swap for 1 year
b) Sell USD against AUD in a 1-year forward foreign exchange contract
c) Sell a 1-year AUD Cap
d) Purchase a 1-year Certificate of Deposit

Example 21-13: FRM Exam 1999—Question 153/Credit Risk

21-13. The amount of potential exposure arising from being long an OTC USD/EUR forward contract will be a function of the

I) Credit quality of the counterparty
II) Tenor of the contract
III) Volatility of the spot USD/EUR exchange rate
IV) Volatility of the USD interest rate
V) Volatility of the EUR interest rate
VI) Nominal amount of the contract

a) All of the above
b) All *except* I
c) I, II, III, and VI
d) III, IV, and V

Example 21-14: FRM Exam 1998—Question 33/Credit Risk

21-14. The amount of potential exposure arising from being long an over-the-counter USD/DEM forward contract will be a function of the

a) Credit quality of the counterparty
b) Credit quality of the counterparty and the tenor of the contract
c) Volatility of the USD/DEM exchange rate and the tenor of the contract
d) Volatility of the USD/DEM exchange rate and the credit quality of the counterparty

21.3 Exposure Modifiers

In a continual attempt to decrease credit exposures, the industry has developed a number for methods to limit exposures. This section analyzes marking-to-market, margins and collateral, recouponing, and netting arrangements.

21.3.1 Marking to Market

The ultimate form of reducing credit exposure is marking-to-market (MTM). **Marking-to-market** involves settling the variation in the contract value on a regular basis, e.g. daily. For OTC contracts, counterparties can agree to longer periods, e.g. monthly or quarterly.

If the MTM treatment is symmetrical across the two counterparties, it is called **two-way marking-to-market**. Otherwise if one party settles losses only, it is called **one-way marking-to-market**.

Marking-to-market has long been used by organized exchanges to deal with credit risk. The reason is that exchanges are accessible to a wide variety of investors, including retail investors, unlike OTC markets where institutions interacting with each other typically will have an on-going relationship. As one observer put it,

> *"Futures markets are designed to permit trading among strangers, as against other markets which permit only trading among friends."*

With daily marking-to-market, the *current* exposure is reduced to zero. There is still, however, *potential* exposure because the value of the contract could change before the next settlement. Potential exposure arises from (i) the time interval between MTM periods and (ii) the time required for liquidating the contract when the counterparty defaults.

In the case of a retail client, the broker can generally liquidate the position fairly quickly, within a day. When positions are very large, as in the case of brokers dealing with long-term capital management (LTCM), however, the liquidation period could be much longer. Indeed LTCM's bailout was motivated by the potential disruption to financial markets should all brokers attempt to liquidate their contracts with LTCM at the same time.

Another issue with marking-to-market is that it introduces other types of risks, in particular

- **Operational risk**, which is due to the need to keep track of contract values and to make or receive payments daily, and
- **Liquidity risk**, because the institution now needs to keep enough cash to absorb variations in contract values.

Margins

Potential exposure is covered by margin requirements. **Margins** represent cash or securities that must be advanced in order to open a position. The purpose of these funds is to provide a buffer against potential exposure.

Exchanges, for instance, require customers to post **initial margin**, whenever establishing a new position. This margin serves as a performance bond to offset possible future losses should the customer default. Contract gains and losses are then added to the posted margin in the **equity account**. Whenever the value of this equity account falls below a threshold, set at a **maintenance margin**, new funds must be provided.

Margins are set in relation to price volatility and to the type of position, speculative or hedging. Margins increase for more volatile contracts. Margins are typically lower for hedgers because a loss on the futures position can be offset by a gain on the physical, assuming no basis risk. Some exchanges set margins at a level that covers the 99th percentile of worst daily price changes, which is a daily VAR system for credit risk.

Collateral

Over-the-counter markets may allow posting securities as **collateral** instead of cash. This collateral protects against current and potential exposure. Typically, the amount of the collateral will exceed the funds owed by an amount known as the **haircut**.

This difference will be a function of market and credit risk. For instance, cash can have a haircut of zero, which means that there is full protection against current exposure. Government securities can require a haircut of 1%, 3%, and 8% for short-term, medium-term, and longer-term maturities. With greater price volatility, there is an increasing chance of losses if the counterparty defaults and the collateral loses value, which explains the increasing haircuts.

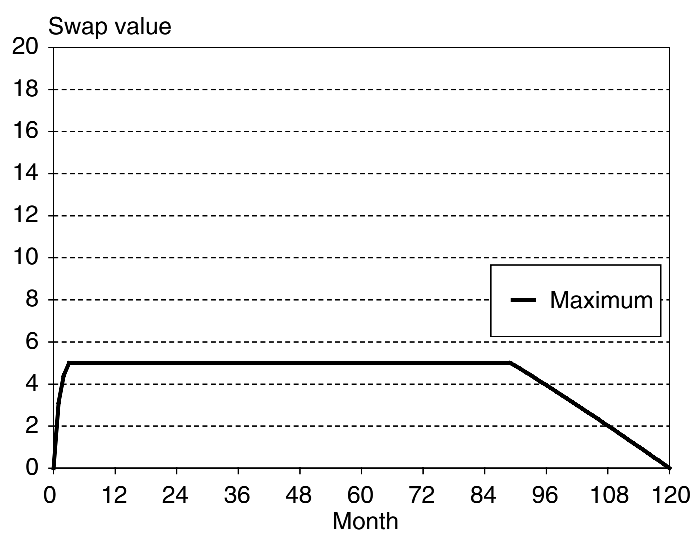
21.3.2 Exposure Limits

Credit exposure can be managed by setting **position limits** on the exposure to a counterparty. Ideally, these should be evaluated in a portfolio context, taking into account all the contracts between an institution and a counterparty.

To enforce limits, information on transactions must be centralized in middle-office systems, which generate an *exposure profile* for each counterparty. The exposure profile is then used to manage credit line usage for several arbitrarily defined maturity buckets. Each proposed additional trade with the same counterparty is then examined for incremental effect on total exposure.

These limits can be also set at the instrument level. In the case of a swap, for instance, an **exposure cap** requires a payment to be made whenever the value of the contract exceeds some amount. Figure 21-11 shows the effect of a $5 million cap on our 10-year swap. If, after two years, say, the contract suddenly moves into a positive value of $11 million, the counterparty would be required to make a payment of $6 million to bring the swap's outstanding value back to $5 million. This now limits the worst exposure to $5 million and also lowers the average exposure.

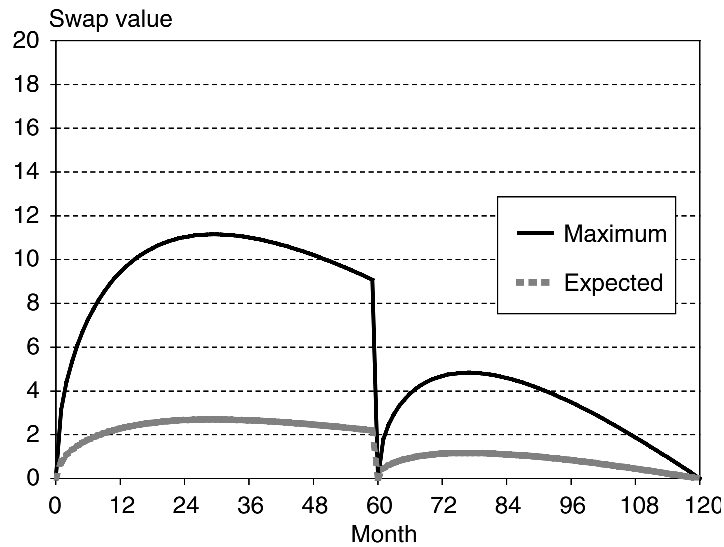
FIGURE 21-11 Effect of Exposure Cap



21.3.3 Recouponing

Another method to control exposure at the instrument level is recouponing. **Recouponing** is a clause in the contract requiring the contract to be marked-to-market

FIGURE 21-12 Effect of Recouponing After 5 Years



at some fixed date. This involves (i) exchanging cash to bring the MTM value to zero and (ii) resetting the coupon or the exchange rate to prevailing market values.

Figure 21-12 shows the effect of 5-year recouponing on our 100-year swap. The exposure is truncated to zero after 5 years. Thereafter, the exposure profile is that of a swap with a remaining 5-year maturity.

21.3.4 Netting Arrangements

Perhaps the most powerful arrangement to control exposures are **netting agreements**. These are by now a common feature of standardized **master swap agreements** such as the one established in 1992 by the **International Swaps and Derivatives Association** (ISDA).

The purpose of these agreements is to provide for the **netting** of payments across a set of contracts. In case of default, a counterparty cannot stop payments on contracts that have negative value while demanding payment on positive value contracts. As a result, this system reduces the exposure to the net payment for all the contracts covered by the netting agreement.

Table 21-2 gives an example with four contracts. Without a netting agreement, the exposure of the first two contracts is the sum of the positive part of each, or $100 million. In contrast, if the first two fall under a netting agreement, their value would offset each other, resulting in a net exposure of $\$100 - \$60 = \$40$ million. If contracts

TABLE 21-2 Comparison of Exposure with and without Netting

Contract	Contract Value	Exposure No Netting	Exposure With Netting for 1 & 2
	Under netting agreement		
1	+\$100	+\$100	
2	−\$60	+\$0	
Total, 1 & 2	+\$40	+\$100	+\$40
	No netting agreement		
3	+\$25	+\$25	
4	−\$15	+\$0	
Grand total, 1 to 4	+\$50	+\$125	+\$65

3 and 4 do not fall under the netting agreement, the exposure is then increased to \$40 + \$25 = \$65 million.

To summarize, the **net exposure** with netting is

$$\text{Net exposure} = \text{Max}(V, 0) = \text{Max}(\sum_{i=1}^{N} V_i, 0) \tag{21.18}$$

Without a netting agreement, the **gross exposure** is the sum of all positive-value contracts

$$\text{Gross exposure} = \sum_{i=1}^{N} \text{Max}(V_i, 0) \tag{21.19}$$

This is always greater than, or at best equal to, the exposure under the netting agreement.

The benefit from netting depends on the number of contracts N and the extent to which contract values covary with each other. The larger the value of N and the lower the correlation, the greater the benefit from netting. It is easy to verify from Table 21-2 that if all contracts move into positive value at the same time, or have high correlation, there will be no benefit from netting.

Figures 21-13 and 21-14 illustrate the effect of netting on a portfolio of two swaps with the same counterparty. In each case, interest rates could increase or decrease with the same probability.

In Figure 21-13, the bank is long both a 10-year and 5-year swap. The top panel describes the worst exposure when rates fall. In this case there is positive exposure for both contracts, which we add to get the total portfolio exposure. Whether there is

FIGURE 21-13 Netting with Two Long Positions

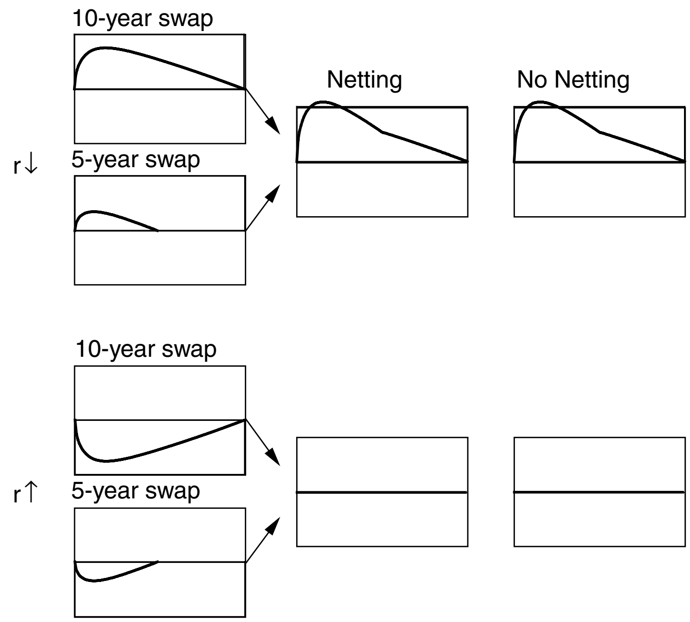
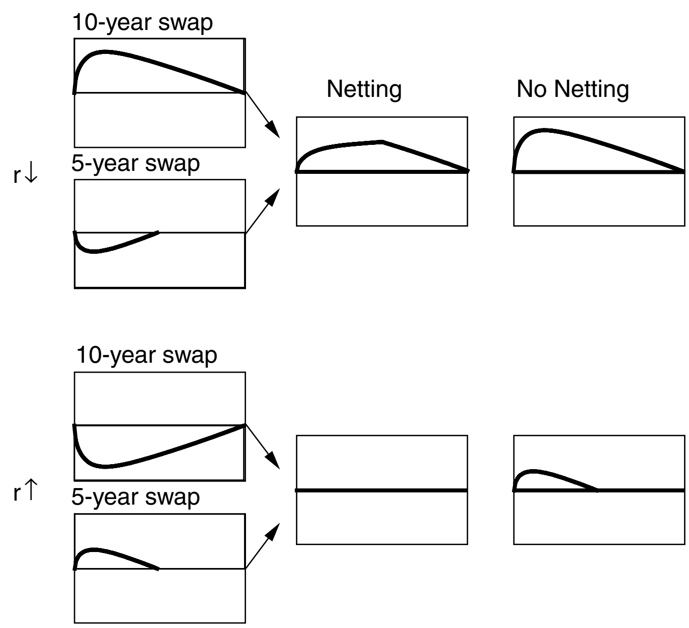


FIGURE 21-14 Netting with a Long and Short Position



netting or not does not matter because the two positions are positive at the same time. The bottom panel describes the worst exposure when rates increase. Both positions as well as the portfolio have zero exposure.

In Figure 21-14, the bank is long the 10-year and short the 5-year swap. When rates fall, the first swap has positive value and the second has negative value. With netting, the worst exposure profile is reduced. In contrast, with no netting the exposure is that of the 10-year swap. Conversely, when rates increase, the swap value is negative for the first and positive for the second. With netting, the exposure profile is zero, whereas without netting it is the same as that of the 5-year swap.

Banks provide some information in their annual report about the benefit of netting for their current exposure. Without netting agreements or collateral, the **gross replacement value** (GRV) is reported as the sum of the worst-case exposures if all counterparties K default at the same time:

$$\text{GRV} = \sum_{k=1}^{K} \text{Gross exposure}_k = \sum_{k=1}^{K} \left[\sum_{i=1}^{N_k} \text{Max}(V_i, 0) \right] \tag{21.20}$$

With netting agreements and collateral, the resulting exposure is defined as the **net replacement value** (NRV). This is the sum, over all counterparties, of the net positive exposure minus any collateral held:

$$\text{NRV} = \sum_{k=1}^{K} \text{Net exposure}_k = \sum_{k=1}^{K} \left[\text{Max}\left(\sum_{i=1}^{N_k} V_i, 0 \right) - \text{Collateral}_k \right] \tag{21.21}$$

Example 21-15: FRM Exam 1998—Question 34/Credit Risk

21-15. A diversified portfolio of OTC derivatives with a single counterparty currently has a net mark-to-market of $20 million. Assuming that there are no netting agreements in place with the counterparty, determine the current credit exposure to the counterparty.

a) Less than $20 million
b) Exactly $20 million
c) Greater than $20 million
d) Unable to be determined

Example 21-16: FRM Exam 1999—Question 131/Credit Risk
21-16. To reduce credit risk, a company can
a) Expose themselves to many different counterparties
b) Take on a variety of positions
c) Set up netting agreements with all of their approved trading partners
d) All of the above

Example 21-17: FRM Exam 1999—Question 154/Credit Risk
21-17. A diversified portfolio of OTC derivatives with a single counterparty currently has a net mark-to-market of USD 20,000,000 and a gross absolute mark-to-market (the sum of the value of all positive value positions minus the value of all negative value positions) of USD 80,000,000. Assuming there are no netting agreements in place with the counterparty, determine the current credit exposure to the counterparty.
a) Less than or equal to USD 19,000,000
b) Greater than USD 19,000,000 but less than or equal to USD 40,000,000
c) Greater than USD 40,000,000 but less than USD 60,000,000
d) Greater than USD 60,000,000

Example 21-18: FRM Exam 1999—Question 123/Credit Risk
21-18. An equity repo is a repo in which common stock is used as collateral in place of the more usual fixed-income instrument. The mechanics of equity repos are effectively the same as fixed-income repos, except that the haircut
a) Is smaller because equities are more liquid than fixed-income instruments
b) Is larger because equity prices are more volatile than those of fixed-income instruments
c) About the same for both equity and fixed-income deals because the counterparty credit risk is the same
d) Cannot be determined in advance because equity prices, in contrast to fixed-income instrument prices, are not martingales

21.4 Credit Risk Modifiers

Credit risk modifiers operate on credit exposure, default risk, or a combination of both. For completeness, this section discusses modifiers that affect default risk.

21.4.1 Credit Triggers

Credit triggers specify that if either counterparty credit rating falls below a specified level, the other party has the right to have the swap cash-settled. These are not exposure modifiers, but instead attempt to reduce the probability of default. For instance, if all outstanding swaps can be terminated when the counterparty rating falls below A, the probability of default is lowered to the probability that a counterparty will default while rated A or higher.

These triggers are useful when the credit rating of a firm deteriorates slowly, because few firms directly jump from investment-grade into bankruptcy. The increased protection can be estimated by analyzing transition probabilities, as discussed in Chapter 19. For example, say a transaction with an AA-rated borrower has a cumulative probability of default of 0.81% over ten years. If the contract can be terminated whenever the rating falls to BB or below, this probability falls to 0.23% only.

21.4.2 Time Puts

Time puts, or **mutual termination options**, permit either counterparty to terminate unconditionally the transaction on one or more dates in the contract. This feature decreases both the default risk and the exposure. It allows one counterparty to terminate the contract if the exposure is large and the other party's rating starts to slip.

Triggers and puts, which are a type of **contingent requirements**, can cause serious trouble, however. They create calls on liquidity precisely in states of the world where the company is faring badly, putting further pressures on the company's liquidity. Indeed triggers in some of Enron's securities forced the company to make large cash payments and propelled it into bankruptcy. Rather than offering protection, these clauses can trigger bankruptcy, affecting all creditors adversely.

21.5 Answers to Chapter Examples

Example 21-1: FRM Exam 1999—Question 130/Credit Risk

a) There is no credit risk from selling options if the premium is received up front.

Example 21-2: FRM Exam 1999—Question 151/Credit Risk

d) The maximum exposure is potentially very large because this is a *long* position in an option, certainly larger than the initial premium. At a minimum, the exposure is the current exposure of EUR 10,000.

Example 21-3: FRM Exam 2001—Question 84

b) Being short an option creates no credit exposure, so answers (c) and (d) are false. With the short forward contract, a gain will be realized if the EUR has depreciated.

Example 21-4: FRM Exam 2000—Question 35/Credit Risk

a) To have a credit loss, we need a combination of positive exposure and default. The swaps with Universal Tools have negative exposure, so they do not create credit risk. Answer (a) is the best because it combines positive exposure and default risk.

Example 21-5: FRM Exam 1999—Question 111/Credit Risk

d) For a loan, the principal is at risk, and the payments depend on the level of rates; the swap needs to be in-the-money for a credit loss to occur.

Example 21-6: FRM Exam 1999—Question 133/Credit Risk

b) MTM and notionals alone do not measure the potential exposure. We need a combination of current MTM plus an add-on for potential exposure.

Example 21-7: FRM Exam 1999—Question 118/Credit Risk

c) The value of the swap must be positive to the dealer to have some exposure. This will happen if current rates are less than the fixed coupon.

Example 21-8: FRM Exam 1999—Question 148/Credit Risk

b) See Equation (21.14).

Example 21-9: FRM Exam 2000—Question 29/Credit Risk

a) This question alters the variance profile in Equation (21.12). Taking now the variance instead of the volatility, we have $\sigma^2 = k(T-t)^4 \times t$, where k is a constant. Differentiating with respect to t,

$$\frac{d\sigma^2}{dt} = k[(-1)4(T-t)^3]t + k[(T-t)^4] = k(T-t)^3[-4t + T - t]$$

Setting this to zero, we have $t = T/5$. Intuitively, because the exposure profile drops off faster than in Equation (21.12), we must have earlier peak exposure than $T/3$.

Example 21-10: FRM Exam 1999—Question 149/Credit Risk

c) We know from the previous question that the maximum is at $t = T/3$. We then plug into $\sigma_{\text{MAX}}(V) = [k(T-t)]\sigma\sqrt{t}$. This is also $[kT(2/3)]\sigma\sqrt{T/3} = [4{,}000 \times 2] \times 5 \times \sqrt{250} =$ 632,456. Multiplying by 2.33, we get 1,473,621.

Example 21-11: FRM Exam 2001—Question 8

a) The question asks about potential exposure for various swaps during their life. Interest rate swaps generally have lower exposure than currency swaps because there is no market risk on the principals. Currency swaps with longer remaining maturities have greater potential exposure. This is the case for the 10-year currency swap, which after 2 years has 8 years remaining to maturity.

Example 21-12: FRM Exam 2000—Question 47/Credit Risk

d) The CD has the whole notional at risk. Otherwise, the next greatest exposure is for the forward currency contract and the interest rate swap. The short cap position has no exposure if the premium has been collected. Note that the question eliminates settlement risk for the forward contract.

Example 21-13: FRM Exam 1999—Question 153/Credit Risk

b) All items have an effect on exposure except (I), which is default risk.

Example 21-14: FRM Exam 1998—Question 33/Credit Risk

c) The credit quality is not involved in the calculation of the potential exposure. It is only taken into account for the computation of the Basel risk weights, or for the distribution of credit losses.

Example 21-15: FRM Exam 1998—Question 34/Credit Risk

d) Without additional information and no netting agreement, it is not possible to determine the exposure from the net amount only. The portfolio could have two swaps with value of \$100 million and −\$80 million, which gives an exposure of \$100 million without netting.

Example 21-16: FRM Exam 1999—Question 131/Credit Risk

d) Credit risk will be decreased with netting, more positions and counterparties.

Example 21-17: FRM Exam 1999—Question 154/Credit Risk

c) Define X and Y as the absolute values of the positive and negative positions. The net value is $X - Y = 20$ million. The absolute gross value is $X + Y = 80$. Solving, we get $X = 50$ million. This is the positive part of the positions, or exposure.

Example 21-18: FRM Exam 1999—Question 123/Credit Risk

b) The haircut on equity repos is greater due to the greater price volatility of the collateral.

Chapter 22

Credit Derivatives

Credit derivatives are the latest tool in the management of portfolio credit risk. From 1996 to 2002, the market is estimated to have grown from about $40 billion to more than $2,300 billion. The market has doubled in each of these years.

Credit derivatives are contracts that pass credit risk from one counterparty to another. They allow credit risk to be stripped from loans and bonds and placed in a different market. Their performance is based on a credit spread, a credit rating, or default status. Like other derivatives, they can be traded on a stand-alone basis or embedded in some other instrument, such as a credit-linked note.

Section 22.1 presents the rationale for credit derivatives. Section 22.2 describes credit default swaps, total return swaps, credit spread forward and option contracts, as well as credit-linked notes. Section 22.3 then provides a brief introduction to the pricing and hedging of credit derivatives. Finally, Section 22.4 discusses the pros and cons of credit derivatives.

22.1 Introduction

Credit derivatives have grown so quickly because they provide an efficient mechanism to exchange credit risk. While modern banking is built on the sensible notion that a portfolio of loans is less risky than single ones, banks still tend to be too concentrated in geographic or industrial sectors. This is because their comparative advantage is "relationship banking," which is usually limited to a clientele banks know best. So far, it has been difficult to lay off this credit exposure, as there is only a limited market for secondary loans. In addition, borrowers may not like to see their bank selling their loans to another party, even for diversification reasons.

In fact, credit derivatives are not totally new. **Bond insurance** is a contract between a bond issuer and a guarantor (a bank or insurer) to provide additional payment should the issuer fail to make full and timely payment. A **letter of credit** is a guarantee by a bank to provide a payment to a third party should the primary credit fail on its obligations. The **call feature** in corporate bonds involves an option on the risk-free interest rate as well as the credit spread; this is generally not considered a credit derivative, however. Indeed the borrower can also call back the bond should its credit rating improve. What is new is the transparency and trading made possible by credit derivatives.

Credit derivatives can also be found on organized exchanges. The value of Eurodollar futures is driven by short-term rates plus a credit spread. Hence a Treasury-Eurodollar (TED) spread is solely exposed to credit risk. The credit risk component can be isolated by buying one type of futures contract and shorting the other.

Example 22-1: FRM Exam 1998—Question 44/Credit Risk

22-1. All of the following can be accomplished with the use of a credit derivative *except*

a) Reducing credit concentration risk
b) Allowing a fund to invest in corporate loans
c) Preventing the bankruptcy of loan counterparty
d) Leveraging credit risk

22.2 Types of Credit Derivatives

Credit derivatives are over-the-counter contracts that allow credit risk to be exchanged across counterparties. They can be classified in terms of

- *The underlying credit*, which can be either a single entity of a group of entities
- *The exercise conditions*, which can be a credit event (such as default or a rating downgrade, or an increase in credit spreads
- *The payoff function*, which can be a fixed amount or a variable amount with a linear or nonlinear payoff

Table 22-1 provides a breakdown of the credit derivatives market by instruments, which will be defined later. The largest share of the market consists of plain-vanilla, credit default swaps, typically with 5-year maturities. The next segment consists of **synthetic securitization**, or collateralized debt obligations (CDOs), where the special purpose vehicle gains exposure to a specified portfolio of credit risk via credit derivatives and the payoffs are redistributed across different tranches. We now define each category in turn.

TABLE 22-1 Credit Derivatives by Type Percentage of Total Notionals

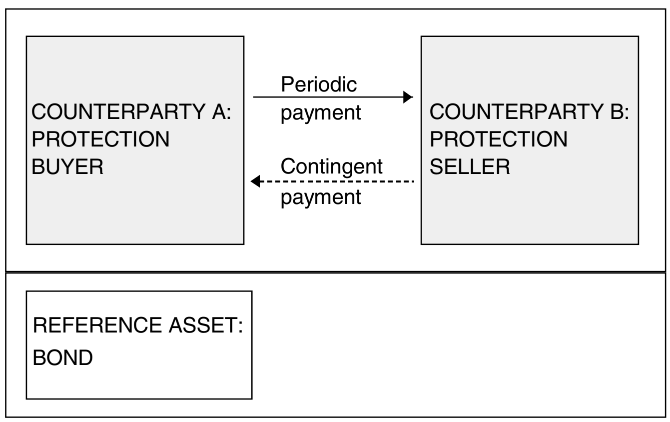
Type	Percent
Credit default swaps	73%
Synthetic securitization	22%
Credit-linked notes	3%
Total return swaps	1%
Credit spread options	1%
Total	100%

Source: *Risk* (February 2003).

22.2.1 Credit Default Swaps

In a **credit default swap** contract, a protection buyer (say A) pays a premium to the protection seller (say B), in exchange for payment if a credit event occurs. The **premium payment** can be a lump sum or periodic. The **contingent payment** is triggered by a credit event (CE) on the underlying credit. The structure of this swap is described in Figure 22-1.

FIGURE 22-1 Credit Default Swap



These contracts represent the purest form of credit derivatives, as they are not affected by fluctuations in market values as long as the credit event does not occur. In the next chapter, we will define this approach as "default mode" marking-to-market (MTM). Also, these contracts are really default options, not swaps. The main difference from a regular option is that the cost of the option is paid in installments instead of up front. When the premium is paid up front, these contracts are called *default put options*.[1]

[1]Default swaps and default options are not identical instruments, however, because a default swap requires premium payments only until a triggering default event occurs.

Example

The protection buyer, call it A, enters a 1-year credit default swap on a notional of $100 million worth of 10-year bonds issued by XYZ. The swap entails an annual payment of 50bp. The bond is called the *reference credit asset*.

At the beginning of the year, A pays $500,000 to the protection seller. Say that at the end of the year, Company XYZ defaults on this bond, which now trades at 40 cents on the dollar. The counterparty then has to pay $60 million to A. If A holds this bond in its portfolio, the credit default swap provides protection against credit loss due to default.

Default swaps are embedded in many financial products: Investing in a risky (credit-sensitive) bond is equivalent to investing in a risk-free bond plus selling a credit default swap.

Say, for instance, that the risky bond sells at $90 and promises to pay $100 in one year. The risk-free bond sells at $95. Buying the risky bond is then equivalent to buying the risk-free bond at $95 and selling a credit default swap worth $5 now. The up-front cost is the same, $90. If the company defaults, the final payoff will be the same.

It is important to realize that entering a credit swap does not eliminate credit risk entirely. Instead, the protection buyer decreases exposure to the reference credit but assumes new credit exposure to seller. To be effective, there has to be a low correlation between the default risk of the underlying credit and of the counterparty.

Table 22-2 illustrates the effect of the counterparty for the pricing of the CDS. If the counterparty is default free, the CDS spread on this BBB credit should be 194bp. The spread depends on the default risk for the counterparty as well as the correlation with the reference credit. In the worst case in the table, with a BBB rating for the counterparty and correlation of 0.8, protection is less effective, and the CDS is only worth 134 bp.

Credit events must be subject to precise definitions. Chapter 19 provided such a list, drawn from the ISDA's Master Netting Agreement. Ideally, there should be no uncertainty about the interpretation of a credit event. Otherwise, credit derivative transactions can create legal risks.

The payment on default reflects the loss to the holders of the reference asset when the credit event occurs. Define Q as this payment per unit of notional. It can take a number of forms.

TABLE 22-2 CDS Spreads for Different Counterparties Reference Obligation is 5-year Bond Rated BBB

Correlation	Counterparty Credit Rating			
	AAA	AA	A	BBB
0.0	194	194	194	194
0.2	191	190	189	186
0.4	187	185	181	175
0.6	182	178	171	159
0.8	177	171	157	134

Source: Adapted from Hull J. and White A. (2001). Valuing credit default swaps II: Modeling default correlations. *Journal of Derivatives*, 8(3), 12–21.

- **Cash settlement**, or a payment equal to the strike minus the prevailing market value of the underlying bond.
- **Physical delivery** of the defaulted obligation in exchange for a fixed payment.
- A **lump sum**, or a fixed amount based on some pre-agreed recovery rate. For instance, if the CE occurs, the recovery rate is set at 40%, leading to a payment of 60% of the notional.

The payoff on a credit default swap is

$$\text{Payment} = \text{Notional} \times Q \times I(\text{CE}) \tag{22.1}$$

where the indicator function $I(\text{CE})$ is one if the credit event has occurred and zero otherwise.

These default swaps have several variants. For instance, the **first of basket to default swap** gives the protection buyer the right to deliver *one, and only one*, defaulted security out of a basket of selected securities. Because the protection buyer has more choices, among a basket instead of just one reference credit, this type of protection will be more expensive than a single credit swap, keeping all else equal. The price of protection also depends on the correlation between credit events. The lower the correlation, the more expensive the swap.

Example 22-2: FRM Exam 1999—Question 122/Credit Risk

22-2. A portfolio manager holds a default swap to hedge an AA corporate bond position. If the counterparty of the default swap is acquired by the bond issuer, then the default swap:

a) Increases in value
b) Decreases in value
c) Decreases in value only if the corporate bond is downgraded
d) Is unchanged in value

Example 22-3: FRM Exam 2000—Question 39/Credit Risk

22-3. A portfolio consists of one (long) $100 million asset and a default protection contract on this asset. The probability of default over the next year is 10% for the asset and 20% for the counterparty that wrote the default protection. The joint probability of default for the asset and the contract counterparty is 3%. Estimate the expected loss on this portfolio due to credit defaults over the next year assuming 40% recovery rate on the asset and 0% recovery rate for the contract counterparty.

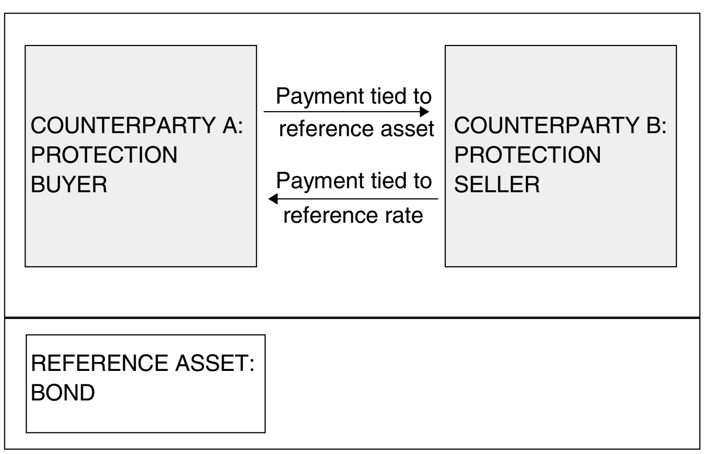
a) $3.0 million
b) $2.2 million
c) $1.8 million
d) None of the above

22.2.2 Total Return Swaps

Total return swaps (TRS) are contracts where one party, called the protection buyer, makes a series of payments linked to the total return on a reference asset. They are also called **asset swaps**. exchange, the protection seller makes a series of payments tied to a reference rate, such as the yield on an equivalent Treasury issue (or LIBOR) plus a spread. If the price of the asset goes down, the protection buyer receives a payment from the counterparty; if the price goes up, a payment is due in the other direction. The structure of this swap is described in Figure 22-2.

This type of swap is tied to changes in the market value of the underlying asset and provides protection against credit risk in an MTM framework. The TRS has the effect of removing all the economic risk of the underlying asset without selling it. Unlike a CDS, however, the swap has an element of market risk because one leg of the payment is a fixed rate.

FIGURE 22-2 Total Return Swap



Example

Suppose that a bank, call it Bank A, has made a $100 million loan to company XYZ at a fixed rate of 10 percent. The bank can hedge its exposure by entering a TRS with counterparty B, whereby it promises to pay the interest on the loan plus the change in the market value of the loan in exchange for LIBOR plus 50bp. If the market value of the loan increases, the bank has to make a greater payment. Otherwise, its payment will decrease, possibly becoming negative.

Say that LIBOR is currently at 9 percent and that after one year, the value of the loan drops from $100 to $95 million. The *net* obligation from Bank A is the sum of

- Outflow of $10\% \times \$100 = \$10$ million, for the loan's interest payment
- Inflow of $9.5\% \times \$100 = \$9.5$ million, for the reference payment
- Outflow of $\frac{(95-100)}{100}\% \times \$100 = -\$5$ million, for the movement in the loan's value

This sums to a net receipt of $-10 + 9.5 - (-5) = \$4.5$ million. Bank A has been able to offset the change in the economic value of this loan by a gain on the TRS.

22.2.3 Credit Spread Forward and Options

These instruments are derivatives whose value is tied to an underlying credit spread between a risky and risk-free bond.

In a **credit spread forward contract**, the buyer receives the difference between the credit spread at maturity and an agreed-upon spread, if positive. Conversely, a payment is made if the difference is negative. An example of the payment formula is

$$\text{Payment} = (S - F) \times \text{MD} \times \text{Notional} \tag{22.2}$$

where MD is the modified duration, S is the prevailing spread and F is the agreed-upon spread. Here, settlement is made in cash.

Alternatively, this could be expressed in terms of prices:

$$\text{Payment} = [P(y + F, \tau) - P(y + S, \tau)] \times \text{Notional} \tag{22.3}$$

where y is the yield-to-maturity of an equivalent Treasury and $P(y+S, \tau)$ is the present value of the security with τ years to expiration, discounted at y plus a spread. Note that if $S > F$, the payment will be positive as in the previous expression.

In a **credit spread option contract**, the buyer pays a premium in exchange for the right to "put" any increase in the spread to the option seller at a predefined maturity:

$$\text{Payment} = \text{Max}(S - K, 0) \times \text{MD} \times \text{Notional} \tag{22.4}$$

where K is the predefined spread. The purchaser of the options buys credit protection, or the right to put the bond to the seller if it falls in value. The payout formula could also be expressed directly in terms of prices, as in Equation (22.3).

Example

A credit spread option has a notional of \$100 million with a maturity of one year. The underlying security is an 8% 10-year bond issued by the corporation XYZ. The current spread is 150bp against 10-year Treasuries. The option is European type with a strike of 160bp.

Assume that, at expiration, Treasury yields have moved from 6.5% to 6% and the credit spread has widened to 180bp. The price of an 8% coupon, 9-year semiannual bond discounted at $y + S = 6 + 1.8 = 7.8\%$ is \$101.276. The price of the same bond discounted at $y + K = 6 + 1.6 = 7.6\%$ is \$102.574. Using the notional amount, the payout is $(102.574 - 101.276)/100 \times \$100,000,000 = \$1,297,237$.

22.2.4 Credit-Linked Notes

Credit-linked notes are not stand-alone derivatives contracts but instead combine a regular coupon-paying note with some credit risk feature. The goal is generally to increase the yield paid to the investor in exchange for taking some credit risk. The simplest form is a corporate, or credit-sensitive, bond.

A general example is provided in Figure 22-3. The investor makes an up-front payment that represents the par value of the credit-linked note. A trustee then invests the funds in a top-rated investment and takes a short position in a credit default swap. The investment could be an AAA-rated Fannie Mae agency note, for instance, that

FIGURE 22-3 Credit-Linked Note

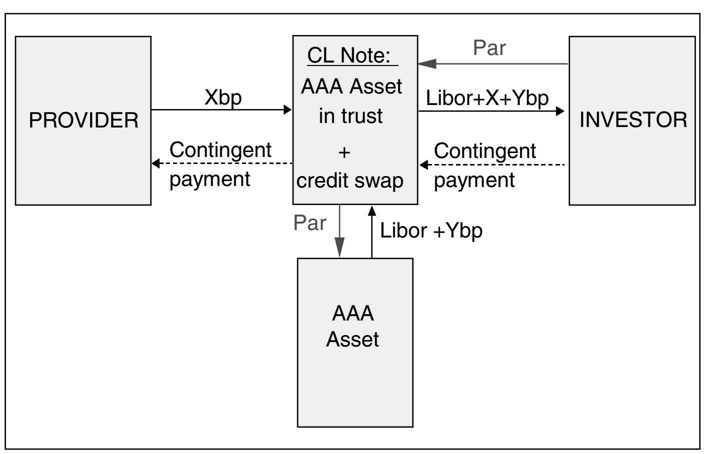


TABLE 22-3 Types of Credit-Linked Notes

Type	Maximum Loss
Asset-backed	Initial investment
Compound credit	Amount from first note's default
Principal protection	None on the principal
Enhanced asset return	Predetermined

pays LIBOR plus a spread of Ybp. The credit default swap is sold by a provider, for example a bank, for an additional annual receipt of Xbp. The total regular payment to the investor is then LIBOR$+Y+X$. In return for this higher yield, the investor must be willing to lose some of the principal should a default event occur.

More generally, credit-linked notes can have exposure to one or more credit risks and increase the yield through leverage. The downside risk may be limited through the features described in Table 22-3.

These structures offer various trade-offs between risk and return. "Asset-backed securities" could lose up to the whole initial investment. The payoffs on "compound credit" notes are linked to various credits and can only lose the amount corresponding to the first credit's default. "Principal protection" notes have their principal guaranteed. "Enhanced asset return" notes have a predetermined maximum loss.

Example 22-4: FRM Exam 2000—Question 33/Credit Risk

22-4. Which one of the following statements is *most* correct?

a) Payment in a total return swap is contingent upon a future credit event.

b) Investing in a risky (credit-sensitive) bond is similar to investing in. a risk-free bond plus selling a credit default swap.

c) In the first-to-default swap, the default event is a default on two or. more assets in the basket.

d) Payment in a credit swap is contingent only upon the bankruptcy of the. counterparty.

Example 22-5: FRM Exam 1999—Question 113/Credit Risk

22-5. Which of the following statements is/are *always* true?

a) Payment in a credit swap is contingent upon a future credit event.

b) Payment in a total rate of return swap is not contingent upon a future credit event.

c) Both (a) and (b) are true.

d) None of the above are true.

Example 22-6: FRM Exam 1999—Question 114/Credit Risk

22-6. In the first-to-default swap, the default event is a default on
a) Any one of the assets in the basket
b) All of the assets in the basket
c) Two or more assets in the basket
d) None of the above

Example 22-7: FRM Exam 1999—Question 144/Credit Risk

22-7. Which of the following is a type of credit derivative?
I) A put option on a corporate bond
II) A total return swap on a loan portfolio
III) A note that pays an enhanced yield in the case of a bond downgrade
IV) A put option on an off-the-run Treasury bond
a) I, II, and III
b) II and III only
c) II only
d) All of the above

Example 22-8: FRM Exam 1998—Question 26/Credit Risk

22-8. The BIS considers all of the following products to be credit derivatives *except*
a) Credit-linked notes
b) Total-return swaps
c) Credit spread options
d) Callable floating-rate notes

Example 22-9: FRM Exam 1998—Question 46/Credit Risk

22-9. Company A and Company B enter into a trade agreement in which Company A will periodically pay all cash flows and capital gains arising from Bond X to Company B. On the same dates Company B will pay Company A LIBOR + 50bp plus any decrease in the market value of Bond X. What type of trade is this?
a) A total return swap
b) A fixed-income-linked swap
c) An inverse floater
d) An interest-rate swap

Example 22-10: FRM Exam 2000—Question 61/Credit Risk

22-10. *(Complex–use the valuation formula with prices)* A credit-spread option has a notional amount of $50 million with a maturity of one year. The underlying security is a 10-year, semiannual bond with a 7% coupon and a $1,000 face value. The current spread is 120 basis points against 10-year Treasuries. The option is a European option with a strike of 130 basis points. If at expiration, Treasury yields have moved from 6% to 6.3% and the credit-spread has widened to 150 basis points, what will be the payout to the buyer of this credit-spread option?

a) $587,352
b) $611,893
c) $622,426
d) $639,023

Example 22-11: FRM Exam 2000—Question 62/Credit Risk

22-11. Bank One has made a $200 million loan to a software company at a fixed rate of 12 percent. The bank wants to hedge its exposure by entering into a total return swap with a counterparty, Interloan Co., in which Bank One promises to pay the interest on the loan plus the change in the market value of the loan in exchange for LIBOR plus 40 basis points. If after one year the market value of the loan has decreased by 3 percent and LIBOR is 11 percent, what will be the net obligation of Bank One?

a) Net receipt of $4.8 million
b) Net payment of $4.8 million
c) Net receipt of $5.2 million
d) Net payment of $5.2 million

22.3 Pricing and Hedging Credit Derivatives

By now, we have developed tools to price and hedge credit risk, which can be extended to credit derivatives. These credit derivatives, however, are complex instruments, as they combine market risk and the joint credit risk of the reference credit and of the counterparty. In general, we need a long list of variables to price these derivatives, including the term structure of risk-free rates, of the reference credit, of the counterparty credit, as well as the joint distribution of default and recoveries. Practitioners use shortcuts that typically ignore the default risk of the counterparty.

22.3.1 Methods

The first approach is the *actuarial approach*, which uses historical default rates to infer the objective expected loss on the credit derivative. For instance, we could use a transition matrix and estimates of recovery rates to assess the actuarial expected loss. This process, however, does not rely on a risk-neutral approach and will not lead to a *fair* price, which includes a risk premium. Neither does it provide a method to hedge the exposure. It only helps to build up a reserve that, in large samples, should be sufficient to absorb the average loss.

The second approach relies on *bond credit spreads* and requires a full yield curve of liquid bonds for the underlying credit. This approach allows us to derive a fair price for the credit derivative, as well as a hedging mechanism, which uses traded bonds for the underlying credit.

The third approach relies on *equity prices* and requires a liquid market for the common stock for the underlying credit as well as information about the structure of liabilities. The Merton model, for instance, allows us to derive a fair price for the credit derivative, as well as a hedging mechanism, which uses the common stock of the underlying credit.

22.3.2 Example: Credit Default Swap

We are asked to value a credit default swap on a $10 million two-year agreement, whereby A (the protection buyer) agrees to pay B (the guarantor, or protection seller) a fixed annual fee in exchange for protection against default of 2-year bonds XYZ. The payout will be the notional times $(100 - B)$, where B is the price of the bond at expiration, if the credit event occurs.

Currently, XYZ bonds are rated A and trade at 6.60%. The 2-year T-note trades at 6.00%.

Actuarial Method

This method computes the credit exposure from the current credit rating and the probability that the company XYZ will default. We use a simplified transition matrix, shown in Table 22-4.

Starting from an A rating, the company could default

- In year 1, with a probability of $P(D_1 \mid A_0) = 1\%$
- In year 2, with a probability of $P(D_2 \mid A_1)P(A_1) + P(D_2 \mid B_1)P(B_1) + P(D_2 \mid C_1)P(C_1) = 0.01 \times 0.90 + 0.02 \times 0.07 + 0.05 \times 0.02 = 1.14\%$

Table 22-4 Credit Ratings Transition Probabilities

Starting State	Ending State A	B	C	D	Total
A	0.90	0.07	0.02	0.01	1.00
B	0.05	0.90	0.03	0.02	1.00
C	0	0.10	0.85	0.05	1.00
D	0	0	0	1.00	1.00

If the recovery rate is 60%, the expected costs are, for the first year, 1%[1 − 60%], and 1.14%[1 − 60%] in the second year. Ignoring discounting, the average annual cost is

$$\text{Annual Cost} = \$10,000,000 \times (1\% + 1.14\%)/2 \times [1 - 60\%] = \$42,800$$

This approach assumes that the credit rating is appropriate and that the transition probabilities and recovery rates are accurately measured.

Credit-Spread Method

Here, we compare the yield on the XYZ bond with that on a default-free asset, such as the T-Note. If all bonds are treated equally, the bonds must have the same term as the maturity of the option. The annual cost of protection is then

$$\text{Annual Cost} = \$10,000,000 \times (6.60\% - 6.00\%) = \$60,000$$

This is higher than the cost from the actuarial approach. The difference can be ascribed to a risk premium, for instance because credit risk is correlated with the general level of economic activity. This approach also assumes that all of the yield spread difference is due to credit risk, when it could be also attributed to other factors, such as liquidity or tax effects.

To hedge, the protection seller would go short the corporate bond and long the equivalent Treasury. Any loss on the default swap because of a credit event would be offset by a gain on the hedge. If the company defaults, the protection buyer could deliver the bond to the protection seller who could then in turn deliver the bond to close out the short sale.

Equity Price Method

This method is more involved. We require a measure of the stock market capitalization of XYZ, of the total value of liabilities, and of the volatility of equity prices.

Using the notations of the chapter on the Merton model, the fair value of the put is

$$\text{Put} = -V[N(-d_1)] + Ke^{-r\tau}[N(-d_2)] \tag{22.5}$$

where d_1 and d_2 depend on V, K, r, σ_V, and the tenor of the put, τ. We could, for example, have a "fair" put option value of $120,000, which, again ignoring discounting, translates into an annual cost of $60,000.

To hedge, the protection seller would go short the stock, in the amount of

$$\frac{\partial \text{Put}}{\partial V} \times \frac{\partial V}{\partial S} = -[N(-d_1)] \times \frac{1}{N(d_1)} = -[1 - N(d_1)] \times \frac{1}{N(d_1)} = 1 - \frac{1}{N(d_1)} \qquad (22.6)$$

which indeed is negative, plus an appropriate position in the risk-free bond.

Example 22-12: FRM Exam 1999—Question 147/Credit Risk

22-12. Which of the following are needed to value a credit swap?
I) Correlation structure for the default and recovery rates of the swap counterparty and reference credit
II) The swap or treasury yield curve
III) Reference credit spread curve over swap or treasury rates
IV) Swap counterparty credit spread curve over swap or treasury rates
a) II, III, and IV
b) I, III, and IV
c) II and III
d) All of the above

Example 22-13: FRM Exam 1999—Question 135/Credit Risk

22-13. The Widget Company has outstanding debt of three different maturities as outlined in the table.

	Widget Company Bonds		Corresponding U.S. Treasury Bonds	
Maturity	Price	Coupon (sa 30/360)	Price	Coupon (sa 30/360)
1 year	100	7.00%	100	6.00%
5 years	100	8.50%	100	6.50%
10 years	100	9.50%	100	7.00%

All Widget Co. debt ranks pari passu, all its debt contains cross default provisions, and the recovery value for each bond is 20. The correct price for a one-year credit default swap (sa 30/360) with the Widget Co., 9.5% 10-year bond as a reference asset is
a) 1.0% per annum
b) 2.0% per annum
c) 2.5% per annum
d) 3.5% per annum

22.4 Pros and Cons of Credit Derivatives

The rapid growth of the credit derivatives market is the best testimony of their usefulness. These instruments are superior risk management tools, allowing the *transfer of risks* to those who can bear them best. Many observers, including bank regulators, have stated that credit risk diversification using credit derivatives helped banks to weather the recession of 2001 and its accompanying increase in defaults, without apparent major problems. This period witnessed the largest-ever corporate bankruptcies (WorldCom and Enron) and sovereign default (Argentina) with barely a ripple in global financial markets. The losses have been spread widely, saving the major U.S. banks from the catastrophic failures typical of previous downturns. In the case of Enron, for instance, exposures amounting to around $2.7 billion were transferred to credit derivatives.

Credit derivatives have another useful function, which is *price discovery*. By creating or extending a market for credit risk, this new market gives market observers a better measure of the cost of credit risk.

Credit derivatives also allow *transactional efficiency*, because they have lower transaction costs than in the cash markets. Counterparties can also take advantage of disparities in the pricing of loans and bonds, making both markets more efficient.

On the downside, this market may be relatively *illiquid*. This is because, unlike interest rate swaps, there is no standardization of the reference credit. By definition, credit risk is specific.

Also, the market still uses *various valuation methods*. This is due to the short supply of data on essential parameters, such as default probabilities and recovery rates. As a result, there is less agreement on the fair valuation of credit derivatives than for other derivatives instruments.

Credit derivatives also introduce a new element of risk, which is *legal risk*. Indeed parties can sometimes squabble over the definition of a credit event. Such disagreement occurred during the Russian default as well as notable debt restructurings and demergers. No doubt this explains why bank regulators are watching the growth of this market with some concern. The question is whether these contracts will be fully effective with widespread defaults.

This is especially so because this market has evolved from *regulatory arbitrage*, that is, attempts to defeat onerous capital requirements mandated by bank regulators. Commercial banks have systematically lowered their capital requirements by laying off loan credit risk through credit derivatives. This can be advantageous if an economically equivalent credit exposure has lower capital requirements (we will discuss regulatory capital requirements in a later chapter). Whether this is a benefit or drawback depends on one's perspective.

Example 22-14: FRM Exam 2000—Question 30/Credit Risk

22-14. Which one of the following statements is *not* an application of credit derivatives for banks?

a) Reduction in economic and regulatory capital usage
b) Reduction in counterparty concentrations
c) Management of the risk profile of the loan portfolio
d) Credit protection of private banking deposits

22.5 Answers to Chapter Examples

Example 22-1: FRM Exam 1998—Question 44/Credit Risk

c) Credit derivatives certainly do not prevent the credit events from happening.

Example 22-2: FRM Exam 1999—Question 122/Credit Risk

b) This is an interesting question that demonstrates that the credit risk of the underlying asset is exchange for that of the swap counterparty. The swap is now worthless; if the underlying credit defaults, the counterparty will default as well (since it is the same).

Example 22-3: FRM Exam 2000—Question 39/Credit Risk

c) The only state of the world with a loss is a default on the asset jointly with a default of the guarantor. This has probability of 3%. The expected loss is $\$100,000,000 \times 0.03 \times (1 - 40\%) = \$1.8$ million.

Example 22-4: FRM Exam 2000—Question 33/Credit Risk

b) Answer (a) is not correct because payment is simply a function of market variables (this is not a credit default swap). Answer (c) is incorrect because the default event in this case is the first default. Answer (d) is incorrect because a credit event is more general than simply bankruptcy. Answer (b) says that a risky bond is the sum of a risk-free bond plus a short position in a credit default swap.

Example 22-5: FRM Exam 1999—Question 113/Credit Risk

c) Payment from the protection seller is contingent upon a credit event for a credit swap and a combination of payment tied to a reference rate and the asset depreciation for a TRS.

Example 22-6: FRM Exam 1999—Question 114/Credit Risk

a) The default event is triggered when there is a first default on necessarily *any* of the assets in the basket.

Example 22-7: FRM Exam 1999—Question 144/Credit Risk

a) Part I, II, and III are correct. An option on a T-bond has no credit component.

Example 22-8: FRM Exam 1998—Question 26/Credit Risk

d) The first three instruments have a major credit component. Callable FRN are not considered credit derivatives. The call option is primarily an interest-rate option.

Example 22-9: FRM Exam 1998—Question 46/Credit Risk

a) The payments are linked to the total return on bond X.

Example 22-10: FRM Exam 2000—Question 61/Credit Risk

c) We need to value the bond with remaining semiannual payments for 9 years using two yields, $y + S = 6.30 + 1.50 = 7.80\%$ and $y + K = 6.30 + 1.30 = 7.60\%$. This gives \$948.95 and \$961.40, respectively. The total payout is then $\$50,000,000 \times [\$961.40 - \$948.95]/\$1000 = \$622,424$.

Example 22-11: FRM Exam 2000—Question 62/Credit Risk

a) The net payment is an outflow of $12\% - 3\%$ minus inflow of $11\% + 0.4\%$, which is a net receipt of -2.4%. Applied to the notional of \$200 million, this gives a receipt of \$4.8 million.

Example 22-12: FRM Exam 1999—Question 147/Credit Risk

d) As a first approximation, the reference credit spread curve may be enough. To be complete, however, we also need information about the credit risk of the swap counterparty, the treasury curve (for discounting), and correlations. The correlation structure enters the pricing through the expectation of the product of the default and loss given default.

Example 22-13: FRM Exam 1999—Question 135/Credit Risk

a) Because all bonds rank equally, all default occur at the same time and have the same loss given default. Therefore the cash flow on the 1-year credit swap can be replicated (including any risk premium) by going long the 1-year Widget bond and short the 1-year T-Bond.

Example 22-14: FRM Exam 2000—Question 30/Credit Risk

d) Credit derivatives are used to reduce regulatory capital usage and counterparty concentrations and to manage the risk profile of the loan portfolio. Private banking deposits are bank liabilities, not assets.

Chapter 23

Managing Credit Risk

The previous chapters have explained how to estimate default probabilities, credit exposures, and recovery rates for individual credits. We now turn to the measurement and management of credit risk for the overall portfolio.

In the past, credit risk was measured on a stand-alone basis, in terms of a "yes" or "no" decision by a credit officer. Some consideration was given to portfolio effect through very crude credit limits at the overall level. Portfolio theory, however, teaches us that risk should be viewed in the context of the contribution to the total risk of a portfolio, not in isolation. Diversification creates what is perhaps the only "free lunch" in finance: The pricing of risk is markedly lower when considering portfolio effects.

The revolution in risk management is now spreading from the portfolio measurement of market risk to credit risk. This is a result of a number of developments.

At the top of the list are technological advances that now enable us to aggregate financial risk in close to real time. Second, the market has witnessed an exponential growth in new products, such as credit derivatives, which allow better management of credit risk. Finally, developments in government policies and financial markets are leading to greater emphasis on credit risk. With the European Monetary Union (EMU), exchange rate risk has disappeared within the Eurozone. This has transformed currency risk into credit risk for European government bonds.[1] Thus, French government debt now carries credit risk, like debt issued by the state of California. Correspondingly, the increasing depth and liquidity of EMU corporate bond markets is leading to a rapid expansion of this market.

[1]In the past there was very little credit risk on European government debt. Although governments could have defaulted on their national-currency denominated debt, it was easier to create inflation to expropriate bondholders. Some have done so with a vengeance, like Italy. Governments do not have this option any more, as the value of the new currency, the euro, is now in the hands of the European Central Bank. Indeed, Chapter 19 has shown that the credit rating of countries is lower when the debt is denominated in foreign currency rather than in the local currency.

Section 23.1 introduces the distribution of credit losses. This contains two major components. The first is the expected credit loss, which is essential information for pricing and reserving purposes, as explained in Section 23.2. The second component is the unexpected credit loss, or worst deviation from the expected loss at some confidence level. Section 23.3 shows how this credit value at risk (VAR), like market VAR, can be used to determine the amount of capital necessary to support a position. The pricing of loans should not only cover expected losses, but also the remuneration of the economic capital set aside to cover the unexpected loss. Finally, Section 23.4 provides an overview of recently developed credit risk models, including CreditMetrics, CreditRisk+, the KMV model, and Credit Portfolio View.

23.1 Measuring the Distribution of Credit Losses

We can now pool together the information on default probabilities, credit exposures, and recovery rates to measure the distribution of losses due to credit risk. For simplicity, we only consider losses in **default mode** (DM), that is, due to defaults instead of changes in market values.

For one instrument, the current or potential credit loss is

$$\text{Credit Loss} = b \times \text{Credit Exposure} \times \text{LGD} \tag{23.1}$$

which involves the random variable b that takes on the value of 1, with probability p, when the discrete state of default occurs, the credit exposure, and the loss given default (LGD).

For a portfolio of N counterparties, the loss is

$$\text{Credit Loss} = \sum_{i=1}^{N} b_i \times \text{CE}_\text{i} \times \text{LGD}_\text{i} \tag{23.2}$$

where CE_i is now the total credit exposure to counterparty i, across all contracts and taking into account netting agreements.

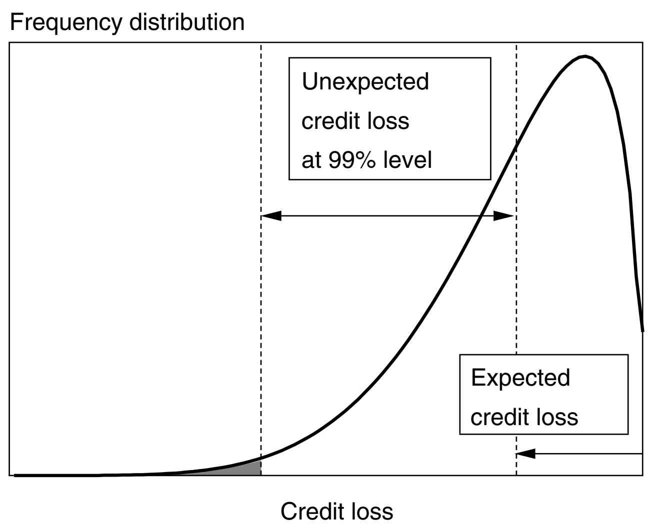
The distribution of credit loss is quite complex. Typically, information about credit is described by the **net replacement value** (NRV), which is also

$$\text{NRV} = \sum_{i=1}^{N} \text{CE}_\text{i} \tag{23.3}$$

evaluated at the current time. This is the worst that could be lost if all parties defaulted at the same time and if there was no recovery. This is not very informative, however. The NRV, which is often disclosed in annual reports, is equivalent to using notionals to describe the risks of derivatives portfolios. It does not take into account the probability of default nor correlations across defaults and exposures.

Chapter 18 gave an example of a loss distribution for a simple portfolio with three counterparties. This example was tractable as we could enumerate all possible states. In general, we need to consider many more credit events. We also need to account for movements and comovements in risk factors, which drive exposures, uncertain recovery rates, and correlations among defaults. This can only be done with the help of *Monte Carlo simulations*. Once this is performed for the whole portfolio, we obtain a distribution of credit losses on a target date. Figure 23-1 describes a typical distribution.

FIGURE 23-1 Distribution of Credit Losses



This leads to a number of fundamental observations.

- **Distribution**

  The distribution of credit losses is *highly skewed to the left,* in contrast to that of market risk factors, which is in general roughly symmetrical. This distribution is actually similar to a short position in an option. This analogy is formalized in the Merton model, which equates a risky bond to a risk-free bond plus a short position in an option.

- **Expected credit loss (ECL)**

  The **expected credit loss** represents the average credit loss. The *pricing* of the portfolio should be such that it covers the expected loss. In other words, the price should be advantageous enough to offset average credit losses. In the case of a bond, the price should be low enough, or yield high enough, to compensate against expected losses. In the case of a derivative, the bank that takes on the credit risk should factor this expected loss into the pricing of its product. Loan loss reserves should also be accumulated as a **credit provision** against expected losses.

- **Worst Credit Loss (WCL)**

  The **worst credit loss** represents the loss that will not be exceeded at some level of confidence. Like a VAR figure, the unexpected credit loss (UCL) is the deviation from the expected loss. The institution should have enough capital to cover the unexpected loss. As we have seen before, the UCL depends on the distribution of joint default rates, among other factors. Notably, the dispersion in the distribution narrows as the number of credits increases and when correlations among defaults decrease.

- **Marginal Contribution to Risk**

  The distribution of credit losses can also be used to analyze the incremental effect of a proposed trade on the total portfolio risk. As in the case of market risk, individual credits should be evaluated not only on the basis of their stand-alone risk, but also of their contribution to the portfolio risk. For the same expected return, a trade that lowers risk should be preferable over one that adds to the portfolio risk. Such trade-offs can only be made with a formal measurement of portfolio credit risk.

- **Remuneration of Capital**

  The measure of worst credit loss is also important for the pricing of credit-sensitive instrument. Say that the distribution has an ECL of $1 billion and UCL of $5 billion. The bank then needs to set aside $5 billion just to cover deviations from expected credit losses. This equity capital, however, will require remuneration. So, the pricing of loans should not only cover expected losses, but also the remuneration of this economic capital. This is what we call a *risk premium* and explains why observed credit spreads are larger than simply to cover actuarial losses.

Example 23-1: FRM Exam 1998—Question 41/Credit Risk
23-1. Credit provisions should be taken to cover all of the following *except*
a) Nonperforming loans
b) The expected loss of a loan portfolio
c) An amount equal to the VAR of the credit portfolio
d) Excess credit profits earned during below average loss years

23.2 Measuring Expected Credit Loss

23.2.1 Expected Loss over a Target Horizon

For pricing purposes, we need to measure the expected credit loss, which is

$$E[\text{CL}] = \int f(b, \text{CE}, \text{LGD})(b \times \text{CE} \times \text{LGD})\, db\, d\text{CE}\, d\text{LGD} \tag{23.4}$$

If the random variables are independent, the joint density reduces to the product of densities. We have

$$E[\text{CL}] = \left[\int f(b)(b)\, db\right]\left[\int f(\text{CE})(\text{CE})\, d\text{CE}\right]\left[\int f(\text{LGD})(\text{LGD})\, d\text{LGD}\right] \tag{23.5}$$

which is the product of the expected values. In other words,

$$\text{Expected Credit Loss} = \text{Prob[default]} \times E[\text{Credit Exposure}] \times E[\text{LGD}] \tag{23.6}$$

As an example, the expected credit loss on a BBB-rated \$100 million 5-year bond with 47% recovery rate is $2.28\% \times \$100,000,000 \times (1 - 47\%) = \$1.2$ million. Note that this expected loss is the same whether the bank has one \$100 million exposure or one hundred exposures worth \$1 million each. The distributions, however, will be quite different.

Example 23-2: FRM Exam 1998—Question 39/Credit Risk
23-2. Calculate the 1-year expected loss of a \$100 million portfolio comprising 10 B-rated issuers. Assume that the 1-year probability of default for each issuer is 6% and the average recovery value for each issuer in the event of default is 40%.
a) \$2.4 million
b) \$3.6 million
c) \$24 million
d) \$36 million

Example 23-3: FRM Exam 1999—Question 120/Credit Risk

23-3. Which loan is more risky? Assume that the obligors are rated the same, are from the same industry, and have more or less same sized idiosyncratic risk. A loan of

a) \$1,000,000 with 50% recovery rate
b) \$1,000,000 with no collateral
c) \$4,000,000 with 40% recovery rate
d) \$4,000,000 with 60% recovery rate

Example 23-4: FRM Exam 1999—Question 112/Credit Risk

23-4. Which of the following conditions results in a higher probability of default?

a) The maturity of the transaction is longer.
b) The counterparty is more creditworthy.
c) The price of the bond, or underlying price in the case of a derivative, is less volatile.
d) Both (a) and (c) result in a higher probability of default.

23.2.2 The Time Profile of Expected Loss

So far, we have focused on a fixed horizon, say a year. For pricing purposes, however, we need to consider the total credit loss over the life of the asset. This should involve the time profile of the exposure, of the probability of default, and the discounting factor. Define PV_t as the present value of a dollar paid at t.

The **present value of expected credit losses** (PVECL) is obtained as the sum of the discounted expected credit losses,

$$\mathrm{PVECL} = \sum_t E[\mathrm{CL_t}] \times \mathrm{PV}_t = \sum_t [k_t \times \mathrm{ECE}_t \times (1-f)] \times \mathrm{PV}_t \tag{23.7}$$

where the probability of default is $k_t = S_{t-1} d_t$, or the probability of defaulting at time t, conditional on not having defaulted before.

Alternatively, we could simplify by using the average default probability and average exposure over the life of the asset

$$\mathrm{PVECL}_2 = \mathrm{Ave}[k_t] \times \mathrm{Ave}[\mathrm{ECE_t}] \times (1-f) \times \left[\sum_t \mathrm{PV}_t\right] \tag{23.8}$$

This approach, however, is only an approximation if default risk and exposure profile change over time in a related fashion. As an example, currency swaps with highly-rated counterparties have an exposure and default probability that both increase with time. Due to this correlation, taking the product of the averages understates credit risk.

Table 23-1 shows how to compute the PVECL. We consider a 5-year interest rate swap with a counterparty initially rated BBB and a notional of $100 million. The discount factor is 6 percent and the recovery rate 45 percent. We also assume default can only occur at the end of each year.

TABLE 23-1 Computation of Expected Credit Loss for a Swap

Year	P(default) (%)			Exposure	LGD	Discount	Total
t	c_t	d_t	k_t	ECE_t	$(1-f)$	PV_t	
1	0.22	0.220	0.220	$1,660,000	0.55	0.9434	$1,895
2	0.54	0.321	0.320	$1,497,000	0.55	0.8900	$2,345
3	0.88	0.342	0.340	$1,069,000	0.55	0.8396	$1,678
4	1.55	0.676	0.670	$554,000	0.55	0.7921	$1,617
5	2.28	0.741	0.730	$0	0.55	0.7473	$0
Total			2.280			4.2124	$7,535
Average			0.456	$956,000	0.55	4.2124	$10,100

In the first column, we have the cumulative default probability c_t for a BBB-rated credit from year 1 to 5, expressed in percent. The second column shows the marginal probability of defaulting during that year d_t and the third the probability of defaulting in each year, conditional on not having defaulted before, $k_t = S_{t-1}d_t$. The end-of-year expected credit exposure is reported in the fourth column ECE_t. The sixth column displays the present value factor PV_t.

The final column gives the product $[k_t\text{ECE}_t(1-f)\text{PV}_t]$. Summing across years gives $7,535 on a swap with notional of $100 million. This is very small, less than 1 basis point. Basically, the expected credit loss is very low due to the small exposure profile. For a regular bond or currency swap, the expected loss is much greater.

The last line shows a shortcut to the measurement of expected credit losses based on averages, from Equation (23.8). The average annual default probability is 0.456. Multiplied by the average exposure, $956,000, the LGD, and the sum of the discount rates gives $10,100. This is on the same order of magnitude as the exact calculation.

Table 23-2 details the computation for a bond assuming a constant exposure of $100 million. The expected credit loss is $1.02 million, about a hundred times larger than for the swap. This is because the exposure is also about a hundred times larger.

As in the previous table, the last line shows results based on averages. Here, the expected credit loss is $1.06 million, very close to the exact number as there is no variation in credit exposures over time.

We could also take the usual shortcut and simply compute an expected credit loss given by the cumulative 5-year default rate times \$100 times the loss given default, which is \$1.254 million. Discounting into the present, we get \$0.937 million, close to the previous result.

TABLE 23-2 Computation of Expected Credit Loss for a Bond

Year	P(default) (%)			Exposure	LGD	Discount	Total
t	c_t	d_t	k_t	ECE_t	$(1-f)$	PV_t	
1	0.22	0.220	0.220	\$100,000,000	0.55	0.9434	\$114,151
2	0.54	0.321	0.320	\$100,000,000	0.55	0.8900	\$156,639
3	0.88	0.342	0.340	\$100,000,000	0.55	0.8396	\$157,009
4	1.55	0.676	0.670	\$100,000,000	0.55	0.7921	\$291,887
5	2.28	0.741	0.730	\$100,000,000	0.55	0.7473	\$300,024
Total			2.280			4.2124	\$1,019,710
Average			0.456	\$100,000,000	0.55	4.2124	\$1,056,461

23.3 Measuring Credit VAR

The other component of the credit loss distribution is the **Credit VAR**, defined as the unexpected credit loss at some confidence level. Using the measure of credit loss in Equation (23.1), we construct a distribution of the credit loss $f(\mathrm{CL})$ over a target horizon. At a given confidence c, the worst credit loss (WCL) is defined such that

$$1 - c = \int_{\mathrm{WCL}}^{\infty} f(x)dx \tag{23.9}$$

The credit VAR is then measured as the deviation away from ECL

$$\mathrm{CVAR} = \mathrm{WCL} - \mathrm{ECL} \tag{23.10}$$

This CVAR number should be viewed as the economic capital to be held as a buffer against unexpected losses. Its application is fundamentally different from the expected credit loss, which aggregates expected losses over time and takes their present values.

Instead, the CVAR is measured over a target horizon, say one year, which is deemed sufficient for the bank to take corrective actions should credit problems start to develop. Corrective action can take the form of exposure reduction or adjustment of economic capital, all of which take considerably longer than the typical horizon for market risk.

Once credit VAR is measured, it can be managed. The portfolio manager can examine the trades that contribute most to CVAR. If these trades are not particularly profitable, they should be eliminated.

The portfolio approach can also reveal correlations between different types of risk. For example, **wrong-way trades** are positions where the exposure is negatively correlated with the probability of default. Before the Asian crisis, for instance, many U.S. banks had lent to Asian companies in dollars, or entered equivalent swaps. Many of these Asian companies did not have dollar revenues but instead were speculating, reinvesting the funds in the local currency. When currencies devalued, the positions were in-the-money for the U.S. banks, but could not be collected because the counterparties had defaulted.

Conversely, **right-way trades** are those where increasing exposure is associated with lower probability of counterparty default. This occurs when the transaction is a *hedge* for the counterparty, for instance when a loss on its side of the trade offsets an operating gain.

Example 23-5: FRM Exam 1998—Question 13/Credit Risk

23-5. A risk analyst is trying to estimate the credit VAR for a risky bond. The credit VAR is defined as the maximum unexpected loss at a confidence level of 99.9% over a one-month horizon. Assume that the bond is valued at $1,000,000 one month forward, and the one-year cumulative default probability is 2% for this bond, what is your best estimate of the credit VAR for this bond assuming no recovery?

a) $20,000
b) $1,682
c) $998,318
d) $0

Example 23-6: FRM Exam 1998—Question 10/Credit Risk

23-6. A risk analyst is trying to estimate the credit VAR for a portfolio of two risky bonds. The credit VAR is defined as the maximum unexpected loss at a confidence level of 99.9% over a one-month horizon. Assume that each bond is valued at $500,000 one month forward, and the one-year cumulative default probability is 2% for each of these bonds. What is your best estimate of the credit VAR for this portfolio, assuming no default correlation and no recovery?

a) $841
b) $1,682
c) $998,318
d) $498,318

23.4 Portfolio Credit Risk Models

23.4.1 Approaches to Portfolio Credit Risk Models

Portfolio credit risk models can be classified according to their approaches.

Top-Down vs. Bottom-Up Models

Top-down models group credit risks using single statistics. They aggregate many sources of risk viewed as homogeneous into an overall portfolio risk, without going into the detail of individual transactions. This approach is appropriate for retail portfolios with large numbers of credits, but less so for corporate or sovereign loans. Even within retail portfolios, top-down models may hide specific risks, by industry or geographic location.

Bottom-up models account for features of each asset/credit. This approach is most similar to the structural decomposition of positions that characterizes market VAR systems. It is appropriate for corporate and capital market portfolios. Bottom-up models are also most useful to take corrective action, because the risk structure can be reverse-engineered to modify the risk profile.

Risk Definitions

Default-mode models consider only outright default as a credit event. Hence any movement in the market value of the bond or in the credit rating is irrelevant.

Mark-to-market models consider changes in market values and ratings changes, including defaults. These fair market value models provide a better assessment of risk, which is consistent with the holding period defined in terms of liquidation period.

Conditional vs. Unconditional Models of Default Probability

Conditional models incorporate changing macroeconomic factors into the default probability. Notably, the rate of default increases in a recession.

Unconditional models have fixed default probabilities and instead tend to focus on borrower or factors-specific information. Some changes in the environment, however, can be allowed by changing parameters.

Structural vs. Reduced-Form Models of Default Correlations

Structural models explain correlations by the joint movements of assets, for example stock prices. For each obligor, this price is the random variable that represents movements in default probabilities.

Reduced-form models explain correlations by assuming a particular functional relationship between default and "background factors". For example, the correlation between defaults across obligors can be modeled by the loadings on common risk factors, say, industrial and country.

Table 23-3 summarizes the essential features of portfolio credit risk models in the industry.

TABLE 23-3 Comparison of Credit Risk Models

	CreditMetrics	CreditRisk+	KMV	CreditPf.View
Originator	JP Morgan	Credit Suisse	KMV	McKinsey
Model type	Bottom-up	Bottom-up	Bottom-up	Top-down
Risk definition	Market value (MTM)	Default losses (DM)	Default losses (MTM/DM)	Market value (MTM)
Risk drivers	Asset values	Default rates	Asset values	Macro factors
Credit events	Rating change/ default	Default	Continuous default prob.	Rating change/ default
−Probability	Unconditional	Unconditional	Conditional	Conditional
−Volatility	Constant	Variable	Variable	Variable
−Correlation	From equities (structural) (structural)	Default process (reduced-form) (reduced-form)	From equities (structural) (structural)	From macro factors (reduced-form)
Recovery rates	Random	Constant within band	Random	Random
Solution	Simulation/ analytic	Analytic	Analytic	Simulation

23.4.2 CreditMetrics

CreditMetrics, published in April 1997 by J.P. Morgan, was the first model to measure portfolio credit risk. The system is a "bottom-up" approach where credit risk is driven by movements in bond ratings. The components of the system are described in Figure 23-2.

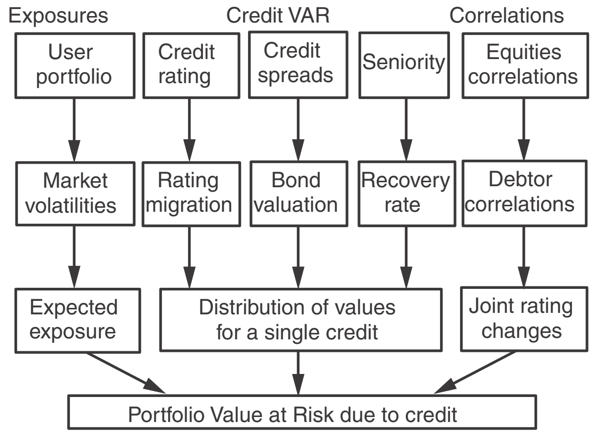
(1) Measurement of exposure by instrument

This starts from the user's portfolio, decomposing all instruments by their exposure and assessing the effect of market volatility on expected exposures on the target date. The range of covered instruments includes bonds and loans, swaps, receivables, commitments, and letters of credit.

(2) Distribution of individual default risk

This step starts with assigning each instrument to a particular credit rating. Credit events are then defined by rating migrations, which include default, through a matrix

FIGURE 23-2 Structure of CreditMetrics



Source: CreditMetrics

of migration probabilities. Thus movements in default probabilities are discrete. After the credit event, the instrument is valued using credit spreads for each rating class. In the case of default, the distributions of recovery rates are used from historical data for various seniority.

This is illustrated in Figure 23-3. We start from a bond or credit instrument with an initial rating of BBB. Over the horizon, the rating can jump to 8 new values, including default. For each rating, the value of the instrument is recomputed, for example $109.37 if the rating goes to AAA, or to the recovery value of $51.13 in case of default. Given the state probabilities and associate values, we can compute an expected bond value, which is $107.09, and standard deviation of $2.99.

FIGURE 23-3 Building the Distribution of Bond Values

		Probability *(pi)*	Value *(Vi)*	Exp. $\Sigma piVi$	Var. $\Sigma pi(Vi-m)^2$
	AAA	0.02%	$109.37	0.02	0.00
	AAA	0.33%	$109.19	0.36	0.01
	AAA	5.95%	$108.66	6.47	0.15
BBB	BBB	86.93%	$107.55	93.49	0.19
	BB	5.30%	$102.02	5.41	1.36
	B	1.17%	$98.10	1.15	0.95
	CCC	0.12%	$83.64	0.10	0.66
	Default	0.18%	$51.13	0.09	5.64
	Sum=	100.00%	m=	$107.09	V= 8.95
				SD=	$2.99

(3) Correlations among defaults

Correlations among defaults are inferred from correlations between asset prices. Each obligor is assigned to an industry and geographical sector, using a factor decomposition. Correlations are inferred from the comovements of the common risk factors, using a database with some 152 country-industry indices, 28 country indices, and 19 worldwide industry indices.

As an example, company 1 may be such that 90% of its volatility comes from the U.S. chemical industry. Using standardized returns, we can write

$$r_1 = 0.90 r_{\text{US,Ch}} + k_1 \epsilon_1$$

where the residual ϵ is uncorrelated with other variables. Next, company 2 has a 74% weight on the German insurance index and 15% on the German banking index

$$r_2 = 0.74 r_{\text{GE,In}} + 0.15 r_{\text{GE,Ba}} + k_2 \epsilon_2$$

The correlation between asset values for the two companies is

$$\rho(r_1, r_2) = (0.90 \times 0.74)\rho(r_{\text{US,Ch}}, r_{\text{GE,In}}) + (0.90 \times 0.15)\rho(r_{\text{US,Ch}}, r_{\text{GE,Ba}}) = 0.11$$

CreditMetrics then uses simulations of the joint asset values, assuming a multivariate normal distribution. Each asset value has a standard normal distribution with cutoff points selected to represent the probabilities of changes in credit ratings.

Table 23-4 illustrates the computations for our BBB credit. From Figure 23-3, there is a 0.18% probability of going from BBB into the state of default. We choose z_1 such that the area to its left is $N(z_1) = 0.18\%$. This gives $z_1 = -2.91$, and so on.. Next, we need to choose z_2 so that the probability of falling between z_1 and z_2 is 0.12%, or that the total left-tail probability is $N(z_2) = 0.18\% + 0.12\% = 0.30\%$. This gives $z_2 = -2.75$. And so on. The cutoff points must be selected for each rating class.

TABLE 23-4 Cutoff Values for Simulations

Rating i	Prob. p_i	Cum.Prob. $N(z_i)$	Cutoff z_i
AAA	0.02%	100.00%	
AA	0.33%	99.98%	3.54
A	5.95%	99.65%	2.70
BBB	86.93%	93.70%	1.53
BB	5.30%	6.77%	−1.49
B	1.17%	1.47%	−2.18
CCC	0.12%	0.30%	−2.75
Default	0.18%	0.18%	−2.91

The simulation generates joint assets values that have a multivariate standard normal distribution with the prespecified correlations. Each realization is mapped into a credit rating and a bond value for each obligor. This gives a total value for the portfolio and a distribution of credit losses over an annual horizon.

These simulations can also be used to compute a correlation among default eventss. Because defaults are much less frequent than rating changes, the correlation is typically much less than the correlation between asset values. CreditMetrics asset correlations in the range of 40% to 60% will typically translate into default correlations of 2% to 4%. This result, however, is driven by the joint normality assumption, which is not totally realistic. Other distributions can generate greater likelihood of simultaneous defaults.

Another drawback of this approach is that it does not integrate credit and market risk. Losses are only generated by changes in credit states, not by market movements. There is no uncertainty over market exposures. For swaps, for instance, the exposure on the target date is taken from the expected exposure. Bonds are revalued using today's forward rate and current credit spreads, applied to the credit rating on the horizon. So, there is no interest rate risk.

23.4.3 CreditRisk+

CreditRisk+ was made public by Credit Suisse in October 1997. The approach is drastically different from CreditMetrics. It is based on a purely actuarial approach found in the property insurance literature.

CreditRisk+ is a default mode (DM) model rather than a mark-to-market (MTM) model. Only two states of the world are considered—default and no-default. Another difference is that the default intensity is time-varying, as it can be modeled as a function of factors that change over time.

When defaults are independent, the distribution of default probabilities resembles a Poisson distribution. The system also allows for some correlation by dividing the portfolio into homogeneous sectors within which obligors share the same systematic risk factors.

The other component of the approach is the severity of losses. This is roughly modeled by sorting assets by severity bands, say loans around $20,000 for the first band, $40,000 for the second band, and so on. A distribution of losses is then obtained for each band. These distributions are then combined across bands to generate an overall distribution of default losses.

Overall, the method provides a quick analytical solution to the distribution of credit losses with minimal data inputs. As with CreditMetrics, however, there is no uncertainty over market exposures.

23.4.4 Moody's KMV

Moody's KMV provides forecasts of estimated default frequencies (EDFs) for approximately 30,000 public firms globally.[2] Much of its technology is considered proprietary and unpublished.

The basic idea, however, is an application of the Merton approach to credit risk. The value of equity is viewed as a call option on the value of the firm's assets

$$S = c(A, K, r, \sigma_A, \tau) \tag{23.11}$$

where K is the value of liabilities, taken as the value of all short-term liabilities (one year and under) plus half the book value of all long-term debt. This has to be iteratively estimated from observable variables, in particular the stock market value S and its volatility σ_S. This model generates an estimated default frequency based on the distance between the current value of assets and the boundary point. Suppose for instance that $A = \$100$ million, $K = \$80$ million, and $\sigma_A = \$10$ million. The normalized distance from default is then

$$z = \frac{A - K}{\sigma_A} = \frac{\$100 - \$80}{\$10} = 2 \tag{23.12}$$

If we assume normally distributed returns, the probability of a standard normal variate z falling below -2 is about 2.3 percent. Hence the default frequency is $EDF = 0.023$.

The strength of this approach is that it relies on what is perhaps the best market data for a company—namely, its stock price. KMV claims that this model predicts defaults much better than credit ratings. The recovery rate and correlations across default are also automatically generated by the model.

[2]KMV was founded by S. Kealhofer, J. McQuown, and O. Vasicek (hence the abbreviation KMV) to provide credit risk services. KMV started as a private firm based in San Francisco in 1989 and was acquired by Moody's in April 2002.

23.4.5 Credit Portfolio View

The last model we consider is **Credit Portfolio View** (CPV), published by the consulting firm McKinsey in 1997. The focus of this top-down model is on the effect of macroeconomic factors on portfolio credit risk.

This approach models loss distributions from the number and size of credits in subportfolios, typically consisting of customer segments. Instead of considering fixed transition probabilities, this model conditions the default rate on the state of the economy, the assumption being that default rates increase during recessions. The default rate p_t at time t is driven by a set of macroeconomic variables x^k for various countries and industries through a linear combination called y_t. It functional relationship to y_t, called *logit model*, ensures that the probability is always between zero and one

$$p_t = 1/[1 + \exp(y_t)], \qquad y_t = \alpha + \sum \beta^k x_t^k \tag{23.13}$$

Using a multifactor model, each debtor is assigned to a country, industry, and rating segment. Uncertainty in recovery rates is also factored in. The model uses numerical simulations to construct the distribution of default losses for the portfolio. While useful for modeling default probabilities conditioned on the state of the economy, this approach is mainly top-down and does not generate sufficient detail of credit risk for corporate portfolios.

23.4.6 Comparison

The International Swaps and Derivatives Association (ISDA) recently conducted a comparative survey of credit risk models. The empirical study consisted of three portfolios of 1-year loans with a total exposure of \$66.3 billion for each portfolio.

A. High credit quality, diversified portfolio (500 names)
B. High credit quality, concentrated portfolio (100 names)
C. Low credit quality, diversified portfolio (500 names)

The models are listed in Table 23-5 and include CreditMetrics, CreditRisk+, two internal models, all with a 1-year horizon and 99% confidence level. Also reported are the charges from the Basel I “standard” rules, which will be detailed in a later chapter. Suffice to say that these rules make no allowance for variation in credit quality or diversification effects. Instead, the capital charge is based on 8% of the loan notional.

TABLE 23-5 Capital Charges from Various Credit Risk Models

	Assuming Zero Correlation		
	Portfolio A	Portfolio B	Portfolio C
CreditMetrics	777	2,093	1,989
CreditRisk+	789	2,020	2,074
Internal Model 1	767	1,967	1,907
Internal Model 2	724	1,906	1,756
Basel I Rules	5,304	5,304	5,304
	Assessing Correlations		
	Portfolio A	Portfolio B	Portfolio C
CreditMetrics	2,264	2,941	11,436
CreditRisk+	1,638	2,574	10,000
Internal Model 1	1,373	2,366	9,654
Basel I Rules	5,304	5,304	5,304

The top of the table first examines the case of zero correlations. The Basel rules yield the same capital charge, irrespective of quality or diversification effects. The charge is also uniformly higher than most others, at $5,304 million, which is approximately 8% of the notional. Generally, the four credit portfolio models show remarkable consistency in capital charges.

Portfolios A and B have the same credit quality but B is more concentrated. A has indeed lower CVAR, $800 million against $2,000 million for B. This reflects the benefit from greater diversification. Portfolios A and C have the same number of names but C has lower credit quality. This increases CVAR from around $800 million to $2,000 million.

The bottom panel assesses empirical correlations. The Basel charges are unchanged, as expected because they do not account for correlations anyway. Internal models show capital charges systematically higher than in the previous case. There is also more dispersion in results across models, however. It is interesting to see, in particular, that the economic capital charge for Portfolio C, with low credit quality, is typically twice the Basel charge. Such results demonstrate that the Basel rules can lead to inappropriate credit risk charges. As a result, banks subject to these capital requirements may shift the risk profile to lower-rated credits until their economic capital is in line with regulatory capital. This shift to lower credit quality was certainly not an objective of the Basel rules.

Example 23-7: FRM Exam 2001—Question 27

23-7. What can be said about default correlations in CreditMetrics?
a) Default correlations can be estimated by ratings changes.
b) Firm-specific aspects are more important than correlation.
c) Past history is insufficient to judge default correlations.
d) Default correlations can be estimated by equity valuation.

Example 23-8: FRM Exam 2001—Question 23

23-8. What is the central assumption made by CreditMetrics?
a) An asset or portfolio should be thought of in terms of its diversification.
b) An asset or portfolio should be thought of in terms of the likelihood of default.
c) An asset or portfolio should be thought of in terms of the likelihood of default and in terms of changes in credit quality over time.
d) An asset or portfolio should be thought in terms of changes in credit quality over time.

Example 23-9: FRM Exam 1999—Question 145/Credit Risk

23-9. J.P. Morgan's CreditMetrics uses which of the following to estimate default correlations?
a) CreditMetrics does not estimate default correlations; it assumes zero correlations between defaults.
b) Correlations of equity returns are used.
c) Correlations between changes in corporate bond spreads to treasury are used.
d) Historical correlation of corporate bond defaults are used.

Example 23-10: FRM Exam 1999—Question 146/Credit Risk

23-10. Which of the following is used to estimate the probability of default for a firm in the KMV model?
I) Historical probability of default based on the credit rating of the firm (KMV has a method to assign a rating to the firm if unrated)
II) Stock price volatility
III) The book value of the firm's equity
IV) The market value of the firm's equity
V) The book value of the firm's debt
VI) The market value of the firm's debt
a) I only
b) II, IV, and V
c) II, III, VI
d) VI only

Example 23-11: FRM Exam 2000—Question 60/Credit Risk

23-11. The KMV credit risk model generates an estimated default frequency (EDF) based on the distance between the current value of assets and the book value of liabilities. Suppose that the current value of a firm's assets and the book value of its liabilities are $500 million and $300 million, respectively. Assume that the standard deviation of returns on the assets is $100 million, and that the returns on the assets are normally distributed. Assuming a standard Merton Model, what is the approximate default frequency (EDF) for this firm?

a) 0.010
b) 0.015
c) 0.020
d) 0.030

Example 23-12: FRM Exam 2000—Question 44/Credit Risk

23-12. Which one of the following statements regarding credit risk models is *most* correct?

a) The CreditRisk+ model decomposes all the instruments by their exposure and assesses the effect of movements in risk factors on the distribution of potential exposure.
b) The CreditMetrics model provides a quick analytical solution to the distribution of credit losses with minimal data input.
c) The KMV model requires the historical probability of default based on the credit rating of the firm.
d) The Credit Portfolio View (McKinsey) model conditions the default rate on the state of the economy.

23.5 Answers to Chapter Examples

Example 23-1: FRM Exam 1998—Question 41/Credit Risk

c) Credit provisions should be made for actual and expected losses. Capital, however, is supposed to provide a cushion against unexpected losses based on CVAR.

Example 23-2: FRM Exam 1998—Question 39/Credit Risk

b) The expected loss is $\$100,000,000 \times 0.06 \times (1 - 0.4) = \$3.6$ million. Note that correlations across obligors does not matter for expected credit loss.

Example 23-3: FRM Exam 1999—Question 120/Credit Risk

c) The exposure times the loss given default is, respectively, $500,000, $1,000,000, $2,400,000, and $1,600,000. Loan (c) has the most to lose.

Example 23-4: FRM Exam 1999—Question 112/Credit Risk

a) The cumulative probability of default increases with the horizon, so answer (a) is correct. Answer (b) should be “less”, not more. Answer (c) deals with exposure, not default.

Example 23-5: FRM Exam 1998—Question 13/Credit Risk

c) First, we have to transform the annual default probability into a monthly probability. Using $(1 - 2\%) = (1 - d)^{12}$, we find $d = 0.00168$, which assumes a constant probability of default during the year. Next, we compute the expected credit loss, which is $d \times \$1,000,000 = \$1,682$. Finally, we calculate the WCL at the 99.9% confidence level, which is the lowest number CL_i such that $P(\text{CL} \leq \text{CL}_i) \geq 99.9\%$. We have $P(\text{CL} = 0) = 99.83\%$; $P(\text{CL} \leq 1,000,000) = 100.00\%$. Therefore, the WCL is $1,000,000, and the CVAR is $1,000,000 − $1,682 = $998,318.

Example 23-6: FRM Exam 1998—Question 10/Credit Risk

d) As in the previous question, the monthly default probability is 0.0168. The following table shows the distribution of credit losses.

Default	Probability (p_i)	Loss L_i	$p_i L_i$	$1 - \sum p_i$
2 bonds	$d^2 = 0.00000282$	$1,000,000	$2.8	100.00000%
1 bond	$2d(1-d) = 0.00335862$	$500,000	$1,679.3	99.99972%
0 bond	$(1-d)^2 = 0.99663854$	$0	$0.0	99.66385%
Total	1.00000000		$1,682.1	

This gives an expected loss of $1,682, the same as before. Next, $500,000 is the WCL at a minimum 99.9% confidence level because the total probability of observing a number equal to, or lower than this, is greater then 99.9%. The CVAR is then $500,000-$1,682=$498,318.

Example 23-7: FRM Exam 2001—Question 27

a) Correlations are important drivers of portfolio risk, so (b) is wrong. In CreditMetrics, correlations in asset values drive correlations in ratings change, which drive default correlations. Answer (d) is not correct as it refers to the Merton model, where default probabilities are inferred from equity valuation, liabilities, and volatilities.

Example 23-8: FRM Exam 2001—Question 23

c) The central assumption in CreditMetrics is that asset values are driven by changes in their credit ratings, including default. So, this is more general than (b) and (d).

Example 23-9: FRM Exam 1999—Question 145/Credit Risk

b) CreditMetrics infers the default correlation from equity correlations.

Example 23-10: FRM Exam 1999—Question 146/Credit Risk

b) KMV uses information about the market value of the stock plus the book value of debt.

Example 23-11: FRM Exam 2000—Question 60/Credit Risk

c) The distance between the current value of assets and that of liabilities is $200 million, which corresponds to twice the standard deviation of $100 million. Hence the probability of default is $N(-2.0) = 2.3\%$, or about 0.020.

Example 23-12: FRM Exam 2000—Question 44/Credit Risk

d) Answer (d) is most correct. Answer (a) is wrong because CreditRisk+ assumes fixed exposures. Answer (b) is also wrong because CreditMetrics is a simulation, not analytical model. Finally, KMV uses the current stock price and not the historical default rate.

PART five

Operational and Integrated Risk Management

Chapter 24

Operational Risk

By now, the financial industry has developed standard methods to measure and manage market risk and credit risk. The industry is turning next to operational risk, which has proved to be an important cause of financial losses. Indeed, most financial disasters can be attributed to a combination of exposure to market risk or credit risk along with some failure of controls, which is a form of operational risk.

As in the case of market and credit risk, the financial industry is being pushed in the direction of better controls of operational risk by bank regulators. For the first time, the Basel Committee is proposing to establish capital charges for operational risk, in exchange for lowering them on market and credit risk. The proposed charge would constitute approximately 12% of the total capital requirement. This charge is focusing the attention of the banking industry on operational risk.

The problem is that operational risk is much harder to identify than market and credit risk. Even the very definition of operational risk is open to debate. A narrow view is that operational risk is confined to transaction processing. Another, much wider definition views operational risk as any financial risk other than market and credit risk.

As we shall see, it is important for an institution to adopt a definition of operational risk. Consider the sequence of logical steps in a risk management process: (1) identification, (2) measurement, (3) monitoring, and (4) control. Without proper risk identification, it is very difficult to manage risk effectively.[1]

Previously, operational risk was managed by internal control mechanisms within business lines, supplemented by the audit function. The industry is now starting to use specific structures and control processes specifically tailored to operational risk.

---

[1]This sequence is appropriate for market or credit risks. Reflecting the different nature of operational risk, the Basel Committee defines this sequence in terms of: (1) identification, (2) assessment, (3) monitoring, and (4) control/mitigation. See Basel Committee on Banking Supervision. (2003). *Sound Practices for the Management and Supervision of Operational Risk*, BIS.

To introduce operational risk, Section 24.1 summarizes lessons from well-known financial disasters. It then compares the relative importance of operational risk to its siblings, market and credit risk, across business lines. Given this information, Section 24.2 turns to definitions of operational risk. Various measurement approaches are discussed in Section 24.3. Finally, Section 24.4 shows how to use the distribution of operational losses to manage this risk better operational risk and offers some concluding comments.

24.1 The Importance of Operational Risk

The Basel Committee recently reported that

> *"[a]n informal survey . . . highlights the growing realization of the significance of risks other than credit and market risks, such as operational risk, which have been at the heart of some important banking problems in recent years."*

These problems are described in case histories next.

24.1.1 Case Histories

- *February 2002–Allied Irish Bank's ($691 million loss):* A rogue trader, John Rusnack, hides 3 years of losing trades on the yen/dollar exchange rate at the U.S. subsidiary. The bank's reputation is damaged.
- *March 1997–NatWest ($127 million loss):* A swaption trader, Kyriacos Papouis, deliberately covers up losses by mis-pricing and over-valuing option contracts. The bank's reputation is damaged: NatWest is eventually taken over by the Royal Bank of Scotland.
- *September 1996–Morgan Grenfell Asset Management ($720 million loss):* A fund manager, Peter Young, exceeds his guidelines, leading to a large loss. Deutsche Bank, the German owner of MGAM, agrees to compensate the investors in the fund.
- *June 1996–Sumitomo ($2.6 billion loss):* A copper trader amasses unreported losses over 3 years. Yasuo Hamanaka, known as "Mr. Five Percent," after the proportion of the copper market he controlled, is sentenced to prison for forgery and fraud. The banks' reputation is severely damaged.

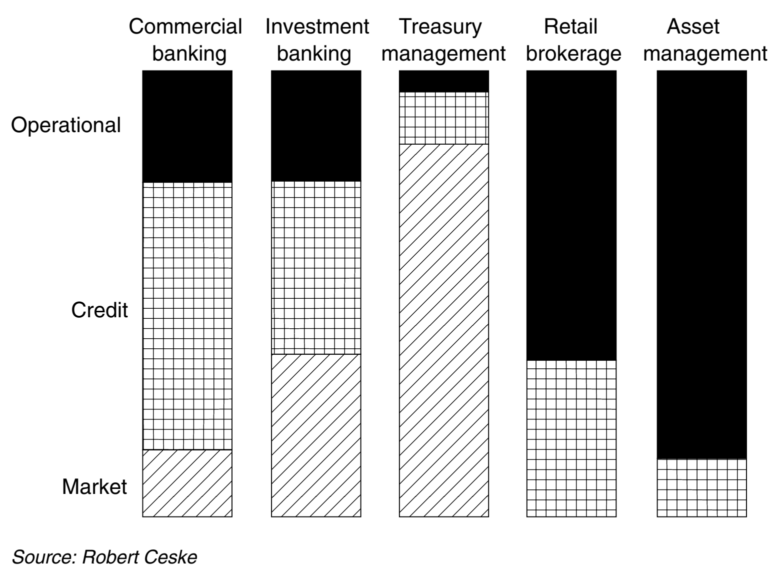
- *September 1995–Daiwa ($1.1 billion loss):* A bond trader, Toshihide Igushi, amasses unreported losses over 11 years at the U.S. subsidiary. The bank is declared insolvent.
- *February 1995–Barings ($1.3 billion loss):* Nick Leeson, a derivatives trader amasses unreported losses over 2 years. Barings goes bankrupt.
- *October 1994–Bankers Trust ($150 million loss):* The bank becomes embroiled in a high-profile lawsuit with a customer that accuses it of improper selling practices. Bankers settles but its reputation is badly damaged. It is later bought out by Deutsche Bank.

The biggest of these spectacular failures can be traced to a **rogue trader**, or a case of internal fraud. They involve a mix of market risk and operational risk (failure to supervise). It should be noted that the cost of these events has been quite high. They led to large direct monetary losses, often to indirect losses due to reputational damage, and sometimes even to bankruptcy.

24.1.2 Business Lines

These failures have occurred across a variety of business lines. Some are more exposed than others to market risk or credit risk. All have some exposure to operational risk. Figure 24-1 provides a typical attribution of risk by business line. This attribution can

FIGURE 24-1 Breakdown of Financial Risks



be interpreted in terms of the amount of economic capital necessary to support each type of risk.

Commercial banking is mainly exposed to credit risk, then to operational risk, then to market risk. Investment banking, trading, and treasury management have greater exposure to market risk. At the other end, business lines such as retail brokerage and asset management are primarily exposed to operational risk. Asset managers, for instance, take no market risk since they act as agents for the investors. If they act in breach of guidelines, however, they may be liable to reimburse clients for their losses, which represents operational risk. Without an appropriate measure of operational risk, institutions may decide to expand into the asset management business if revenues are not properly adjusted for risk.

Similarly, Table 24-1 presents a partial list of risks for market banks that are primarily involved in trading, and credit banks that specialize in lending activities. The table shows that different lines of business are characterized by very different exposures to the listed risks. Credit banks deal with relatively standard products, such as mortgages, with little trading. Hence they have medium operations risk and low operational settlement risk. This is in contrast with trading banks, with constantly changing products and large trading volume, for which both risks are high. Trading banks also have high model risk, because of the complexity of products and high

TABLE 24-1 Examples of Operational Risks

Type of Risk	Definition	Market Bank	Credit Bank
Operations risk	losses due to complex systems and processes	High risk	Medium risk
Ops. settlement risk	lost interest/fines due to failed settlements	High risk	Low risk
Model risk	losses due to imperfect model or data	High risk	Low risk
Fraud risk	reputational/financial damage due to fraud	High risk	Low risk
Misselling risk	losses due to unsuitable sales	Medium risk	Medium risk
Legal risk	reputational/financial damage due to fraud	High risk	Medium risk

Source: Financial Services Authority. (1999). "Allocating Regulatory Capital for Operational Risk," FSA: London.

fraud risk, because of the autonomy given to traders. In contrast, these two risks are low for credit banks.

For trading banks that deal with so-called sophisticated investors, misselling risk has low probability but high value; hence it is a medium risk. (A good example is that of Merrill Lynch settling with Orange County for about $400 million following allegations that the broker had sold the county unsuitable investments.) For credit banks that deal with retail investors, this risk has higher probability but lower value, hence it is a medium risk. Legal risks are high for market banks and medium for credit banks due to the more litigious environment of corporations relative to retail investors.

24.2 Identifying Operational Risk

Operational risk has no clear-cut definition, unlike market risk and credit risk. We can distinguish three approaches, ranging from a broad to a narrow definition.

The first definition is the broadest. It defines operational risk as *any financial risk other than market and credit risk.* This definition is perhaps too broad, as it also includes business risk, which the firm must assume to create shareholder value. This includes poor strategic decision making, such as entering a line of business where margins are too thin. Such risks are not directly controllable by risk managers. Also, a definition in the negative makes it difficult to identify and measure all risks. This opens up the possibility of double counting or gaps in coverage. As a result, this definition is usually viewed as too broad.

At the other extreme is the second definition, which is the narrowest. It defines operational risk as *risk arising from operations*. This includes back office problems, failures in transaction processing and in systems, and technology failures in transaction processing and in systems, and technology breakdowns. This definition, however, just focuses on operations, which is a subset of operational risk, and does not include other significant risks such as internal fraud, improper sales practices, or model risk. As a result, this definition is usually viewed as too narrow.

The third definition is intermediate and seems to be gaining industry acceptance. It defines operational risk as

> *the risk of loss resulting from inadequate or failed internal processes, people and systems, or from external events*

This excludes business risk but includes external events such as external fraud, security breaches, regulatory effects, or natural disasters. Indeed it is now the official Basel Committee definition. It includes legal risk, which arises when a transaction proves unenforceable in law, but excludes strategic and reputational risk.

The British Bankers' Association provides further detail for this definition. Table 24-2 breaks down operational risk into categories of **people risk**, **process risk**, **system risk**, and **external risk**. Among these risks, a notable risk for complex products is **model risk**, which is due to using wrong models for valuing and hedging assets. This is an internal risk that combines lack of knowledge (people) with product complexity/valuation errors (process) and perhaps programming errors (technology).

These classifications are still not totally rigorous, as they confuse the primary source of risks with exposures. Fundamental risks are due to people, technology, and

TABLE 24-2 Operational Risk Classification

Internal Risks		
People	Processes	Systems
Employee collusion/fraud	Accounting error	Data quality
Employee error	Capacity risk	Programming errors
Employee misdeed	Contract risk	Security breach
Employers liability	Misselling/suitability	Strategic risks (platform/suppliers)
Employment law	Product complexity	System capacity
Health and safety	Project risk	System compatibility
Industrial action	Reporting error	System delivery
Lack of knowledge/skills	Settlement/payment error	System failure
Loss of key personnel	Transaction error	System suitability
	Valuation error	

External Risks	
External	Physical
Legal	Fire
Money laundering	Natural disaster
Outsourcing	Physical security
Political	Terrorist
Regulatory	Theft
Supplier risk	
Tax	

Source: British Bankers' Association survey.

external factors. Exposures, for instance systems and controls, do not represent risks but rather means of mitigating risk. Controls can be of two types, **preventative controls** and **damage limitation controls**. The former attempt to decrease the probability of a loss happening; the latter try to limit the size of losses when they occur.

The choice of the appropriate definition is important as the industry starts to tackle operational risk. It is impossible to measure operational risk without a definition, or identification. Measurement, as in the case of market and credit risk, is necessary for better management of operational risk. Also, the function of operational risk manager cannot be properly defined without a definition of the risks that the manager is supposed to oversee. The lack of a precise definition would most likely create conflicts between different categories of risk managers, who would be tempted to attribute losses to somebody else's area of responsibility.

Example 24-1: FRM Exam 2001—Question 48

24-1. Which of the following most reflect an operational risk faced by a bank?
a) A counterparty invokes force majeure on a swap contract.
b) The Federal Reserve unexpectedly cuts interest rates by 100 bps.
c) A power outage shuts down the trading floor indefinitely with no back-up facility.
d) The rating agencies downgrade the sovereign debt of the bank's sovereign counterparty.

Example 24-2: FRM Exam 1998—Question 3/Oper.&Integr.Risk

24-2. Which of the following risks are not related to operational risk?
a) Errors in trade entry
b) Fluctuation in market prices
c) Errors in preparing Master Agreement
d) Late confirmation

Example 24-3: FRM Exam 1999—Question 173/Oper.&Integr.Risk

24-3. A definition of *operational risk* is
I. All the risks that are not currently captured under market and credit risk
II. The potential for losses due to a failure in the operational processes or in the systems that support them
III. The risk of losses due to a failure in people, process, technology or due to external events
a) I only
b) II only
c) II and III only
d) I, II, and III

Example 24-4: FRM Exam 1997—Question 32/Regulatory

24-4. Which of the following is *not* an example of model risk in the context of value at risk measurement models?

a) Model assumptions are adjusted on an annual basis regardless of market and political conditions.
b) The model is developed by a small group of quantitative professionals who are the only personnel who understand its strengths and limitations.
c) Models are validated by an independent risk professional employed by the institution, but who works in another division.
d) Risk managers who use the models are not familiar with underlying model assumptions.

Example 24-5: FRM Exam 1998—Question 5/Oper.&Integr.Risk

24-5. Which of the following may result in an operational risk?

a) Changing a spreadsheet's calculation mode from manual to automatic (Autocalc)
b) Automatic filtering of outliers in historical data
c) Increasing the memory of computers
d) Increasing the CPU speed of computers

Example 24-6: FRM Exam 1998—Question 6/Oper.&Integr.Risk

24-6. Which of the following steps should be done first during risk management processes?

a) Risk measurement
b) Risk control
c) Risk identification
d) Limit setting

24.3 Assessing Operational Risk

Once identified, operational risk should be measured, or assessed if it is less amenable to precise quantification than market or credit risks. Various approaches can be broadly classified into top-down models and bottom-up models.

24.3.1 Comparison of Approaches

Top-down models attempt to measure operational risk at the broadest level, that is, firm-wide or industry-wide data. Results are then used to determine the amount of capital that needs to be set aside as a buffer against this risk. This capital is allocated to business units.

Bottom-up models start at the individual business unit or process level. The results are then aggregated to determine the risk profile of the institution. The main benefit of such approaches is that they lead to a better understanding of the causes of operational losses.

Tools used to manage operational risk can be classified into six categories:

- **Audit oversight**, which consist of reviews of business processes by an external audit department.
- **Critical self assessment**, where each business unit identifies the nature and size of operational risk. These *subjective* evaluations include their expected frequency and severity of losses, as well as a description of how risk is controlled. The tools used for this type of process include checklists, questionnaires, and facilitated workshops.
- **Key risk indicators**, which consist of simple measures that provide an indication of whether risks are changing over time. These *early warning signs* can include audit scores, staff turnover, trade volumes, and so on. The assumption is that operational risk events are more likely to occur when these indicators increase. These *objective* measures allow the risk manager to forecast losses through the application of regression techniques, for example.
- **Earnings volatility** can be used, after stripping the effect of market and credit risk, to assess operational risk. The approach consists of taking a time-series of earnings adjusted for trends, and computing its volatility. This measure is simple to use. It has numerous problems, unfortunately. This risk measure also includes fluctuations due to business and macroeconomic risks, which fall outside of operational risk. Also, such measure is backward-looking and does not account for improvement or degradation in the quality of controls.
- **Causal networks** describe how losses can occur from a cascade of different causes. Causes and effects are linked through conditional probabilities. This process is explained in the appendix. Simulations are then run on the network, generating a distribution of losses. Such bottom-up models improve the understanding of losses since they focus on drivers of risk.
- **Actuarial models**, which combine the distribution of frequency of losses with their severity distribution to produce an *objective* distribution of losses due to operational risk. These can be either bottom-up or top-down models.

24.3.2 Actuarial Models

Actuarial models estimate the objective distribution of losses from historical data and are widely used in the insurance industry. Such models combine two distributions, loss frequencies and loss severities. The **loss frequency distribution** describes the number of loss events over a fixed interval of time. The **loss severity distribution** describes the size of the loss once it occurs.

Loss severities can be tabulated from historical data, for instance measures of the loss severity y_k, at time k. These measures can be adjusted for inflation and some measure of current business activity. Define P_k as the consumer price index at time k and V_k as a business activity measure such as the number of trades. We could assume that the severity is proportional to the volume of business V and to the price level. The *scaled* loss is measured as of time t as

$$x_t = y_k \times \frac{P_t}{P_k} \times \frac{V_t}{V_k} \tag{24.1}$$

Next, define the loss frequency distribution by the variable n, which represents the number of occurrences of losses over the period. The density function is

$$\text{pdf of loss frequency} = f(n), \; n = 0, 1, 2, \ldots \tag{24.2}$$

If x (or X) is the loss severity when a loss occurs, its density is

$$\text{pdf of loss severity} = g(x \mid n = 1), \; x \geq 0 \tag{24.3}$$

Finally, the total loss over the period is given by the sum of individual losses over a random number of occurrences:

$$S_n = \sum_{i=1}^{n} X_i \tag{24.4}$$

Table 24-3 provides a simple example of two such distributions. Our task is now to combine these two distributions into one, that of total losses over the period.

Assuming that the frequency and severity of losses are independent, the two distributions can be combined into a distribution of aggregate loss through a process known as convolution. **Convolution** can be implemented, for instance, through tabu-

TABLE 24-3 Sample Loss Frequency and Severity Distributions

Frequency Distribution		Severity Distribution	
Probability	Frequency	Probability	Severity
0.6	0	0.5	\$1,000
0.3	1	0.3	\$10,000
0.1	2	0.2	\$100,000
Expectation	0.5	Expectation	\$23,500

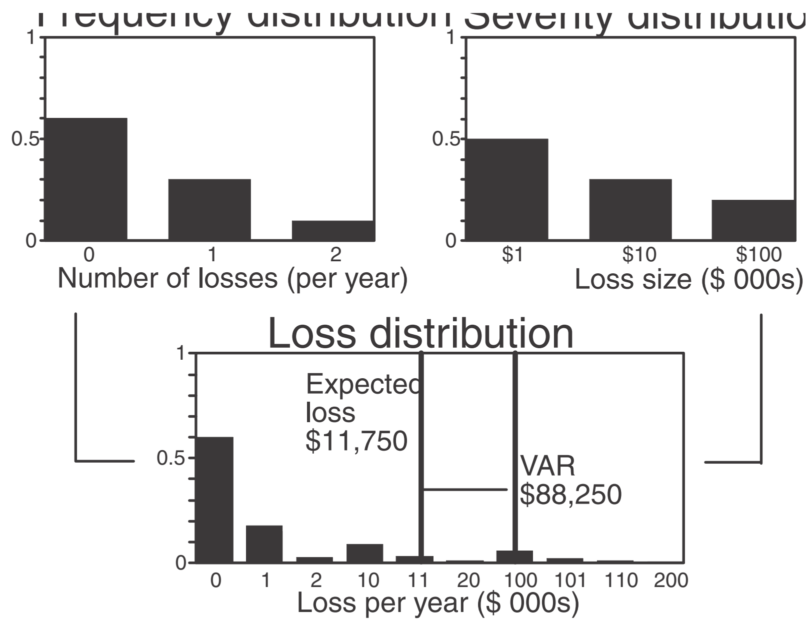
lation. **Tabulation** consist of systematically recording all possible combinations with their probability and is illustrated in Table 24-4.

We start with the obvious case with no loss, which has probability 0.6. Next, we go through all possible realizations of one loss only. From Table 24-3, we see that a loss of \$1,000 can occur with total probability of $P(n = 1) \times P(x = \$1{,}000) = 0.3 \times 0.5 = 0.15$. Similarly for the probability of a one time-loss of \$10,000 and \$100,000, the probability is 0.09 and 0.06, respectively. We then go through all occurrences of two losses, which can result from many different combinations. For instance, a loss of \$1,000 can occur twice, for a total of \$2,000, with a probability of $0.1 \times 0.5 \times 0.5 = 0.025$. We can have a loss of \$1,000 and \$10,000, for a total of \$11,000, with probability of $0.1 \times 0.5 \times 0.3 = 0.015$. And so on until we exhaust all combinations.

TABLE 24-4 Tabulation of Loss Distribution

Nb of losses	First Loss	Second Loss	Total Loss	Probability
0	0	0	0	0.6
1	1000	0	1000	0.15
1	10000	0	10000	0.09
1	100000	0	100000	0.06
2	1000	1000	2000	0.025
2	1000	10000	11000	0.015
2	1000	100000	101000	0.010
2	10000	1000	11000	0.015
2	10000	10000	20000	0.009
2	10000	100000	110000	0.006
2	100000	1000	101000	0.010
2	100000	10000	110000	0.006
2	100000	100000	200000	0.004
Expectation			11750	

FIGURE 24-2 Construction of the Loss Distribution



The resulting distribution is displayed in Figure 24-2. It is interesting to note that the very simple distributions in Table 24-3, with only three realizations, create a complex distribution. We can compute the expected loss, which is simply the product of expected values for the two distributions, or $E[S] = E[N] \times E[X] = 0.5 \times \$23{,}500 = \$11{,}750$. The risk manager can also report the lowest number such that the probability is greater than 95 percent quantile. This is $100,000 with a probability of 96.4%. Hence the unexpected loss, or **operational VAR**, is $\$100,000 - \$11,750 = \$88,250$.

More generally, convolution must be implemented by numerical methods, as there are too many combinations of variables for a systematic tabulation.

Example 24-7: FRM Exam 2000—Question 64/Operational Risk Mgt.

24-7. Which statement about operational risk is *true*?

a) Measuring operational risk requires both estimating the probability of an operational loss event and the potential size of the loss.

b) Measurement of operational risk is well developed, given the general agreement among institutions about the definition of this risk.

c) The operational risk manager has the primary responsibility for management of operational risk.

d) Operational risks are clearly separate from other risks, such as credit and market.

Example 24-8: FRM Exam 1999—Question 166/Oper.&Integr.Risk

24-8. When measuring operational risk, the complete distribution of potential losses for each risk type is formed using

a) An insurance-based volatility distribution

b) Back office distributions of transaction size and number of transactions per day

c) An operational and catastrophic distribution

d) A frequency and severity distribution

Example 24-9: FRM Exam 1999—Question 167/Oper.&Integr.Risk

24-9. A particular operational risk event is estimated to occur once in 200 years for an institution. The loss for this type of event is expected to be between HKD 25 million and HKD 100 million with equal probability of loss in that range (and zero probability outside that range). Based on this information, determine the fair price of insurance to protect the institution against a loss of over HKD 80 million for this particular operational risk.

a) HKD 133,333

b) HKD 90,000

c) HKD 120,000

d) HKD 106,667

Example 24-10: FRM Exam 1999—Question 169/Oper.&Integr.Risk

24-10. The measurement of exposure to operational risk should be based on the assessment of

I. The probability of an operational failure

II. The extent of insurance coverage

III. The probability distribution of losses in case of failure

a) I only

b) II only

c) I and III only

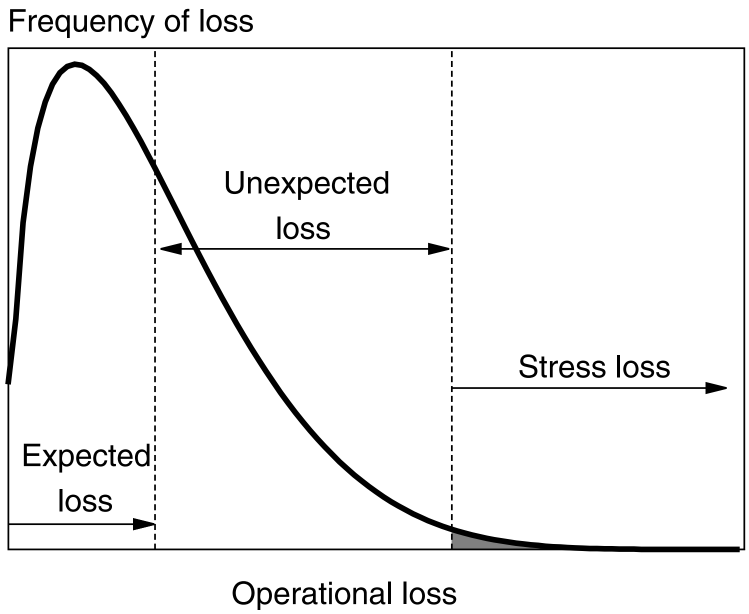
d) I, II, and III

24.4 Managing Operational Risk

24.4.1 Capital Allocation and Insurance

Like market VAR, the distribution of operational losses can be used to estimate expected losses as well as the amount of capital required to support this financial risk. Figure 24-3 highlights important attributes of a distribution of losses, taken as positive values, due to operational risk.

FIGURE 24-3 Distribution of Operational Losses



The **expected loss** represents the size of operational losses that should be expected to occur. Typically, this represents high frequency, low severity events. This type of loss is generally absorbed as an ongoing cost and managed through internal controls. Such losses are rarely disclosed. systems.

The **unexpected loss** represents the deviation between the quantile loss at some confidence level and the expected loss. Typically, this represents lower frequency, higher severity events. This type of loss is generally offset against capital reserves or transferred to an outside insurance company, when available. Such losses are sometimes disclosed publicly but often with little detail.

The **stress loss** represents a loss in excess of the unexpected loss. By definition, such losses are very infrequent but extremely damaging to the institution. The Barings bankruptcy can be attributed, for instance, in large part to operational risk. This type of loss cannot be easily offset through capital allocation, as this would require too much capital. Ideally, it should be transferred to an insurance company. Due to their severity, such losses are disclosed publicly.

Even so, purchasing insurance is no panacea. The insurance payment would have to be very quick and in full. The bank could fail while waiting for payment, or arguing over the size of compensation. because, once the insurance is acquired, the purchaser has less incentives to control losses. This problem is called **moral hazard**. The insurer will be aware of this and increase the premium accordingly. The premium may also be high because of the **adverse selection** problem. This describes a situation where banks vary in the quality of their controls. Banks with poor controls are more likely to purchase insurance than banks with good controls. Because the insurance company does not know what type of bank it is facing, it will increase the average premium.

Example 24-11: FRM Exam 2001—Question 49

24-11. Which of the term below is used within the insurance industry to refer to the effect of a reduction in control of losses by an individual insured insured due to the protection provided by insurance?

a) Control trap
b) Moral hazard
c) Adverse selection
d) Control hazard

Example 24-12: FRM Exam 2001—Question 51

24-12. Which of the terms below refers to the situation where the various buyers of insurance have different expected losses, but the insurer (or the capital market, as the seller of insurance) is unable to distinguish between the different types of hedge buyer and is therefore unable to charge differentiated premiums?

a) Moral hazard
b) Average insurance
c) Adverse selection
d) Control hazard

24.4.2 Mitigating Operational Risk

The approach so far has consisted of taking operational risk as given. Such measures are extremely useful because they highlight the size of losses due to operational risk. Armed with this information, the institution can then decide whether it is worth spending resources on decreasing operational risk.

Say that a bank is wondering whether to install a **straight-through processing** system, which automatically captures trades in the front office and transmits them to the back office. Such system eliminates manual intervention and the potential for human errors, thereby decreasing losses due to operational risk. The bank should purchase the system if its cost is less than its operational risk benefit.

More generally, reduction of operational risk can occur in the frequency of losses and/or in the size of losses when they occur. Operational risk is also contained by a firm-wide risk management framework. In a later chapter, we will discuss *best practices* in risk management, which are designed to provide some protection against operational risk.

Consider for instance a transaction in a plain-vanilla, 5-year, interest rate swap. This simple instrument generates a large number of cash flows, each of which have the potential for errors. At initiation, the trade needs to be booked and confirmed with the counterparty. It needs to be valued so that a P&L can be attributed to the trading unit. With biannual payments, the swap will generate ten cash flows along with ten rate resets and net payment computations. These payments need to be computed with absolute accuracy, that is, to the last cent. Errors can range from minor issues, such as paying a day late, to major problems, such as failure to hedge or fraudulent valuation by the trader.

The swap will also create some market risk, which may need to be hedged. The position needs to be transmitted to the market risk management system, which will monitor the total position and risk of the trader and of the institution as a whole. In addition, the current and potential credit exposure needs to be regularly measured and added up to all other trades with the same counterparty. Errors in this risk measurement process can lead to excessive exposure to market and/or credit risk.

Operational risk can be minimized in a number of ways.[2] Internal control methods consist of

- *Separation of functions*: Individuals responsible for committing transactions should not perform clearance and accounting functions.
- *Dual entries*: Entries (inputs) should be matched from two different sources, that is, the trade ticket and the confirmation by the back office.
- *Reconciliations*: Results (outputs) should be matched from different sources, for instance the trader's profit estimate and the computation by the middle office.
- *Tickler systems*: Important dates for a transaction (e.g., settlement, exercise dates) should be entered into a calendar system that automatically generates a message before the due date.
- *Controls over amendments*: Any amendment to original deal tickets should be subject to the same strict controls as original trade tickets.

[2]See also Brewer. (1997). Minimizing Operations Risk. In Schwartz, R. & Smith C. (Eds.). *Derivatives Handbook*. New York: Wiley.

External control methods consist of

- *Confirmations*: Trade tickets need to be confirmed with the counterparty, which provides an independent check on the transaction.
- *Verification of prices*: To value positions, prices should be obtained from external sources. This also implies that an institution should have the capability of valuing a transaction in-house before entering it.
- *Authorization*: The counterparty should be provided with a list of personnel authorized to trade, as well as a list of allowed transactions.
- *Settlement*: The payment process itself can indicate if some of the terms of the transaction have been incorrectly recorded, for instance, as the first cash payments on a swap are not matched across counterparties.
- *Internal/external audits*: These examinations provide useful information on potential weakness areas in the organizational structure or business process.

24.5 Conceptual Issues

The management of operational risk, however, is still beset by conceptual problems. First, unlike market and credit risk, operational risk is largely internal to financial institutions. This makes it difficult to collect data on operational losses which ideally should cover a large number of operational failures, because institutions are understandably reluctant to advertise their mistakes. Another problem is that losses may not be directly applicable to another institution, as they were incurred under possibly different business profiles and internal controls.

Second, market and credit risk can be conceptually separated into exposures and risk factors. Exposures can be easily measured and controlled. In contrast, the link between risk factors and the likelihood and size of operational losses is not so easy to establish. Here, the line of causation runs through internal controls.

Third, very large operational losses, which can threaten the stability of an institution, are relatively rare (thankfully so). This leads to a very small number or observations in the tails. Such “thin tails” problem makes it very difficult to come up with a robust “value for operational risk” (VOR) at a high confidence level. As a result, there is still some skepticism as to whether operational risk can be subject to the same quantification as market and credit risks.

Example 24-13: FRM Exam 1999—Question 170/Oper.&Integr.Risk

24-13. Operational risk capital (ORC) should provide a cushion against

I. Expected losses
II. Unexpected losses
III. Catastrophic losses

a) I only
b) II only
c) I and II only
d) I, II, and III

Example 24-14: FRM Exam 1998—Question 4/Oper.&Integr.Risk

24-14. What can be said about the impact of operational risk on both market risk and credit risk?

a) Operational risk has no impact on market risk and credit risk.
b) Operational risk has no impact on market risk but has impact on credit risk.
c) Operational risk has impact on market risk but no impact on credit risk.
d) Operational risk has impact on market risk and credit risk.

24.6 Answers to Chapter Examples

Example 24-1: FRM Exam 2001—Question 48

c) A power outage is an example of system failure, which is part of the operational risk definition. Answer (d) is a case of credit risk. Answer (b) is a case of market risk. Answer (a) is a mix of credit and legal risk.

Example 24-2: FRM Exam 1998—Question 3/Oper.&Integr.Risk

b) Fluctuations in market prices reflect market risk.

Example 24-3: FRM Exam 1999—Question 173/Oper.&Integr.Risk

d) All the three definitions have been used and highlight a different aspect of operational risk.

Example 24-4: FRM Exam 1997—Question 32/Regulatory

c) Model risk includes model assumptions that are too rigid (a), that are only understood by a small group of people (b) or not understood by risk managers (d). Having the models validated by independent reviewers decreases model risk.

Example 24-5: FRM Exam 1998—Question 5/Oper.&Integr.Risk

b) Automatic filtering of outliers may weed out bad data points but also reject real observations, which may bias downward forward-looking measures of risk. Also, changing a spreadsheet's calculation mode from automatic to manual can create operational risk.

Example 24-6: FRM Exam 1998—Question 6/Oper.&Integr.Risk

c) We need to identify risk, before measuring, controlling and managing them.

Example 24-7: FRM Exam 2000—Question 64/Operational Risk Mgt.

a) Constructing the operational loss requires the probability, or frequency, of the event as well as estimates of potential loss sizes. Answer (b) is wrong as measurement of op risk is still developing. Answer (c) is wrong as the business unit is also responsible for controlling operational risk. Answer (d) is wrong as losses can occur as a combination of operational and market or credit risks.

Example 24-8: FRM Exam 1999—Question 166/Oper.&Integr.Risk

d) The distribution of losses due to operational risk results from the combination of loss frequencies and loss severities.

Example 24-9: FRM Exam 1999—Question 167/Oper.&Integr.Risk

c) The expected loss severity is, with a uniform distribution from 80 to 100 million, 90 million. The frequency of this happening would be once every 200 years times the ratio of the [80, 100] range to the total range [25, 100], which is $(20/75)/200 = 0.001333$. The expected loss is $90,000,000 \times 0.00133 =$ HKD120,000.

Example 24-10: FRM Exam 1999—Question 169/Oper.&Integr.Risk

c) The distribution of losses due to operational risk is derived from the loss frequency (I) and loss severity distributions (III).

Example 24-11: FRM Exam 2001—Question 49

b) Moral hazard arises when insured individuals have no incentive to control their losses because they are insured.

Example 24-12: FRM Exam 2001—Question 51
b) Adverse selection refers to the fact that individuals buy insurance knowing that they have greater risk than the average, but that the insurer charges the same premium to all.

Example 24-13: FRM Exam 1999—Question 170/Oper.&Integr.Risk
b) Capital can only provide protection against unexpected losses at a high confidence level. Above that, insurance can pick up the risk.

Example 24-14: FRM Exam 1998—Question 4/Oper.&Integr.Risk
d) As seen in the example of the effect of a failure to record the terms of the swap correctly, operational risk can create both market and credit risk.

Appendix: Causal Networks

Causal networks explain losses in terms of a sequence of random variables. Each variable itself can be due to the combination of other variables. For instance, settlement losses can be viewed as caused by a combination of (1) exposure and (2) time delay. In turn, exposure depends on (a) the value of the transaction and (b) whether it is a buy or sell. Next, the causal factor for time delay can be chosen as (a) the exchange, (b) the domicile, (c) the counterparty, (d) the product, and (e) daily volume.

These links are displayed through graphical models based on process work flows. One approach is the **Bayesian network**. Here, each node represents a random variable; each arrow represents a causal link.

Causes and effects are related through conditional probabilities, an application of Bayes' theorem. For instance, suppose we want to predict the probability of a settlement failure, or *fail*. Set $y = 1$ if there is a failure and zero otherwise. The causal factor is, say, the quality of the back-office team, which can be either good or bad. Set $x = 1$ if the team is bad. Assume there is a 20 percent probability that the team is bad. If the team is good, the conditional probability of a fail is $P(y = 1 \mid x = 0) = 0.1$. If the team is bad, this probability is higher, $P(y = 1 \mid x = 1) = 0.7$. We can now construct the unconditional probability of a fail, which is

$$P(y = 1) = P(y = 1 \mid x = 0)P(x = 0) + P(y = 1 \mid x = 1)P(x = 1) \qquad (24.5)$$

which is here $P(y = 1) = 0.1 \times (1 - 0.20) + 0.7 \times 0.20 = 0.22$. Armed with this information, we can now evaluate the benefit of changing the team from bad to good through training, for example, or new hires. Or, we could assess the probability that the team is bad given that a fail has occurred. Using Bayes' rule, this is

$$P(x = 1 \mid y = 1) = \frac{P(y = 1, x = 1)}{P(y = 1)} = \frac{P(y = 1 \mid x = 1)P(x = 1)}{P(y = 1)} \tag{24.6}$$

which is here $P(x = 1 \mid y = 1) = \frac{0.7 \times 0.20}{0.22} = 0.64$. In other words, the probability that the team is bad has increased from 20 percent to 64 percent based on the observed fail. Such observation is useful for process diagnostics.

Once all initial nodes have been assigned probabilities, the Bayesian network is complete. The bank can now perform Monte Carlo simulations over the network, starting from the initial variables and continuing to the operational loss to derive a distribution of losses.

Chapter 25

Risk Capital and RAROC

The methodologies described so far have covered market, credit, and operational risk. In each case, the distribution of profits and losses reveals a number of essential insights. First, the expected loss is a measure of reserves necessary to guard against future losses. At the very least, the pricing of products should provide a buffer against expected losses. Second, the unexpected loss is a measure of the amount of economic capital required to support the bank's financial risk. This capital, also called **risk capital**, is basically a value-at-risk (VAR) measure.

Armed with this information, institutions can now make better informed decision about business lines. Each activity should provide sufficient profit to compensate for the risks involved. Thus, product pricing should account not only for expected losses but also for the remuneration of risk capital.

Some activities may require large amounts of risk capital, which in turn requires higher than otherwise returns. This is the essence of **risk-adjusted return on capital** (RAROC) measures. The central objective is to establish benchmarks to evaluate the economic return of business activities. This includes transactions, products, customer trades, business lines, as well as the entire business.

RAROC is also related to concepts such as shareholder value analysis and economic value added. In the past, performance was measured by yardsticks such as **return on assets** (ROA), which adjusts profits for the associated book value of assets, or **return on equity** (ROE), which adjusts profits for the associated book value of equity. None of these measures is satisfactory for evaluating the performance of business lines as they ignore risks.

Section 25.1 introduces RAROC measures for performance evaluation. The section also demonstrates the link between RAROC and other concepts such as shareholder value analysis and economic value added. Section 25.2 then shows how to use risk-adjusted returns to evaluate products and business lines.

25.1 RAROC

RAROC was developed by Bankers Trust in the late 1970s. The bank was faced with the problem of evaluating traders involved in activities with different risk profiles.

25.1.1 Risk Capital

RAROC is part of the family of **risk-adjusted performance measures** (RAPM). Consider, for instance, two traders that each returned a profit of $10 million over the last year. The first is a foreign currency trader, the second a bond trader. The question is, How do we compare their performance? This is important in order to provide appropriate compensation as well as to decide in which line of activity to expand.

Assume the FX and bond traders have notional amount and volatility as described in Table 25-1. The bond trader deals in larger amounts, $200 million, but in a market with lower volatility, at 4 percent per annum, against $100 million and 12 percent for the FX trader. The **risk capital** (RC) can be computed as a VAR measure, say at the 99 percent level over a year, as Bankers Trust did. Assuming normal distributions, this translates into a risk capital of

$$\text{Risk Capital (RC)} = \text{VAR} = \$100{,}000{,}000 \times 0.12 \times 2.33 = \$28\text{million}$$

for the FX trader and $19 million for the bond trader. More precisely, Bankers Trust computes risk capital from a weekly standard deviation σ_w as

$$\text{RC} = 2.33 \times \sigma_w \times \sqrt{52} \times (1 - \text{tax rate}) \times \text{Notional} \tag{25.1}$$

which includes a tax factor that determines the amount required on an after-tax basis.

TABLE 25-1 Computing RAPM

	Profit	Notional	Volatility	VAR	RAPM
FX trader	$10	$100	12	$28	36%
Bond trader	$10	$200	4	19%	54%

The risk-adjusted performance is then measured as the dollar profit divided by the risk capital

$$\text{RAPM} = \frac{\text{Profit}}{\text{RC}} \tag{25.2}$$

and is shown in the last column. Thus the bond trader is actually performing better than the FX trader as the activity requires less risk capital. More generally, risk capital should account for credit risk, operational risk, as well as any interaction.

It should be noted that this approach views risk on a stand-alone basis, i.e. using each product's volatility. In theory, for capital allocation purposes, risk should be viewed in the context of the bank's whole portfolio and measured in terms of marginal contribution to the bank's overall risk. In practice, however, it is best to charge traders for risks under their control, which means the volatility of their portfolio.

25.1.2 RAROC Methodology

RAROC measures proceed in three steps.

- *Risk measurement.* This requires the measurement of portfolio exposure, of the volatility and correlations of the risk factors.
- *Capital allocation.* This requires the choice of a confidence level and horizon for the VAR measure, which translates into an economic capital. The transaction may also require a regulatory capital charge if appropriate.
- *Performance measurement.* This requires the adjustment of performance for the risk capital.

Performance measurement can be based on a RAPM method or one of its variants. For instance, **economic value added (EVA)** focuses on the creation of value during a particular period in excess of the required return on capital. EVA measures residual economic profits as

$$\text{EVA} = \text{Profit} - (\text{Capital} \times k) \tag{25.3}$$

where profits are adjusted for the cost of economic capital defining k as a discount rate. Assuming the whole worth is captured by EVA, the higher the EVA, the better the project or product.

RAROC is formally defined as

$$\text{RAROC} = \frac{[\text{Profit} - (\text{Capital} \times k)]}{\text{Capital}} \tag{25.4}$$

This is a *rate of return*, obtained by dividing the dollar EVA return by the dollar amount of capital.[1]

Another popular performance measure is **shareholder value analysis** (SVA), whose purpose is to maximize the total value to shareholders. The framework is that of a net present value (NPV) analysis, where the worth of a project is computed by taking the present value of future cash flows, discounted at the appropriate interest rate k, minus the up-front capital. A project that has positive NPV creates positive shareholder value.

Although SVA is a prospective multiperiod measure whereas EVA is a one-period measure, EVA and SVA are consistent with each other provided the same inputs are used. Consider, for instance, a one-period model where capital is fully invested or excess capital has zero return. The next period payoff is then the profit plus the initial capital; we discount this payoff at the cost of capital and subtract the initial capital. We seek to maximize the NPV, or SVA, which is

$$\text{NPV} = \frac{[\text{Profit} + \text{Capital}]}{(1+k)} - \text{Capital} = \frac{[\text{Profit} - \text{Capital} \times k]}{(1+k)} \tag{25.5}$$

which is equivalent to maximizing the numerator, or EVA.

If the risk capital can be invested at the rate r, the final payoff must account for the return on capital. The numerator is then modified to

$$\text{EVA} = [\text{Profit} - \text{Capital} \times (k - r)] \tag{25.6}$$

25.1.3 Application to Compensation

This system allows the trader's compensation to be adjusted for the risk of the activities. The goal is not to decrease total compensation, however. This is illustrated in Table 25-2. Under the old bonus system, the bonus is 20 percent of the profit, or $2 million for the FX trader. We assume that the FX trader has control over the average volatility and want to encourage him or her to lower risk.

[1] This measure is sometimes called RARORAC, or risk-adjusted return on risk-adjusted capital. Some definitions of RAROC use regulatory capital in the denominator. Another measure is RORAC, or return on risk-adjusted capital, which omits the adjustment in the denominator.

The benchmark, or target risk, is set at $20 million and described in the last row. The new bonus scheme pays a percentage of the EVA using a cost of capital of 15 percent. Thus for the FX trader, the EVA is $\$10 - 15\% \times \$28 = \$5.8$ million. We now calibrate the multiplier so that a target RC of $20 million would result in a bonus of $2 million. Hence, the total compensation is still the same if the risk capital is equal to that of the benchmark. This yields a multiplier of 29 percent. Note that the benchmark compensation is the same under the old and new system.

Table 25-2 shows that the new bonus system would result in a payment of $29\% \times \$5.8 = \$1.7$ million to the FX trader. This is less than under the old system due to the fact that the risk capital was higher than the benchmark. Such a system will immediately capture the attention of the trader, who will now focus on risk as well as profits. The other trader, with the same profit but lower capital, has a higher bonus than under the old system, at $2.1 million instead of $2 million.

TABLE 25-2 Risk-Adjusted Compensation ($ Millions)

	Profit	Capital (VAR)	Bonus old	Capital Charge	EVA	Bonus new
	(1)	(2)	(3)	(4)	(5)	(6)
			20% × (1)	15% × (2)	(1) − (4)	29% × (5)
FX trader	$10	$28	$2.0	$4.2	$5.8	$1.7
Bond trader	$10	$19	$2.0	$2.8	$7.2	$2.1
Benchmark	$10	$20	$2.0	$3.0	$7.0	$2.0

Example 25-1: FRM Exam 1999—Question 159/Oper.&Integr.Risk

25-1. To calculate risk-adjusted return on capital (RAROC), what information is required?

a) 1-year holding period, 99% confidence interval loss for the portfolio
b) Tax rate
c) Both (a) and (b)
d) None of the above

Example 25-2: FRM Exam 2000—Question 70/Operational Risk Mgt.

25-2. A bond trader deals in $100 million in a market with very high volatility of 20 percent per annum. He yields $10 million profit. The risk capital (RC) is computed as a value-at-risk (VAR) measure at the 99 percent level over a year. Assuming normal distribution of return, calculate the risk-adjusted performance measure (RAPM).

a) 15.35%
b) 19.13%
c) 21.46%
d) 25.02%

25.2 Performance Evaluation and Pricing

We now give the example of the analysis of the risk-adjusted return for an interest rate swap. All revenue and cost items should be attributed to the product.

- *Gross revenue* consists of the present value of the bid and ask spread plus any fees.
- *Hedging costs* can be traced to the need to hedge out market risk, as incurred.
- *Expected credit costs* measure the statistically expected losses due to credit risk (also known as **credit provision**) and operational risk.
- *Operating costs* reflect direct, indirect, and overhead expenses.
- *Tax costs* measure tax expenses.

The sum of revenues minus all costs can be called *expected net income*. It still does not account for the remuneration of risk capital. This is the purpose of EVA, as in Equation (25.3). EVA and RAROC allow the institution to evaluate an existing product or business line.

This application is still passive. The same methodology can be inverted to make **pricing decisions**, i.e. to determine the minimum revenue required for a transaction to be viable. Consider the EVA formula, Equation (25.3). This can also be viewed as a minimum amount of revenues that covers costs and the cost of risk capital:

$$\text{Revenue} = \text{Costs} + [\text{Capital} \times (k - r)] \tag{25.7}$$

As an example, we illustrate the pricing of a 5-year interest rate swap for various credit counterparties, which is shown in Table 25-3.[2] Assuming there is only credit risk or that the swap is hedged against market risk, we can compute various costs expressed in basis points (bp) of the notional, including the expected credit loss. This corresponds to the actuarial estimate of credit loss, from the combination of credit exposure, probability of default, and loss given default. For the Aaa credit, for example, this amounts to 0.29bp of principal, which is very low, reflecting the low probability of default.[3]

The next step is to compute the amount of risk capital required to support the transaction. This can be derived from the unexpected loss, or credit VAR. For the Aaa credit, this is 4.00bp. Assume that the cost of capital is 15 percent but that capital is invested at 8 percent, which yields a net cost of capital of 7 percent. The required net income is then 7 percent of 4.00bp, or 0.28bp.

The rest of the table then works backward, starting with taxes of 40% which requires a pretax net income of $0.28/(1-40\%) = 0.47$bp. To this we add operating costs, the credit provision, and hedging costs for a total of 2.25 bp in required revenues. For a Baa credit counterparty, the required revenue would be higher, at 8.50bp, due to higher credit provisions and a higher risk capital.

TABLE 25-3 Pricing a Swap (Basis Points)

	Aaa	Aa	A	Baa
Capital at Risk	4.00	8.00	15.00	25.00
Cost of capital (7%)				
Required Net Income	0.28	0.56	1.05	1.75
Tax (40%)	0.19	0.37	0.70	1.17
Pretax net income	0.47	0.93	1.75	2.92
Operating costs	1.00	1.50	2.00	2.50
Credit provision	0.29	0.56	1.05	2.58
Hedging costs	0.50	0.50	0.50	0.50
Required revenue	2.25	3.50	5.30	8.50

[2]See also Lam. (1997). Firmwide Risk Management. In Schwartz, R. and Smith, C. (Eds.). *Derivatives Handbook.* New York: Wiley.

[3]This should be obtained using the methodology presented in Chapter 23 for computing the PVECL. For instance, with a 5-year cumulative default rate of 0.29%, average credit exposure of 1% of notional, 100 percent loss given default, and no discounting, we get exactly a PVECL of 29bp.

25.3 Answers to Chapter Examples

Example 25-1: FRM Exam 1999—Question 159/Oper.&Integr.Risk

c) Bankers RAROC computes the risk capital using the quantitative parameters in (a) plus a tax factor. So, the answer is both (a) and (b).

Example 25-2: FRM Exam 2000—Question 70/Operational Risk Mgt.

c) VAR is $\$100,000,000 \times 0.2 \times 2.33 = \$46,600,000$. hence RAPM is $\$10/\$46 = 21.46\%$.

Chapter 26

Best Practices Reports

Best practices in the industry have evolved from the lessons of financial disasters. Some well-publicized losses in the early 1990s led to the threat of regulatory action against derivatives. Indeed, a warning shot was fired on January 1992 by Gerald Corrigan, then president of the New York Federal Reserve Bank:

> *High-tech banking and finance has its place, but it's not all it's cracked up to be. I hope this sounds like a warning, because it is.*

Financial institutions then realized that it was in their best interests to promote a set of best practices to forestall regulatory action. This led to the Group of Thirty (G-30) report, which was issued in July 1993.

The 1995 Barings failure was followed by an in-depth report from the Bank of England in July. Similarly, the 1998 near-failure of Long-Term Capital Management (LTCM) was analyzed in a report produced by the Counterparty Risk Management Policy Group (CRMPG) in June 1999. These reports added to the collective wisdom about best practices.

This chapter reviews the lessons from reports that have shaped the risk management profession. Section 26.1 summarize the G-30 report, Section 26.2 the Bank of England report, and Section 26.3 the CRMPG report.

26.1 The G-30 Report

The Group of Thirty (G-30) is a private, nonprofit association, consisting of senior representatives of the private and public sector and of academia. In the wake of the derivatives disasters of the early 1990s, the G-30 issued a report in 1993 that has become a milestone document for risk management.[1]

[1]Group of Thirty. (1993). *Derivatives: Practices and Principles*. New York: Group of Thirty. The report is available at the IFCI Web site, risk.ifci.ch , maintained by the *International Finance and Commodities Institute* (ICFI), a nonprofit Swiss foundation.

The report provides a set of 24 sound management practices, which are summarized as follows. These recommendations have implications for (1) the culture of the organization, (2) the systems used, and (3) the required expertise at all levels.

1. Role of Senior Management

Dealers and end-users should use derivatives in a manner consistent with the overall risk management and capital policies approved by their boards of directors.... Policies governing derivatives use should be clearly defined, including the purposes for which these transactions are to be undertaken. Senior management should approve procedures and controls to implement these policies, and management at all levels should enforce them.

In other words, derivatives policies should be set by top management.

2. Marking-to-Market

Dealers should mark their derivatives positions to market, on at least a daily basis, for risk management purposes.

In other words, marking to market is the most appropriate valuation technique. Countless mistakes have happened when institutions valued instrument using a historical, accrual method.

3. Market Valuation Methods

Derivatives portfolios of dealers should be valued based on mid-market levels less specific adjustments, or on appropriate bid or offer levels.

In addition, adjustments should be made for expected credit losses and administrative costs.

4. Identifying Revenue Sources

Dealers should measure the components of revenue regularly and in sufficient detail to understand the sources of risk.

In other words, users should understand the drivers of profit and losses as well as their major risks.

5. Measuring Market Risk

Dealers should use a consistent measure to calculate daily the market risk of their derivatives positions and compare it to market risk limits.

- *Market risk is best measured as "value at risk" using probability analysis based upon a common confidence interval (e.g., two standard deviations) and time horizon (e.g., a one-day exposure).*

- *Components of market risk that should be considered across the term structure include: absolute price or rate change (delta); convexity (gamma); volatility (vega); time decay (theta); basis or correlation; and discount rate (rho).*

This recommendation endorsed VAR as the "best" measure of market risk.

6. Stress Simulations

Dealers should regularly perform simulations to determine how their portfolios would perform under stress conditions.

In other words, VAR measures should be complemented by stress test simulations to examine the effect of rare and extreme events.

7. Investing and Funding Forecasts

Dealers should periodically forecast the cash investing and funding requirements arising from their derivatives portfolios.

In other words, liquidity requirements should be closely watched.

8. Independent Market Risk Management

Dealers should have a market risk management function, with clear independence and authority, to ensure that the following responsibilities are carried out:

- *Risk limits (recommendation 5)*
- *Stress tests (recommendation 6)*
- *Revenue reports (recommendations 4 and 5)*
- *Backtesting VAR*
- *Review of pricing models and reconciliation procedures*

This recommendation stresses the need for a market risk management function with "clear independence and authority" (of the trading function).

9. Practices by End-Users

(. . .) end-users should adopt the same valuation and market risk management practices that are recommended for dealers.

In other words, these recommendations also apply to end-users.

10. Measuring Credit Exposure

Dealers and end-users should measure credit exposure on derivatives in two ways:

- *Current exposure, which is the replacement cost of derivatives transactions, that is, their market value*
- *Potential exposure, which is an estimate of the future replacement cost of derivatives transactions (. . .)*

Credit exposure is a function of the current market value of the asset and of potential further increases.

11. Aggregating Credit Exposure

Credit exposures on derivatives, and all other credit exposures to a counterparty, should be aggregated taking into consideration enforceable netting arrangements. Credit exposures should be calculated regularly and compared with credit limits.

In other words, exposure should be controlled at the counterparty level.

12. Independent Credit Risk Management

Dealers and end-users should have a credit risk management function with clear independence and authority, and with analytical capabilities in derivatives, responsible for:

- *Approving credit exposure measurement standards*
- *Setting credit limits and monitoring their use*
- *Reviewing credits and concentrations of credit risk*
- *Reviewing and monitoring risk reduction arrangements*

This also endorses the need for a credit risk management function. Here again, the emphasis is on "clear independence."

13. Master Agreements

Dealers and end-users are encouraged to use one master agreement as widely as possible with each counterparty to document existing and future derivatives transactions, including foreign exchange forwards and options.

These master agreements are extremely useful as they reduce legal risks. In addition, they can substantially reduce credit exposures if they allow netting.

14. Credit Enhancement

Dealers and end-users should assess both the benefits and costs of credit enhancement and related risk-reduction arrangements.

Credit enhancement methods should be expanded as they can substantially reduce credit exposures.

15. Promoting Enforceability

Dealers and end-users should work together on a continuing basis to identify and recommend solutions for issues of legal enforceability, both within and across jurisdictions, as activities evolve and new types of transactions are developed.

This prods the industry into developing solutions to reduce the uncertainty about enforceability of contracts.

16. Professional Expertise

Dealers and end-users must ensure that their derivatives activities are undertaken by professionals in sufficient number and with the appropriate experience, skill levels, and degrees of specialization.

Thus, promotion of knowledge and good practices in risk management is important, as advocated by GARP.

17. Systems

Dealers and end-users must ensure that adequate systems for data capture, processing, settlement, and management reporting are in place so that derivatives transactions are conducted in an orderly and efficient manner in compliance with management policies.

Derivatives activities can only be safely conducted if supported by the requisite technology.

18. Authority

Management of dealers and end-users should designate who is authorized to commit their institutions to derivatives transactions.

In other words, authority to trade should be granted only to specific individuals.

Finally, recommendations 19 to 24 deal with accounting and disclosure issues. Perhaps the most important principle behind these recommendations is the separation of the risk management functions from those of trading.

Example 26-1: FRM Exam 1997—Question 4/Regulatory

26-1. What did the Group of 30 develop?

a) A set of risk management principles
b) A regulatory framework for the Federal Reserve and the BIS
c) A manual for derivatives users
d) A set of recommendations for international futures exchanges

26.2 The Bank of England Report on Barings

Indeed, violation of this fundamental principle or separation of functions was the primary cause of the Barings failure. Nick Leeson had control over both the front office and the back office. This organizational structure allowed him to falsify trading entries, hiding losses in a special account.

But new lessons were also described in the main report on Barings, produced by the Bank of England (BoE).[2] The report mentioned for the first time "reputational risk." **Reputational risk** is the risk of indirect losses to earnings arising from negative public opinion. These losses are distinct from the direct monetary loss ascribed to an event.

As an example, Bankers Trust became embroiled in a dispute with Procter & Gamble, a U.S. consumer product company, over losses in a swap contract. This feud damaged the reputation of Bankers Trust and caused indirect reputational losses over and above the amount that the bank eventually paid.

The BoE report also reiterated several lessons from this disaster.

- **Duty to understand**

  Management teams have a duty to understand fully the businesses they manage. Senior Barings management later claimed they did not fully understand the nature of their business (which is equivalent to claiming financial insanity, or that one is not responsible for financial losses due to a lack of understanding).

- **Clear responsibility**

  Responsibility for each business activity must be clearly established. Barings had a *matrix* structure, with responsibilities assigned by product and region, which made it harder to assign responsibility on one person.

- **Relevant internal controls**

  Internal controls, including clear segregation of duties, is fundamental to any effective risk control system.

- **Quick resolution of weaknesses**

  Any weakness identified by an internal or external audit must be addressed quickly. In the Barings case, an internal audit report in the summer of 1994 had identified the lack of segregation of duties as a significant weakness. Yet this was not addressed by Barings top management.

---

[2]Bank of England. (1995). *Report of the Board of Banking Supervision Inquiry into the Circumstances of the Collapse of Barings*, London: HMSO Publications. The report is available at the IFCI Web site, risk.ifci.ch.

26.3 The CRMPG Report on LTCM

The near-failure of the hedge fund Long-Term Capital Management (LTCM) also led to useful lessons for the industry. The Counterparty Risk Management Policy Group (CRMPG) was established in the wake of the LTCM near-failure to strengthen practices related to the management of financial risks.

The CRMPG consists of senior-level practitioners from the financial industry, including many banks that provided funding to LTCM. The industry came under criticism for allowing LTCM to build up so much leverage. Apparently, loans to LTCM were fully collateralized as to their current, but not potential exposure. In fact, it was the fear of disruption of markets and the potential for large losses that led the New York Federal Reserve Bank to orchestrate a bailout of LTCM.

In response, the CRMPG report provides a set of recommendations, summarized as follows.[3]

1. Information Sharing

Financial institutions should obtain more information from their counterparties, especially when significant credit exposures are involved. These include the capital condition and market risk of the counterparty.

2. Confidentiality

As some of this information is considered confidential, institutions should safeguard the use of proprietary information.

3. Leverage, Market Risk, and Liquidity

Financial risk managers should monitor the risks of large counterparties better, focusing on the interactions between leverage, liquidity, and market risk.

4. Risk Management Expertise

Financial institutions should ensure that risk managers have the appropriate level of experience and skills.

5. Liquidation-Based Estimates of Exposure

When exposures are large, information on exposures based on marked-to-market values should be supplemented by liquidation-based values. This should include current and potential exposures.

[3]Counterparty Risk Management Policy Group. (1999). *Improving Counterparty Risk Management Practices,* New York: CRMPG. At www.counterparty.org

6. Stress-Testing

Institutions should stress test their market and credit exposure, taking into account the concentration risk to groups of counterparties and the risk that liquidating positions could move the markets.

7. Collateralization

Loans to highly leveraged institutions should require appropriate collateral, taking into account liquidation costs.

8. Valuation and Exposure Management

Institutions should recognize the cost of credit risk in capital charges and continuously monitor their exposures using, if possible, external valuation services.

9. Management Responsibilities

Senior management should convey clearly its tolerance for risk, expressed in terms of potential losses. The function of risk managers is then to design a reporting system that enables senior management to monitor the risk profile.

10. Large Exposure/Risk Reporting

Senior management should receive regular reports on large exposures.

11. Concentration Analysis

Senior management should be informed about concentrations of market and credit risk due to positive correlations between the firm's own principal positions and counterparties's positions.

12. Contextual Information

Senior management should be able to assess key assumptions behind the analysis.

In addition, the report makes a number of other recommendations related to market practices and conventions, as well as regulatory reporting. In particular, the report identifies areas for improvements in standard industry documents, which should help to ensure that netting arrangements are carried out in a timely fashion.

Perhaps the most important lesson from LTCM for brokers is the relationship between market risk and credit risk. The G-30 report recommended the establishment of market and credit risk functions, but did not discuss integration of these functions. When LTCM was about to fail, brokers realized that they had no protection for potential exposure and that many of their positions were similar to those of LTCM. Had LTCM defaulted (a credit event), brokers could have lost billions of dollars due to market risk.

The required integration of market and credit risk seems recognized in a recent survey by Capital Markets Risk Advisors, which revealed that the proportion of institutions having integrated the two functions rose from 9 percent before 1998 to 64 percent after the crisis.

The second lesson is the need for risk managers to make adjustments for large or illiquid positions. The third lesson from LTCM is that institutions should perform systematic stress tests, as VAR models based on recent history can fail to capture the extent of losses in a disrupted market. This seems obvious, as VAR only purports to give a first-order magnitude of the size of losses in a normal market environment.

26.4 Answers to Chapter Examples

Example 26-1: FRM Exam 1997—Question 4/Regulatory

a) The G-30 developed best-practice risk management principles.

Chapter 27

Firmwide Risk Management

This chapter turns to best practices for firmwide management of financial risks. The financial industry has come to realize that risk management should be implemented on a global basis, across business lines and types of risk. This is due to a number of factors, including (1) increased exposures to more global sources of risk as institutions become more global, (2) interactions between risk factors, and (3) linkages in products across types of market risks as well as types of financial risks. These linkages make it important to consider correlations among risks and products.

Interactions between types of risk bear emphasis, as they are too often ignored. The industry has made great strides in recent years in the measurement of market and credit risk. Once measured, risk can be penalized, as with a RAROC measure. The danger is that this creates an incentive to move risk to areas where it is not well measured or controlled. For instance, collateral payments in swaps decrease credit risk by marking-to-market on a regular basis but create a greater need for cash flow management, which increases operational and liquidity risk. The reverse can also occur as an operational failure, such as an incorrect confirmation of a trade can lead to inappropriate hedging or greater market risk. Incorrect data entry of swap terms can create incorrect market risk measurement as well as incorrect credit exposures. As we have seen in the previous chapter, many banks were not aware of their exposure to LTCM due to the separation of their credit risk and market risk functions.

The industry has also recognized that, to benefit from diversification effects, various risks have to be measured and compared. This explains the trend toward integrated, or firmwide, risk management.

Section 27.1 first reviews different types of financial risks. Section 27.2 discusses the three pillars of global risk management, consisting of best-practice policies, methods, and infrastructure. Section 27.3 then turns to a description of organizational structures that are consistent with these best practices. Finally, Section 27.4 shows how traders can be controlled through compensation adjustment and limits.

27.1 Types of Risk

We first briefly review various types of financial risks.

- **Market risk** arises from movements in the level or volatility of market prices.
- **Liquidity risk** takes two forms, asset liquidity risk and funding liquidity risk. **Asset liquidity risk**, also known as **market/product liquidity risk**, arises when a transaction cannot be conducted at prevailing market prices due to the size of the position relative to normal trading lots. **Funding liquidity risk**, also known as **cash-flow risk**, refers to the inability to meet payments obligations. Asset liquidity risk generally falls under the market risk management function.
- **Credit risk** originates from the fact that counterparties may be unwilling or unable to fulfill their contractual obligations.
- **Operational risk**, as we have seen, is generally defined as the risk of loss resulting from failed or inadequate internal processes, systems and people, or from external events.

As we have seen in the chapter on operational risk, these risk categories do not fit into neat, separate *silos*. Operational risk can create market and credit risk and vice versa. This is why it is essential to view financial risks on a firmwide basis.

Integrated risk management provides a consistent and global picture of risk across the whole institution. This requires measuring risk across all business units and all risk factors, using consistent methodologies, systems, and data.

Example 27-1: FRM Exam 2001—Question 130

27-1. Liquidity risk is the risk that
I. The markets get less active, making it difficult to exit
II. The offices get flooded
III. It becomes difficult to borrow money
IV. The process for settlement becomes less smooth
a) I and II
b) II and III
c) I and III
d) I and IV

Example 27-2: FRM Exam 1999—Question 160/Oper.&Integr.Risk

27-2. The risk that one of the parties will fail to meet its obligation to make payments in a swap agreement is called

a) Counterparty risk
b) Operational risk
c) Market risk
d) Notional risk

Example 27-3: FRM Exam 1998—Question 10/Oper.&Integr.Risk

27-3. What are the driving forces of integrated risk management?

I. The increasing complexity of products
II. Linkages between markets
III. The potential benefits offered by portfolio effects

a) I only
b) II only
c) II and III only
d) I, II, and III

27.2 Three-Pillar Framework

Firmwide risk management is best viewed as resting on three pillars, all equally as important as the others.[1] These pillars include policies, methodologies, and infrastructure.

27.2.1 Best-Practice Policies

Best-practice policies should reflect the mission statement of the corporation. In many cases, this is framed in terms of increasing shareholder value, which is equivalent to providing a return that is consistent with the risks assumed. Thus, strategic decisions to enter or exit a business should be made after appropriate consideration of expected returns as well as risks involved.

In practice, the institution also needs to specify the extent of the risks that it will feel comfortable taking, expressed on a worst-loss basis. This can be translated into a target credit rating, for instance. The resulting risk tolerance will provide the philosophy for firmwide risk management policies.

[1] See also Mark. (1997). "Risk Oversight for the Senior Managers: Controlling Risk in Dealers," in Schwartz, R. and Smith, C. (Eds.). *Derivatives Handbook,* New York: Wiley.

These policies need to be established at the highest level of the organization, that is, board of directors and senior management. Policies need to set limits on market risk, for instance through a worst loss expressed in terms of VAR or stress-testing analysis. Similarly, these policies need to be translated into credit and operational risk VAR measures, along with internal risk controls. Along with limit policies comes the need to define the extent of disclosures, both internal and external.

27.2.2 Best-Practice Methodologies

These policies could not be implemented without appropriate analytical methods to measure, control, and manage financial risks. These require state-of-the-art techniques to value portfolios and to measure their risks. Clearly, risk needs to be measured and priced at the portfolio level, using the most appropriate method.

Risk measurement methodologies also provide tools to set and monitor risk-based limits for traders and business units, as well as to adjust profits and losses for the relevant cost of risk capital.

27.2.3 Best-Practice Infrastructure

These policies and methodologies can only be implemented with the appropriate infrastructure. This includes an organizational design that reflects a firmwide risk management philosophy, people with the requisite training, expertise, and compensation, and systems that can support risk-management decisions. These will be examined in greater detail next.

Example 27-4: FRM Exam 1998—Question 11/Oper.&Integr.Risk

27-4. The best-practice risk management approach is a three-pillar framework. The three pillars are best-practice policy, best-practice infrastructure, and best-practice methodologies. Which of the following aspects of a financial institution are highly dependent upon the *best-practice* policies?

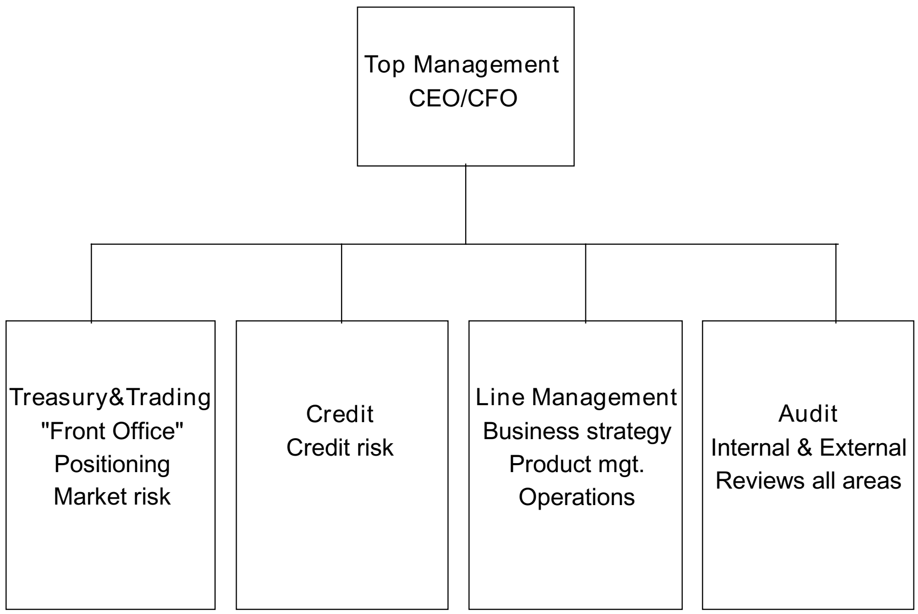
I. Business strategies
II. Risk tolerance
III. Disclosure

a) I only
b) I and II only
c) II and III only
d) I, II, and III

27.3 Organizational Structure

To be effective, the organizational structure must be designed to reflect the policy of effective firmwide risk management. Figure 27-1 reflects a typical organizational structure of an old-style commercial bank.

FIGURE 27-1 Old-Style Organizational Structure



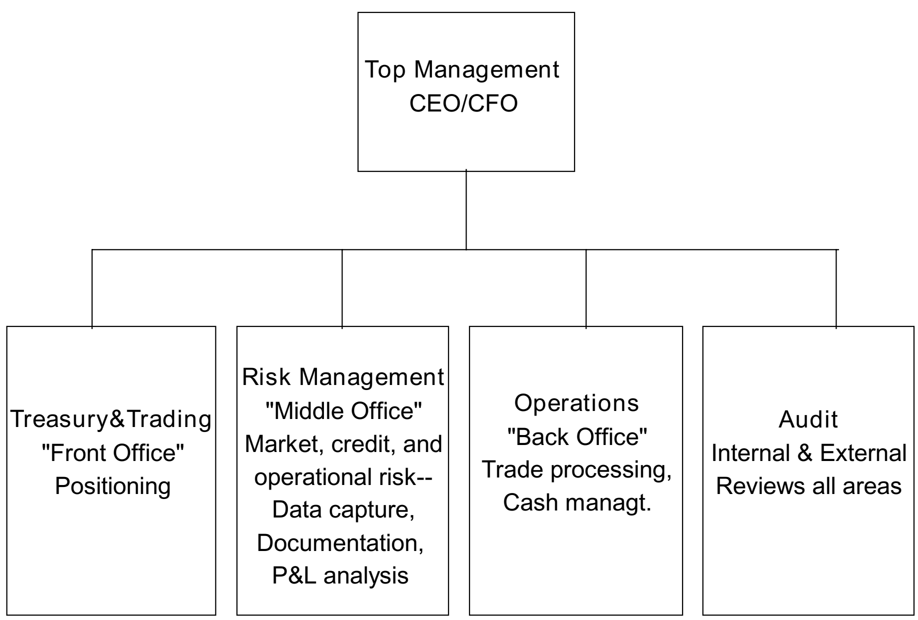
Here, risk is mainly monitored by the business lines. The risk manager approves transactions, sets exposure limits, and monitors the exposure limits as well as the counterparty's financial health. Treasury and trading implement proprietary trading and hedging. At the same time, this unit measures and monitors position and perhaps VAR limits. Line management deals with business and product strategy. It also controls operations. Finally, the audit function, external or internal, provides an independent review of business processes.

There are numerous problems with such structure. Perhaps the main one is that market risk management reports to trading, which violates the principle of independence of risk management. In addition, the decentralization of risk management among separate lines lead to a lack of coordination and the failure to capture correlations between different types of risk. The credit risk manager, for instance, will prefer an instrument that transforms credit risk into operational risk, which is under another manager's watch. Situations where credit risk and market risk exacerbate each other (as in the case of LTCM) will also be missed. Finally, models and databases may be inconsistent across lines.

To maintain independence, risk managers should not report to traders but instead directly to top management. Ideally, the risk management function should be a firmwide function, covering market, credit, and operational risks. Such structure will avoid situations where risks are pushed from one area where they are well measured toward other areas. Firmwide risk management should also be able to capture interactions between different types of risks.

The philosophy of separation of functions and independence of risk management must be embodied in the organizational structure of the institution. Figure 27-2, for instance, describes one such implementation. The most important aspect of this flowchart is that the risk management unit is independent of the trading unit.

FIGURE 27-2 Modern Organizational Structure



The **front office** is concerned with positioning, and perhaps some local hedging, subject to position and VAR limits established by risk management. The **back office** deals with trade processing and reconciliation as well as cash management. Here, the **middle office** has expanded functions, which include risk measurement and control.

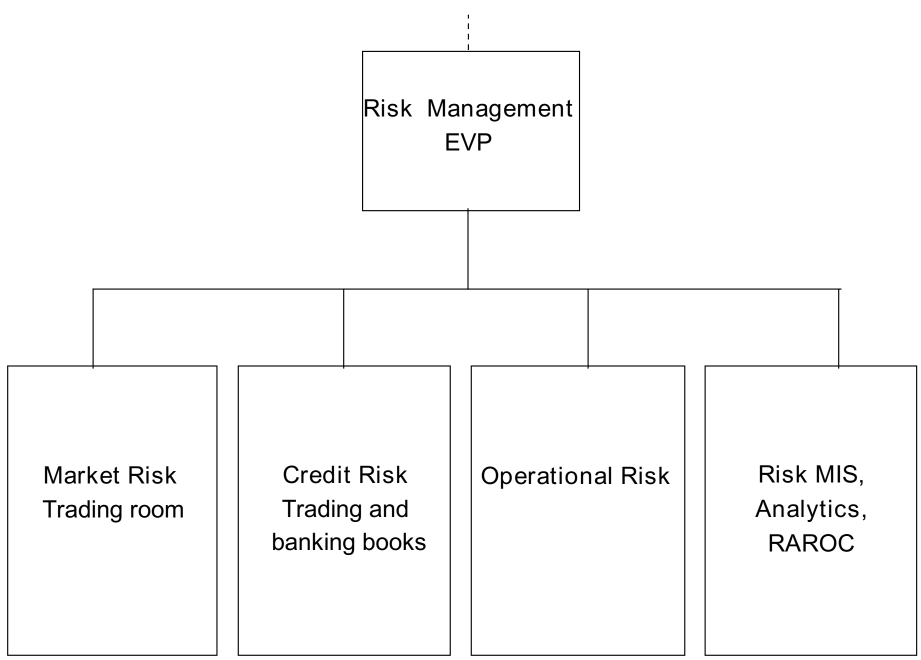
The **chief risk officer** is responsible for

- Establishing risk management policies, methodologies, and procedures consistent with firmwide policies
- Reviewing and approving models used for pricing and risk measurement
- Measuring risk on a global basis as well as monitoring exposures and movements in risk factors
- Enforcing risk limits with traders
- Communicating risk management results to senior management

Figure 27-3 describes the centralization of the risk management function under an executive vice president or chief risk officer. The figure shows the units reporting to this new function.

To this officer report *market risk management*, which monitors risk in the trading book; *credit risk management*, which monitors risk in the banking and trading books; *operational risk management*, which monitors operational risks; and systems. The latter unit deals with *risk management information systems (MIS)*, which include hardware, software, and data capture; *analytics*, which develops and tests risk management methodologies; and RAROC, which ensures that economic capital is allocated according to risk.

FIGURE 27-3 Risk Management Organizational Structure



Example 27-5: FRM Exam 1999—Question 171/Oper.&Integr.Risk

27-5. The Operational Risk Manager should report to
I. The Chief Executive Officer
II. The Chief Operating Officer
III. The Chief Risk Officer
a) I only
b) II only
c) III only
d) I and III only

Example 27-6: FRM Exam 1999—Question 164/Oper.&Integr.Risk

27-6. When would it be prudent for a trader to direct accounting entries?
a) Never
b) When senior management of the firm and the Board of Directors are aware and have approved such on an exception basis
c) When audit controls are such that the entries are reviewed on a regular basis to ensure detection of irregularities
d) Solely during such times as staffing turnover requires the trader to back-fill until additional personnel can be hired and trained

Example 27-7: FRM Exam 1998—Question 7/Oper.&Integr.Risk

27-7. Independent credit risk management should be responsible for
I. Approving credit exposure measurement standards
II. Setting credit limits and monitoring adherence to such limits
III. Reviewing counterparty creditworthiness and concentration of credit risk
a) I only
b) II only
c) I and II only
d) I, II, and III

Example 27-8: FRM Exam 1998—Question 9/Oper.&Integr.Risk

27-8. The members of the board of directors should have which of the following responsibilities related to risk management
I. The board must approve the firm's risk management policies and procedures.
II. The board must be able to evaluate the performance of risk management activities.
III. The board must maintain oversight of risk management activities.
a) I and II only
b) II and III only
c) I and III only
d) I, II, and III

Example 27-9: FRM Exam 2000—Question 63/Operational Risk Mgt.
27-9. Which one of the following statements about operations risk is *not* correct?
a) The operations unit for derivatives activities, consistent with other trading and investment activities, should report to an independent unit and should be managed independently of the business unit.
b) It is essential that operational units be able to capture all relevant details of transactions, identify errors, and process payments or move assets quickly and accurately.
c) Because the business unit is responsible for the profitability of a derivatives function, it should be responsible for ensuring proper reconciliation of front and back office databases on a regular basis.
d) Institutions should establish a process through which documentation exceptions are monitored, resolved, and appropriately reviewed by senior management and legal counsel.

Example 27-10: FRM Exam 1997—Question 3/Regulatory
27-10. To develop an effective risk management function within a large financial institution, the head of risk management should report to whom?
a) The head of trading
b) The head of IT
c) The board of directors
d) Depends on the institution

27.4 Controlling Traders

27.4.1 Trader Compensation

The compensation structure for traders should also be given due thought. Usually, traders are paid a bonus that is directly related to their performance, for instance 20 percent of profits, when positive. Note that the design of this compensation contract is asymmetric, like an option. If the trader is successful, he or she can become a millionaire at a very young age. If the trader loses money, he or she is simply fired. In many cases, the trader will find another employment since he or she now has experience.

Such a contract is designed to attract the very best talents into trading. The downside is that the trader, who is now long an option, has an incentive to increase the value of this option by increasing the risk of the positions. This, however, may not be in the best interests of the company.

Such tendency for risk taking can be controlled in a variety of fashions:
(1) by modifying the structure of the compensation contract to align the interests of the trader and of the company better (e.g. by paying with company stock, or tying compensation to longer-term performance),
(2) by subtracting a risk-based capital charge from trading profits, as in a RAROC-type system, or
(3) by appointing an independent risk manager.

To be effective, it is essential that the compensation structure for *risk managers* be independent of how well traders perform. The compensation for risk managers needs to be attractive enough to draw talented individuals, however.

27.4.2 Trader Limits

To some extent, trading risks can be managed by appropriately altering the incentives of traders. Alternatively, this risk can also be controlled by imposing limits. These can be separated into backward-looking and forward-looking limits. The former consist of stop-loss limits. The latter consist of exposure or VAR limits.

Stop-loss limits are restrictions on traders' positions that are imposed after a trader has accumulated losses. Because their design is backward-looking, they cannot prevent losses from occurring. What they do prevent, however, are attempts by traders who lose money to recover their losses by "doubling their bets," that is, taking bigger bets in the hope that a future gain will be sufficient to wipe out a string of previous losses. These limits may also be useful if markets are trending, as losses would then be amplified if positions were not changed.

Exposure limits are systematically imposed on traders as a means to control losses before they occur. These are defined in terms of notional principal. For example, the maximum position for a yen trader could be set at the equivalent of $10 million. These limits are typically set by considering the worst loss a unit could absorb, combined with an extreme move in the risk factor.

The problem with such limits is that they do not account for diversification nor movements in market risks. Also, complex products for which the notional does not represent the worst loss lend themselves to a form of limit "arbitrage," where the trader abides by the limit guideline but not its spirit. For instance, a trader may have a $10 million limit on notes with maturities up to five years. Typically, such notes will have duration of, say, 4 years. The spirit of the limit is to cap the interest rate exposure. The trader, however, may circumvent the spirit of the limit by investing in inverse floaters with a duration of 12 years.

VAR limits are now becoming a more common addition to conventional limits. These account for diversification and time variation in risk. For example, the VAR limit for a business unit may be less than the sum of the VAR limits for individual desks due to diversification. In practice, VAR limits are also susceptible to arbitrage, so that they are used together with exposure limits.

A potential drawback of VAR limits is that their effect may be highly influenced by the volatility of underlying risk factors. Consider for instance a bond trader with a $10 million position with duration of 10 years. If the daily volatility of the 10-year zero is 0.41 percent, the 95 percent confidence VAR is about $\$10,000,000 \times 0.41\% \times 1.65 = \$67,000$. Say the VAR limit is set at $70,000. The next day, markets become more volatile and the forecast volatility, using an EWMA model, jumps to 0.60 percent. The position's VAR now becomes $99,000, which is in excess of the VAR limit by $29,000. Without an increase in the limit, the trader now has to cut down the position in order to satisfy the VAR requirement.

While such a system usefully anticipates a forward-looking increase in volatility, it is worth making a number of points. The first is that the estimate of increased volatility is not perfectly measured. A GARCH model may produce slightly different results, say, an increase in volatility to 0.50 instead of 0.60. If the statistical models cannot be distinguished from each other, who is to say that the correct number is 0.60? Also, the higher VAR may be offset by an increased return. Indeed, periods of high volatility often reflect falling asset prices due to a higher risk premium. In other words, future expected returns may be higher. One has to be careful about systematically allowing traders to invoke this interpretation, though. Finally, cutting down positions may not be feasible or acceptable in the face of large liquidation costs.

Example 27-11: FRM Exam 1999—Question 165/Oper.&Integr.Risk

27-11. All of the following would strengthen the internal controls for sales personnel *except*

a) Tape recording of incoming and outgoing calls
b) Prompt confirmation of trades and acquisition of completed legal agreements
c) Compensation schemes directly linked to calendar year revenues
d) Independent credit department personnel reviewing and approving, as deemed appropriate, all over-line requests

Example 27-12: FRM Exam 1999—Question 162/Oper.&Integr.Risk

27-12. The best example of an effective risk control function would be a unit that

a) Uncovers numerous control exceptions, violations of law, and procedural errors, while maintaining a non-controversial relationship with risk taking personnel

b) Is staffed by competent personnel who report to the head of the trading department while maintaining independence from front office personnel

c) Conveys issues regarding control mechanisms, risk levels, and the quality of managerial governance; achieves timely and constructive action by responsible personnel; and thereby has few repeat criticisms

d) Efficiently skews review coverage towards areas experiencing high losses or mediocre performance, thereby reducing resource requirements

Example 27-13: FRM Exam 1998—Question 13/Oper.&Integr.Risk

27-13. Which of the following roles should not reside within an independent global risk management function?

a) Establishing risk management policies and procedures

b) Reviewing and approving risk management methodologies and models, in particular those used for pricing and valuation

c) Executing trading strategies to hedge out global market risk d) Communicating risk management results to executive management and the board of directors, as well as investors, rating agencies, stock analysts, and regulators

Example 27-14: FRM Exam 2000—Question 69/Operational Risk Mgt.

27-14. Which of the following strategies can contribute to minimizing operational risk?

I. Individuals responsible for committing to transaction should perform clearance and accounting functions.

II. To value current positions, price information should be obtained from external sources.

III. Compensation schemes for traders should be directly linked to calendar revenues.

IV. Trade tickets need to be confirmed with the counterparty.

a) I and II

b) II and IV

c) III and IV

d) I, II, and III

27.5 Answers to Chapter Examples

Example 27-1: FRM Exam 2001—Question 130

c) Liquidity risk arises as asset liquidity risk, when transactions cannot be conducted at prevailing market prices (exiting positions is difficult, i.e. costly, to liquidate) and as funding liquidity risk, when losses cannot be funded easily by borrowing.

Example 27-2: FRM Exam 1999—Question 160/Oper.&Integr.Risk

a) This also belongs to the credit risk category.

Example 27-3: FRM Exam 1998—Question 10/Oper.&Integr.Risk

d) Integrated risk management is driven by linkages between products and markets, as well as correlations.

Example 27-4: FRM Exam 1998—Question 11/Oper.&Integr.Risk

d) Policies are derived from business strategies and include risk tolerance and disclosure.

Example 27-5: FRM Exam 1999—Question 171/Oper.&Integr.Risk

c) To have integrated management of market, credit, and operational risk, all three managers should report to the Chief Risk Officer, who then reports to the CEO.

Example 27-6: FRM Exam 1999—Question 164/Oper.&Integr.Risk

a) As one risk manager has said, this is one of the few instances where never means *absolutely never*. Allowing traders to tabulate their profit and losses themselves is a recipe for disaster.

Example 27-7: FRM Exam 1998—Question 7/Oper.&Integr.Risk

d) The credit risk manager goes through all the steps in the risk management process; he participates in approving standards and sets and monitors risk limits.

Example 27-8: FRM Exam 1998—Question 9/Oper.&Integr.Risk

d) The board must approve policies, be able to evaluate and maintain oversight of risk management.

Example 27-9: FRM Exam 2000—Question 63/Operational Risk Mgt.

c) Answers (a), (b), and (d) are all reasonable. Answer (c) violate the separation of trading and back office functions.

Example 27-11: FRM Exam 1997—Question 3/Regulatory

c) The G-30 recommends an independent risk control function for market and credit risk. As a result, the head of risk management should report directly to the board of directors, or senior management, but certainly *not* to the head of trading.

Example 27-11: FRM Exam 1999—Question 165/Oper.&Integr.Risk

c) Linking compensation to revenues provides incentives for better performance but, unfortunately, for avoiding controls as well.

Example 27-12: FRM Exam 1999—Question 162/Oper.&Integr.Risk

c) Having too many exceptions indicates that the control function is not working properly, so (a) is wrong. Risk managers cannot report to the head of trading, so (b) is wrong. Reducing personnel requirement is not an end in itself, so (d) is wrong. The goal is to create an environment that is conducive to controlled risk-taking.

Example 27-13: FRM Exam 1998—Question 13/Oper.&Integr.Risk

c) Risk management cannot implement any trading activity due to the potential conflict of interest, even for hedging.

Example 27-14: FRM Exam 2000—Question 69/Operational Risk Mgt.

b) Answer I violates the principle of separation of functions. Answer III may create problems of traders taking too much risk. Answer II advises to use external sources for valuing positions, as traders may affect internal price data.

PART six

Legal, Accounting, and Tax Risk Management

Chapter 28

Legal Issues

Part VI of this manual now turns to legal, accounting, and tax issues in risk management. **Legal risk** can be defined as the risk that contracts are not legally enforceable or documented correctly. More generally, this is "the risk that a transaction cannot be consummated because of some legal barrier, such as inadequate documentation, a regulatory prohibition on a specific counterparty, and non-enforceability of bilateral and multilateral close-out netting and collateral arrangements in bankruptcy."[1] This includes changes in law, mistakes, liabilities of agents, and political risks.

Legal risk invariably arises when the counterparty lost money on a transaction. Legal risk is also intimately related to credit risk, as situations of default require enforcement of contracts, which creates legal uncertainty.

This chapter will focus on legal risk for derivatives, although many of the concepts developed here also apply to legal risks for other financial instruments, such as loans or bonds. This chapter is structured as follows. Section 28.1 briefly reviews the history of legal risks in the derivatives markets. Section 28.2 discusses netting, an important feature of swaps that has been developed to control market, credit, and legal risk. Next, Section 28.3 summarizes the master netting agreement established by the **International Swaps and Derivatives Association (ISDA)** in 1992. Readers, however, should also read the full text of the agreement.[2]

Otherwise, the legal environment has drastically changed in the wake of corporate scandals such as Enron and WorldCom. This has led to new regulations that apply to all public companies listed on U.S. exchanges. Section 28.4 presents the main provisions of the Sarbanes-Oxley Act, which aims at strengthening firmwide risk management practices. Finally, Section 28.5 contains a glossary of useful legal terms.

[1]See for instance the Federal Reserve Board's in-depth guide, *Trading and Capital Markets Activities Manual.* (1998). [On-line]. Available at: http://www.federalreserve.gov.

[2]For instance in "Legal and Documentation Issue of Swaps and Financial Derivatives," by Ian Wallace in *Swap and Derivative Financing*, S. Das, Editor (1994).

28.1 Legal Risks with Derivatives

While legal risks have always existed in derivatives contracts, they became more significant inception of the swap markets. Unlike exchange-traded futures which are standardized, the essence of the over-the-counter market is to *tailor* contracts to the counterparty. This, however, requires not only customizing financial terms (prices, quantities, maturities) but also the legal documentation to the counterparty, which creates additional risk.

Legal risks are also intermingled with market and credit risks. When a counterparty loses a large amount of money on a transaction, reflecting market risk, there may be a tendency to resort to legal action as a means to recover some of the losses. For example, when Procter & Gamble lost $157 million on swaps arranged by Bankers Trust, the consumer company sued its bank and recovered its losses.

Another famous example of legal risk is the case of **Hammersmith & Fulham**. This concerned a series of interest rate swaps entered by city councils in Britain. The municipalities had taken large positions in interest rate swaps that turned out to produce large losses as British interest rates almost doubled from 1988 to 1989.

The swaps were later ruled invalid by the British High Court. The court decreed that the city councils did not have the authority to enter these transactions, which were found to be **ultra vires** (or "beyond the power" of the cities to enter). All the contracts were deemed void and hence the cities were not responsible for the losses. As a result, losses of $178 million had to be absorbed by their counterparty banks.

After this experience, banks have tried to control their legal risks by verifying that their counterparties indeed have the right to enter into the proposed transactions. Even so, this is not always easy to assess. Before the Hammersmith verdict, for instance, many lawyers were convinced the swaps in question were indeed legal.

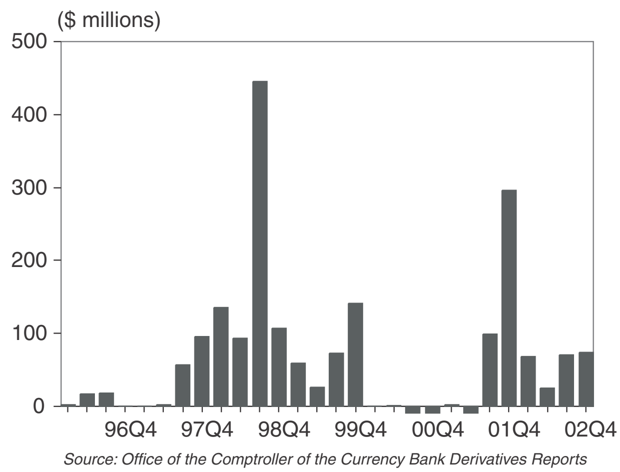
Up until recently, the Hammersmith loss was the greatest single credit loss in the swap markets. For instance, a study by the ISDA noted that total losses amounted to only $358 million by the end of 1991. About 50 percent of this sum was due to the Hammersmith case.

Even so, these losses are relatively small compared to the size of the market. The total of $358 million represents only 0.012% of the notional amount of $4.3 trillion at the time. As we have learned, however, notionals provide an exaggerated measure of the size of derivatives markets. A more relevant measure would be the credit exposure, which is closest in concept to the face value of bonds. Current exposure is measured by mark-to-market values, which amounted to $77.5 billion in 1991. Compared to this, the loss percentage is still only 0.46%.

For more recent data, we can turn to information provided by the **Office of the Comptroller of the Currency (OCC)** for U.S. commercial banks.[3] The OCC provide quarterly reports on the charge-offs from derivatives (or credit losses).[4] Figure 28-1 presents quarterly charge-offs since 1996. By the end of the sample, these losses had accumulated to $1,900 million.

The peak quarterly losses occurred in the third quarter of 1998, as a result of the Asian financial crisis and the Russian default. Even this number of $445 million represents only 0.0014% of total notional of $33 trillion, or 0.11% of the total credit exposure. Another perspective would be to compare this peak number to the charge-offs on loans, which was 0.49% in the same quarter. Overall, derivatives credit losses are very small relative to the size of these markets. More often than not, these involve litigation, however.

FIGURE 28-1 Charge-offs on Derivatives: U.S. Commercial Banks



Legal risks can arise from a number of sources.

- *A failure in contracting*. This can happen if the contract is not properly authorized or executed, as in the Hammersmith case. Even in the United States, there was some legal uncertainty as to the legal status of swaps until recently. The

[3]The OCC is an agency overseeing U.S. commercial banks. Chapter 30 will present an overview of bank regulators.

[4][On-line]. Available at http://www.occ.treas.gov/deriv/deriv.htm .

Commodity Exchange Act did not make it clear that swaps are legally distinct from futures contracts. If swaps had been ruled to be futures contract, they could have been found illegal and thus void. This only changed with the passage of the **Commodity Futures Modernization Act** of 2000, which secured legal certainty for OTC derivatives transactions.

- *A failure in contract documentation.* Mistakes can arise in contract documentation, such as incorrect number entries.
- *Bankruptcy risks.* By nature, the bankruptcy process is fraught with uncertainties. For instance, the bankruptcy court could "cherry pick" the contracts, or choose to honor the contracts having the greatest value for the defaulting party only, to the detriment of counterparties.

Special protection is accorded, however, for the set-off of margin payments and liquidation of collateral under securities contracts and commodities contracts. In the United States, close-out netting agreements (to be defined in the next section) are specifically exempted from the automatic stay provision that applies upon the filing of a bankruptcy petition. This protection was adopted by the **Financial Institutions Reform Recovery and Enforcement Act (FIRREA)** of 1989, which also confirmed the right to access the collateral posted by the defaulting counterparty.

Even so, there is often uncertainty in the application of these laws. The case of Long-Term Capital Management (LTCM) is a good example, because LTCM was chartered in the Cayman Islands. Had LTCM declared bankruptcy in the Cayman Islands, there was legal uncertainty as to whether counterparty banks would have had the right to liquidate their collateral under the U.S. Bankruptcy Code. This uncertainty is reported to have been one reason why the same banks wanted to avoid a messy bankruptcy scenario and agreed to bail out LTCM.

- *Changes in laws and regulations.* Contracts may contain clauses protecting, for instance, one party against changes in tax or regulatory treatments. For instance, coupons on Eurobonds are exempt from withholding taxes. If the country of the bond issuer imposes new taxes, the issuer may be subject to a so-called **gross-up clause** that requires it to pay the investor additional money to make up for the new tax.[5] Changes in the regulatory environment may also induce changes in the value of contracts.

[5]Additional complications may arise as the issuer may have the right to redeem the bond at par. If the bond is trading at a premium, this provides a windfall profit for the issuer.

28.2 Netting

As we have seen when analyzing credit risk, netting has developed over time as a powerful mechanism to reduce credit exposure. The purpose of **netting** is to offset transactions between two parties with settlement of the *net* difference in cash flows across all contracts covered by a netting agreement. In the case of bankruptcy, however, netting is only fully beneficial when enforced by the courts.

28.2.1 G-30 Recommendations

The **Group of Thirty** (G-30) report, issued in July 1993, emphasizes that the "netting of contractual payments... is the most important means of mitigating credit risk." As a result, it makes a separate set of recommendations for legislators, regulators, and supervisors.

Recommendation 21: Recognizing Netting

Regulators and supervisors should recognize the benefits of netting arrangements where and to the full extent that they are enforceable, and encourage their use by reflecting these arrangements in capital adequacy standards. Specifically, they should promptly implement the recognition of the effectiveness of bilateral close-out netting in bank capital regulations.

Recommendation 22: Legal and Regulatory Uncertainties

Legislators, regulators, and supervisors, including central banks, should work in concert with dealers and end-users to identify and remove any remaining legal and regulatory uncertainties with respect to

- *The form of documentation required to create legally enforceable agreements (statute of frauds)*
- *The capacity of parties, such as government entities, insurance companies, pension funds, and building societies, to enter into transactions (ultra vires)*
- *The enforceability of bilateral close-out netting and collateral arrangements in bankruptcy*
- *The enforceability of multibranch netting arrangements in bankruptcy*
- *The legality/enforceability of derivatives transactions*

ISDA keeps track of countries that have adopted or are considering changes in legislation to allow netting. It has obtained legal opinions that netting would be upheld in most leading jurisdictions.[6] Similarly, the Bank for International Settlements has issued a report that concludes that bilateral netting is likely to be effective in G-10 countries.[7] Doubts, however, remain, especially in **common law**[8] countries (such as the United Kingdom, Australia, and Canada) which rely on case laws to establish legal principles, as there have been no previous cases establishing precedents for netting.

28.2.2 Netting under the Basel Accord

In 1995, the Basel Committee on Banking Supervision (BCBS) lowered capital charges to recognize, and encourage, netting agreements.[9] The BCBS recognizes netting under **novation**, which substitutes outstanding debt payments for new ones that provide for *net* payment obligations. Under novation, any obligation between a bank and its counterparty to deliver a given currency on a given value date is automatically amalgamated with all other obligations for the same currency and value date, legally substituting one single amount for the previous gross obligations.

Another form is the **close-out netting agreement**, which is a bilateral contract that specifies that upon default, the non-defaulting party nets gains and losses with the defaulting counterparty to a single payment for all covered transactions.

The ability to **terminate** financial market contracts upon an event of default is central to the effective management of financial risk. Without a close-out or termination clause, counterparties would be helplessly watching their contracts fluctuating in value during the bankruptcy process, which could take years.

The Basel Accord recognizes netting, as long as the bank can satisfy its national supervisor that it has

(1) A netting contract or agreement with the counterparty which creates a single legal obligation, covering all included transactions, such that the bank would have either a claim to receive or obligation to pay only the net sum of the positive and negative mark-to-market values of included individual transactions in the event a counterparty fails to perform due to any of the following: default, bankruptcy, liquidation or similar circumstances

[6]Refer to the site www.isda.org.

[7]Bank for International Settlements. (1990). *Report of the Committee on Interbank Netting Schemes of the Central Banks of the Group of Ten Countries (Lamfalussy Report)*.

[8]The glossary defines some of these legal terms.

[9]See the BCBS. (1995). *Basel Capital Accord: Treatment of Potential Exposure for Off-Balance Sheet Items*. [On-line]. Available: http://www.bis.org/publ/bcbs18.pdf.

(2) Written and reasoned legal opinions that, in the event of a legal challenge, the relevant courts and administrative authorities would find the bank's exposure to be such a net amount under
- the law of the jurisdiction in which the counterparty is chartered and, if the foreign branch of a counterparty is involved, then also under the law of the jurisdiction in which the branch is located;
- the law that governs the individual transactions; and
- the law that governs any contract or agreement necessary to effect the netting.
The national supervisor, after consultation when necessary with other relevant supervisors, must be satisfied that the netting is enforceable under the laws of each of the relevant jurisdictions
(3) Procedures in place to ensure that the legal characteristics of netting arrangements are kept under review in the light of possible changes in relevant law

28.2.3 Walk-Away Clauses

Netting, however, only attracts a favorable capital treatment for contracts without **walk-away clauses**. These clauses, also known as **limited two-way payment provisions**, allow both parties to walk away from the contract in case of default.

Consider, for example, the case of the collapse of the Drexel Burnham Lambert Group (DBL Group) in 1990, which placed its swap subsidiary, DBL Products, in default. Some swaps were out-the-money for DBL Products, in which case counterparties had a claim against DBL Products. This placed them in the same position as other unsecured senior creditors, which seems normal.

Other swaps, however, were in-the-money for DBL Products, which means that counterparties owed money. In theory, the walk-away clause would have permitted them to reap a windfall profit, thus randomly benefiting from the misfortune of others, which seems questionable.

Even so, nearly all in-the-money contracts were fully paid. Counterparties settled to avoid expensive litigation about the enforceability of these contracts. Financial institutions also recognized that walk-away clauses create uncertainty for financial markets. Contracts have now evolved to contain a **full two-way payment provision**, which provide for full payment to the counterparty, subject to a bankruptcy distribution rule.

The final nail in the coffin of walk-away clause is the ruling by the Basel Committee that such contracts are not provided any regulatory relief in terms of lower capital requirement.

28.2.4 Netting and Exchange Margins

Netting also applies to the credit risk that futures traders face from their brokers. Clients deposit margins with their brokers. Assuming the broker is a clearing member, the broker in turn deposits margins with the clearinghouse.

If a broker goes bankrupt, clients could lose the part of their margins held by the broker. In the United States, two clearinghouses (CME and NYMEX) collect *gross margins*, that is, a separate margin for all client positions. Others collect *net margins*, that is, allow the broker to offset long and short positions by different customers. This netting decreases the margin held by the clearinghouse. In theory, a gross margin system is safer for the client because a greater fraction of the margin is held by the clearinghouse. The risk of a net margin system is lessened, however, if the broker properly *segregates* client accounts by holding them separately from its own accounts.

28.3 ISDA Master Netting Agreement

At the beginning of the 1980s, swaps were tailor-made financial contracts that required documentation to be drafted on a case-by-case basis. This was very time consuming, costly, and introduced a time lag between the commercial agreement and the signing of the legally-binding contract.

In response, the industry started to develop standardized terms for swaps. Like futures, this made it easier to offset the contracts, increasing liquidity and decreasing legal uncertainty. Out of this effort came the **master netting agreement** established by the ISDA in 1987 and revised in 1992. This form establishes a template for a standardized contract, which is supplemented by a **schedule to the master agreement** and the actual **confirmation of contract**. Parties have the flexibility to select parts of the agreement or to amend the base document through the schedule. The more specific clauses (e.g., confirmation) override more general clauses.

The ISDA master agreement contains the following provisions, as in any contract for payment.

1. A list of *obligations*, detailing the mechanics of payment conditions (section 2 in the ISDA agreement), including the netting of obligations
2. A list of *credit provisions*, which describe events of default and termination (section 5), early termination (section 6), and credit support provisions (e.g. the system of collateral payments). The event of default includes
   (i) Failure to pay
   (ii) Breach of agreement
   (iii) Credit support default (e.g. failure to provide collateral when due)
   (iv) Misrepresentation
   (v) Default under a specified transaction
   (vi) Cross-default, which is optional
   (vii) Acts pertaining to bankruptcy or liquidation
   (viii) Mergers without the successor assuming the obligation to perform under the swap
   Termination includes
   (i) An illegality in which a party is unable to perform due to a change in law or regulation
   (ii) A tax event such as a change in tax law that causes a party to make an additional payment (called gross-up)
   (iii) A tax event upon merger
   (iv) A credit event upon merger where the creditworthiness of the successor is materially weaker than the original entity
3. A list of contractual *boilerplate statements* including representations (section 3), agreements (section 4), transfer provisions (section 7), governing law (section 13), and so on

Although the ISDA forms attempt to provide comprehensive and standardized coverage of swap events, they cannot anticipate every eventuality. When Russia defaulted on its domestic-currency debt on August 17, 1998, it imposed a moratorium on foreign-currency debt payments as well as a 90-day freeze on forward foreign exchange contracts. It has maintained payment on its foreign debt, however. Whether this constitutes a credit event on the foreign debt was not clearly defined by the swap agreements in place. This has created considerable arguments over the interpretation of standard contracts.

By 1999, the ISDA had published a revised set of definitions for credit derivatives that considers both sovereign and non-sovereign entities. This list is provided in Chapter 19.

Example 28-1: FRM Exam 2001—Question 124

28-1. Most credit derivatives contracts
a) Are based upon English law
b) Are written on a one-off basis
c) Have a clause about restructuring
d) Are based upon the ISDA agreement

Example 28-2: FRM Exam 1999—Question 175/Legal & Other Risks

28-2. The ISDA Master Agreement and other similar agreements for derivative contracts address primarily
a) Legal and credit risk
b) Market and legal risk
c) Legal and operational risk
d) Liquidity and legal risk

Example 28-3: FRM Exam 1999—Question 176/Legal & Other Risks

28-3. The framework in which the ISDA Master Agreement is used includes the Master Agreement, schedule, and confirmation. What is the order of precedence of these if any clauses conflict?
a) Master Agreement, Schedule, confirmation
b) Schedule, Master Agreement, confirmation
c) Master Agreement, confirmation, Schedule
d) Confirmation, Schedule, Master Agreement

Example 28-4: FRM Exam 1998—Question 14/Credit Risk

28-4. All of the following are considered events of default under the ISDA Master Agreement *except*
a) Failure by a party to make a payment or delivery
b) Misrepresentation
c) Bankruptcy
d) None of the above

Example 28-5: FRM Exam 2000—Question 22/Legal & Other Risk

28-5. A typical master netting agreement as established by ISDA will contain all of the following *except* a list of
a) Obligations
b) Historical market prices
c) Credit provisions
d) Contractual boilerplate statements

Example 28-6: FRM Exam 1999—Question 179/Legal & Other Risks

28-6. Which of the following are considered to be termination events by the ISDA Master Agreement?

I. An illegality, in which a party is unable to perform, based on changes in law or regulation

II. A tax event that causes a party to make an additional payment (gross-up) or to have an amount withheld from a payment because of a change in the tax law

III. A credit event upon merger in which the credit worthiness of the merged entity becomes materially weaker than that of the original entity

a) I and II

b) I and III

c) II and III

d) All of the above.

Example 28-7: FRM Exam 1999—Question 180/Legal & Other Risks

28-7. In 1998, many credit derivatives contracts dependent on Russian credits faced legal problems because

a) Netting was unenforceable with Russian counterparties.

b) Collateral posted by a Russian counterparty and held in Russia could not be kept if the counterparty defaulted.

c) There were disputes over whether credit events had occurred or not, because the definitions of credit events were not sufficiently rigorous in credit derivative contracts.

d) A Russian court ruled that it was illegal to enter a credit derivative contract.

Example 28-8: FRM Exam 1998—Question 8b/Oper.&Integr.Risk

28-8. Which of the following is considered to be the responsibility of the legal risk manager?

I. Inadequate documentation of OTC derivatives transactions

II. The enforceability of netting agreements in bankruptcy

III. Deciding whether default has occurred

a) I only

b) II only

c) I and II only

d) I, II, and III

Example 28-9: FRM Exam 1998—Question 24/Regulatory

28-9. If a bank executes a derivatives contract with a client for whom the transaction is not appropriate, the bank has

a) Booked an illegal transaction

b) Placed the bank's reputation at risk due to potential litigation and credit risks

c) An obligation to reverse the trade

d) To closely monitor the market value to ensure pre-settlement risk does not exceed the customer's internal credit limit

28.4 The 2002 Sarbanes-Oxley Act

U.S. Congress passed the **Sarbanes-Oxley Act** in the wake of Enron, WorldCom, and Global Crossing, the three largest bankruptcies in recent corporate history.[10] The Act tries to restore investor confidence in public corporations by improving the structure for corporate governance and control.[11] This legislation applies to all companies with a public listing in the United States and contains these key provisions.

- *Creation of a new regulator:* The **Public Company Accounting Oversight Board (PCAOB)** now registers and oversees public accounting firms. PCAOB is under the supervision of the **Securities and Exchange Commission**. PCAOB has resources and muscles: It is well funded and can impose penalties. Previously, the industry was self-regulated, which critics claimed led to lax controls.
- *Certification by CEOs and CFOs:* This provision requires them to sign off on the company's financial statements and internal controls. Penalties for false statements include fines or jail time. As a result, top management will require middle managers to certify the information they provided is accurate. This should give middle management a strong incentive to resist pressures from above to cook the books.
- *Ban on non-audit consulting services:* The company's auditor is barred from performing several kinds of additional services due to perceived conflicts of interest.

[10] The Act, sometimes called SOX, is named after Paul Sarbanes and Michael Oxley, a U.S. senator and congressman, respectively.

[11] One definition of **corporate governance** is the process of high level control of an organization. It involves the combination of board of directors, management, and controls that guide the firm.

In the case of **Enron**, for instance, the consulting services provided by Arthur Andersen were so profitable that it became lax in auditing its client. In 2000, Andersen earned $25 million from audit services and $27 million from consulting services to Enron alone.

- *Independence of audit committee:* This requires that all audit committee members be *outside* directors, who are not employed by the company. The audit committee, which is part of the board of directors, appoints and supervises outside accounting firms. At least one member of this committee must be a *financial expert*. This provision should minimize management influence over the audit process.

A major goal of the Act is to minimize the possibility of harmful conflicts of interest, such as those that led to false disclosures for Enron and WorldCom. The spirit of the Act is actually very much in line with the best practices for risk management delineated in Chapter 27. Separation of duties and independent oversight are essential for effective governance.

On the downside, the Act will create more paperwork and rising audit fees. There may be greater reluctance for qualified individuals to serve on corporate boards due to the perception of greater legal liabilities. Another issue is that the Act applies to foreign companies listed on U.S. exchanges. This can create conflicts with foreign laws, such as board composition in some countries. In response, the SEC has exempted foreign companies from some of the provisions of the Sarbanes-Oxley Act.

28.5 Glossary

28.5.1 General Legal Terms

Common law: System of law derived from the English system of laws "common to the population," produced primarily by a group of judges to harmonize their decisions with those in other parts of the country. It was introduced after the Norman conquest of England as a means of unifying the country. Common law builds on precedents. This is in contrast to the French-type system of civil law.

Civil law: Legal system whose law is centered around a comprehensive legislative code (e.g., such as that established by Napoléon in France).

Civil law: In the United States, law under which a person (the plaintiff) may sue another person (the defendant) to obtain redress for a wrong committed by the defendant, for example a breach of contract. This is in contrast with criminal law.

Criminal law: Law that defines public offenses against the state or government and prescribes their punishment. This is a part of public law, which also includes constitutional and administrative law.

28.5.2 Bankruptcy Terms

Absolute priority rule (APR): Hierarchical rule for the distribution of the firm's assets: payments go first to secured creditors, then to priority creditors (e.g., to cover taxes and bankruptcy costs), then to unsecured creditors (such as bondholders and bank depositors), then to subordinated-debt holders, and finally to stockholders. See also Chapter 19.

Automatic stay: In bankruptcy, a suspension of legal actions (other than the bankruptcy proceeding itself) until the bankruptcy case is over.

Bankruptcy: A legal process under which (1) a financially troubled debtor is declared to be insolvent, or incapable of meeting debt payments, (2) the assets of the debtor are distributed to creditors according to bankruptcy law, and (3) the debtor, if honest, is discharged from liabilities for remaining unpaid debt.

The word "bankruptcy" comes from the Italian *banca rotta*, or broken bench. The tradition was that when a medieval trader failed to pay his creditors, his trading bench was broken.

Liquidating proceeding: A bankruptcy proceeding in which the debtor's assets are converted to cash and distributed to creditors. In the United States, liquidation is covered under *Chapter 7* of the U.S. Bankruptcy Code.

Reorganization proceeding: A bankruptcy proceeding in which the troubled firm may stay in business as it is reorganized in a process of financial rehabilitation. In the United States, reorganization is covered under *Chapter 11* of the U.S. Bankruptcy Code. A majority of creditors and equity holders must approve the plan, otherwise liquidation proceeds under *Chapter 7*.

28.5.3 Contract Terms

Acceleration clause: A provision in a promissory note permitting the debtor to make, or the creditor to receive, payment before the due date.

Close-out, or termination clause: A provision that gives the right to terminate a contract upon certain specified events and to calculate a termination amount due to, or due from, the defaulting party.

Cross-default clause: A contractual provision, whereby default on a contract occurs whenever the counterparty defaults on *any* other obligation.

Covenant: A contractual provision, whereby one party promises to take certain specific actions (positive covenant) or to refrain from taking certain actions (negative covenant). Bond covenants contain clauses prohibiting, for instance, the creditor from selling major assets or paying too large a dividend to stockholders.

Negative pledge clause: A provision that prevents the subordination of a contract to secured creditors, by pledging assets for new debt, for instance.

Netting: A provision that gives the right to *set off*, or net, claims or payment obligations between two or more parties, with the goal of arriving at a single net payment.

Novation: The extinguishment of a party's obligation (e.g. the debt of the obligee) through an agreement between the old obligor, a new obligor, and the obligee to substitute the old obligor for a new one.

Pari passu: Equal ranking (from Latin), meaning that all creditors within the same class will be treated equally. Term often used in bankruptcy proceedings where creditors are paid pro rata in accordance with the amount of their claims.

Security agreement: An agreement between a debtor and a creditor whereby the creditor receives security interest, or property, to secure debt payments.

Secured transaction: An arrangement such that the creditor is provided with a backup source of payment if the debtor defaults.

Ultra vires: Outside the power of a person or corporation (from Latin). This is in contrast to *intra vires*.

28.6 Answers to Chapter Examples

Example 28-1: FRM Exam 2001—Question 124

d) Most derivatives contracts are based on the standard form provided by the ISDA, which provides uniformity in contracts and reduces legal uncertainty.

Example 28-2: FRM Exam 1999—Question 175/Legal & Other Risks

a) The Master Agreement primarily deals with legal issues in case of default.

Example 28-3: FRM Exam 1999—Question 176/Legal & Other Risks

d) The general principle is that specific amendments overrule general contract terms. So, the order of precedence is from specific to general.

Example 28-4: FRM Exam 1998—Question 14/Credit Risk

d) All of these satisfy the definition of a default event.

Example 28-5: FRM Exam 2000—Question 22/Legal & Other Risk

b) A master agreement will contain a list of obligations, credit provisions, and boilerplate statements. There is no reason to have historical market prices.

Example 28-6: FRM Exam 1999—Question 179/Legal & Other Risks

d) All of these satisfy the definition of a termination event. For precise definitions, see the ISDA Master Agreement.

Example 28-7: FRM Exam 1999—Question 180/Legal & Other Risks

c) The ISDA form did not cover events from sovereign entities such as a moratorium. Russia had defaulted on its ruble-denominated debt, but continued to make payment on its foreign-currency debt.

Example 28-8: FRM Exam 1998—Question 8b/Oper.&Integr.Risk

d) The legal risk manager is responsible for documenting derivatives transactions, deciding whether there is default on payments and if so, helping to enforce netting agreements under bankruptcy.

Example 28-9: FRM Exam 1998—Question 24/Regulatory

b) The transaction may not be appropriate but in general will be legal. This places the bank's reputation at risk but there is no obligation to reverse the trade. Nor does the bank know the customer's credit limit.

Chapter 29

Accounting and Tax Issues

We now turn to general issues related to reporting, or accounting. This includes **internal reporting**, which is essential for performance evaluation and attribution, as well as **external reporting**, which is required for shareholders and for tax purposes.

While risk management is essentially a forward-looking process, accounting focuses on past performance and current positions. spite of the increased emphasis given to risk management, Obviously, reporting remains a fundamental component of doing business as it provides a measure of performance. It also drives the compensation of business units and strategic decisions to enter or exit markets. Bonuses are distributed based on the performance of business units. Likewise, decisions to allocate capital and resources to various units are generally driven by an extrapolation of their past performance. Hence, it is essential that reporting rules provides transparent, reliable, and comparable measures of performance.

Accounting risk arises when inappropriate accounting methods could cause losses. This risk is subsumed within operational risk. It is also related to **tax risk**, which is the risk of loss due to inappropriate tax computations, or changes in tax regulations.

Section 29.1 reviews the organizing principles for accounting for financial assets with a view toward internal reporting. As in the previous chapter, we place particular emphasis on derivatives given their importance and recent changes in regulatory requirements. It is also widely believed that the development of derivatives has outpaced accounting standards. The accounting treatment of special-purpose entities (SPEs) is also examined because of their importance in the Enron debacle.

Section 29.2 then discusses external financial reporting, or disclosure rules, for derivatives. We primarily focus on pronouncements by the **Financial Accounting Standards Board** (FASB), which is an independent agency responsible for developing **Generally Accepted Accounting Principles** (GAAP) for U.S. corporations. Required disclosures by the **International Accounting Standards Board** (IASB), which develops international accounting standards, are explained in Section 29.3. Section 29.4 then briefly summarizes relevant tax issues. Finally, Section 29.5 provides some concluding comments. Given the complexity of these topics, the purpose of this chapter is only to provide a summary of the issues.

Overall, the biggest issue in accounting for financial instruments is whether to report their value on an accrual/historical cost basis or by marking to market. The unmistakable trend is toward the transparency provided by mark-to-market prices.

29.1 Internal Reporting

Internal reporting was already discussed in Chapter 25, capital (RAROC). The central objective of RAROC measures is to evaluate the economic return of business activities, specifically focusing on the return to risk-adjusted capital.

29.1.1 Purpose of Internal Reporting

At an even more basic level, however, the purpose of internal accounting is to measure the raw performance of various business units. This may involve conflicts between the business units, which will argue in favor of showing large profits, and the accounting unit, which should strive for objectivity, transparency, and conservativeness. In practice, this translates into an asymmetry in the potential for accounting errors. Profits are usually not understated as traders have a strong incentive to scrutinize their performance numbers and will complain if profits appear too low. More often than not, errors end up producing *overstatements* of profits that must be corrected later, if caught.

Reporting rules can have an effect on real decisions and create or aggravate real financial losses. A good example is that of two Japanese trading companies, Showa Shell and Kashima Oil, that lost more than $1 billion each in the currency markets in 1993 and 1994. Apparently, some employees entered forward contracts to purchase the dollar in excess of the company's limit. The problem was compounded by Japanese accounting rules, which allowed traders to roll over their forward contracts into new ones, without having to realize losses (no marking-to-market). As the dollar started to depreciate, the losses were not visible and were allowed to grow to very large amounts.

Ideally, the *accounting* treatment of transactions and positions should reflect their *economic* substance. Sometimes this is defined as a "true and fair view." This is easier said than done, however.

29.1.2 Comparison of Methods

Consider first the problem of valuing outstanding assets and liabilities at a point in time. Many, if not most, of items on balance sheets are recorded at historical cost, that is, at their original purchase price with some predefined adjustments such as depreciation. This is the **historical cost method**.

For other items, economic values can be assessed from **market prices**, which are widely viewed as providing fair value. Indeed, the FASB formally defines **fair value** as the "amount at which an asset could be bought or sold in a current transaction between willing parties, that is, other than in a forced or liquidation sale." This is also called the **mark-to-market (MTM) method**. The main advantage of this method is its *transparency*.

Various methods also exist when dealing with profits and losses over a time period. The **cash method** recognizes profits and losses when the actual cash flow occurs. Another method is the **accrual method**, which recognizes revenues when earned and expenses when incurred; this matches income with expenses in an account period. More generally, accrual accounting calculates profits at the time of the trade but recognizes them over the life of the transaction. The method is also conservative in that losses are recognized as soon as they occur. Finally, profits and losses can also be computed from changes in MTM values, which is again the MTM method.

Table 29-1 compares the various methods for the case of a hedge of a long position in oil. The spot rate is $S = \$13.00$ and the oil is sold 6-month forward at a price of $K = \$15.00$. The time to maturity is $\tau = 6/12 = 1/2$. Storage costs are \$2 per barrel-year, payable at the end of the period, and the interest rate is 6%. Based on this information, the forward price should be $F = S(1 + r\tau) + C\tau = \$13(1 + 6/200) + \$2/2 = (\$13 + \$0.39) + \$1 = \$14.39$. Since the delivery price is actually higher than the forward rate, the contracts should generate a profit.

The first column in the table illustrates the historical-cost method. A profit of \$1,000 is recognized at expiration only. The cost of capital is not considered as it is included in the overall borrowing costs for the institution.

The second column displays the accrual method. This accounts for the capital cost of \$390, which brings a net profit to \$610. Here, the profit is spread evenly over the period of the deal.

TABLE 29-1 Comparison of Accounting Methods

Spot price ($/bbl)	$13.00
Delivery price ($/bbl)	$15.00
Quantity (bbl)	1000
Storage ($/bbl-year)	$2.00
Interest rate (% pa)	6%
Future value factor	1.03

		Method		
		Historical	Accrual Expiration	MTM Inception
Cost of purchase of oil		($13,000)	($13,000)	
Current value of oil inventory				($13,000)
Receivable for oil delivery		$15,000	$15,000	
Current value of oil contract				$14,563
Storage costs at expiration		($1,000)	($1,000)	
Storage costs at inception				($971)
Interest costs			($390)	
Profit		$1,000	$610	$592
Profit each month	0			$592
	1	$0	$101.7	$3
	2	$0	$101.7	$3
	3	$0	$101.7	$3
	4	$0	$101.7	$3
	5	$0	$101.7	$3
	6	$1,000	$101.7	$3
Total profit		$1,000	$610	$610

Source: Adapted from Sage, Richard. (2000). Accounting Risk. In *The Professional's Handbook of Financial Risk Management.* Lore, M. & Borodovsky, L. Eds. London: Butterworth Heinemann.

The right column considers the MTM method. At inception, the profit is the present value of the locked-in future profit. We have seen in Chapter 5 that the value of an outstanding long position in a forward contract was $(F - K)$PV, where PV is the present value factor. Here, the position is worth $(K - F)Q/(1 + r\tau) =$ $(\$15 - \$14.39)1000/1.03 = \$610/1.03 = \$592$. This would be booked just at the beginning of the first month. The monthly interest on $592 is $\$592 \times 6/1200 = \$3$. Adding this monthly interest gives a final profit of $610, which is the same as under the accrual method.

This example points to a number of potential issues with the internal accounting of trades. The trader will want to book the profit as early as possible, in order to get a bonus early. From the viewpoint of the company, however, the full capture of the cash flow occurs at the end only.

The danger is that early recognition could fail to account for other costs or uncertainties. For instance, there may be some operational expense to administering the deal. Or, the trade could involve credit risk. It is therefore important to allocate all expenses properly.

Problems with Marking-to-Market

When marking-to-market, another set of problems arises, which is the proper choice of market prices.

- *Objective market prices.* The most important problem is that accounting or risk management must ensure that obtained market prices are truly *objective*, that is, not affected by the trader whose position is being evaluated. This is not so easy as it seems for exotic deals. There have been many situations of traders pushing up market prices when they had a long position in order to inflate their profits artificially.

  Such manipulation befell National Westminster Bank, which announced a loss of $127 million due to mispricing derivatives in 1997. Apparently, a junior interest rate trader "created false profits over a period of two years" by providing prices that were too high and thus overestimated the value of his option positions.

  Similarly, using model prices to assess value is fraught with danger. David Askin, for instance, used his proprietary valuation models to value his $600 million CMO portfolio. When the mortgage market dropped in February 1994, he initially reported a loss of 2 percent that was later revised to 28 percent. As a result, he was sanctioned by the Securities and Exchange Commission for misrepresentation.
- *Bid or ask prices.* Another problem is whether we should use bid or ask prices. We could decide to use bid (lower) prices for all long positions and ask (higher) prices for all short positions. For a dealer bank, however, many of these positions may be *crossed* across desks and will not involve a transaction with external markets. In this case, one could use mid-market quotes with perhaps a provision for the spread.
- *Nonsimultaneous quotes.* The fact that markets around the world close at different times imparts additional noise to the MTM value of hedged portfolios. Suppose that a trading desk is long a futures contract in London and short a contract with the same specifications in Chicago. Eventually, the contracts will converge to the same value, capturing arbitrage profits. Marking-to-market using London

and Chicago closing times will create artificial volatility. This can be handled by recording market prices at the same time (e.g. London at 4 P.M. and Chicago at 9 A.M., both in local times) or by taking some price interpolation (e.g. forecasting the London price at 4 P.M. Chicago time).

29.1.3 Historical Cost versus Marking-to-Market

It is widely recognized that marking-to-market imposes a powerful discipline and should be used whenever possible. This general principle, however, is tempered by the fact that some items in financial statements are valued using historical cost data, or with cash flows amortized using the accrual method. If so, the accounting method for the derivative should be *consistent* with that of the hedged item. Otherwise, marking-to-market one side of the hedge only will produce an artificial impression of volatility that does not reflect the economic reality of the hedge.

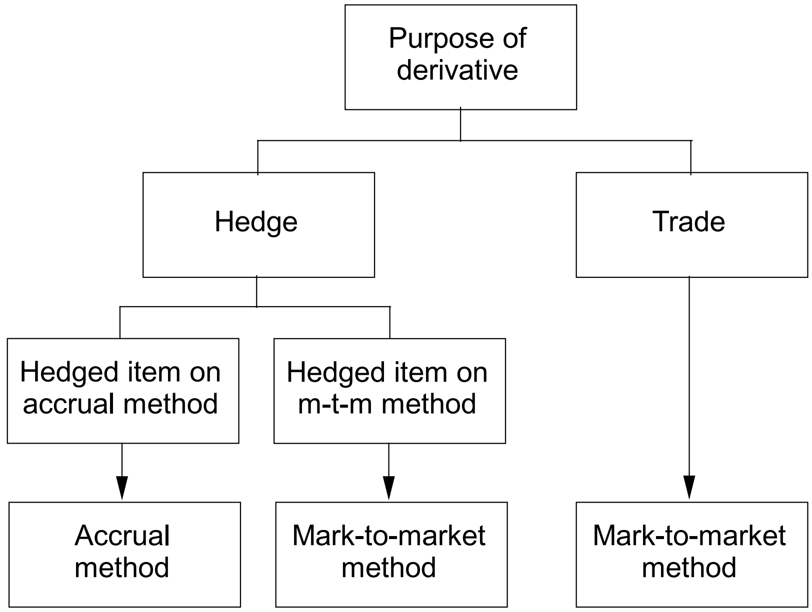
In fact, much of the discussion of the appropriate accounting method for derivatives centers on this issue of excess volatility. Corporations apparently strive to smooth out their earnings by active management of their accounting numbers. Marking-to-market financial instruments on the balance sheet does introduce some additional volatility, which is typically disliked by corporate financial officers (CFOs). The counterargument is that this volatility represents fluctuations in actual economic value.

Figure 29-1 describes general principles for choosing an accounting method for derivatives. The crucial factor is management *intent* for holding the derivative. The issue is, is it held for trading or hedging purposes?

In trading portfolios where financial assets are valued using MTM prices, derivatives should be valued using the MTM method as well. In contrast, when the derivative is used as a hedge, the appropriate method depends on the accounting method used for the hedged item. In the ideal case, the hedged item is marked-to-market, in which case the derivative should be marked-to-market also. With an effective hedge, market fluctuations should cancel out.

The problem is that most often the hedged item (say a foreign-currency denominated debt) is booked on an accrual basis. It then makes little sense to use MTM valuation for the derivative (say a currency swap designed to take out currency risk), as this would create artificial volatility. Instead, the derivative should be booked using the same accrual basis. This is still not ideal because the hedge could be imperfect, creating residual volatility that remains hidden.

FIGURE 29-1 Hedge versus Mark-to-Market Accounting



Example 29-1: FRM Exam 1998—Question 8/Regulatory

29-1. Which of the following price sources for derivative transactions is the most prudent for financial reporting purposes?
a) Trader marks
b) Valuation models
c) Directly observable market prices
d) Broker quotes

Example 29-2: FRM Exam 1997—Question 21/Regulatory

29-2. Which of the following does *not* represent a sound policy for the periodic revaluation of trading assets for corporate profit and loss reporting purposes?
a) Revaluations should be performed independent of risk-takers, including any derived factors used in the valuation (e.g., volatilities for option products).
b) For highly structured or illiquid deals, an end-user should avoid obtaining the valuation from the dealer that originated the transaction.
c) Volatilities used in options revaluation should always be obtained from at least two years of historical data.
d) Frequency of revaluation should be consistent with the significance of the activity. For example, dealers should revalue their positions daily and end-users should generally revalue monthly, but no less frequently than quarterly.

Example 29-3: FRM Exam 2000—Question 25/Legal & Other Risks

29-3. Marking-to-market on a futures contract that is long in London and short in Chicago can be handled by which of the following?

I. Recording the close price in both locations
II. Recording market prices at the same instant, regardless of time zones
III. Recording market prices at the same local time in both locations
IV. Forecasting the London price at 4 P.M. Chicago time

a) I or II
b) II or IV
c) I, II or IV
d) I, II, III or IV

29.2 External Reporting: FASB

For a long time, derivatives were considered as off-balance sheet items that did not appear in the financial statements, except perhaps in a footnote. This may have been acceptable when derivatives were marginal items. Over time, however, the market for derivatives has grown to enormous amounts, over 100 trillion dollars. For most financial institutions, derivatives by now dwarf balance sheet items. The notional amount of derivatives held by U.S. commercial banks, for instance, is now around $56,000 billion, which is ten times their assets of $5,600 billion. Even nonfinancial corporations have also become heavy users of derivatives. As a result, it has become increasingly important to reflect derivatives in financial statements.

29.2.1 FAS 133

The **Financial Accounting Standards Board** (FASB) has long struggled to set standards for the disclosure and accounting treatment of derivatives. The latest view of the FASB is that derivatives are, in effect, assets or liabilities, like other balance sheet items. Keeping them off balance sheet can conceal their risk.

In June 1998, the FASB passed a new set of standards, No. 133, *Accounting for Derivative Instruments and Hedging Activities* that unifies derivatives accounting, hedge accounting, and disclosure in a single statement. **FAS 133** represents a radical change in the accounting of derivatives and supersedes a hodgepodge of previous rules. Effective June 15, 2000, it basically requires derivatives to be recorded on the balance sheet at *fair value*, that is, at market prices.

For the first time, changes in the market value of derivatives must be reported in earnings. A major exception remains, however, for derivatives used (and designated) as a hedge. In this situation, FAS 133 allows the gain or loss to be recognized in earnings at the same time as the hedged item.

29.2.2 Definition of Derivative

FAS 133 provides a formal and complete definition of a **derivative instrument**. This is defined as a contract with all three of the following characteristics.

(i) It has one or more *underlyings* and one or more *notional amounts*. The underlying is that from which the contract derives its value (e.g., an asset price, reference rate or index—such as a stock, bond, currency, or a commodity). The underlying is a market-related characteristic that gives rise to changes in value. As an example, the value of a futures contract for oil will change as the price of oil changes; the underlying is the price of oil, not oil itself. The notional amount is a number of currency units, shares, or other physical units specified in the contract. The payoff on the derivative instrument is a function of the notional and the underlying. For instance, a NYMEX oil futures contract has a notional of 1,000 barrels. The dollar payment is the change in price per barrel times the notional.

(ii) It requires an *initial investment* of zero or “smaller” than would be required for an equivalent cash position (that has the same response to market factors). For instance, the initial investment in a forward or swap contract is zero. For an option, the premium is much less than the cost of taking a delta-equivalent position in the underlying. More precisely, for an option-based contract,it has an initial net investment equal to the fair value of the option component; for other types of derivatives, it requires an initial net investment that is less than 5 percent of the fully prepaid amount.[1]

(iii) Its terms require or permit *net settlement* (e.g. interest rate swaps). Alternatively, there is a market mechanism for net settlement (e.g., liquidating a futures contract by going back to the exchange). Or, the asset to be delivered is readily convertible into cash or is itself a derivative instrument (e.g., an option on futures).

Notwithstanding these conditions, these contracts do not fall in this category

[1] As of this writing, these additional precisions are part of an amendment proposed in May 2002. The amendment is highly likely to be added to FAS 133.

(a) *Regular-way* securities trades (e.g., a transaction to purchase a stock to be settled in the normal 3-day period)

(b) Normal purchases and sales (of non-financial instruments such as machinery, in the course of normal business)

(c) Traditional insurance contracts (such as life insurance or property and casualty insurance, where the payoff is the result of an identifiable event instead of the change in the underlying)

(d) Certain financial guarantee contracts (where the payoff is a credit event instead of the change in the underlying, but only when the buyer of the guarantee is exposed to a loss on the underlying asset)

(e) Certain over-the-counter contracts, such as weather derivatives, options on real estate and capital goods (which are not readily convertible into cash)

(f) Derivatives that serve as an impediment to sales accounting

(g) Contracts indexed to an entity's own stock

(h) Executive stock options

(i) Investments in life insurance, some investment contracts, and loan commitments[2]

29.2.3 Embedded Derivative

Another provision of FAS 133 deals with the treatment of **embedded derivatives**. These are derivatives that are included in the provisions of other contracts. An example is a structured note where the payoff is a function of the return on the S&P index. Under FAS 133, such hybrid instrument should be split between the host contract and the embedded derivative if and only if these conditions are met.

(1) The economic characteristics of the contract and embedded derivative are not "clearly and closely related."

(2) The fair market value for the hybrid contract otherwise would not be reported on the balance sheet.

(3) The embedded derivative would meet the definition of a derivative on a stand-alone basis.

When the split occurs, the embedded derivative component is subject to the FAS 133 rules. A few examples illustrate these rules. Hybrid securities held in the trading

[2] This category is also part of the May 2002 proposed amendments.

portfolio do not need to be separated, because they are marked-to-market anyway. Condition (2) is not satisfied.

In other situations, one has to interpret the terms "clearly and closely related." Consider a corporate callable bond. Conditions (2) and (3) are satisfied. Condition (1), however, is not satisfied because the host contract and derivative are closely related. The call option to the issuer involves an underlying (the interest rate) that also drives the value of the host contract. So, there is no need to separate the components.

Consider next a convertible bond. Conditions (2) and (3) are satisfied. The option feature is driven by the stock price, which is not related to the interest rate in the host bond contract. As a result, condition (1) is satisfied. FAS 133 thus requires an investor in a convertible bond to separate the option feature from the host contract. On the other hand, this does not apply to the issuer of a convertible bond, since the derivative is indexed to the entity's own stock (condition (3) is not satisfied due to exclusion (g)).

29.2.4 Disclosure Rules

The FAS 133 **disclosure** method depends on the purpose of the derivative, which is consistent with the reasoning behind Figure 29-1. Gains and losses from derivatives are accounted according to one of these methods:

a. *No hedging designation.* The gain or loss should flow into *earnings*.

b. *Fair-value hedge.* This applies when the derivative is designated and qualifies as a hedge. The gain or loss on the derivative should go into *earnings*, along with the offsetting loss or gain on the hedged instrument. Because this offset may not be perfect, earnings may be affected by some residual risk, called **hedge ineffectiveness**.

This category applies when the derivative is used to hedge changes in the *fair value* of a recognized asset or liability or of an unrecognized firm commitment. These represent hedges of existing or anticipated *positions*.

c. *Cash-flow hedge.* The gain or loss on the derivative should go into **other comprehensive income (OCI)** (outside earnings). Otherwise, it should be reclassified into *earnings* at the same time that the hedged transaction affects earnings.

This category applies when the derivative is used to hedge changes in the *cash* *flows* of a recognized asset or liability or of a forecasted transaction. These represent hedges of existing or anticipated *cash flows*.

d. *Foreign currency hedge.* This category applies when the derivative is used to hedge the foreign currency exposure of

- *An unrecognized firm commitment* (using the fair-value method)
- *An available-for-sale security* (using the fair-value method)
- *A forecasted transaction* (using the cash-flow method)
- *A net investment in a foreign operation* (using a translation adjustment in an equity account)

29.2.5 Hedge Effectiveness

To obtain hedge treatment, the derivative should be designated as a hedge at inception. Users are required to create documentation that supports the business purpose and effectiveness of the hedge at inception and on an ongoing basis. The hedge must be monitored regularly, at least on a quarterly basis, in line with the financial reporting cycle.

FAS 133 does not prescribe a particular method for measuring hedging effectiveness.[3] This can be done using a *prospective (forward-looking) approach*, using regression or other statistical analysis of past changes in fair values or cash flows. A *retrospective (backward-looking) evaluation* is also allowed, where the user must assess whether the hedge did achieve offsetting changes in fair value or cash flows. The application of statistical techniques is complex and requires a good understanding of statistics.

As an example, consider again the hedge of oil in Table 29-1. The company designates the derivative as a hedge of the changes in the fair value of the inventory of oil. The hedging relationship qualifies for fair value hedge accounting. In this case, the company expects no ineffectiveness because (a) the notional amount of the derivative matches the amount of the hedged inventory and (b) the underlying of derivative is the price of the same grade and location as the inventory.

Assume that after three months, the price of oil falls from \$13 to \$10 per barrel. Ignoring time effects, the value of the short derivative position should increase by \$3.

[3]See the Derivatives Implementation Group. (2000). *Statement 133 Implementation Issue No. E7: Hedging–General: Methodologies to Assess Effectiveness of Fair Value and Cash Flow Hedges.* FASB: Connecticut. Available at: http://www.fasb.org .

We would then have an entry into earnings that reflects a loss of $3,000 in the inventory and is offset by a gain of $3,000 on the derivative.

Let us say now that the terms of the derivative do not match exactly those of the inventory and that the derivative position increases in value by only $1 per barrel. The entries in earnings would then reflect a loss of $3,000 in the inventory and a gain of $1,000 on the derivative, for a net loss of $2,000. In this case, there is some fluctuation in earnings that reflects the ineffectiveness of the hedge. This difference reflects a true economic loss, however.

29.2.6 General Evaluation of FAS 133

FAS 133 is widely viewed as a complex set of standards. The initial rules were published in a 245-page document, which is comparable to a course textbook. Later amendments are also very long. When the standards were initially proposed, there was some opposition, in part due to the complexity of the rules but also coming from banks that feared that derivatives usage would be aversely affected. Far from it, however, derivatives have continued their unabated growth.

Another source of concern was that FAS 133 would increase the volatility of reported earnings. This is not always the case, however. When constructed as effective hedges, adding derivatives has a minimal impact on earnings volatility. FAS 133, however, does penalize **macro hedges**, which are hedges applied at the portfolio level as opposed to individual transaction level. Macro hedges reduce the number and volume of hedging transactions but do not benefit from hedge treatment.

As an example, consider a car manufacturer that has a yen exposure not because it exports to Japan but because its competitors are Japanese. The firm has no yen transactions on its books, but would reduce its risk by hedging its yen exposure. Such a hedge does not qualify for hedge accounting. Derivatives profits and losses have to be shown in earnings. On the other hand, the hedge should offset movements in operating cash flows. When the yen depreciates, domestic sales and profits should suffer but at the same time gains should accrue on the hedges. So, there should be some economic offsets in earnings.

29.2.7 Accounting Treatment of SPEs

The **Enron** failure has highlighted deficiencies in the application of U.S. financial reporting standards. Enron made extensive use of **special-purpose entities** (SPEs), which are financial vehicles used to convert income-producing assets, such as loans, bonds, credit-card receivables, or pipelines into cash. In a clean securitization process, a company transfers assets to an SPE, in return for cash, accounting for the deal as a sale, thus removing the assets from the balance sheet.

In the case of Enron, however, the company still kept an equity stake in the SPE. Even so, Enron was not required to *consolidate*, that is, include its interests in the SPE on its balance sheet. This was allowed because the SPE was structured to have *sufficient independent economic substance,* which was defined as a situation where outside investors have an equity stake of at least 3% of the SPE's capital. Enron only had to show *equity in the SPE affiliate* on its balance sheet. The end result was that Enron was able to move assets and debt out of its balance sheet, artificially lowering its leverage. This increased Enron's credit rating and made its stock look more desirable than it really was.

The problem was that Enron gave outside investors guarantees of the SPE's performance. In most cases, such support operations are optional. Problems arise with explicit guarantees, however. Some SPEs carried guarantees that effectively placed all the risk on Enron itself without being reflected on Enron's balance sheet. When the SPEs began to perform poorly, Enron was obligated to prop them up with cash or its own shares. As the size of those liabilities became clear, Enron's stock collapsed and the company was forced into bankruptcy. Compounding the scandal were conflicts of interest created by some Enron executives' personal holdings in the SPEs.

The FASB has revised its rules to make it harder for companies to keep SPEs off the books. The new guidance, called interpretation 46, is based on two provisions. First, to qualify for off-balance sheet treatment, a SPE must contain at least 10% of outside equity, up from the current 3%. Second, the outside equity should be at risk, and as such cannot be protected by side agreements with the parent company.

Example 29-4: FRM Exam 1998—Question 10/Regulatory

29-4. All of the following instruments are considered to be derivatives under FAS 133 *except*

a) Futures contracts
b) Total return swaps
c) Credit default swaps
d) Option contracts

Example 29-5: FRM Exam 2000—Question 24/Legal & Other Risks

29-5. According to a provision in FAS 133, under which of the following conditions should embedded derivatives be split between the host contract and the embedded derivative?

I. The economic characteristics of the contract and embedded derivative are not "clearly and closely related."

II. The fair market value for the hybrid contract otherwise would not be reported on the balance sheet.

III. The embedded derivative would meet the definition of a derivative on a stand-alone basis.

IV. The payoff is not a function of the return on a linked instrument.

a) I and II

b) II and III

c) I, II and III

d) I, II, III, and IV

Example 29-6: FRM Exam 1998—Question 11/Regulatory

29-6. Under FAS 133, which of the following instruments would require bifurcation of the cash instrument and the embedded derivative instrument?

a) Inverse floater

b) Inflation indexed bond

c) Indexed amortizing notes

d) Callable debt

Example 29-7: FRM Exam 1998—Question 12/Regulatory

29-7. Which type of derivative contract is least appropriate for a manufacturing company trying to hedge a rise in the cost of its raw materials?

a) Long futures

b) Long call option

c) Short put option

d) Floating-price payer on commodity swap

Example 29-8: FRM Sample Question

29-8. Adoption of FAS 133 derivative hedge accounting requires which of the following?

a) Hedges must be declared on an ongoing basis.

b) Correlation must be analyzed at least every three months to coincide with the financial reporting cycle.

c) Hedges must be marked to market and booked into income.

d) FAS 133 must be adopted as the global standard for booking derivative.

Example 29-9: FRM Sample Question

29-9. Which of the following approaches for measuring the effectiveness of hedges are permissible under FAS 133 hedge accounting rules?

a) Statistical techniques
b) Cash flow analysis
c) Dollar offset
d) Any of the above

29.3 External Reporting: IASB

The **International Accounting Standards Committee** (IASC) was set up in 1973 to champion global accounting standards. IASC was superseded by the **International Accounting Standards Board** (IASB) in 2001. International securities regulators gave IASB a mandate to devise common reporting standards acceptable for listing on any stock exchange. The European Union, in particular, will require all EU companies to comply with IASB standards by 2005.

IASB publishes its Standards in a series of pronouncements called International Financial Reporting Standards (IFRS). It has also adopted the body of Standards issued by the IASC. Those pronouncements continue to be designated **International Accounting Standards** (IAS). IASB has started the revision process for 12 of the 34 active standards.

The FASB is not bound to adopt IASB's standards, although each has agreed to try to converge to the highest quality accounting treatment. There are also differences of opinion with respect to the philosophy of accountings standards. Should they be guided by *principles* or by *detailed rules*? Both approaches have strengths and weaknesses. U.S. regulators tend to emphasize detailed rules, which may encourage companies to exploit loopholes in the system. Indeed, Enron devoted much effort to game its financial reporting system. Enron may have followed the letter, but certainly not the spirit of the system. On the other hand, guiding principles may give too much leeway in interpretation. A proper balance between the two approaches will be required.

29.3.1 IAS 37

IAS 37, which came in force after July 1999, deals with **provisions**, contingent liabilities and assets. The key principle is that a provision should be recognized only when there is a liability, that is, a present obligation resulting from past events. IAS 37 thus aims to ensure that only genuine obligations are recognized in the financial statements, unlike contingent liabilities.

IAS 37 requires a firm to make provisions on the balance sheet if, and only if three conditions are met: (i) a present obligation (legal or constructive) has arisen as a result of a past event (the obligating event), (ii) payment is probable ("more likely than not"), and (iii) the amount can be estimated reliably.

An obligating event is an event that creates a legal or constructive obligation and, therefore, results in a company having no realistic alternative but to settle the obligation. A constructive obligation arises if past practice creates a valid expectation on the part of a third party, for example, a retail store that has a long-standing policy of allowing customers to return merchandise within, say, a 30-day period. A legal obligation arises when a lawsuit is filed as a result of a past event and there is a high probability of a settlement.

29.3.2 IAS 39

IAS 39, which came in force after January 2001, deals with these financial instruments: cash; demand and time deposits; commercial paper; accounts, notes, and loans receivable and payable; debt and equity securities; asset-backed securities, such as collateralized mortgage obligations, repurchase agreements; derivatives; leases, right and obligations of insurance contracts and pension contracts.

The key principle behind IAS 39 is that all financial instruments must be recognized on the balance sheet. Note that this is broader than FAS 133, which only applies to derivatives. Initial measurement is at cost, which is the fair value of whatever was paid or received. Subsequent measurement depends on the category. For derivatives, changes in values must flow into earnings, except for hedges.

Finally, IAS 39 also deals with **hedge accounting**. For accounting purposes, hedging means designating a derivative financial instrument as an offset in net profit or loss to the change in fair value or cash flows of a hedged item. The designation must be in writing, up front, and be consistent with an established risk management strategy. As with FAS 133, IAS 39 recognizes fair-value hedges and cash-flow hedges.

Example 29-10: FRM Exam 2001—Question 57

29-10. Unlike credit risk, when the calculated expected credit losses might be covered by general and/or specific provisions in the balance sheet, in operational risk, due to its multidimensional nature, the treatment of expected losses is more complex and restrictive. Recently, with the issuing of IAS 37 by the IASB, the rules have become clearer as to what can (or cannot) be subject to provisions. Which of the operational risk types below can clearly be provisioned (given that a figure can be reasonably estimated)?
a) Transaction processing risk
b) Legal risk
c) Systems risk
d) Interest expenses

29.4 Tax Considerations

The taxation of financial instruments is a complex topic that evolves over time, differs across jurisdiction, and is often not consistent across economically equivalent assets.

In fact, financial innovations are often viewed as a response to changes in the *tax code* and *regulation.* One example is the differential treatment of capital gains and ordinary income, which can lead to arbitrage opportunities. For instance, zero-coupon bonds were initially created to take advantage of the fact that their return was entirely viewed as capital gains, which are taxed at a lower rate than income. Since then, tax authorities have changed the tax code to bring in line the taxation of zeros and coupon-paying bonds.

Even though their tax advantages have faded, however, zeros are still widely used as they provide effective hedges for investors with fixed liabilities. The continued growth of derivatives is explained by the fact that they make markets more complete by increasing opportunities for *risk sharing* among investors. Even so, avoidance of taxes and regulation often have provided the impetus for the creation of new financial instruments.

Generally, key issues in taxation are the

- *Nature*, or *character* of taxable gains and losses (i.e. capital or ordinary income)
- *Timing* of their recognition (i.e. at inception, during the life, or at expiration of the transaction)

- *Source* of revenues, which determines whether income will bear tax (i.e., U.S. income of non-U.S. persons attracts a U.S. withholding tax and foreign source income of U.S. persons are subject to U.S. federal income taxes)

Consider, for instance, U.S. tax rules. The character issue is more important for non-corporate taxpayers, who face lower tax rates on capital gains than on ordinary income. For corporate taxpayers, in contrast, capital gains and income are taxed at the same rate.

Futures contracts fall under Section s1256 of the Internal Revenue Code. Positions in futures are marked-to-market and treated as if they are closed out on the last day of the tax year. Gains and losses are of a *capital* nature, except for foreign exchange gains and losses, which are treated as ordinary income, falling under Section s988.

Hedging transactions, however, are treated differently. These are defined as transactions entered for one of these reasons:

1. To reduce the risk of price changes with respect to assets held or to be held for the purpose of producing ordinary income
2. To reduce the risk of price changes (e.g., interest rate changes or currency fluctuations) with respect to borrowings

Hedging transactions are taxed as *ordinary income*, with recognition of gains or losses matching the recognition of that of the hedged item. Note that the definition of hedge transaction for tax purposes differs from that for accounting purposes, requiring a different set of books.

Example 29-11: FRM Exam 2000—Question 21/Legal & Other Risks

29-11. Hedging transactions are taxed as

a) Capital gains

b) Dividend income

c) Ordinary income

d) Interest income

29.5 Answers to Chapter Examples

Example 29-1: FRM Exam 1998—Question 8/Regulatory

c) Directly observable market quotes are least susceptible to price manipulation.

Example 29-2: FRM Exam 1997—Question 21/Regulatory

c) Traders have access to recent historical volatilities and could bias their position toward those assets with low historical risk but high implied risk. This would understate their true risk.

Example 29-3: FRM Exam 2000—Question 25/Legal & Other Risks

b) The prices should be recorded at the same time using actual quotes or, possibly, forecasting the price of the closed market based on information from the other market.

Example 29-4: FRM Exam 1998—Question 10/Regulatory

c) Credit default swaps do not necessarily satisfy the third condition, which is to allow net settlements.

Example 29-5: FRM Exam 2000—Question 24/Legal & Other Risks

c) Answers I, II, and III are correct. The derivative should have a payoff that does depend on an underlying.

Example 29-6: FRM Exam 1998—Question 11/Regulatory

a) The inverse floater is a fixed-rate bond plus a long position in a receive-fixed swap. Thus it is a hybrid instrument. We need to check whether the three conditions for separation are satisfied. The swap is not closely related to the host contract and hence satisfies condition (1). The inverse floater is not marked-to-market on the balance sheet, which satisfies condition (2). Finally, the swap is a derivative and hence satisfies condition (3). Answer (b) is incorrect because the coupon payments and the inflation index are clearly and closely related. Same for answers (c) and (d). Indexed amortizing notes repay the principal according to a schedule that depends on the value of a reference index.

Example 29-7: FRM Exam 1998—Question 12/Regulatory

d) The company has a natural *short* position in the product. Price increases can be hedged by taking a long futures or long call position, so answers (a) and (b) are appropriate hedges. Selling a put does not provide a hedge against price increases, but offsets the benefit of falling prices. This is not a hedge, but is consistent with the natural short position. Finally, paying the floating price on the swap means that the company will have to pay *more* if the commodity price increases. This is the opposite of what the company should do to hedge.

Example 29-8: FRM Sample Question

b) The performance of hedges must be reviewed at least every three months. Answer (a) is not correct because hedges must be declared at inception and on an ongoing basis. Answer (c) is not correct because cash-flow hedges go into OCI. Finally, answer (d) refers to the IASB standards, which do not apply to U.S. firms.

Example 29-9: FRM Sample Question

d) Any of the methods is permissible. The approach should be chosen at inception, however, and should not vary during the hedging period.

Example 29-10: FRM Exam 2001—Question 57

b) Interest expenses are not an obligating event, so (d) is wrong. Systems risk and transaction processing risk are not past events. Legal risk can be provisioned as a result of a *past* event that is likely to lead to a lawsuit and settlement.

Example 29-11: FRM Exam 2000—Question 21/Legal & Other Risks

c) As stated in the text, hedging transactions are taxed as ordinary income.

PART
seven

Regulation and Compliance

Chapter 30

Regulation of Financial Institutions

We now tackle the last part of this manual, which deals with regulatory capital. Banks and securities houses must now comply with risk-based capital requirements. These regulatory capital requirements have been the catalyst for the revolution in risk management. They have spurred the industry into better understanding and management of their risks. In turn, regulators are now forced to upgrade their regulatory requirements to keep up with modern developments in risk management. Analyzing the rationale behind these regulations yields interesting insights into broader issues that we have not addressed yet, such as systemic risk.

This chapter is structured as follows. Section 30.1 provides a broad classification of financial institutions subject to regulation. Section 30.2 then discusses systemic risk, which is viewed as a major rationale for regulation. Next, Sections 30.3 and 30.4 describes the regulation of commercial banks and securities houses, respectively. Finally, Section 30.5 concludes with a summary of the tools and objectives of financial regulation.

30.1 Definition of Financial Institutions

Financial institutions are fundamentally different from other corporations. When an industrial corporation goes bankrupt, shareholders, bondholders, and other creditors suffer financial losses. The overall effects of the failure, however, are limited to direct stakeholders.

In contrast, the failure of a financial institution can be potentially much more harmful. Financial institutions include

- **Commercial banks**, whose primary function is to hold customer deposits and to extend credit to businesses, households, or governments.[1]

[1] Similar intermediaries are **savings institutions**, which specialize in residential mortgages, and **credit unions**, which extend mortgage and consumer credit. These are generally local and relatively small institutions whose failure is unlikely to destabilize financial markets.

- **Securities houses**, whose primary function is to intermediate in securities markets. These include **investment banks**, which specialize in the initial sale of securities in the primary markets,[2] and **broker-dealers**, whose primary function is to assist in the trading of securities in the secondary markets.[3]
- **Insurance companies**, which provide property and casualty (P&C) or life insurance coverage.

In some countries, the first two types are separated and subject to different regulators. This was the case in the United States until the recent repeal of the Glass-Steagall Act, which separated banking and securities functions. This is an example of **asset restrictions** on financial institutions. In other countries with a so-called **universal bank** model, a bank can engage in traditional banking and securities activities.

Financial institutions also include other intermediaries that constitute the "buy side" of Wall Street, in contrast with banks and brokers, the "sell side" that intermediates in financial markets. The buy side consists of professional (as opposed to private) investors, called **institutional investors**, which include insurance companies, pension and endowment funds, investment companies (e.g., mutual and closed-end funds), and hedge funds. These are subject to different regulatory requirements from banks and securities houses.

At the outset, we should ask the question of whether regulation of financial institutions is at all necessary. After all, other industries are not regulated (except for antitrust reasons, i.e., to avoid monopolies such as in the recent Microsoft case). Private corporations already have their own governance mechanism, which is shareholder supervision. Shouldn't shareholders decide on the appropriate risk-return profile for the company in which they have invested their own funds? Why should governments intervene in free markets? Why do we need regulators?

[2] The term "bank" in "investment bank" is a misnomer, since these institutions do not extend credit like commercial banks.

[3] Brokers act as pure intermediaries and simply match buyers with sellers. As a result, they take no market risk. In contrast, dealers stand ready to buy and sell securities at given prices. Therefore, they must maintain an inventory of securities and are exposed to market risk.

30.2 Systemic Risk

Unlike other entities, banks and securities houses play a special role of intermediation. They facilitate payment flows across customers and maintain markets for financial instruments. This very role, however, can also make bank failures much more disruptive for the economy than the failure of other entities. The threat is that of systemic risk.

Systemic risk may be defined as the risk of a sudden shock, which would damage the financial system to such an extent that economic activity in the wider economy would suffer.

Systemic risk involves contagious transmission of the shock due to actual or suspected exposure to a failing bank. This is usually accompanied by a **flight to quality**, which reflects an increased demand for government securities, pushing up the relative cost of capital to the corporate sector. If prolonged, this can lead to a fall in investment spending and dampen consumption.

Indeed, failures in the domestic banking system have been particularly damaging. Among emerging markets, domestic financial collapses have often cost more than 10 percent of a country's Gross Domestic Product. In each case, the government (rather, the taxpayer) has covered the cost of the failure in the belief that this would be less costly than letting a domestic banking failure spread to the rest of the domestic economy.

Systemic risk can come from two sources:

- *Panicky behavior of depositors or investors.* This can arise from the failure of an institution or a political shock. In a **bank run**, depositors become worried about the stability of their bank (when there is no deposit insurance) and demand an immediate return of their funds, which may lead to a failure of the bank. Similarly, a sudden drop in securities prices may lead to margin calls, forcing leveraged investors to liquidate their positions, which puts further pressure on prices. Some institutions may fail, resulting in a loss of liquidity and a credit crunch.
- *Interruptions in the payment system.* This can arise from the failure of an institution or from a technological breakdown in the payment system. Banks and securities houses are central to the payment system by which transactions for goods, services, and assets are cleared and settled. When an institution cannot pay, it may expose the payment system to a breakdown.

30.3 Regulation of Commercial Banks

Our experience with systemic risk is profoundly marked by the banking crisis of the 1930s in the United States. The banking system was subject to **bank runs**, when depositors lost faith in the ability of their deposit bank to make full payment and "ran to the bank" to withdraw their funds.

The problem is that the bank may be perfectly solvent, that is, have assets (e.g., loans, real estate) whose value exceeds its liabilities (e.g., demand deposits). Because such assets are illiquid, however, the bank may not be able to meet redemptions immediately, leading to default. Indeed during the U.S. banking crisis of the 1930s, one bank in three failed, causing a severe contraction of credit.

In response, the United States established federal **deposit insurance** in 1933. The insurance fund protects investors if their bank fails, thereby eliminating the need for a bank run. This scheme was widely credited for stopping bank runs. By now, most countries have a compulsory deposit insurance program.

The problem with deposit insurance, however, is that some of the financial risk is now passed on to the deposit insurance fund (i.e., ultimately the government or taxpayer). This creates a need for regulation of insured institutions.

Turning next to the other source of systemic risk, the prime example of a breakdown in the payment system was the June 1974 failure of Bankhaus Herstatt, a small German bank active in the foreign exchange market. The bank was shut down by noon, U.S. time, after having received payments in German marks. In exchange, the counterparty banks were due to receive payment in the same afternoon in U.S. dollars. These payments never came, however, creating substantial losses and a serious liquidity squeeze for counterparties. This event caused severe disruption in the payment system and was perhaps the most extreme shock experienced in the foreign exchange market. What has become known as **Herstatt risk** has led to a concerted effort by bank regulators to try to avoid such situations, which ultimately gave birth to the **Basel Committee on Banking Supervision (BCBS)**.

The BCBS consists of central bankers from the Group of Ten (G-10) countries.[4] Its primary objective is to promote the **safety and soundness** of the global financial

[4]The Basel Committee's members are senior officials from the G-10 (Belgium, Canada, France, Germany, Italy, Japan, the Netherlands, Sweden, United Kingdom, and the United States plus Luxembourg and Switzerland), who meet four times a year, usually in Basel, under the aegis of the Bank for International Settlements. Its Web site is at http://www.bis.org.

system, that is, to try to control systemic risk. Another objective is to create a system that ensures a **level playing field** for global financial institutions.[5]

The Basel Committee has established minimum risk-based capital standards that apply to so-called **core institutions**. These represent internationally active *commercial* banks, which are major players in large-value payment systems. The capital adequacy rules are described in a series of documents known as the **Basel Accord**, which will be analyzed in the following chapters.

It should be emphasized that core institutions are ultimately regulated by their domestic banking regulators. Although pronouncements of the Basel Committee are not legally binding, member countries have implemented them. Even countries that are not part of the Basel Committee often feel obligated to abide by the same regulations. By now, over 100 countries have adopted the framework of the Basel Accord.

In the United States, for instance, commercial banks are regulated by the **Board of Governors of the Federal Reserve System** (the "Fed"),[6] the **Office of the Comptroller of the Currency** (OCC),[7] and the **Federal Deposit Insurance Corporation (FDIC)**.[8] This fragmentation of supervision is somewhat puzzling but is common among U.S. agencies.

In the United Kingdom, the regulatory framework is more logical, with only one regulator for banks, securities markets, and insurance firms, the **Financial Services Authority** (FSA).[9] This all-powerful regulator was created in October 1997, taking over banking supervision from the Bank of England.

In Japan, supervision of financial markets, including banking, securities business, and insurance, rests with the **Financial Services Agency** (FSA), established in July 2000. This responsibility is shared with the central bank, or **Bank of Japan**, which

---

[5]At that time, one concern was that Japanese banks were expanding into global markets and were able to undercut their competitors due to more lenient Japanese regulations.

[6]The Federal Reserve supervises all bank holding companies and state-chartered banks that are members of the Federal Reserve System. Its Web site is at http://www.bog.frb.fed.us.

[7]The principal function of the OCC is to supervise U.S. national banks and branches and agencies of foreign banks in the United States. National banks are defined as those chartered by the federal government, as opposed to state banks. The OCC is a bureau of the Treasury Department. Its Web site is at http://www.occ.treas.gov.

[8]The FDIC is a U.S. government agency whose mission is to maintain the stability and public confidence in the nation's financial system. It has provided deposit insurance since 1933. Its Web site is at http://www.fdic.gov.

[9]Its Web site is at www.fsa.gov.uk.

conducts monetary policy and ensures the stability of the financial system by monitoring financial institutions.[10]

Banks in the **European Union** (EU) are subject to minimum standards, which are binding over all member countries.[11] The **Solvency Ratio Directive** was published in December 1989 and implements the 1988 Basel rules for credit risk. To this was added the **Capital Adequacy Directives (CAD)** in March 1993, which implements the building-block approach for market risk. This was updated in 1998 to allow the use of internal models. France and Germany have different regulators for retail and wholesale markets. In France, this is split up between the Commission des Operations de Bourse and the Commission Bancaire (Banque de France), respectively.[12] In Germany, the agencies are the Federal Securities Supervisory Office, formally Bundesaufsichtsamt für den Wertpapierhandel (BAWe), and the Federal Banking Supervisory Office, formally Bundesaufsichtsamt für das Kreditwesen (BAKred).[13] There is now discussion of having one single pan-European regulator to have a truly integrated financial market.

The Basel Accord sets *minimum* risk-based levels of capital for core institutions (it will be examined in detail in the next chapter). National authorities, however, are free to adopt arrangements that set higher levels or other criteria. The Federal Reserve board, for example, has an additional requirement based on the bank's **leverage ratio**.[14] This places a constraint on the degree to which a banking organization can leverage its equity capital base.

To summarize, the regulation of commercial banks is motivated by two objectives

- *Minimizing systemic risk*
- *Protecting the deposit insurance fund*

[10]The Web sites for the FSA and the Bank of Japan are at http://www.fsa.go.jp and http://www.boj.or.jp.

[11]The EU includes Austria, Belgium, Denmark, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, the Netherlands, Portugal, Spain, Sweden, and the United Kingdom. This covers all countries in Western Europe except for Switzerland and Norway.

[12]The Web sites are at http://www.cob.fr and http://www.banque-france.fr.

[13]The Web sites are at http://www.bwade.de http://www.bakred.de.

[14]The ratio of (tier 1) capital to total consolidated assets must be greater than 3 percent plus an additional cushion of 100 to 200 basis points. Tier 1 capital will be defined in the chapter on the Basel Accord.

Failure to meet the capital-adequacy requirements triggers regulatory action, affecting the types of activities in which institutions can engage and requiring **prompt corrective action** (PCA), including the possible appointment of a receiver.

30.4 Regulation of Securities Houses

The regulation of securities houses substantially differs from that of commercial banks. Broker-dealers hold securities on the asset and liability side (usually called long and short) of their balance sheet. Because securities are much more liquid than bank loans, there is no rationale for bank runs.

The objectives of regulation for securities houses are

- *Protecting the customer.* One goal is to protect the firm's customers against a default of their broker-dealer. The rationale here is that small investors (e.g., the traditional "widows and orphans") are less capable of informed investment decisions. Another goal is to protect consumers against excessive prices or opportunistic behavior by financial intermediaries.
- *Ensuring the integrity of markets.* The goal is to ensure that failure by one institution does not destabilize financial markets, causing systemic risk.

Let us first examine the consumer protection argument. First, it must be emphasized that investors are risk takers by definition. As Philip McBride Johnson, former chairman of the Commodities Futures Trading Commission, has put it,

> *Regulation is not meant to insulate investors from the consequences of free economic forces, or from their own poor judgment, but rather from abuses perpetrated by other persons.*

Regulation, however, is generally considered necessary when the market fails in two respects, either through **excessive prices** or **opportunistic behavior**.

In a free market with informed customers, prices can be excessive only if sellers collude to maintain high prices. This is why there is a need for **antitrust legislation** to prevent collusion among financial intermediaries.

Opportunistic behavior can arise if sellers have more information than buyers because, for instance, of access to inside information. This justifies laws against trading on **inside information**. Or, brokers may have **conflicts of interest** that push them to give bad advice to their clients for the brokers' personal profit. Likewise, accounting standards and **disclosure rules** help to reduce asymmetries of information in financial markets, which is ultimately socially beneficial as it increases participation in financial markets.

Finally, brokers are subject to **suitability standards**. Broker-dealers are obligated, when making recommendations to clients, to recommend only transactions that are suitable to the client's financial situation, investment objectives, and sophistication. Unsuitable recommendations may constitute fraud, which is punishable by law.

Securities regulators require a prudent capital reserve to achieve the goals of protecting consumers and markets. The purpose of this capital is to ensure an orderly *liquidation* of the institution, in contrast to banks, for which capital is measured on an ongoing basis. These minimum reserves are calculated using different methods that use the total amount of debt, the total amount of money owed customers, and, more recently, measures of market risk based on VAR.

As with commercial banks, securities regulators meet in a global forum, the **International Organization of Securities Commissions (IOSCO)**, based in Montréal.[15] Its Technical Committee addresses regulatory problems related to international securities transactions. The IOSCO and the Basel Committee collaborate on common regulatory issues. Likewise, regulatory authority rests with a domestic supervisor, for example the **Securities and Exchange Commission** (SEC) in the United States.[16]

Securities regulation is based on either the "comprehensive approach" or the "simplified approach." The **comprehensive approach** is a system of capital charges detailed by the regulator. In contrast, the **simplified approach** uses a VAR model.

In the United States, the SEC uses the comprehensive approach with its **net capital rule**, Rule 15c3-1 under the Securities Exchange Act of 1934. A broker-dealer must satisfy a minimum capital ratio based on the calculated ratio of capital to debt or receivables. This ratio is 6.67 percent of aggregate debt, or 2 percent of the total

[15]Its Web site is at http://www.iosco.org.

[16]The SEC is a U.S. federal agency that has wide authority to oversee the nation's security markets. Among other responsibilities, it regulates the financial reporting practices of public corporations. To make information reporting more transparent, the SEC now requires registrants to disclose quantitative information on market risks using one of three possible alternatives: (i) a tabular presentation of expected cash flows and contract terms summarized by risk category, (ii) a sensitivity analysis expressing possible losses for hypothetical changes in market prices, and (iii) a VAR measure. Its Web site is at http://www.sec.gov.

amount of money owed by customers. To compute net capital, only liquid assets are considered, minus **haircuts**, which provide a further margin of safety in case of default and reflect market risk, liquidity risk, and counterparty risk.

The SEC's net capital rule, however, is widely viewed as quite conservative. As a result, it has become too expensive to operate derivatives activities under these rules. In January 1999, the SEC issued a ruling that created a class of **OTC derivatives dealers**, which are dealers active in OTC derivative markets. To bring their regulatory requirements in line with foreign firms and U.S. banks, the SEC created risk-based capital rules based on internal VAR models, which parallel the Basel rules.

30.5 Tools and Objectives of Regulation

Table 30-1 provides a summary of the tools and objectives of financial regulation. Systemic risk is controlled through capital adequacy rules, asset restrictions, and disclosure standards. Consumer protection is achieved through capital standards, disclosure rules, and conflict of interest rules.

TABLE 30-1 Tools and Objectives of Financial Regulation

	Objectives	
Tools	Systemic Risk	Consumer Protection
Capital standards	✓	✓
Disclosure standards	✓	✓
Asset restrictions	✓	
Antitrust enforcement		✓
Conflict rules		✓

Source: Herring and Litan (1995), *Financial regulation in the global economy*. Washington, DC: Brookings Institution.

Capital adequacy and disclosure rules can help to achieve both objectives. Disclosure reduces asymmetries in capital markets, protecting consumers. In addition, more disclosure can also stabilize capital markets. Firms that fail to reveal much information about their activities may be susceptible to market rumors, possibly resulting in loss of business or funding difficulties. Indeed, the turmoil that surrounded the near-failure of Long-Term Capital Management illustrates the panic behavior of banks that suspect that a financial institution with large positions similar to theirs may fail.

Indeed the Basel Committee has stated that disclosure

can reinforce the efforts of supervisors to foster financial market stability in an environment of rapid innovation and growing complexity. If provided with meaningful information, investors, depositors, creditors and counterparties can impose strong market discipline on financial institutions to manage their trading and derivatives activities in a prudent fashion and in line with their stated business objectives.

Example 30-1: FRM Exam 1999—Question 187/Regulation

30-1. The Basel (Basle) Capital Accord applies to these entities:
a) National banks chartered in the United States
b) All internationally active commercial banks
c) All banks and securities firms in the G-10 countries plus Luxembourg
d) Banks regulated by the Swiss banking regulatory authorities

Example 30-2: FRM Exam 1999—Question 186/Regulation

30-2. Which statement best defines "suitability" as it relates to a dealer's recommendation of a security transaction to a customer?
a) Customer suitability requires that a securities dealer run stress test simulations against a customer's portfolio, before recommending a particular transaction.
b) Customer suitability suggests that a securities dealer should verify that a proposed financial transaction is suitable for a customer's stated cash resources objectives.
c) Customer suitability requires that a securities dealer have reasonable grounds for believing that its recommendations are suitable based on customer information regarding the customer's securities holdings, financial status, and needs.
d) Customer suitability suggests that a securities dealer should make a quantitative assessment of the customer's level of sophistication.

Example 30-3: FRM Exam 1999—Question 188/Regulation

30-3. Which of the following financial institutions needs to comply with the provisions of CAD, the Capital Adequacy Directive? This question concerns the main home-country operations of these banks, not certain overseas subsidiaries or branches.
a) J. P. Morgan (an American Bank)
b) Credit Suisse First Boston (a Swiss Bank)
c) Deutsche Bank (a German Bank)
d) Sumitomo Bank (a Japanese Bank)

Example 30-4: FRM Exam 2000—Question 129/Regulation

30-4. The Bank of Japan

a) Is the primary Japanese bank authorized to review risk management practices of foreign investment banks and brokers in Japan

b) Shares its bank supervisory and audit role with the FSA

c) Has no supervisory or audit responsibilities with regard to financial institutions

d) Is authorized to supervise broker-dealer entities only

30.6 Answers to Chapter Examples

Example 30-1: FRM Exam 1999—Question 187/Regulation

b) The capital accord applies to commercial banks with international activities.

Example 30-2: FRM Exam 1999—Question 186/Regulation

c) Customer suitability does not require specific actions, such as running a stress test, verifying cash balances, or computing quantitative measures. Rather, the dealer must reasonably believe that the transaction is wel suited to the objectives of the customer.

Example 30-3: FRM Exam 1999—Question 188/Regulation

c) The Capital Adequacy Directive applies to banks within the European Union. Of the four countries listed, only Germany belongs to the EU.

Example 30-4: FRM Exam 2000—Question 129/Regulation

b) The BOJ is a central bank with responsibility over stability of financial markets and regulates commercial banks. This responsibility is shared with the Financial Services Agency (FSA).

Chapter 31

The Basel Accord

The **Basel Capital Accord**, concluded on July 15, 1988, represents a landmark financial agreement for the regulation of internationally active commercial banks. It instituted for the first time minimum levels of capital to be held by international banks against financial risks.

Initially, the capital charges were based on a set of standard, rigid rules defined by the Basel Committee on Banking Supervision (BCBS), These risk-based capital adequacy requirements evolved over time, first covering credit risk, then market risks. The latest rules by the Basel Committee, called Basel II, 2001, represent an extensive revision of the capital charges that allow more flexibility as well as greater reliance on the banks' internal methodologies. The new rules also add a charge against operational risks.

This chapter is structured as follows. Section 31.1 provides a broad overview of the Basel Accord. Section 31.2 explains the original Basel capital requirements, with particular emphasis on credit risk. Market risk is a complex subject in itself and will be developed in the next chapter. Section 31.3 illustrates the application of capital adequacy ratios for Citigroup. Finally, Section 31.4 discusses major drawbacks of the original Basel Accord and describes the main components of the New Accord.

31.1 Steps in The Basel Accord

31.1.1 The 1988 Accord

The original goal of the 1988 Basel Accord, which came into force in 1992, was to provide a set of minimum capital requirements for commercial banks. Its primary objective was to promote the safety and soundness of the global financial system and to create a level-playing field for internationally-active banks. The **risk-based capital charges** roughly attempted to create a greater penalty for riskier assets.

Initially, the 1988 Basel Accord only covered credit risk. The Accord set a minimum level of capital expressed as a ratio of the total risk-weighted (RW) assets, which include on-balance-sheet and off-balance-sheet items. Banks have to hold capital that covers at least 8 percent of their risk-weighted assets. The purpose of this capital is to serve as a buffer against unexpected financial losses, thereby protecting depositors and financial markets.

31.1.2 The 1996 Amendment

In 1996, the Basel Committee amended the Capital Accord to incorporate market risks. This amendment, which came into force at the end of 1997, added a capital charge for market risk. Banks are allowed to use either a standardized model or an **internal models approach**, based on their own risk management system.

The amendment separates the bank's assets into two categories, the trading book and banking book. The **trading book** represents the bank portfolio with financial instruments that are intentionally held for short-term resale and typically marked-to-market. The **banking book** consists of other instruments, mainly loans, that are held to maturity.

The 1996 amendment adds a capital charge for (i) the market risk of trading books, and (ii) the currency and commodity risk of the banking book. In exchange, the credit risk charge excludes debt and equity securities in the trading book and positions in commodities. As before, it still includes all OTC derivatives, whether in the trading or banking books.

31.1.3 The New Basel Accord

Capital markets have witnessed enormous changes since the initial Capital Accord of 1988. Increasingly, the original credit risk charges have appeared outdated and, even worse, may be promoting unsound behavior by some banks.

In January 2001, the Basel Committee issued a comprehensive revision to the Basel Accord. The Committee is expected to finalize the Accord by year-end 2003. Based on that release date, the implementation date has been set for January 2007 to allow for domestic rule-making processes and time to prepare for the new rules.

The new framework is based on **three pillars**, viewed as mutually reinforcing:

- *Pillar 1: Minimum capital requirement.* These are meant to cover credit, market, and operational risk. Relative to the 1988 Accord, banks have now a wider choice of models for computing their risk charges.
- *Pillar 2: Supervisory review process.* Relative to the previous framework, supervisors are given an expanded role. Supervisors need to ensure that
  - Banks have in place a process for assessing their capital in relation to risks.
  - Banks indeed operate above the minimum regulatory capital ratios.
  - Corrective action is taken as soon as possible when problems develop.
- *Pillar 3: Market discipline.* The New Accord emphasizes the importance of risk disclosures in financial statements. Such disclosures enable market participants to evaluate banks' risk profile and the adequacy of their capital positions. The new framework sets out disclosure requirements and recommendations. Banks that fail to meet disclosure requirements will not qualify for using internal models. As internal models generally lead to lower capital charges, this provides a strong incentive for complying with disclosure requirements. In essence, the trade-off for greater reliance on a bank's own models is greater transparency.

The New Accord provides for finer measurement of credit risk, which will generally lead to lower capital requirements. In order to maintain the overall level of bank capital, however, new capital charges are set against **operational risk**. Capital adequacy will be measured as follows:

$$\frac{\text{Total Capital}}{\text{Credit Risk} + \text{Market Risk} + \text{Operational Risk}} = \text{Bank's Capital Ratio} > 8\% \qquad (31.1)$$

As before, credit risk in the denominator is measured by the sum of risk-weighted assets for credit risk. The other items are measured from the multiplication of the **market risk charge** (MRC) and **operational risk charge** (ORC) by $(1/8\%) = 12.5$. For instance, if a bank has \$875 in risk-weighted assets and MRC $=$ \$10 and ORC $=$ \$20, the denominator would be computed as $\$875 + [(\$10 + \$20) \times 12.5] = \$1,250$. The bank then has to hold at least $8\% \times \$1,250 = \$100$ in capital to satisfy the minimum requirement. This is equivalent to saying that the total charge must be at least $8\% \times \$875 + \$10 + \$20 = \$70 + \$10 + \$20 = \$100$.

Figure 31-1 summarizes the coverage of credit, market, and operational risk charges for the banking and trading books. Banks will also have access to a menu of methods to compute their risk charges. These are described in Table 31-1.

FIGURE 31-1 Summary of Basel II Risk Charges

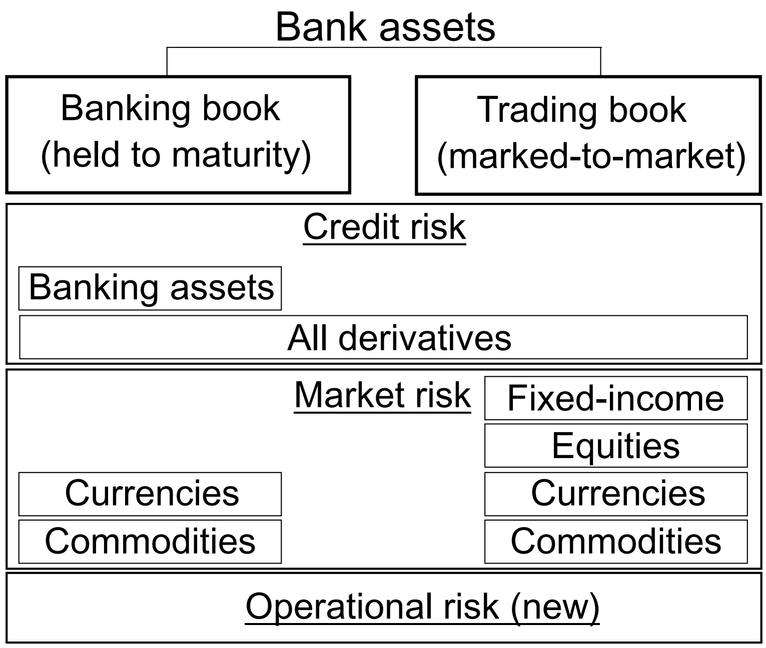


TABLE 31-1 Menu of Approaches to Measure Risk

Risk Category	Allowed Approach
Credit	Standardized Approach (based on the 1988 Accord) Foundation Internal Rating Based Approach Advanced Internal Rating Based Approach
Market	Standardized Approach Internal Models Approach
Operational	Basic Indicator Approach Standardized Approach Advanced Measurement Approach

Example 31-1: FRM Exam 1997—Question 17/Regulatory

31-1. For regulatory capital calculation purposes, what market risks must be incorporated into a bank's VAR estimate?

a) Risks in the trading account relating to interest rate risk, and equity risk

b) Risks in the trading account relating to interest rate risk and equity risk and risks throughout the bank related to foreign exchange and commodity risks

c) Risk throughout the bank related to interest rate risk, equity risk, foreign exchange risk, and commodity risk

d) Interest rate risk, equity risk, foreign exchange risk, and commodity risk in the trading account only

31.2 The 1988 Basel Accord

31.2.1 Risk Capital

The 1988 capital adequacy rules require any **internationally active bank** to carry capital of at least 8 percent of its total risk-weighted assets. This applies to commercial banks on a consolidated basis. So, for instance, holding companies that are parents of banking groups have to satisfy the capital adequacy requirements.

In the Basel Accord, "capital" has a broader interpretation than the book value of equity. act as a buffer to protect is its ability to absorb losses, providing some protection to creditors and depositors. Hence, to be effective, capital must be permanent, cannot impose mandatory fixed charges against earnings, and must allow for legal subordination to the rights of creditors and depositors.

The Basel Accord recognizes three forms of capital.

(1) Tier 1 capital, or "core" capital

Tier 1 capital includes equity capital and disclosed reserves, most notably after-tax retained earnings. Such capital is regarded as a buffer of the highest quality.

- **Equity capital** consists of issued and fully paid common stock and nonredeemable, noncumulative preference shares.
- **Disclosed reserves** correspond to share premiums, retained profits, and general reserves.[1]

(2) Tier 2 capital, or "supplementary" capital

Tier 2 capital includes components of the balance sheet that provides some protection but ultimately must be redeemed or contain a mandatory charge against future income. These include

- **Undisclosed reserves**, or hidden reserves that are allowed by the accounting standards of some countries. These are reserves that passed through the earnings statement but remain unpublished. Due to this lack of transparency, as well as the fact that many countries refuse to recognize undisclosed reserves, undisclosed reserves are not part of core capital.

[1]The term "reserve" refers to a part of equity capital. It should not be confused with "provision," which is a type of liability, and "allowance" for loan impairment, which is a reduction in the value of the loan recorded on the balance sheet. Impairment is established when "it is probable that the bank will not be able to collect." In contrast, a "charge-off" (or "write-off") occurs when there is no realistic prospect of recovery.

- **Asset revaluation reserves** which arise, for instance, from long-term holdings of equity securities that are valued at historical acquisition costs. Such capital could be used to absorb losses on a going-concern basis, subject to some discount to reflect market volatility and future taxes in case of sales.
- **General provision/loan loss reserves**, which are held against future unidentified losses. These reduce tier 1 capital but may qualify as tier 2 capital to the extent that they do not reflect a known deterioration in particular assets (in which case they are "specific.")
- **Hybrid debt capital instruments**, which combine some characteristics of equity and of debt. When they are unsecured, subordinated, and fully paid-up, they are allowed into supplementary capital. These include, for instance, **cumulative preference shares**.
- **Subordinated term debt**, with a minimum original maturity of five years, and subject to a discount of 20 percent during the last five years. Subordinated debt would be junior in right of payment to all other indebtedness in the event of liquidation.

(3) Tier 3 capital, for market risk only

Tier 3 capital consists of short-term subordinated debt with a maturity of at least two years. This is only eligible to cover market risk.

There are additional restrictions on the relative amount of various categories. Of the 8 percent capital charge for credit risk, at least 50 percent must be covered by tier 1 capital. Next, the amount of tier 3 capital is limited to 250 percent of tier 1 capital allocated to support market risks (tier 2 capital can be substituted for tier 3 capital if needed). Other restrictions apply to various elements of the three tiers.

Finally, some items are deducted from the capital base, including goodwill and investments in financial entities. The latter is motivated by the need to discourage cross-holding and double-counting of capital.

For credit risk, the eligible capital must exceed the regulatory capital, or

$$\text{Eligible Tier 1 Capital for CR} + \text{Allowed Tier 2 Capital} \geq \text{CRC} \tag{31.2}$$

A similar constraint applies to market risk capital, or

$$\text{Eligible Tier 1 Capital for MR} + \text{Allowed Tier 3 (or 2) Capital} \geq \text{MRC} \tag{31.3}$$

A worked-out example later will be given later. Next, we look at the construction of risk charges.

Example 31-2: FRM Exam 1999—Question 189/Regulation

31-2. Banks are required to maintain a percentage of of their assets as tier 1 capital. Which of the following count towards this capital requirement?

I. Shareholders equity
II. Sovereign debt held in the trading book
III. Common stock of other banks
IV. Subordinated debt issued by the bank in question (subject to certain qualifying rules)

a) I, II, and IV
b) II and III
c) I and IV
d) I only

Example 31-3: FRM Exam 2000—Question 139/Regulation

31-3. Tier 1 capital includes all of the following *except*

a) Asset revaluation reserves
b) Common stock
c) Noncumulative preferred shares
d) Disclosed reserves

31.2.2 On-Balance-Sheet Risk Charges

We first examine on-balance sheet assets, which consist principally of loans for most credit institutions. Ideally, the capital charges should provide some recognition of variation in asset quality.

Indeed, the 1988 Basel Accord applies to the notional of each asset a **risk capital weight** taken from four categories, as described in Table 31-2. Each dollar of risk-weighted notional exposure must be covered by 8 percent capital.

These categories provide an extremely rough classification of credit risk. For instance, claims on Organization for Economic Cooperation and Development (OECD) central governments, such as holdings of U.S. Treasuries, are assigned a weight of zero since these assets have presumably no default risk.[2] Cash held is also assigned a zero weight. At the other extreme, claims on corporations, including loans, bonds, and equities, receive a 100 percent weight, whatever the risk of default or maturity of the loan.

[2]The OCED currently consists of Austria, Belgium, Canada, Denmark, France, Germany, Greece, Iceland, Ireland, Italy, Luxembourg, The Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, Turkey, United Kingdom, United States, Japan, Finland, Australia, New Zealand, Mexico, Czech Republic, Hungary, Korea, and Poland, in order of accession.

TABLE 31-2 Risk Capital Weights by Asset Class

Weights	Asset Type
0%	Cash held
	Claims on OECD central governments
	Claims on central governments in national currency
20%	Cash to be received
	Claims on OECD banks and regulated securities firms
	Claims on non-OECD banks below 1 year
	Claims on multilateral development banks
	Claims on foreign OECD public-sector entities
50%	Residential mortgage loans
100%	Claims on the private sector (corporate debt, equity, ...)
	Claims on non-OECD banks above 1 year
	Real estate
	Plant and equipment

The **credit risk charge** (CRC) is then defined for balance-sheet items (BS) as

$$\text{CRC(BS)} = 8\% \times (\text{Risk} - \text{Weighted Assets}) = 8\% \times \left(\sum_i w_i \times \text{Notional}_i \right) \qquad (31.4)$$

where w_i is the risk weight attached to asset i.

Example 31-4: FRM Exam 2001—Question 38

31-4. A bank subject to the Basel I Accord makes a loan of $100m to a firm with a risk weighting of 50%. What is the basic on-balance credit risk charge?

a) $8m

b) $4m

c) $2m

d) $1m

31.2.3 Off-Balance-Sheet Risk Charges

By the late 1980s, focusing on balance-sheet items only missed an important component of the credit risk of the banking system, which is the exposure to swaps. The first swaps were transacted in 1981. By 1990, the outstanding notional of open positions had grown to $3,500 billion, which seems enormous. Some allowance had to be made for the credit risk of swaps. Unlike loans, however, the notional amount does not represent the maximum loss.

To account for such **off-balance sheet** (OBS) items, the Basel Accord computes a "credit exposure" that is equivalent to the notional for a loan, through **credit conversion factors**. The Accord identifies five broad categories.

- Instruments that substitute for loans (e.g. guarantees, bankers' acceptances, and standby letters of credit serving as guarantees for loans and securities) carry the full 100% weight (or credit conversion factor). The rationale is that the exposure is not different from a loan. Take a **financial letter of credit** (LC), for instance, which provides irrevocable access to bank funds for a client. When the client approaches credit distress, it will almost assuredly draw down the letter of credit. Like a loan, the full notional is at risk. This category also includes asset sales with recourse, where the credit risk remains with the bank, and forward asset purchases.
- Transaction-related contingencies (e.g., performance bonds or **commercial letters of credit** related to particular transactions) carry a 50% factor. The rationale is that a performance letter of credit is typically secured by some income stream and has lower risk than a general financial LC.
- Short-term, self-liquidating trade-related liabilities (e.g., documentary credits collateralized by the underlying shipments) carry a 20% factor.
- Commitments with maturity greater than a year (such as credit lines), as well as note issuance facilities (NIFs), carry a 50% credit conversion factor. Shorter-term commitments or revocable commitments have a zero weight. Note that this applies to the unfunded portion of commitments only, as the funded portion is an outstanding loan and appears on the balance sheet.
- Other derivatives, such as swaps, forwards and options on currency, interest rate, equity, and commodity products are given special treatment given the complexity of their exposures.

The Basel Accord computes the credit exposure for derivatives as the sum of the current, **net replacement value** (NRV) plus an **add-on** that is supposed to capture future or **potential exposure**:

$$\text{Credit Exposure} = \text{NRV} + \text{Add-on} \tag{31.5}$$

$$\text{Add-on} = \text{Notional} \times \text{Add-on Factor} \times (0.4 + 0.6 \times \text{NGR})$$

Here, the add-on factor depends on the **tenor** (maturity) and type of contract, as listed in Table 31-3 (NGR will be defined later). It roughly accounts for the maximum credit exposure which, as we have seen before, depends on the volatility of the risk factor and the maturity. This explains why the add-on factor is greater for currency, equity, and commodity swaps than for interest rate instruments, and also increases with maturity.

TABLE 31-3 Add-on Factors for Potential Credit Exposure (Percent of Notional)

Residual Maturity (tenor)	Contract				
	Interest Rate	Exchange Rate, Gold	Equity	Precious Metals	Other Commodities
$<$ 1 year	0.0	1.0	6.0	7.0	10.0
1–5 year	0.5	5.0	8.0	7.0	12.0
$>$ 5 year	1.5	7.5	10.0	8.0	15.0

More precisely, the numbers have been obtained from simulation experiments (such as those in Chapter 21) that measure the 80th percentile worst loss over the life of a matched pair of swaps. The matching of pairs reflects the hedging practice of swap dealers and effectively divides the exposure in two, since only one swap can be in-the-money. Take, for instance, a currency swap with 5-year initial maturity. Assuming exchange rates are normally distributed and ignoring interest rate risk, the maximum credit exposure as a fraction of the notional should be

$$\text{WCE} = \frac{1}{2} \times 0.842 \times \sigma \sqrt{5}, \tag{31.6}$$

where the $\frac{1}{2}$ factor reflects swap matching and the 0.842 factor corresponds to a one-sided 80 percent confidence level. Assuming a 10% annual volatility, this gives WCE = 9.4%. This is in line with the add-on of 7.5% in Table 31-3.

Further simulations by the Bank of England and the New York Fed have shown that these numbers also roughly correspond to a 95th percentile loss over a six-month horizon. In the case of a new 5-year interest-rate swap, for instance, the worst exposure over the life at the 80th percent level is 1.49%; the worst exposure over six months at the 95th percent level is 1.58%. This is in line with the add-on of 1.5% for this category.

Next, the NGR factor in Equation (31.5) represents the **net-to-gross ratio**, or ratio of current net market value to gross market value, which is always between 0 and 1.

The purpose of this factor is to reduce the capital requirement for contracts that fall under a legally valid netting agreement.

Take a situation where a bank has two swaps with the same counterparty currently valued at +100 and at −60. The gross replacement value is the sum of positive values, which is 100. The net value is 40, creating a NGR ratio of 0.4. Without netting agreements in place, i.e. with $NGR = 1$, the multiplier $(0.4 + 0.6 \times \text{NGR})$ is equal to one. There is no reduction in the add-on.

At the other extreme, if all contracts currently net out to zero, $\text{NGR} = 0$, and the multiplier $(0.4 + 0.6 \times \text{NGR})$ will be equal to 0.4. The purpose of this constant of 0.4 is to provide protection against *potential movements* in the NGR which, even if currently zero, could change over time.

The computation of risk-weighted assets is then obtained by applying counterparty risk weights to the credit exposure in Equation (31.5). Since most counterparties for such transactions tend to be excellent credit, the risk weights from Table 31-3 are multiplied by 50%. The **credit risk charge** for OBS items is defined as

$$\text{CRC(OBS)} = 8\% \times \left(\sum_i w_i \times 50\% \times \text{Credit Exposure}_\text{i} \right). \tag{31.7}$$

Example: The credit charge for a swap

Consider a \$100 million interest-rate swap with a domestic corporation. Assume a residual maturity of four years and a current market value of \$1 million. What is the credit risk charge?

Answer

Since there is no netting, the factor $(0.4 + 0.6 \times \text{NGR}) = 1$. From Table 31-3, we find an add-on factor of 0.5. The computed credit exposure is then $\text{CE} = \$1,000,000 + \$100,000,000 \times 0.5\% \times 1 = \$1,500,000$. This number must be multiplied by the counterparty-specific risk weight and one-half of 8 percent to derive the minimum level of capital needed to support the swap. This gives \$60,000.

Example 31-5: FRM Exam 2001—Question 45

31-5. The Basel Accord computes the credit exposure of derivatives using both replacement cost and an "add-on" to cover potential future exposure. Which of the following is the correct credit risk charge for a purchased 7 year OTC equity index option of $50m notional with a current mark to market of $15m with no netting and a counterparty weighting of 100%?

a) $1.6m
b) $1.2m
c) $150,000
d) $1m

Example 31-6: FRM Exam 2000—Question 134/Regulation

31-6. BIS capital requirement for an unfunded, short-term (under one year) credit commitment is

a) 0%
b) 4%
c) 8%
d) 100%

Example 31-7: FRM Exam 2000—Question 137/Regulation

31-7. The BIS requirement for capital charge of an unfunded commitment of original maturity of greater than one year, as compared to an equivalent funded commitment (or loan) is

a) The same
b) Half
c) A quarter
d) Zero

31.2.4 Total Risk Charge

Finally, the total risk charge is computed as the sum of the credit risk charges, both for balance-sheet and off-balance sheet items, plus the market risk charge. Define MRC as the market risk charge, which will be detailed in the next chapter.

To translate all numbers into similar risk-adjusted assets, the MRC is transformed into a risk-adjusted asset equivalent, by dividing the MRC by 8%. For instance, if MRC is computed as $1,832 million, the risk-adjusted asset number would be $22.9 billion, which is taken as equivalent to the notional of corporate bonds.

We can then simply sum the risk-adjusted assets across all risk categories to find the total risk charge (TRC) of

$$\text{TRC} = \text{CRC} + \text{MRC} = 8\% \times (\text{Total Risk} - \text{Adjusted Assets}), \qquad (31.8)$$

subject to various restrictions on the use of different tiers. The New Accord adds an operational risk charge to this.

Table 31-4 gives an example.[3] The total risk-adjusted assets for credit risk are 7,500. The market risk charge is 350 which translates into 350/8% = 4,375 in risk assets. The credit risk charge is 8% of 7,500, or 600. Of this, no more than 50% can be accounted by tier 2 capital. So, we could have 300 in tier 1 capital plus 300 in tier 2 capital covering credit risk. For market risk, we know the maximum ratio of tier 3 to tier 1 capital is 250-to-100. Hence, with a 350 market risk charge, we can have a maximum allocation of 250 for tier 3 for every 100 of tier 1.

The next step is to match these numbers with the available capital. Assume the bank has capital available of 700, 100, and 600 in tiers 1, 2, and 3, respectively. For credit risk, we only have 100 in tier 2 capital, so that the remaining 500 must be in the form of tier 1 capital. For market risk, we apply the maximum of 250 in tier 3 capital, so that the remainder of 100 comes from tier 1 capital.

TABLE 31-4 Computation of Capital Requirements

Category	Risk Assets	Capital Charge (8%)	Minimum Capital, Required	Available Capital	Minimum Capital, Actual	Eligible Capital
Credit risk	7,500	600	Tier 1: 300		Tier 1: 500	
			Tier 2: 300		Tier 2: 100	
Market risk	4,375	350	Tier 1: 100		Tier 1: 100	
			Tier 3: 250		Tier 3: 250	
Tier 1				700		700
Tier 2				100		100
Tier 3				600		250
Total	11,875	950		1,400	950	1,050
Capital ratio						8.8%

[3]This expands on the BCBS publication (January 1996) on page 50.

This leaves a buffer of excess capital. We can compute the capital ratio using all eligible capital. All of tier 1 capital is eligible, plus 100 in tier 2, plus 250 in tier 3. This sums to a total of 1,050, which translates into an "eligible" capital ratio of 1,050/11,875 = 8.8%. The bank has also 600 − 250 = 350 in unused tier 3 capital.

Example 31-8: FRM Exam 1999—Question 134/Credit Risk
31-8. A risk analyst is asked to prepare a BIS credit risk report based on accounting data. He receives a report that shows the mark-to-market value of the following instruments by client: Interest Rate Caps Bought, Interest Rate Caps Sold, Interest Rate Swaps. The analyst's system contains the following additional information:
I. The time to maturity of the instruments
II. The presence or absence of a netting agreement
III. The amount of "add-on" [for each instrument]
IV. The credit rating of the client
Which items does the analyst need in order to create the report?
a) I and IV only
b) II and III only
c) II and III and IV
d) All the above

31.3 Illustration

As an illustration, let us examine the capital adequacy requirements for Citigroup, which is the biggest global bank.

Table 31-5 summarizes on-balance-sheet and off-balance-sheet items as of December 2002. The bank has total assets of $1,097 billion, consisting of cash equivalents, securities, loans, trading assets, and other assets. The notional for each asset is assigned to one of the four risk weight categories, ranging from 0% to 100%. For example, out of the $301.9 billion in securities, $161.3 have a zero risk weight, presumably because these represent OECD government bonds. Of the remainder, $73 billion has a 20% weight, $5.1 billion has a 50% weight, and $56.9 has a 100% weight. Most of the loans carry a risk weight of 100%. Trading assets are excluded from this computation because they carry a market risk charge only.

The second panel of the table displays off-balance-sheet information. The second column displays the notional, the third the conversion factor, and the fourth the credit equivalent, which is the product of the previous two. As described in the previous sec-

TABLE 31-5 Citigroup's Risk-Weighted Assets

On-Balance-Sheet Assets ($ Billion)							
	Notional	Not		Risk Weight Category			
Item		Cov'd		0%	20%	50%	100%
Cash and due	33.7	0.0		12.6	20.4	0.0	2.0
Securities	301.9	5.6		161.3	73.0	5.1	56.9
Loans and leases	465.8	−12.8		8.7	33.5	97.6	338.9
Trading assets	155.2	155.2		0.0	0.0	0.0	0.0
All other assets	140.6	34.5		25.3	9.7	2.4	68.7
Total on-BS	1097.2	182.4		208.0	136.6	105.1	466.4
Off-Balance-Sheet Items ($ Billion)							
	Notional	Conv.	Credit	Risk Weight Category			
Item		Factor	Equiv.	0%	20%	50%	100%
Financial standby LC	32.5	1.00	32.5	10.0	3.2	0.3	19.0
Performance standby LC	7.3	0.50	3.7	0.2	0.3	0.0	3.1
Commercial LC	5.0	0.20	1.0	0.1	0.2	0.0	0.7
Securities lent	38.0	1.00	38.0	37.9	0.1	0.0	0.0
Other credit substitutes	3.0	12.50	26.5	0.0	0.0	0.0	26.5
Other off-balance sheet	1.7	–	2.4	0.0	0.1	0.6	1.7
Unused commit. gt. 1 yr	82.1	0.50	41.0	0.9	1.3	1.1	37.7
Derivative contracts	2380.9		96.2	8.5	41.3	46.4	0.0
Total off-BS	2550.4		241.2	57.5	46.5	48.4	88.8

tion, the conversion factors are 1.00 for financial LCs and securities lent, 0.50 for performance LCs and unused commitments greater than one year, and 0.20 for commercial LCs. Note that other credit substitutes represent residual interests subject to a dollar-for-dollar capital requirement, which implies a conversion factor of $(1/8\%) = 12.50$.

Finally, note the huge size of the notional derivatives position. At $2,381 billion, it is more than twice Citigroup's total assets of $1,097 billion and dwarfs its equity of $87 billion. The notional amounts, however, give no indication of the risk. The credit equivalent amount, which consists of net replacement value plus the add-on, is $96.2 billion, a much lower number.

From this information, we can compute the total risk-weighted assets and capital adequacy ratios. This is shown in Table 31-6. The first line adds up on-balance-sheet and off-balance-sheet items for each category. Multiplication by the risk weights gives the second line. The total RW assets for credit risk are $668.6 billion, which consists of $546.3 billion for on-BS items and $122.3 billion for off-BS items. To this, we add the RW assets for market risk, or $30.6 billion. Thus, most of Citigroup's regulatory risk capital covers credit risk. Market risk represents only 4% of the total.

TABLE 31-6 Citigroup's Capital Requirements

Risk-Weighted Assets ($ Billion)					
	Risk Weight Category				
Item	0%	20%	50%	100%	Total
On-BS and off-BS items	265.4	183.1	153.5	555.2	
Credit RW assets	0.0	36.6	76.8	555.2	668.6
Market RW assets					30.6
Others					−2.8
Total RW assets					696.3

Capital	Amount ($ Billion)	Ratio (Percent)
Tier 1	59.0	8.5%
Tier 2	19.3	2.8%
Total	78.3	11.2%

The total RW assets add up to $696.3 billion. Applying the 8% ratio, we find a minimum regulatory capital of $55.7 billion. In fact, the available risk capital adds up to $78.3 billion, which represents a 11.2% ratio, comfortably above the regulatory minimum. The ratio for a **well capitalized bank** would be 10%. Apparently, the regulatory constraint is not binding.

The bank could decide itself on the optimal capital ratio, based on a careful consideration of the trade-off between increasing expected returns and increasing risks. If the current capital ratio is viewed as too high, the bank could shrink its capital base through dividend payments or share repurchases. Like other major banks, Citigroup has decided to hold more capital than the minimum regulatory standard of 8%, which would correspond to a BBB rating.

31.4 The New Basel Accord

The Basel Accord has been widely viewed as successful in raising banking capital ratios. As a result of the Accord, the aggregate tier 1 ratio increased from $840 to $1,500 billion from 1990 to 1998 for the 1,000 largest banks. Indeed the banking system now seems to have enough capital to weather most storms, including the Asian crisis of 1997 and the recession of 2001-2002.

31.4.1 Issues with the 1988 Basel Accord

Over time, however, these regulations have shown their age. The system has led to **regulatory arbitrage**, which can be broadly defined as bank activities aimed at getting around these regulations. Lending patterns have been transformed, generally in the direction of taking on more credit risk to drive the economic capital up to the level of regulatory capital.

To illustrate, consider a situation where a bank can make a loan of $100 million to an investment-grade company rated AAA or to a speculative-grade company rated CCC. The bank is forced to hold regulatory capital of $8 million, so it has to borrow $92 million. Suppose the rate of return on the AAA loan is 6 percent, after expenses. The cost of borrowing is close, at 5.7 percent. The dollar return to shareholders is then $\$100,000,000 \times 6\% - \$92,000,000 \times 5.7\% = \$756,000$. Compared to a capital base of $8 million, this represents a rate of return of 9.5% only, which may be insufficient for shareholders. The bank could support this loan with a much smaller capital base. For instance, a capital base of $2 million would require borrowing $98 million and would yield a return of $\$100,000,000 \times 6\% - \$98,000,000 \times 5.7\% = \$414,000$, assuming the cost of debt remains the same. This translates into a rate of return of 20.7%, which is much more acceptable. The bank, however, is unable to lower its capital due to the binding regulatory requirement.

Suppose now the rate of return on the CCC loan is 7 percent, after expenses and expected credit losses. The dollar return to shareholders is now $1.756 million, which represents a 22.0% rate of return. In this situation, the bank has an incentive to increase the risk of its loan in order to bring the economic capital more in line with its regulatory capital. This simple example has shown that regulation may perversely induce banks to shift lending to lower-rated borrowers.

There are four major flaws in the 1988 Accord.

- *Inadequate differentiation between credit risks.* The risk weights for corporate loans are set at 100% and are the same for an AAA-rated borrower and for low-rated companies. In addition, there is no adjustment for loan maturity. This led to distortions in lending patterns. The risk weights are 20% only for short-term loans to non-OECD banks such as Thai banks and for loans to OECD banks such as in Mexico or Korea. Because of these low risk weights, banks have been lending more to relatively weak banks. In 1997, 60% of $380 billion bank lending to Asia had a maturity less than one year, which aggravated the liquidity crisis in Asia.

- *Securitization.* Because of the high regulatory cost of keeping loans on their balance sheets, banks have transformed loans into tradeable securities. This process has generally eliminated the better-quality loans from balance sheets, leaving banks with lower-quality loans on their books. This deterioration of the remaining loan book was certainly not the original purpose of the Capital Accord.
- *Nonrecognition of credit risk mitigation techniques.* The Accord does not fully recognize the benefits of credit risk mitigation techniques, which include collateral, guarantees, netting, and **credit derivatives**. Such exchange of risk is prudent but not fully rewarded by the regulatory system.
- *Nonrecognition of diversification of credit risk.* The rules do not account for portfolio risk, or attempts to diversify across regions or industries. This is probably the most basic flaw in the Accord. By summing measures of individual credit risk, the Accord totally ignores the benefit of diversification. This is important as the banking system has time and again failed due to lack of diversification.

Example 31-9: FRM Exam 1998—Question 3/Regulatory

31-9. A bank which funds itself at LIBOR−5 bp., purchases an A+ rated corporate floating coupon loan paying LIBOR+15 bp. Based on the Basel I minimum capital requirements, what is the annualized return on regulatory capital for this loan?

a) 2.5%
b) 5.0%
c) 11%
d) None of the above

31.4.2 The New Basel Accord: Credit Risk Charges

To address these issues, the Basel Committee released a new set of proposals, dubbed **Basel II**, in June 1999. The proposals were revised in January 2001 and are expected to be finalized by 2003. For the credit risk charges, banks have now a choice of three approaches.

(1) Standardized Approach

This is an extension of the 1988 Accord, but with finer classification of categories for credit risk, based on external credit ratings, provided by **external credit assessment institutions.** Table 31-7 describes the new weights, which now fall into 5 categories for banks and sovereigns, and 4 categories for corporates. For sovereigns, OECD membership is no longer given preferential status. For banks, two options are available.

The first assigns a risk weight one notch below that of the sovereign; the other uses an external credit assessment. The new Accord also removes the 50% risk weight cap on derivatives.

TABLE 31-7 New Basel Risk Weights: Standardized Approach

Claim	Credit Rating					
	AAA/ AA−	A+/ A−	BBB+/ BBB−	BB+/ B−	Below B−	Unrated
Sovereign	0%	20%	50%	100%	150%	100%
Banks-option 1	20%	50%	100%	100%	150%	100%
Banks-option 2	20%	50%	50%	100%	150%	50%
Short-term	20%	20%	20%	50%	150%	20%
Claim	AAA/ AA−	A+/ A−	BBB+/ BB−		Below BB−	Unrated
Corporates	20%	50%	100%		150%	100%

Note: Under option 1, the bank rating is based on the sovereign country in which it is incorporated. Under option 2, the bank rating is based on an external credit assessment. Short-term claims are defined as having an original maturity less then three months.

(2) Foundation Internal Rating Based Approach

Under the **internal rating based approach** (IRB), banks are allowed to use their internal estimate of creditworthiness, subject to regulatory standards. Under the foundation approach, banks estimate the **probability of default** (PD) and supervisors supply other inputs, which carry over from the standardized approach. Table 31-8 illustrates the link between PD and the capital requirement for various asset classes.[4] For instance, an unsecured senior corporate loan with a 1.00% probability of default would be assigned a CRC of 8.0% of the notional, which implies a risk weight of 100%.

(3) Advanced Internal Rating Based Approach

Under the advanced approach, banks can supply other inputs as well. These include **loss given default** (LGD) and **exposure at default** (EAD). The combination of PDs and LGDs for all applicable exposures are then mapped into regulatory risk weights. The capital charge is obtained by multiplying the risk weight by EAD by 8%. The advanced IRB approach applies only to sovereign, bank, and corporate exposures and not to retail portfolios.

It has been estimated that under the standardized approach capital requirements for the industry will be about 3% higher. Under foundation IRB, they will fall around 12%, with a larger reduction of 17% under advanced IRB. The reduction in credit risk charges under foundation IRB will be offset by a new risk charge of 12% for operational risk.

[4]For more detail, see the BCBS documents.

TABLE 31-8 IRB Capital Requirements (Modified)

Probability of Default	Corporate	Residential Mortgage	Other Retail
0.03%	1.40%	0.40%	0.40%
0.10%	2.70%	1.00%	0.90%
0.25%	4.30%	2.00%	1.80%
0.50%	5.90%	3.40%	2.80%
0.75%	7.10%	4.50%	3.60%
1.00%	8.00%	5.50%	4.20%
1.25%	8.70%	6.40%	4.70%
1.50%	9.30%	7.30%	5.10%
2.00%	10.30%	8.80%	5.70%
2.50%	11.10%	10.20%	6.20%
3.00%	11.90%	11.50%	6.60%
4.00%	13.40%	13.70%	7.10%
5.00%	14.80%	15.70%	7.40%
10.00%	21.00%	23.20%	8.50%
20.00%	30.00%	32.50%	10.60%

While there is still no acceptance of internal **portfolio credit risk models**, these changes allow better differentiation of credit risk. There is also improved recognition of credit mitigation techniques. Similarly, securitization is also explicitly dealt with. These two items will be further detailed in the next section.

31.4.3 Securitization and Credit Risk Mitigation

The New Accord also deals explicitly with **securitization**, which involves the economic or legal transfer of assets to a third party, typically called **special purpose vehicle** (SPV).

A bank can remove these assets from its balance sheet only after a true sale, which is defined using **clean break** criteria. These are satisfied if (a) the transferred assets are legally separated from the seller, (b) the holders of the SPV have the right to pledge or exchange those interests, and (c) the seller does not maintain control over the assets. Otherwise, the Accord imposes risk weights for securitization tranches that are described in Table 31-9. The risk weight for a BBB-rated tranche is 100%. For the lowest-rated tranches, the bank must hold capital equal to the notional amount, which implies a risk weight of $(1/8\%) = 1250\%$.

TABLE 31-9 New Basel Risk Weights: Securitization Tranches

	AAA/ AA−	A+/ A−	BBB+/ BBB−	BB+/ BB−	B+ and below or unrated
Tranche	20%	50%	100%	350%	1250% (deduction)

Next, **collateralized credit exposures** are those where the borrower has posted cash or securities as collateral. Recognition is only given to cash, listed corporate equities, investment-grade debt, and sovereign securities rated BB- or better.

Two treatments are possible. In the simple approach, the risk of the collateral is simply substituted for that of the counterparty. In contrast, the comprehensive approach is more accurate and will lead to lower capital charges.

Even if the exposure is exactly matched by the collateral, there is some credit risk due to the volatility of asset values during a default. This is measured by a **haircut** parameter (H) that is instrument-specific and approximates the 99% VAR over a ten-day period. For equities in an index, for instance, $H = 20\%$.

The adjusted value of collateral is then

$$C_A = \frac{C}{1 + H} \tag{31.9}$$

where C is the current market value of the collateral held. Defining E as the value of the uncollateralized exposure and r as the associated risk weight, the risk-weighted assets are given by $r \times (E - C_A)$, if positive.

Finally, **guarantees** and **credit derivatives** are a form of protection against obligor default provided by a third party, called the guarantor. Capital relief, however, is only granted if there is no uncertainty as to the quality of the guarantee. Protection must be direct, explicit, irrevocable, and unconditional. In such situation, one can apply the principle of **substitution**. In other words, if Bank A buys credit protection against a default of Company B from Bank C, it substitutes C's credit risk for B's risk.

31.4.4 The Basel Operational Risk Charge

One of the most significant, and controversial, addition to the New Accord is the operational risk charge (ORC). The Basel Committee expects that the ORC will represent on average 12% of the total capital charge.

The new rules give three alternatives methods. The simplest is called the **basic indicator approach**. This is based on an aggregate measure of business activity. fee income, operating costs, or assets. The capital charge equals a fixed percentage (*alpha factor*) of the exposure indicator (EI), defined as gross income

$$\mathrm{ORC}^{BIA} = \alpha \times \mathrm{EI} \tag{31.10}$$

Currently, alpha has been set at approximately 17-20%. The advantage of this method is that it is simple, transparent, and uses readily available data. The problem is that it does not account for the quality of controls. As a result, this approach is expected to be mainly used by non-sophisticated banks.

The second method is the **standardized approach**. This divides the bank's activities into a number of standardized business units. Each business line is then characterized by an exposure indicator, taken as gross income for simplicity. The capital charge is obtained by multiplying each exposure indicator by a fixed percentage (*beta factor*) and summing across business lines

$$ORC^{SA} = \sum_i \beta_i \times EI_i \tag{31.11}$$

As before, the beta factors are set by supervisors. This approach is still simple but better reflects varying risks across business lines. It can only be used if the bank demonstrates effective management and control of operational risk.

The third class of method is the **advanced measurement approach** (AMA). This allows banks to use their own internal models in the estimation of required capital. One example is the **internal measurement approach** (IMA). In the first step, banks classify their business units along the same lines as the standardized approach. Banks then measure, based on their own internal loss data, a probability of loss event (PE) and a loss given that event (LGE), as for credit risk. The expected loss is given as the product of EI, PE, and LGE. Based on *gamma factors*, the capital charge is obtained as the summation of expected loss times gamma across business lines

$$\mathrm{ORC}^{IMA} = \sum_i \gamma_i (\mathrm{EI}_i \times \mathrm{PE}_i \times \mathrm{LGE}_i) \tag{31.12}$$

More generally, the AMA approach involves a **value for operational risk** (VOR) measured at the 99.9% confidence level over a one-year horizon. As with market risk, internal models are allowed only for banks that satisfy qualitative criteria, such as effective risk management and control, as well as sound measurement and validation of operational risk models. This approach offers the most refined measurement of operational risk and is expected to be used by more sophisticated institutions.

Example 31-10: FRM Exam 2000—Question 135/Regulation
31-10. As of November 2000, which one of the following will generally receive 8% BIS capital charge (100% asset weight)?
a) Investment in a publicly traded stock for trading purposes
b) Investment in a U.S. government bond
c) Investment in a Venture Capital fund for speculation purposes
d) None of the above

Example 31-11: FRM Exam 2000—Question 131/Regulation
31-11. The June 1999 Basel Committee on Banking Supervision issued proposals for reform of its 1988 Capital Accord (the Basel II proposals). An implication of these proposed reforms is the possibility of
a) Using internal models or external ratings in the computation of minimum capital requirements
b) Allocating capital based on an internal VAR model
c) Including credit risk in the overall internal model framework to compute capital requirements
d) All of the above

Example 31-12: FRM Exam 1998—Question 21/Regulatory
31-12. Which of the following risks is most difficult to measure and manage?
a) Credit risk, because returns are not normally distributed
b) Market risk, because of the optionality of many positions
c) Interest-rate risk, because no one can consistently predict directional changes
d) Operational risk, because sufficient data does not exist

31.5 Answers to Chapter Examples

Example 31-1: FRM Exam 1997—Question 17/Regulatory

b) In addition to all the risks in the trading book (interest rate, equity, forex, commodity), the market capital charges also include forex and commodity risks in the bank book.

Example 31-2: FRM Exam 1999—Question 189/Regulation
d) Allowable tier 1 capital includes equity (book equity) and disclosed reserves only. Subordinated debt with maturity greater than 5 years is only for tier 2.

Example 31-3: FRM Exam 2000—Question 139/Regulation
a) Tier 1 capital includes common stock, disclosed reserves, and noncumulative preferred shares.

Example 31-4: FRM Exam 2001—Question 38
b) Under the Basel I rules, the charge is $\$100 \times 50\% \times 8\% = \$4$ million.

Example 31-5: FRM Exam 2001—Question 45
a) From Table 31-3, the add-on factor is 10%. This gives a credit exposure of $\$15 + \$50 \times 10\% = \$20$ million, and a credit risk charge of $\$20 \times 8\% = \$1.6$ million.

Example 31-6: FRM Exam 2000—Question 134/Regulation
a) Unfunded commitments are off-balance-sheet items (unlike funded commitments, which are loans). Below a year, the credit conversion factor is zero, which means zero BIS weight.

Example 31-7: FRM Exam 2000—Question 137/Regulation
b) Unfunded commitments with maturities greater than a year (and irrevocable) have a 50% conversion factor, or 4% BIS weight instead of the usual 8%.

Example 31-8: FRM Exam 1999—Question 134/Credit Risk
b) The BIS method does not take into account the credit rating of the counterparty. The add-on already incorporates the type of instrument and maturity. The analyst only needs items II and III.

Example 31-9: FRM Exam 1998—Question 3/Regulatory
a) An 8% capital charge applies to this bond. We buy \$100 worth of the bond, which is funded at the bank rate, for a net dollar return of $\$100[(L + 0.15\%) - (L - 0.05\%)] = \$0.20$. We need to keep \$8 in capital, which we assume is not invested. The rate of return is then $\$0.20/\$8 = 2.5\%$. (Also note that the capital adequacy rules are from the Basel Committee on Banking Supervision, not the BIS).

Example 31-10: FRM Exam 2000—Question 135/Regulation
c) The capital charges for the trading portfolio do not follow the 8% credit risk charges, so that (a) does not apply. A U.S. government bond held in the banking book has a zero weight, so that (b) is false. An investment in a Venture Capital fund, however, is typically not marked to market and as a result will be classified into the banking book with the usual 8% risk charge.

Example 31-11: FRM Exam 2000—Question 131/Regulation
a) The 1999 and revised 2001 proposals differentiate more finely across credit ratings, using external or internal ratings. Internal *portfolio* credit risk, or VAR, models are still not allowed across all risk categories.

Example 31-12: FRM Exam 1998—Question 21/Regulatory
d) By now there is some consensus on measuring market and credit risk. Operational risk is more difficult to measure because of the lack of data and standardized methodology.

31.6 Further Information

The following documents are available at the BIS Web site, www.bis.org, Basel, Switzerland: BIS. BCBS (1988) is the original Basel Accord, that has been subsequently amended for market risk (1996a) and (1996b). Good risk management practices are described in BCBS (1994), (1995), and (1998a). The initial Basel II proposal is described in BCBS (1999a). BCBS (1999b) deals with credit risks. BCBS (1998) and (2001b) deal with operational risks. The 2001 revision is in BCBS (2001a).

The documents listed next are *essential* readings for risk managers. All BCBS publications are at www.bis.org/publ/ .

Basel Committee on Banking Supervision. (1988). *International Convergence of Capital Measurement and Capital Standards*. www.bis.org/publ/bcbs04a.pdf

Basel Committee on Banking Supervision. (1996a). *Supervisory Framework for the Use of 'Backtesting' in Conjunction with the Internal Models Approach to Market Risk Capital Requirements*. bcbs22.pdf

Basel Committee on Banking Supervision. (1996b). *Amendment to the Basel Capital Accord to Incorporate Market Risk*. bcbs24.pdf and bcbs24a.pdf

Basel Committee on Banking Supervision. (1999a). *A New Capital Adequacy Framework*. bcbs50.pdf

Basel Committee on Banking Supervision. (2001a). *The New Basel Capital Framework*. Ten documents, or 541 pages, at bcbsca01.pdf to bcbsca10.pdf. These are, in order, *The New Basel Capital Accord: An Explanatory Note, Overview of The New Basel Capital Accord, The New Basel Capital Accord, The Standardised Approach to Credit Risk, The Internal Ratings-Based Approach, Asset Securitisation, Operational Risk, Pillar 2 (Supervisory Review Process), Principles for the Management and Supervision of Interest Rate Risk, Pillar 3 (Market Discipline).* The third document is known as the *rules*. The last seven documents provide supporting technical details.

The following documents are important.

Basle Committee on Banking Supervision. (1994). *Risk Management Guidelines for Derivatives*. bcbsc211.pdf

Basel Committee on Banking Supervision. (1995). *Public Disclosure of the Trading and Derivatives Activities of Banks and Securities Firms*. bcbsc213.pdf

Basel Committee on Banking Supervision. (1998a). *Framework for the Evaluation of Internal Control Systems*. bcbs33.pdf

Basel Committee on Banking Supervision. (1998b). *Operational Risk Management*. bcbs42.pdf

Basel Committee on Banking Supervision. (1999b). *Credit Risk Modelling: Current Practices and Applications*. bcbs49.pdf

Basel Committee on Banking Supervision. (2001b). *Working Paper on the Regulatory Treatment of Operational Risk*. bcbs_wp8.pdf

Regulatory documents from the European Union are as follows: EU (1989), known as the **Solvency Ratio Directive**, adopts the 1988 Basel Accord for credit risk; EU (1993), known as the **Capital Adequacy Directive**, adopts the standardized approach to market risk; EU (1998) adopts the internal models approach.

European Union. (1989). *Council Directive 89/647/EEC of 18 December 1989 on a solvency ratio for credit institutions*. EU: Brussels.
At europa.eu.int/eur-lex/en/lif/dat/1989/en_389L0647.html

European Union. (1993). *Council Directive 93/6/EEC of 15 March 1993 on the capital adequacy of investment firms and credit institutions*. EU: Brussels. At europa.eu.int/eur-lex/en/lif/dat/1993/en_393L0006.html

European Union. (1998). *Council Directive 98/31/EC of the European Parliament and of the Council of 22 June 1998 amending Council Directive 93/6/EEC on the capital adequacy of investment firms and credit institutions*. EU: Brussels. At europa.eu.int/eur-lex/en/lif/dat/1998/en_398L0031.html

The Federal Reserve Board papers on credit derivatives are

Board of Governors of the Federal Reserve System. (1996). *Supervisory Guidance for Credit Derivatives*. Washington, DC: Board of Governors of the Federal Reserve System. At www.federalreserve.gov/boarddocs/SRLETTERS/1996/SR9617.htm

Board of Governors of the Federal Reserve System. (1997). *Application of Market Risk Capital Requirements to Credit Derivatives*. Washington, DC: Board of Governors of the Federal Reserve System. At www.federalreserve.gov/boarddocs/SRLETTERS/1997/SR9718.htm

The SEC's risk-based rules for OTC derivatives dealers is at www.sec.gov/rules/final/34-40594.txt.

Chapter 32

The Basel Market Risk Charges

After the credit risk charges were instituted in 1988, regulators turned their attention to market risk in response to the increased proprietary trading activities of commercial banks. The Capital Accord was amended in 1996 to include a capital charge for market risk, to be implemented by January 1, 1998, at the latest. The capital charge can be computed using two methods. The first is based on a "standardized" method, similar to the credit risk system with add-ons determined by the Basel rules. This method provides a rough but conservative measure of the capital charge market risk.

The second method is called the **internal models approach** (IMA) and is based on the banks' own risk management systems, which are more precise and adaptable than the rigid set of standardized rules. This approach must be viewed as a breakthrough in financial regulation. For the first time, regulators relied on the banks' own systems to determine the capital charge. Since banks would have an incentive to understate their market risk, however, the internal models approach also includes a strong system of verification, based on backtesting.

Other regulatory requirements evolved in parallel. The European Union's Capital Adequacy Directive (CAD) introduced the standardized model in 1993 and was extended to the internal models in 1998.

This chapter discusses the implementation of capital charges for market risk. Section 32.1 summarizes the standardized method; more detail is provided in the appendix. The application of the internal model approach is described in Section 32.2. Section 32.3 then turns to stress testing. Finally, the framework for backtesting is presented in Section 32.4.

32.1 The Standardized Method

The objective of the market risk amendment was "to provide an explicit capital cushion for the price risk to which banks are exposed." This was viewed as important in further strengthening the soundness and stability of the international banking system and of financial markets. The original proposal was issued in April 1993 and was based on a prespecified **building block approach**. Essentially, this consists of attaching add-ons to all positions, which are added up across the portfolio.

The bank's market risk is first computed for portfolios exposed to interest-rate risk (IR), equity risk (EQ), foreign currency risk (FX), commodity risk (CO), and option risk (OP), using specific guidelines. The bank's total risk is then obtained from the summation of risks across the four categories. Because the construction of the risk charge follows a highly structured and standardized process, this approach is sometimes called the **standardized method**. The market risk charges for these categories are further detailed in the appendix.

The bank's total risk is obtained from the summation of risks across different types of risks, i, on each day, t:

$$MRC_t^{\mathrm{STD}} = \sum_{j=1}^{5} MRC_t^j = MRC_t^{\mathrm{IR}} + MRC_t^{\mathrm{EQ}} + MRC_t^{\mathrm{FX}} + MRC_t^{\mathrm{CO}} + MRC_t^{\mathrm{OP}} \quad (32.1)$$

The standardized model is relatively easy to implement. It is also robust to model misspecification. The building-block approach, however, has been criticized on several grounds. First, the risk classification is arbitrary. For instance, a capital charge of 8 percent is applied uniformly to equities and currencies without regard for their actual return volatilities. Different currencies have different volatilities relative to the dollar that also can change over time.

Second, the approach leads to very conservative capital requirements because risk charges are systematically added up across different sources of risk, which ignores diversification. For instance, fixed-income charges are computed for each currency separately, then added up across markets. Implicitly, this approach is a worst-case scenario that assumes that the worst loss will occur at the same time across all sources of risk. In practice, these markets are not perfectly correlated, which means that the worst loss will be less than the sum of individual worst losses. Thus the standardized model fails to recognize the benefits of diversification.

Recognition of this incentive problem has led to another, more flexible approach based on internal models.

32.2 The Internal Models Approach

In contrast to the simplistic standardized approach, the **internal models approach** (IMA) relies on internal risk management systems developed by banks themselves as the basis for the market risk charge.

This approach must be considered a watershed in financial regulation. For the first time, regulators implicitly recognized that banks had developed sophisticated risk management systems, in many cases far more sophisticated than simple standardized rules. Indeed, the complexity and speed of development of innovations in financial markets is such that rigid rules can be easily skirted with new products. Perhaps the other motivation for the IMA is that, if this approach leads to lower capital charges, bank will have an incentive to develop sound risk management systems.

Regulators, however, have not totally given up their authority. A bank can use internal models only after it has been explicitly approved by the supervisory authority. The bank must satisfy qualitative requirements, its model needs to be sufficiently detail and subject to a rigorous backtesting process.

32.2.1 Qualitative Requirements

Not any bank can use internal models, though. Regulators first must have some general reassurance that the bank's risk management system is sound. As a result, banks have to satisfy first various **qualitative standards**:

(a) *Independent risk control unit.* The bank must have a risk control unit that is independent of trading and reports to senior management. This structure minimizes potential conflicts of interest.

(b) *Backtesting.* The bank must conduct a regular backtesting program, which provides essential feedback on the accuracy of internal VAR models.

(c) *Involvement.* Senior management and the board need to be involved in the risk control process and devote sufficient resources to risk management.

(d) *Integration.* The bank's internal risk model must be integrated with day-to-day management. This is to avoid situations where a bank could compute its VAR simply for regulatory purposes and otherwise ignore it.

(e) *Use of limits.* The bank should use its risk measurement systems to set internal trading and exposure limits.

(f) *Stress testing.* The bank should conduct stress tests on a regular basis. Stress tests results should be reviewed by senior management and be reflected in policies and limits set by management and the board of directors.

(g) *Compliance.* The bank should ensure compliance with a documented set of policies.

(h) *Independent review.* An independent review of the trading units and of the risk control unit should be performed regularly, at least once a year. This includes verification with backtesting.

32.2.2 The Market Risk Charge

In addition to these requirements, the bank's risk model must contain a sufficient number of risk factors, where the definition of *sufficient* depends on the extent and complexity of trading activities.

For material exposures to interest rates, there should be at least six factors for yield curve risk plus separate factors to model spread risk. For equity risk, the model should at least consist of beta mapping on an index; a more detailed approach would have industry and even individual risk factor modeling. For active trading in commodities, the risk model should account for movements in spot rates plus convenience yields.

Banks should also capture the nonlinear price characteristics of option positions, including vega risk. Correlations *within* broad risk categories are recognized explicitly. Regulators can also recognize correlations *across* risk categories provided the model is sound.

Once these requirements are satisfied, the market risk charge is computed according to these rules:

- *Quantitative parameters.* The computation of daily VAR shall be based on a set of uniform quantitative inputs:
  a. A horizon of 10 trading days, or two calendar weeks; banks can, however, scale their daily VAR by the square root of time
  b. A 99 percent confidence interval

c. An observation period based on at least a year of historical data or, if a non-equal weighting scheme is used, an average time lag of at least six months[1]

d. At least quarterly updating, or whenever prices are subject to material changes (so that sudden increases in risk can be picked up)

- *Market risk charge.* The general market capital charge shall be set at the higher of the previous day's VAR, or the average VAR over the last 60 business days, times a "multiplicative" factor k. The exact value of this **multiplicative factor** is to be determined by local regulators, subject to an absolute floor of 3. The purpose of this factor is twofold. Without this risk factor, a bank would be expected to have losses that exceed its capital in one ten-day period out of a hundred, or about once in four years. This does not seem prudent.
  Second, the factor serves as a buffer against model misspecifications, for instance assuming a normal distribution when the distribution has "fat" tails.
- *Plus factor.* A penalty component, called **plus factor**, shall be added to the multiplicative factor, k, if verification of the VAR forecasts reveals that the bank systematically underestimates its risks. We will discuss this further in the context of backtesting, which will be developed in a further section.
  The purpose of this factor is to penalize a bank that provides an overly optimistic projection of its market risk. It provides a feedback mechanism that rewards truthful internal monitoring and should provide incentives to build sound risk management systems.

In summary, the market risk charge on any day t is

$$MRC_t^{\text{IMA}} = Max\left(k\frac{1}{60}\sum_{i=1}^{60}\text{VAR}_{t-i},\ \text{VAR}_{t-1}\right) + \text{SRC}_t, \tag{32.2}$$

where VAR_{t-i} is the bank's VAR over a 10-day horizon at the 99 percent level of confidence. Here, the factor k reflects both the multiplicative and the plus factors.

The first term consists of a multiplier k times the average VAR over the last 60 days. The second term uses solely yesterday's VAR, and will be binding if markets have experienced a sharp increase in risk. This would be unusual, as yesterday's VAR

[1]This is similar to a duration computation. For instance, with equal weights over the last 250 trading days, this average time lag is $\sum_{t=1}^{N} t(1/N) = N(N+1)/2\,(1/N) = (N+1)/2 = 125.5$ days, or six months.

would have to go up to at least three times the average over the last quarter. Alternatively, this term could become binding if the bank suddenly increases the size of its positions.

Finally, SRC represents the **specific risk charge**, which represents a buffer against idiosyncratic factors, including default and event risk, related to individual bond and equity issuers. Banks that use internal models can incorporate specific risk in their VAR, as long as they (i) satisfy additional criteria and (ii) can demonstrate that they can deal with event and default risk.[2]

32.2.3 Combination of Approaches

The banks' market risk capital requirement will be either (a) the risk charge obtained by the standardized methodology, obtained from an arithmetic summation across the five risk categories, or (b) the risk charge obtained by the internal models approach, or (c) a mixture of (a) and (b) summed arithmetically.

Example 32-1: FRM Exam 2001—Question 40

32-1. What is the Internal Models Approach?

a) A method of calculating regulatory capital using a firms own internal market risk model and data
b) Using standardized models from the regulatory to calculate capital
c) Making forecasts on credit ratings using inside information
d) Using the Fed's own proprietary risk model to calculate capital requirements

Example 32-2: FRM Exam 2001—Question 42

32-2. Which of the following best describes the quantitative parameters of the Internal Models Approach?

a) 10-day trading horizon, 99% confidence interval, minimum 1 years of data, minimum quarterly updates
b) 1-day trading horizon, 95% confidence interval, 5 years of data, updated weekly
c) 1-day trading horizon, 99% confidence interval, minimum 1 years of data, updated monthly
d) 10-day trading horizon, 97.5% confidence interval, minimum 5 years of data, updated daily

[2] The difficulty with event and default risk is that it is typically not reflected in historical data. When a bank cannot satisfy (ii), a prudential surcharge is applied to the measure of specific risk. (This is detailed in the September 1997 modification of the Market Risk Amendment, available at: www.bis.org/publ/bcbs24a.pdf .)

Example 32-3: FRM Exam 1999—Question 184/Regulation

32-3. You are given that the RiskMetrics VAR for a portfolio is $1,000,000. What is the approximate Basel Committee VAR?

a) $4,450,000
b) $225,000
c) $1,000,000
d) $1,412,121

Example 32-4: FRM Exam 1999—Question 190/Regulation

32-4. The Amendment to the Capital Accord requires that internal models

a) Utilize at least six months of historical data
b) Utilize at least one year of equally weighted historical data
c) Utilize enough historical data so that the weighted average age of the data is at least six months
d) Utilize two years of historical data, unequally weighted

Example 32-5: FRM Exam 1999—Question 196/Regulation

32-5. Under the Amendment to the Capital Accord to Incorporate Market Risks, value at risk

a) Must be calculated using a 99th-percentile one-tailed confidence interval and a 10-day holding period
b) Must be calculated using a 99th-percentile one-tailed confidence interval, but may use a shorter holding period and a square root of time scaling
c) May use any percentile (e.g., 95th as used in RiskMetrics) scale to the 99th percentile using normal distribution assumptions, may use a shorter or longer holding period than 10 days, and scale using the square root of time
d) May use any percentile or holding period as long as backtesting results are satisfactory

Example 32-6: FRM Exam 1998—Question 4/Regulatory

32-6. A trading book has interest rate VAR of 200 million, equity VAR of 15 million, and F/X VAR of 50 million. The VAR has been computed based on a 99% confidence level and a 10-day holding period. Assuming normal distributions and no correlation among the asset classes, determine the required regulatory capital based on the current Basel minimum capital requirements for the market risk in this book.

a) 150 million
b) 207 million
c) 620 million
d) 795 million

Example 32-7: FRM Exam 1999—Question 197/Regulation

32-7. The capital requirement specified in the Amendment to the Capital Accord to Incorporate Market Risks is

a) The previous day's VAR number multiplied by a multiplication factor

b) The greatest of (i) the previous day's VAR number multiplied by a multiplication factor, and (ii) the average of the daily VAR over the last 60 business days multiplied by a multiplication factor

c) The greatest of (i) the previous day's VAR risk number, and (ii) the average of the daily VAR over the last 60 business days multiplied by a multiplication factor

d) The greatest of (i) the previous day's VAR number multiplied by a factor, and (ii) the maximum of the daily VAR over the last 60 business days

Example 32-8: FRM Exam 1999—Question 194/Regulation

32-8. According to the current version of the Amendment to the Capital Accord to Incorporate Market Risks, a specific risk method must meet certain criteria if a bank is to be allowed to use it for calculating capital requirements. Which of the following statements are *true*?

I. If the method does not meet the criteria, the capital figure produced for specific risk is subject to a lower limit of 50% of the capital figure under the standardized methodology.

II. If the method does not meet the criteria, the bank must use the figure produced by the standardized methodology instead.

III. If the method does meet the criteria, but the bank has no methodologies in place that adequately capture event and default risk for its traded debt and equity positions, the specific risk capital charge is subject to a prudential surcharge.

IV. The specific risk charge is not affected by any methodologies the bank may have for measuring default or event risk, as these risks are currently covered by credit risk capital charges.

a) I and IV

b) I and III

c) II and IV

d) II and III

Example 32-9: FRM Exam 1998—Question 19/Regulatory
32-9. Which one of the following statements is *false* regarding the calculation of the specific risk charge for the market risk capital rule?
a) If the bank can demonstrate that its specific risk modeling captures all aspects of specific risk, a surcharge will not be required.
b) If a bank's model captures the idiosyncratic variation in its debt and equity portfolios, but does not measure default and event risk, a model calculated surcharge should be added to the capital charge.
c) Specific risk includes default and event risk but not idiosyncratic variation.
d) If a bank's model does not measure specific risk, the surcharge for specific risk should be 100% of the standardized specific risk charge.

Example 32-10: FRM Exam 1998—Question 18/Regulatory
32-10. What would be the market risk capital requirement for a bank with a one day VAR of $100 and a specific risk surcharge of $30, based on the current BIS minimum capital requirements?
a) $300
b) $316
c) $949
d) $979

32.3 Stress-Testing

Stress-testing is one of the qualitative requirements for a bank to use internal models. The purpose of stress testing is to identify events that could greatly impact the bank and are presumably not captured in VAR measures. A major goal of stress-testing is to "evaluate the capacity of the bank's capital to absorb large potential losses."

Stress-testing can be described as a process to identify and manage situations that could cause extraordinary losses. This can be done with a set of tools, including (i) scenario analysis, (ii) stressing models, volatilities- and correlations, and (iii) policy responses.

Scenario analysis consists of evaluating the portfolio under various states of the world. Stress-testing also involves evaluating the effect of changes in valuation models, as well as in inputs such as volatilities and correlations. Policy responses consist of identifying steps the bank can take to reduce its risk and conserve capital.

Stress tests fall into three categories:

- *Scenarios requiring no simulation.* These consist of analyzing large past losses over a recent reporting period to gain a better understanding of the vulnerabilities of the bank. While providing useful information, this approach is backward-looking and does not account for changes in portfolio composition.
- *Scenarios requiring a simulation.* These consists of running simulations of the current portfolio subject to large historical shocks, for example, the stock market crash of 1987, the ERM crisis of September 1992, the bond market rout of 1994, and so on.
- *Bank-specific scenarios.* These scenarios would be driven by the current position of the bank, instead of historical experience. For instance, a strategy of going long the off-the-run bond while shorting the equivalent on-the-run bond (as LTCM did) may appear safe based on recent historical patterns. Its risk, however, critically depends on correlations remaining high. In this particular case, the institution should evaluate the effect of a correlation breakdown.

The assessment of stress-testing is essential to evaluate the risk profile of institutions. Results should be reported routinely to senior management and periodically to the board of directors. When stress-test results reveal a particular vulnerability, corrective action should be taken, by reducing or hedging the position.

In practice, stress-testing is much more subjective than VAR measures. The Basel guidelines are suitably vague. First, there is no systematic method to identify scenarios of interest. Second, the process assigns no probability to the extraordinary loss that has been identified. As a result, it is often difficult to know how to follow up on stress test results. In particular, it would be impractical to guard against every single potential disaster. Perhaps the most useful aspect of stress-testing is that it can help to identify undetected weaknesses in the bank's portfolio.

Example 32-11: FRM Exam 1999—Question 195/Regulation

32-11. According to the current version of the Amendment to the Capital Accord to Incorporate Market Risks in relation to stress testing, which of of the following statements is *true*?

I. Stress-testing results should be communicated to traders.
II. Stress-testing results should be communicated to senior management.
III. Stress-testing results should be communicated to the bank's board of directors.
IV. Limits should be set on the loss indicated by stress tests.
V. The levels of limits (e.g., VAR limits) should reflect the results of stress testing.

a) I, II, III, and IV
b) I, II, and V
c) II, III, and V
d) II, III, and IV

Example 32-12: FRM Exam 1998—Question 20/Regulatory

32-12. Value at risk (VAR) measures should be supplemented by portfolio stress-testing because

a) VAR does not indicate how large the losses will be beyond the specified confidence level.
b) Stress-testing provides a precise maximum loss level.
c) VAR measures are correct only 95% of the time.
d) Stress-testing scenarios incorporate reasonably probable events.

Example 32-13: FRM Exam 1997—Question 15/Regulatory

32-13. Which one of the following is *not* an explicitly permitted VAR modeling technique of the Amendment to the Capital Accord to Incorporate Market Risk?

a) Historical simulation
b) Variance/covariance matrices
c) Monte Carlo simulation
d) Scenario analysis

32.4 Backtesting

Internal models were allowed by the Basel Committee in large part because they were amenable to verification. **Verification** is the general process of checking whether the model is adequate. This can be made with a set of tools, including backtesting, stress-testing, and independent review and oversight. This section focuses on backtesting techniques for verifying the accuracy of VAR models. **Backtesting** is a statistical testing framework that consists of checking whether actual trading losses are in line with VAR forecasts. Each exceedence is called an **exception**.

32.4.1 Measuring Exceptions

But first, we have to define the **trading outcome**. One definition is the actual profit or loss over the next day. This return, however, does not exactly correspond to the previous day's VAR. All VAR measures assume a *frozen* portfolio from the close of a trading day to the next, and ignore fee income. In practice, trading portfolios do change. Intraday trading will generally increase risk. Fee income is more stable and decreases risk. Although these effects may offset each other, the actual portfolio may have more or less volatility than implied by VAR.

This is why it is recommended to construct **hypothetical portfolios**, which are constructed so as to match the VAR measure exactly. Their returns are obtained from fixed positions applied to the actual returns on all securities, measured from close to close.

The Basel framework recommends using both hypothetical and actual trading outcomes in backtests. The two approaches are likely to provide complementary information on the quality of the risk management system.

32.4.2 Statistical Decision Rules

The Basel backtesting framework consists of recording *daily* exceptions of the 99 percent VAR over the last year. Note that even though capital requirements are based on a 10-day period, backtesting uses a daily interval, which entails more observations. On average, we would expect 1% of 250, or 2.5 instances of exceptions over the last year. Too many exceptions indicate that either the model is understating VAR or the bank is unlucky. How do we decide which explanation is most likely?

Such statistical testing framework must account for two types of error:

- **Type 1 errors**, which describe the probability of rejecting a correct model, due to bad luck
- **Type 2 errors**, which describe the probability of not rejecting a model that is false

Ideally, one would want to structure a test that has low type 1 and type 2 error rates. In practice, one has to trade off one type of error against another. Most statistical tests fix the type 1 error rate, say at 5%, and structure the test so as to minimize the type 2 error rate, or to maximize its power. The **power of a test** is also one minus the type 2 error rate.

Define x as the number of exceptions, n as the total number of observations, and p as the confidence level. The random variable x then has a binomial distribution. Armed with this information, we can find the cutoff point for a type 1 error rate.

32.4.3 The Penalty Zones

The Basel Committee has decided that up to 4 exceptions is acceptable, which defines a "green" light zone. If the number of exceptions is 5 or more, the bank falls into a "yellow" or "red" zone and incurs a progressive penalty where the multiplicative factor, k, is increased from 3 to 4. The "plus factor" is described in Table 32-1.

An incursion into the red zone generates an *automatic*, nondiscretionary penalty. This is because it would be extremely unlikely to observe more than 10 exceptions if the model was indeed correct.

TABLE 32-1 The Basel Penalty Zones

Zone	Number of Exceptions	Potential Increase in k
Green	0 to 4	0.00
Yellow	5	0.40
	6	0.50
	7	0.65
	8	0.75
	9	0.85
Red	≥ 10	1.00

If the number of exceptions falls within the yellow zone, the supervisor has discretion to apply a penalty, depending on the causes for the exceptions. The Basel Committee uses these categories:

- *Basic integrity of the model:* The deviation occurred because the positions were incorrectly reported or because of an error in the program code. This is a very serious flaw. In this case, a penalty "should" apply and corrective action should be taken.

- *Deficient model accuracy:* The deviation occurred because the model does not measure risk with enough precision (e.g., does not have enough risk factors). This is a serious flaw too. A penalty "should" apply and the model should be reviewed.
- *Intraday trading:* Positions changed during the day. Here, a penalty "should be considered." If the exception disappears with the hypothetical return, the problem is not in the bank's VAR model.
- *Bad luck:* Markets were particularly volatile or correlations changed. These exceptions "should be expected to occur at least some of the time" and may not suggest a deficiency of the model but rather bad luck.

To understand the dilemma facing supervisors, Table 32-2 presents type 1 and type 2 error rates for various numbers of exceptions, with a correct model (i.e., with 99% coverage) and incorrect models (e.g., with 97% or 95% coverage). With 5 exceptions or more, the cumulative probability, or type 1 error rate, is 10.8%. This represents the probability of penalizing a bank that has a correct model due to bad luck. With 10 exceptions, however, this type 1 error rate falls to zero.

We can also examine the type 2 error rate with a VAR model that only provides 97% coverage. Assuming a normal distribution, this implies that the VAR should be

TABLE 32-2: Basel Rules for Backtesting
Probabilities of Obtaining Exceptions (T = 250)

Zone	Number of Exc.	Model is correct Coverage = 99%		Model is incorrect Coverage = 97%		Coverage = 95%	
		Prob.	Cumul. (Type 1 error)	Prob.	Cumul. (Type 2 error)	Prob.	Cumul. (Type 2 error)
	N	$P(X = N)$	$P(X \geq N)$	$P(X = N)$	$P(X < N)$	$P(X = N)$	$P(X < N)$
Green	0	8.1	100.0	0.0	0.0	0.0	0.0
	1	20.5	91.9	0.4	0.0	0.0	0.0
	2	25.7	71.4	1.5	0.4	0.0	0.0
	3	21.5	45.7	3.8	1.9	0.1	0.0
Green	4	13.4	24.2	7.2	5.7	0.3	0.1
Yellow	5	6.7	10.8	10.9	12.8	0.9	0.5
	6	2.7	4.1	13.8	23.7	1.8	1.3
	7	1.0	1.4	14.9	37.5	3.4	3.1
	8	0.3	0.4	14.0	52.4	5.4	6.5
Yellow	9	0.1	0.1	11.6	66.3	7.6	11.9
Red	10	0.0	0.0	8.6	77.9	9.6	19.5
	11	0.0	0.0	5.8	86.6	11.1	29.1

Source: Basel Committee on Banking Supervision, January 1996, *Supervisory Framework for the Use of 'Backtesting' in Conjunction with the Internal Models Approach to Market Risk Capital Requirements.*

higher by a ratio of 2.33 to 1.88, or 1.24. Thus the true risk charge should be higher by 24%.

The table shows that the type 2 error rate for less than 5 exceptions is 12.8%. This represents the probability of not catching a bank that willfully understates its risk. This is not very high. However, this probability falls as the true model deviates more from the target 99 percent coverage. With a 95% coverage, the type 2 error rate is only 0.5%. Thus it is very unlikely that the supervisor would miss a bank that substantially understates its VAR.

Example 32-14: FRM Exam 1999—Question 192/Regulation

32-14. The Amendment to the Capital Accord recommends that backtesting compares VAR to
a) Actual P&L
b) Hypothetical P&L, i.e. P&L based on end-of-day positions
c) Both actual and hypothetical P&L
d) Does not specify a choice

Example 32-15: FRM Exam 1999—Question 193/Regulation

32-15. The Amendment to the Capital Accord defines the "yellow zone" as the following range of exceptions out of 250 observations
a) 3 to 7
b) 5 to 9
c) 6 to 9
d) 6 to 10

Example 32-16: FRM Exam 1999—Question 191/Regulation

32-16. For purposes of backtesting, a VAR internal model, the Amendment to the Capital Accord requires
a) Comparing one year of daily P&L to a 99% one-tail confidence one-day VAR with an exception produced whenever P&L < −VAR
b) Comparing one year of daily P&L to a 98% two-tail confidence one-day VAR with an exception produced whenever P&L is outside the interval (−VAR, +VAR)
c) Comparing one year of rolling ten-day P&L to a 99% one-tail confidence ten-day VAR with an exception produced whenever P&L < −VAR
d) Comparing one year of rolling ten-day P&L to a 99% one-tail confidence ten-day VAR with an exception produced whenever P&L < −3VAR

Example 32-17: FRM Exam 1998—Question 1/Regulatory

32-17. According to the Basel backtesting framework guidelines, penalties start to apply if there are five or more exceptions during the previous year The Type 1 error rate of this test is 11%. If the true coverage is 97% of exceptions instead of the required 99%, the power of the test is 87%. This implies that there is a

a) 89% probability regulators will reject the correct model
b) 11% probability regulators will reject the incorrect model
c) 87% probability regulators will not reject the correct model
d) 13% probability regulators will not reject the incorrect model

32.5 Answers to Chapter Examples

Example 32-1: FRM Exam 2001—Question 40

a) The IMA is based on the banks's internal VAR system for market risk. It does not use a standardized approach, nor the Fed's model.

Example 32-2: FRM Exam 2001—Question 42

a) The IMA is based on a 10-day horizon, 99% confidence level, one year of data, with at least quarterly updates.

Example 32-3: FRM Exam 1999—Question 184/Regulation

a) Assuming normally and independently distributed returns, the RM VAR needs to be adjusted from 95% to 99% confidence and from 1 day to 10 days. This gives $\$1,000,000 \times (2.326/1.645) \times \sqrt{10} = \$4.5$ million.

Example 32-4: FRM Exam 1999—Question 190/Regulation

c) Answer (b) is correct if the bank uses fixed weights only. Otherwise, the average time lag of the observations cannot be less than 6 months.

Example 32-5: FRM Exam 1999—Question 196/Regulation

b) Under the IMA, VAR must be computed at the 99 percent confidence level, either over a 10-day period or over a 1-day period with appropriate time scaling.

Example 32-6: FRM Exam 1998—Question 4/Regulatory

c) This is obtained as $3 \times \sqrt{200^2 + 15^2 + 50^2} = 3 \times 207 = 620$. If this was a banking book only, the charge would apply to the currency component only, or \$150 million.

Example 32-7: FRM Exam 1999—Question 197/Regulation

c) See Equation (32.2).

Example 32-8: FRM Exam 1999—Question 194/Regulation

d) Banks can use their internal models if they satisfy a list of criteria; otherwise, they have to use the standardized approach. Even so, if they do not account for default and event risk, a prudential surcharge applies.

Example 32-9: FRM Exam 1998—Question 19/Regulatory

c) Specific risk includes (i) idiosyncratic risk plus (ii) default/event risk.

Example 32-10: FRM Exam 1998—Question 18/Regulatory

d) The total MRC is $3 \times \$100 \times \sqrt{10} + \$30 = \$949 + \$30 = \$979$.

Example 32-11: FRM Exam 1999—Question 195/Regulation

c) Stress-test results should be reported to senior management and the board, who have control over traders. So, (II) and (III) are correct. (V) is also correct, because it describes a situation where the stress-test exercise leads to a reduction in the position. (IV) is wrong. The loss indicated by stress tests is too large to establish stop-loss limits; it would then be too late to save the bank.

Example 32-12: FRM Exam 1998—Question 20/Regulatory

a) VAR only gives an indication of the worst loss under normal conditions (e.g., 95% confidence). It does not address the behavior in the tails. Stress-test results are certainly not precise.

Example 32-13: FRM Exam 1997—Question 15/Regulatory

d) Scenario analysis is not a probabilistic description of potential losses, unlike the covariance matrix approach or historical or Monte Carlo simulations.

Example 32-14: FRM Exam 1999—Question 192/Regulation

c) Both measures are informative.

Example 32-15: FRM Exam 1999—Question 193/Regulation

b) See Table 32-1.

Example 32-16: FRM Exam 1999—Question 191/Regulation

a) Backtesting is based on daily data at the one-tail 99 percent level.

Example 32-17: FRM Exam 1998—Question 1/Regulatory

d) The power is also one minus the type 2 error rate, which implies a 13% probability of not rejecting an incorrect model.

Appendix: Details of The Standardized Model

A.1 Interest Rate Risk

The purpose of the interest-rate rules is to provide a robust measure of interest-rate risk, taking into account the portfolio duration as well as basis risk across maturities. In addition to this measure of **general market risk**, we need to account for **specific risk**, which is issuer specific, or **idiosyncratic**.

For general market risk, the rules first define a set of maturity bands, within which net positions are identified across all on- and off-balance-sheet items.[3] These bands are shown in Table 32-3 for instruments with a coupon greater than 3 percent.

A risk weight is then assigned to each of the 13 bands, varying from zero for positions under one month to 6 percent for positions over 20 years. The sum of all

TABLE 32-3 General Market Charge for Interest Rate Risk

	Zone 1 (months)				Zone 2 (years)			Zone 3 (years)						Total
Maturity	0–1	1–3	3–6	6–12	1–2	2–3	3–4	4–5	5–7	7–10	10–15	15–20	> 20	
Weight (%)	0.00	0.20	0.40	0.70	1.25	1.75	2.25	2.75	3.25	3.75	4.50	5.25	6.00	
Vertical dis.:	10%													
Horiz. dis.: within zone	40%				30%			30%						
across 1, 2	40%							xxxx	xxxx	xxxx	xxxx	xxxx	xxxx	
across 2, 3	xxxx	xxxx	xxxx	xxxx	40%									
across 1, 3	100%													

Example

	0–1	1–3	3–6	6–12	1–2	2–3	3–4	4–5	5–7	7–10	10–15	15–20	> 20	Total
Position		75	−50	150			50			−150 +13.33				
Total		75	−50	150			50			−136.67				88.33
Position × wgt	0	0.150	−0.200	1.050	0	0	1.125	0	0	−5.625 0.500	0	0	0	
Total	1.000				1.125			−5.125						3.000
Vertical dis.										0.5× 10%				0.050
Horiz. dis.: within zone	0.2 × 40%													0.080
across 1, 2	None							xxxx	xxxx	xxxx	xxxx	xxxx	xxxx	
across 2, 3	xxxx	xxxx	xxxx	xxxx	1.125 × 40%									0.450
across 1, 3	1.000 × 100%													1.000
Total charge:														4.580

Notes: The vertical disallowance comes from the minimum of the absolute values of −5.625 and 0.50, times the 10% factor. The horizontal disallowance within zone 1 comes from the minimum of the absolute values of 0.15 + 1.05 and −0.20, times the 40% factor. The 2–3 zones horizontal disallowance comes from the minimum of the absolute values of 1.125 and −5.125, times the 40% factor. The 1–3 zones horizontal disallowance comes from the minimum of the absolute values of 1.000 and −5.125, times the 100% factor.

[3] An alternative method uses duration bands.

weighted net positions then yields an overall interest rate risk indicator, or market risk charge, based on duration only. The netting of positions within a band and aggregation across bands assumes perfect correlations across debt instruments.

In addition, the rules have vertical and horizontal **disallowance** that increase the risk charge. Within each band, these disallowances are given by the product of a weight applied to the minimum of the absolute value of the sum of long and short positions.

To understand the base risk weight, consider for instance the 7–10 year band, which carries a weight of 3.75 percent. The Basel document reports that this assumes a change in yield of 0.65 percent, which corresponds to a modified duration of $3.75\%/0.65\% = 5.8$ years. Indeed this is consistent with the duration of an 8 percent coupon, 8-year bond.

What does this yield change represent? We know that the standard deviation of annual yield changes is about 1 percent. We can set $\sigma\sqrt{T} = 0.65\%$ and solve for T. Hence, this is a one-standard-deviation movement over a horizon of $T = (0.65/1.00)^2 \times 12 = 5$ months. Alternatively, we could ask how many standard deviations this number represents over a 2-week interval. Since the volatility over this period is $\sigma\sqrt{T} = 1\%\sqrt{2/52} = 0.20\%$, the change in yield represents a movement of $0.65\%/0.20\% = 3.3$ times the 2-week volatility.

Table 32-3 gives a worked-out example for a sample portfolio.[4] The portfolio consists of

- A bond with residual maturity of 8 years and market value of $13.33 million.
- A bond with residual maturity of 2 months and market value of $75 million.
- A pay-fixed swap with notional of $150 million, maturity of 8 years and a reset in 9 months.
- A long bond futures position with maturity of 3.5 years for the underlying, delivery date for the futures in six months, and notional value of $50 million.

We first slot the positions in each band. For the swap, this involves a short position of $150 million on the 7–10 year band accompanied by a long position of $150 million on the 6–12 month band. The latter corresponds to the reset of the floating coupon. For the futures, this involves a long position of $50 million on the 3–4 year band

[4] This expands on the information on page 52 of BCBS publication (January 1996).

(for the underlying) accompanied by a short position on the 3–6 month band (for the futures delivery).

Each position is then multiplied by the risk weight and summed across all assets. This gives a "base" market risk charge of \$3.0 million. This would be appropriate with a one-factor term-structure model. In fact, we have more than one source of risk, which explains the introduction of "disallowances" for the netting, that is, when two positions have different signs.

First, we have a vertical disallowance for the 7–10 year band due to the two positions. This is computed as

$10\% \times \min(|-5.625|, |0.500|) = \$0.05$ million. Second, we factor in a horizontal disallowance within zone 1, which is $40\% \times \min(|-0.200|, |1.200|) = \$0.08$ million. Third, we factor in a horizontal disallowance across zones 2 and 3, which is $40\% \times \min(|1.125|, |-5.125|) = \$0.45$ million. Finally, we factor in a horizontal disallowance across zones 1 and 3, which is $100\% \times \min(|1.000|, |-5.125|) = \$1.00$ million. Note that there is no disallowance between zones 1 and 2 since the two net positions have the same sign. Adding up the base risk charge to the disallowance, we have a general market risk charge of \$4.58 million.

To this we must add a charge for specific risk. Table 32-4 details the categories and associated charges. Among those, the **qualifying category** refers to securities issued by (i) public-sector entities, (ii) multilateral development banks, and (iii) other issuers rated investment grade. For the "other" category, we note that the specific risk charge is the same as the 8% charge for credit risk in the banking book.

TABLE 32-4 Specific Market Charge for Interest Rate Risk

Category	Charge
Government	0.00%
Qualifying:	
residual maturity less than 6 months	0.25%
residual maturity less of 6 to 24 months	1.00%
residual maturity above 24 months	1.60%
Other	8.00%

The market risk charge (MRC) for interest rate risk at each time t is the sum of the general and specific risk charges,

$$MRC_t^{IR} = \mathrm{GMRC}_t^{IR} + \mathrm{SMRC}_t^{IR} \tag{32.3}$$

These are summed across different national markets, without diversification benefits.

A.2 Equity Risk

As before, the equity risk charge is divided into a general and specific risk charge. Specific risk is defined as the portfolio's **gross equity positions**, that is, the sum of the absolute value of all the long and of the short positions. General risk is defined as the portfolio's **net equity positions**, that is, the difference between the sum of all the long and of the short positions. In each case, long and short positions must be calculated for each national market separately. Table 32-5 lays out the categories and associated charges.

TABLE 32-5 Market Charge for Equity Risk

Category	Charge
General risk	8.00%
Specific risk	
normal	8.00%
portfolio is liquid and diversified	4.00%
equity index derivatives	2.00%

The market risk charge for equity risk is then obtained as

$$MRC_t^{\mathrm{EQ}} = \mathrm{GMRC}_t^{\mathrm{EQ}} + \mathrm{SMRC}_t^{\mathrm{EQ}} \tag{32.4}$$

A.3 Currency Risk

Currency positions consist of outright spot and forward positions, as well as the net delta of option positions. Gold is included in the currency risk category as it display similar volatility.

Banks have the choice to use a simplified method or internal models, which better account for correlations. The first step is to value all positions in dollars or the reference currency. Define the value for currency i as V_i. Under the simplified model, we first classify all positions into long or short. The exposure is then taken as the maximum of the absolute value of the total long or short positions. To this is added the absolute value of the gold position.

The market risk charge for currency risk is then obtained by applying an 8% weight to the exposure,

$$\mathrm{MRC}_t^{\mathrm{FX}} = 8\% \times \left[\mathrm{Max}\left(\sum_i^{\mathrm{LONG}} V_i, \sum_j^{\mathrm{SHORT}} V_j\right) + |V_{\mathrm{GOLD}}|\right] \tag{32.5}$$

A.4 Commodity Risk

Commodities typically have greater volatility than currencies or gold. In addition, forward prices do not move in parallel at different maturities, reflecting greater basis risk than for currencies. Banks can use one of three approaches. The first is the internal models approach. The second is a **maturity ladder** approach, which is similar to that of interest rate risk. The third is a simplified approach where the risk charge is obtained by applying a 15% weight to the net exposure and 3% weight to the gross exposure

$$MRC_t^{CO} = 15\% \times |V^{LONG} - V^{SHORT}| + 3\% \times (|V^{LONG}| + |V^{SHORT}|) \qquad (32.6)$$

The maturity approach is described in Table 32-6. The basic charge for the net position is 15 percent of notional. Positions can be offset within each band, subject to a spread

TABLE 32-6 Market Charge for Commodity Risk

Maturity	0–1m	1–3m	3–6m	6–12m	1–2y	2–3y	> 3y	Total
Risk weights (%)								
Spread	1.50	1.50	1.50	1.50	1.50	1.50	1.50	
Carry forward	0.60	0.60	0.60	0.60	0.60	0.60	0.60	
Net position								15.0
Example								
Position			+800		+600		−600	
			−1000					
Total			−200		+600		−600	−200
Spread, 3–6m			800					
Carry forward			−200		200			
Spread, 1–2y					200			
Carry forward					400		−400	
Spread, > 3y							400	
Net position							−200	
Spread × wgt			2 × 800 × 1.5% = 24		2 × 200 × 1.5% = 6		2 × 400 × 1.5% = 12	42.0
Carry fwd × wgt			2 × 200 × 0.6% = 2.4		2 × 400 × 0.6% = 4.8			7.2
Net × wgt							200 × 15% = 30	30.0
Total charge:								79.2

weight of 1.5%. Positions also can be offset across bands, subject to a carry-forward weight of 0.6%.

Normally, the market risk charge applies separately to each commodity. If we have a total of N commodities, the market risk charge is then

$$MRC_t^{CO} = \sum_{i=1}^{N} MRC_{i,t}^{CO} \tag{32.7}$$

A.5 Option Risk

The market risk charge can be computed using one of three approaches: a simplified, an intermediate, or an internal models approach, in order of increasing sophistication. Banks that have significant positions in options are expected to use more sophisticated approaches. For instance, banks that simply purchase options can use the first method (since there is less downside risk).

1. The Simplified Approach

The risk charge is explained in Table 32-7. For outright long positions in options, the charge is the minimum of the MRC for the underlying asset or the value of the option. For typical long option positions, the worst loss is indeed the premium.

TABLE 32-7 Market Charge for Option Risk: Simplified Method

Position	Charge
Protective put (long spot + long put) or Covered call (short spot + long call)	GMRC + SMRC for underlying minus in-the-money amount of the option
Long call or long put	Minimum of: (1) MRC for underlying asset (2) market value of option

2. The Intermediate Approach

This approach accounts for optionality and can be implemented with either of two methods. The first, **delta-plus method**, corresponds to an analytical decomposition of option risk, as with the "Greeks." The net delta of option position is computed first. This is factored into the standard risk charge for the relevant category (e.g., currency for currency options), as explained in the previous sections.

Second, an additional charge is computed for gamma and vega risk. For each underlying asset, the charges are defined as in Table 32-8. The total gamma charge is taken as the sum of the absolute values of all negative gamma charges (since only negative gamma adds to the risk). To this is added the total vega charge, which is the sum of absolute values of all vega charges.

TABLE 32-8 Gamma/Vega Charge for Option Risk: Delta-Plus Method

Risk Type	Charge	Underlying	Movement
Gamma	$\text{GRC} = 1/2 \times \Gamma \times (\Delta P)^2$		$\Delta P/P$
		Fixed-income	Weights in Table 32-1
		Equity	8%
		Currency	8%
		Commodity	15%
Vega	$\text{VRC} = \Lambda \times (\Delta\sigma)$		$\Delta\sigma/\sigma$
		All	±25%

Example: The delta-plus approach for a short commodity call option

Consider a European call option with parameters $K = 490$, $T = 12$ months, $F = 500$, $r = 8\%$, $\sigma = 20\%$, and current value $c = 65.48$. The "Greeks" are $\Delta = -0.721$, $\Gamma = -0.0034$, and $\Lambda = -1.68$ (with σ expressed in percent). What is the capital charge for this position?[5]

Answer

We first compute the delta equivalent, which is $F \times \Delta = 500 \times -0.721 = -360.5$. This becomes one of the inputs into the market risk charge. If there is no other position, a risk weight of 15% applies, leading to a linear market risk charge of $MRC^L = |-360.5| \times 15\% = 54.075$.

Next, the gamma risk charge is $\text{GRC} = 1/2 \times |-0.0034| \times (500 \times 15\%)^2 = 9.5625$. Finally, the vega risk charge is obtained from an increase in volatility (since we are short the option) of 25 percent of 20%, which is 5%. This gives $\text{VRC} = |-1.68| \times 5 = 8.40$. The nonlinear market risk charge for this position is then $9.56 + 8.40 = 17.96$.

To summarize, the nonlinear portion of the market risk charge is obtained across underlying assets as:

$$MRC_t^{\text{OP(NL)}} = \sum_i^N |\text{Min}[\text{GRC}_{i,t}, 0]| + \sum_i^N |\text{VRC}_{i,t}|, \tag{32.8}$$

with the linear MRC being subsumed in the other categories.

[5] This expands on the example on page 54 of the BCBS publication (January 1996).

The second method is a **scenario approach**. For each portfolio of options on the same underlying, the bank should construct a grid of movements in risk factors, using the ranges in Table 32-8. The capital charge for each underlying asset is obtained from the worst loss in the grid. The total market risk charge is then aggregated across underlying assets:

$$MRC_t^{\mathrm{OP}} = \sum_i^N MRC_{i,t} \tag{32.9}$$

Index